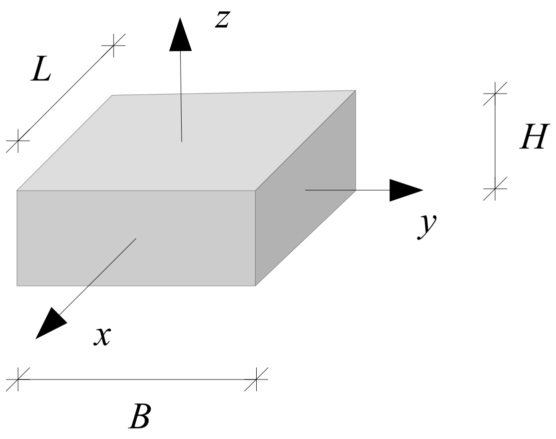
EXERCISE 1

Find the tensor of moment of inertia of a cuboid about axes containing its centroid and parallel to its edges.



SOLUTION:

Moment of intertia about *x* axis:

$$I_x = \iiint_V \rho(y^2+z^2)dV =$$
$$= \int_{-L/2}^{L/2}\left[\int_{-B/2}^{B/2}\left[\int_{-H/2}^{H/2}\rho(y^2+z^2)\,\mathrm{d}z\right]\mathrm{d}y\right]\mathrm{d}x =$$
$$= \rho\int_{-L/2}^{L/2}\mathrm{d}x\left[\int_{-B/2}^{B/2}\left[y^2 z+\frac{z^3}{3}\right]_{-H/2}^{H/2}\mathrm{d}y\right] = \rho[x]_{-L/2}^{L/2}\cdot\int_{-B/2}^{B/2}\left(H y^2+\frac{H^3}{12}\right)\mathrm{d}y = \rho L\left[H\frac{y^3}{3}+\frac{H^3}{12}y\right]_{-B/2}^{B/2} =$$
$$= \rho L\left(\frac{HB^3}{12}+\frac{BH^3}{12}\right) = \frac{\rho LBH}{12}(B^2+H^2) = \frac{m}{12}(B^2+H^2)$$

Similarly: $\quad I_y = \frac{m}{12}(L^2+H^2) \qquad I_z = \frac{m}{12}(L^2+B^2)$

Deviation moment of inertia about axes *x* and *y*:

$$D_{yz} = \iiint_V \rho\, yz\, dV = \int_{-L/2}^{L/2}\left[\int_{-B/2}^{B/2}\left[\int_{-H/2}^{H/2}\rho\, yz\,\mathrm{d}z\right]\mathrm{d}y\right]\mathrm{d}x = \rho\int_{-L/2}^{L/2}\mathrm{d}x\int_{-B/2}^{B/2}\left[\frac{yz^2}{2}\right]_{-H/2}^{H/2}\mathrm{d}y = 0$$

Similarly: $\quad D_{zx} = 0 \qquad D_{xy} = 0$

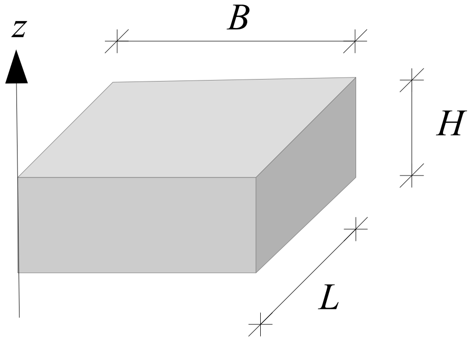
It is due to fact that the deviation moment of inertia are calculated about symmetry axes of the cuboid.

Tensor of moment of intertia:

$$\mathbf{I} = \begin{bmatrix} I_x & -D_{xy} & -D_{zx} \\ -D_{xy} & I_y & -D_{yz} \\ -D_{zx} & -D_{yz} & I_z \end{bmatrix} = \frac{m}{12}\begin{bmatrix} (B^2+H^2) & 0 & 0 \\ 0 & (L^2+H^2) & 0 \\ 0 & 0 & (L^2+B^2) \end{bmatrix}$$

EXERCISE 2

Find the moment of inertia of a cuboid rotating about an axis containing one of its edges.



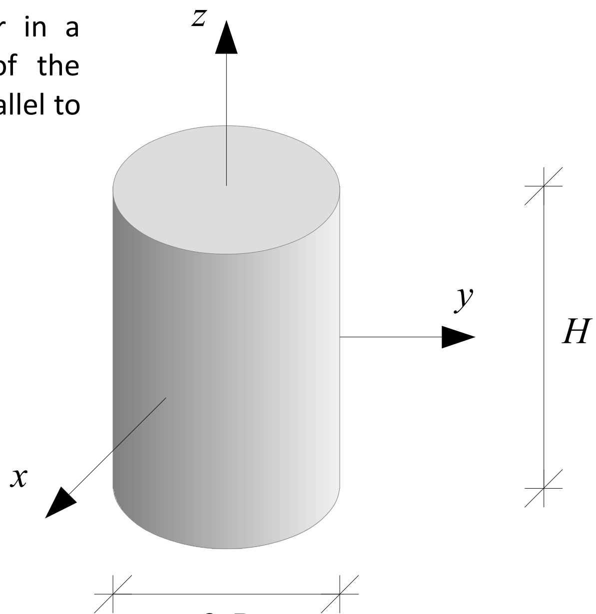
SOLUTION:

Let's consider the case when the rotation axis is parallel to the *z* axis. Moment of inertia will be found with the use of I_z component determined previously and with the use of parallel axis theorem.

$$I' = I + m d^2 = \frac{m}{12}(L^2+B^2) + m\left[\left(\frac{L}{2}\right)^2+\left(\frac{B}{2}\right)^2\right] = \frac{m}{3}(L^2+B^2)$$

EXERCISE 3

Find the tensor of moment of inertia for a cylinder in a coordinate system with its origin in the centroid of the cylinder. One of the axes of the coordinate system is parallel to the axis of cylinder.



SOLUTION:

Since chosen coordinate axes are principal axes (they contain symmetry axes of cylinder) all deviation moment of inertia are equal 0. We need only to find moment of inertia about axis of cylinder and about any axis perpendicular to it. Let's suppose that the axis of cylinder it the *z* axis:

Moment of inertia about the axis of cylinder:

$$I_z = \iiint_V \rho(x^2+y^2)dV = \rho\iiint (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \rho\iiint \left[(x^2+y^2)J\right]\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}z =$$

$$= \rho\int_{r=0}^{R}\left[\int_{\phi=0}^{2\pi}\left[\int_{z=-H/2}^{H/2}\left[(r^2\cos^2\phi+r^2\sin^2\phi)r\right]\mathrm{d}z\right]\mathrm{d}\phi\right]\mathrm{d}r = \rho\int_0^R r^3\,\mathrm{d}r\int_0^{2\pi}(\cos^2\phi+\sin^2\phi)\,\mathrm{d}\phi\int_{-H/2}^{H/2}\mathrm{d}z =$$

$$= \rho\left[\frac{r^4}{4}\right]_0^R\cdot[\phi]_0^{2\pi}\cdot[z]_{-H/2}^{H/2} = \frac{1}{2}\rho\pi R^4 H = \frac{1}{2}mR^2$$

Moment of inertia about an axis perpendicular to the axis of cylinder:

$$I_x = \iiint_V \rho(y^2+z^2)dV = \rho\iiint (y^2+z^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \rho\iiint \left[(y^2+z^2)J\right]\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}z =$$

$$= \rho\int_{r=0}^{R}\left[\int_{\phi=0}^{2\pi}\left[\int_{z=-H/2}^{H/2}\left[(r^2\sin^2\phi+z^2)r\right]\mathrm{d}z\right]\mathrm{d}\phi\right]\mathrm{d}r = \rho\int_0^R\int_0^{2\pi}\left[z\,r^3\sin^2\phi + r\frac{z^3}{3}\right]_{-H/2}^{H/2}\mathrm{d}\phi\,\mathrm{d}r =$$

$$= \rho\int_0^R\int_0^{2\pi}\left[H\,r^3\sin^2\phi + r\frac{H^3}{12}\right]\mathrm{d}\phi\,\mathrm{d}r = \rho\int_0^R\left[H\,r^3\left(\frac{\phi}{2}-\frac{1}{4}\cos 2\phi\right)+r\,\phi\frac{H^3}{12}\right]_0^{2\pi}\mathrm{d}r =$$

$$= \rho\int_0^R\left[\pi H\,r^3 + r\,\pi\frac{H^3}{6}\right]\mathrm{d}r = \rho\pi H\left[\frac{r^4}{4}+r^2\frac{H^2}{12}\right]_0^R = \rho\pi R^2 H\left[\frac{R^2}{4}+\frac{H^2}{12}\right] = \frac{m}{12}\left[3R^2+H^2\right]$$

Tensor bezwładności:

$$\mathbf{I} = \begin{bmatrix} I_x & -D_{xy} & -D_{zx} \\ -D_{xy} & I_y & -D_{yz} \\ -D_{zx} & -D_{yz} & I_z \end{bmatrix} = \frac{m}{12}\begin{bmatrix} (3R^2+H^2) & 0 & 0 \\ 0 & (3R^2+H^2) & 0 \\ 0 & 0 & 6R^2 \end{bmatrix}$$

EXERCISE 4

Find the tensor of moment of inertia of a ball in any coordinate system.

SOLUTION:

Due to symmetry of a ball any coordinate system is a principal system, so deviation moment of inertia are always equal 0. Similarly, moment of inertia about any axis containing centroid will always be the same:

$$I_z = \iiint_V \rho(x^2+y^2)dV = \rho\iiint (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \rho\iiint\left[(x^2+y^2)J\right]\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}\psi =$$

$$= \rho\int_{r=0}^{R}\left[\int_{\phi=0}^{2\pi}\left[\int_{\psi=-\pi/2}^{\pi/2}\left[(r^2\cos^2\phi\sin^2\psi + r^2\sin^2\phi\sin^2\psi)r^2\sin\psi\right]\mathrm{d}\psi\right]\mathrm{d}\phi\right]\mathrm{d}r =$$

$$= \rho\int_0^R r^4\,\mathrm{d}r\int_0^{2\pi}(\cos^2\phi+\sin^2\phi)\,\mathrm{d}\phi\int_{-\pi/2}^{\pi/2}\sin^3\psi\,\mathrm{d}\psi = 2\rho\int_0^R r^4\,\mathrm{d}r\int_0^{2\pi}(\cos^2\phi+\sin^2\phi)\,\mathrm{d}\phi\int_0^{\pi/2}\sin^3\psi\,\mathrm{d}\psi =$$

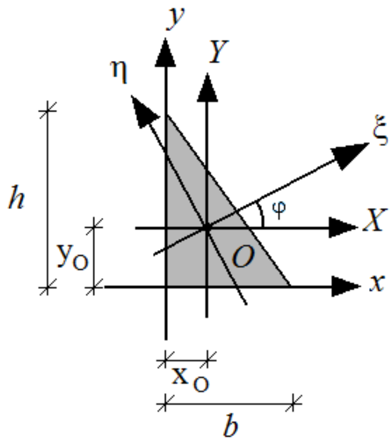
$$= 2\rho\left[\frac{r^5}{5}\right]_0^R\cdot[\phi]_0^{2\pi}\cdot\left[\frac{1}{3}\cos^3\psi-\cos\psi\right]_0^{\pi/2} = \frac{8\pi R^5\rho}{15} = \frac{2}{5}mR^2$$

EXERCISE 5

There is a triangle of sides b and h. Find the centroid, central moment of inertia and deviation moments of inertia, orientation of principle axes of inertia and values of principle moments of inertia.

SOLUTION:

The area of triangle may be described as follows: $A=\left\{x,y:\ x\in(0,b),\ y\in\left(0,h-\frac{h}{b}x\right)\right\}$



Surface area:

$$A=\iint_A \mathrm{d}A=\int_{x=0}^{b}\int_{y=0}^{h-\frac{h}{b}x}\mathrm{d}y\,\mathrm{d}x=\int_{x=0}^{b}[y]_{y=0}^{h-\frac{h}{b}x}\mathrm{d}x=\int_0^b\left(h-\frac{h}{b}x\right)\mathrm{d}x=$$
$$=h\left[x-\frac{1}{2b}x^2\right]_0^b=\frac{1}{2}bh$$

Statical moments and location of centroid:

$$S_x=\iint_A y\,\mathrm{d}A=\int_{x=0}^{b}\int_{y=0}^{h-\frac{h}{b}x}y\,\mathrm{d}y\,\mathrm{d}x=\int_{x=0}^{b}\left[\frac{y^2}{2}\right]_{y=0}^{h-\frac{h}{b}x}\mathrm{d}x=\frac{1}{2}\int_0^b\left(h-\frac{h}{b}x\right)^2\mathrm{d}x=$$
$$=\frac{1}{2}\int_0^b\left[h^2-2\frac{h^2}{b}x+\frac{h^2}{b^2}x^2\right]\mathrm{d}x=\frac{h^2}{2}\left[x-\frac{1}{b}x^2+\frac{1}{3b^2}x^3\right]_0^b=\frac{bh^2}{6}$$

$$S_y=\iint_A x\,\mathrm{d}A=\int_{x=0}^{b}\int_{y=0}^{h-\frac{h}{b}x}x\,\mathrm{d}y\,\mathrm{d}x=\int_{x=0}^{b}x[y]_{y=0}^{h-\frac{h}{b}x}\mathrm{d}x=\int_0^b\left[hx-\frac{h}{b}x^2\right]\mathrm{d}x=$$
$$=\left[\frac{h}{2}x^2-\frac{h}{3b}x^3\right]_0^b=\frac{hb^2}{6}$$

$$x_O=\frac{S_y}{A}=\frac{b}{3}\qquad\qquad y_O=\frac{S_x}{A}=\frac{h}{3}$$

Moments of inertia in global coordinate system:

$$I_x=\iint_A y^2\mathrm{d}A=\int_{x=0}^{b}\int_{y=0}^{h-\frac{h}{b}x}y^2\mathrm{d}y\,\mathrm{d}x=\int_{x=0}^{b}\left[\frac{y^3}{3}\right]_{y=0}^{h-\frac{h}{b}x}\mathrm{d}x=\frac{1}{3}\int_0^b\left(h-\frac{h}{b}x\right)^3\mathrm{d}x=$$
$$=\frac{1}{3}\int_0^b\left[h^3-3\frac{h^3}{b}x+3\frac{h^3}{b^2}x^2-\frac{h^3}{b^3}x^3\right]\mathrm{d}x=\frac{h^3}{3}\left[x-\frac{3}{2b}x^2+\frac{1}{b^2}x^3-\frac{1}{4b^3}x^4\right]_0^b=\frac{bh^3}{12}$$

$$I_y=\iint_A x^2\mathrm{d}A=\int_{x=0}^{b}\int_{y=0}^{h-\frac{h}{b}x}x^2\mathrm{d}y\,\mathrm{d}x=\int_{x=0}^{b}x^2[y]_{y=0}^{h-\frac{h}{b}x}\mathrm{d}x=\int_0^b\left[hx^2-\frac{h}{b}x^3\right]\mathrm{d}x=$$
$$=\left[\frac{h}{3}x^3-\frac{h}{4b}x^4\right]_0^b=\frac{hb^3}{12}$$

$$D_{xy} = \iint_A xy \,\mathrm{d}A = \int_{x=0}^{b} \int_{y=0}^{h-\frac{h}{b}x} xy \,\mathrm{d}y \,\mathrm{d}x = \int_{x=0}^{b} x \left[\frac{y^2}{2}\right]_{y=0}^{h-\frac{h}{b}x} \mathrm{d}x = \frac{1}{2}\int_0^b x\left(h-\frac{h}{b}x\right)^2 \mathrm{d}x =$$

$$= \frac{1}{2}\int_0^b \left[h^2 x - 2\frac{h^2}{b}x^2 + \frac{h^2}{b^2}x^3\right]\mathrm{d}x = \frac{h^2}{2}\left[\frac{1}{2}x^2 - \frac{2}{3b}x^3 + \frac{1}{4b^2}x^4\right]_0^b = \frac{b^2h^2}{24}$$

Central moments of inertia:

$$I_X = I_x - A\cdot y_O^2 = \frac{bh^3}{12} - \frac{1}{2}bh\cdot\left(\frac{h}{3}\right)^2 = \frac{bh^3}{36}$$

$$I_Y = I_y - A\cdot x_O^2 = \frac{hb^3}{12} - \frac{1}{2}bh\cdot\left(\frac{b}{3}\right)^2 = \frac{hb^3}{36}$$

$$D_{XY} = D_{xy} - A\cdot x_O y_O = \frac{b^2h^2}{24} - \frac{1}{2}bh\cdot\frac{b}{3}\cdot\frac{h}{3} = -\frac{b^2h^2}{72}$$

Polar moment of inertia:

$$I_0 = \iint_A (X^2+Y^2)\,\mathrm{d}A = I_X + I_Y = \frac{bh}{36}(h^2+b^2)$$

Tensor of moment of inertia: $\mathbf{I} = \begin{bmatrix} I_X & -D_{XY} \\ -D_{XY} & I_Y \end{bmatrix} = \frac{bh}{72}\begin{bmatrix} 2h^2 & bh \\ bh & 2b^2 \end{bmatrix}$

Invariants of the above tensor: $\alpha = \mathrm{tr}(\mathbf{I}) = I_X + I_Y = \frac{bh}{36}(h^2+b^2)$

$\beta = \det(\mathbf{I}) = I_X I_Y - D_{XY}^2 = \frac{b^4h^4}{1728}$

Secular equation: $I^2 - \alpha I + \beta = 0$

Principal moments of inertia – roots of secular equation:

$$I_\xi = I_{max} = \frac{I_X+I_Y}{2} + \sqrt{\left(\frac{I_X-I_Y}{2}\right)^2 + D_{XY}^2} = \frac{bh}{72}\left[b^2+h^2+\sqrt{b^4-b^2h^2+h^4}\right]$$

$$I_\eta = I_{min} = \frac{I_X+I_Y}{2} + \sqrt{\left(\frac{I_X-I_Y}{2}\right)^2 + D_{XY}^2} = \frac{bh}{72}\left[b^2+h^2-\sqrt{b^4-b^2h^2+h^4}\right]$$

The angle between axis X of central coordinate system and axis of maximum central moment of inertia is equal:

$$\mathrm{tg}\,\varphi = \frac{D_{XY}}{I_Y - I_{max}} = \frac{bh}{h^2-b^2+\sqrt{h^4-h^2b^2+b^4}}$$

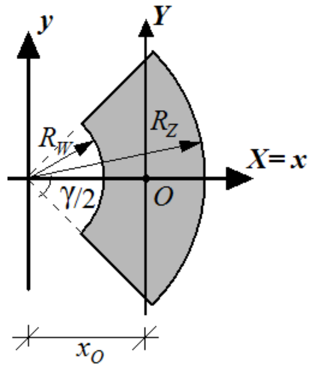
EXERCISE 6

There is a sectort of a ring of internal radius R_W and external radius R_Z, corresponding to central angle γ. Find the centroid and values of principal central moments of inertia.

SOLUTION:

Global coordinate system may be chosen in such a way that its x axis is a symmetry axis of this shape. Centroid lies on this axis. It is also one of the principal central axis of inertia. The second principal axis must be perpendicular to it. The area of this shape may be described with the use of polar coordinates as follows:

$$A=\left\{r,\psi:\ \ r\in(R_W,R_Z),\psi\in\left(-\frac{\gamma}{2},\frac{\gamma}{2}\right)\right\}$$



Surface area:

$$A=\iint_A \mathrm{d}A=\int_{r=R_w}^{R_z}\int_{\psi=-\gamma/2}^{\gamma/2} J\,\mathrm{d}\psi\,\mathrm{d}r=\int_{R_W}^{R_Z} r\,\mathrm{d}r\int_{-\gamma/2}^{\gamma/2}\mathrm{d}\psi=$$
$$=\left[\frac{r^2}{2}\right]_{R_W}^{R_Z}[\psi]_{-\gamma/2}^{\gamma/2}=\frac{\gamma}{2}(R_Z^2-R_W^2)$$

Statical moments and location of centroid:

$$S_y=\iint_A x\,\mathrm{d}A=\int_{r=R_w}^{R_z}\int_{\psi=-\gamma/2}^{\gamma/2} r\cos\psi\,J\,\mathrm{d}\psi\,\mathrm{d}r=\int_{R_W}^{R_Z} r^2\mathrm{d}r\int_{-\gamma/2}^{\gamma/2}\cos\psi\,\mathrm{d}\psi=\left[\frac{r^3}{3}\right]_{R_W}^{R_Z}[\sin\psi]_{-\gamma/2}^{\gamma/2}=$$
$$=\frac{2}{3}(R_Z^3-R_W^3)\cdot\sin\frac{\gamma}{2}$$

$$x_O=\frac{S_y}{A}=\frac{4}{3\gamma}\frac{(R_Z^3-R_W^3)}{(R_Z^2-R_W^2)}\sin\left(\frac{\gamma}{2}\right)\qquad y_O=0$$

Moments of inertia in the global coordinate system:

$$I_x=I_X=\iint_A y^2\mathrm{d}A=\int_{r=R_w}^{R_z}\int_{\psi=-\gamma/2}^{\gamma/2}(r\sin\psi)^2 J\,\mathrm{d}\psi\,\mathrm{d}r=\int_{R_W}^{R_Z} r^3\mathrm{d}r\int_{-\gamma/2}^{\gamma/2}\sin^2\psi\,\mathrm{d}\psi=$$
$$\left[\frac{r^4}{4}\right]_{R_W}^{R_Z}\left[\frac{1}{2}(x-\sin\psi\cos\psi)\right]_{-\gamma/2}^{\gamma/2}=\frac{1}{8}(R_Z^4-R_W^4)[\gamma-\sin\gamma]$$

$$I_y=\iint_A x^2\mathrm{d}A=\int_{r=R_w}^{R_z}\int_{\psi=-\gamma/2}^{\gamma/2}(r\cos\psi)^2 J\,\mathrm{d}\psi\,\mathrm{d}r=\int_{R_W}^{R_Z} r^3\mathrm{d}r\int_{-\gamma/2}^{\gamma/2}\cos^2\psi\,\mathrm{d}\psi=$$
$$\left[\frac{r^4}{4}\right]_{R_W}^{R_Z}\left[\frac{1}{2}(x+\sin\psi\cos\psi)\right]_{-\gamma/2}^{\gamma/2}=\frac{1}{8}(R_Z^4-R_W^4)[\gamma+\sin\gamma]$$

Central moments of inertia:

$$I_X = I_x - A\cdot y_O^2 = \frac{1}{8}(R_Z^4 - R_W^4)[\gamma - \sin\gamma]$$

$$I_Y = I_y - A\cdot x_O^2 = \frac{1}{8}(R_Z^4 - R_W^4)[\gamma + \sin\gamma] - \frac{8}{9\gamma}\frac{(R_Z^3 - R_W^3)^2}{(R_Z^2 - R_W^2)}\sin^2\left(\frac{\gamma}{2}\right)$$

Polar moment of inertia:

$$I_0 = I_X + I_Y = \frac{\gamma}{4}(R_Z^4 - R_W^4) - \frac{8}{9\gamma}\frac{(R_Z^3 - R_W^3)^2}{(R_Z^2 - R_W^2)}\sin^2\left(\frac{\gamma}{2}\right)$$

Special cases:

<u>Sectors of a circle</u> $R_Z = R,\quad R_W = 0$

- <u>Circle</u> $(\gamma = 2\pi)$:

$$A = \pi R^2 \qquad x_O = 0 \qquad I_X = I_Y = \frac{\pi R^4}{4}$$

- <u>Half of a circle</u> $(\gamma = \pi)$:

$$A = \frac{\pi R^2}{2} \qquad x_O = \frac{4}{3}\frac{R}{\pi} \qquad I_X = \frac{\pi R^4}{8} \qquad I_Y = R^4\left(\frac{\pi}{8} - \frac{8}{9\pi}\right)$$

- <u>Quarter of a circle</u> $\left(\gamma = \frac{\pi}{2}\right)$:

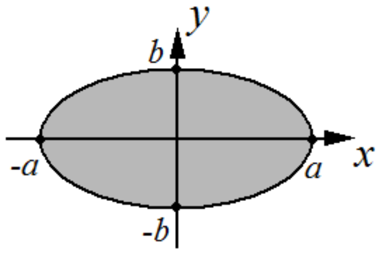
$$A = \frac{\pi R^2}{4} \qquad x_O = \frac{4\sqrt{2}}{3}\frac{R}{\pi} \qquad I_X = \frac{\pi R^4}{4} = \frac{(\pi - 2)}{16}R^4 \qquad I_Y = \frac{(9\pi^2 + 18\pi - 128)}{144}R^4$$

<u>Ring</u> $R_Z = R,\quad R_W = r,\quad (\gamma = 2\pi)$:

$$A = \pi(R^2 - r^2) \qquad x_O = 0 \qquad I_X = I_Y = \frac{\pi(R^4 - r^4)}{4}$$

EXERCISE 7

Find principal central moments of inertia of an ellipse of half-axes of length *a* and *b* (*a* > *b*). Find the central moments of inertia and deviation moments of inertia about axes which are rotated about half-axes of this ellipse with an angle equal α.



SOLUTION:

Area of an ellipse may be described as follows:
$$A=\left\{x,y:\ \frac{x^2}{a^2}+\frac{y^2}{b^2}\leqslant 1\right\}$$

As the integration domain is symmetric and the integrand is an even function it is enough to calculate the integrals only in one quarter of the plane and then multiply the result with four.

Moment of inertia about *x* axis:

$$I_x=\iint_A y^2\,dA=4\int_{x=0}^{a}\int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}} y^2\,dy\,dx=4\int_0^a\left[\frac{y^3}{3}\right]_0^{b\sqrt{1-\frac{x^2}{a^2}}}dx=\frac{4b^3}{3}\int_0^a\left(1-\frac{x^2}{y^2}\right)^{3/2}dx=\left|\begin{matrix}u=\dfrac{x}{a}\\ dx=a\cdot du\end{matrix}\right|=$$
$$=\frac{4ab^3}{3}\int_0^1(1-u^2)^{3/2}du=\frac{4ab^3}{3}\int\frac{u^4-2u^2+1}{\sqrt{1-u^2}}du$$

Obtained integral is calculated with the use of the following substitution:
$$\int\frac{u^4-2u^2+1}{\sqrt{1-u^2}}du\equiv(au^3+bu^2+cu+d)\sqrt{1-u^2}+A\int\frac{du}{\sqrt{1-u^2}}$$

After differentiating both sides and finding a common denominator on the right side, we state that polynomials on both sides are equal, so coefficients by corresponding terms on one and on the other side must be the same:

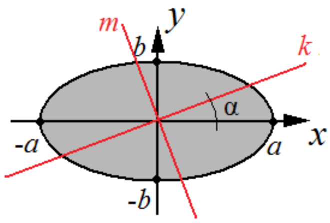
$$\int_0^1(1-u^2)^{3/2}du=\left[\frac{1}{8}u(5-2u^2)\sqrt{1-u^2}+\frac{3}{8}\arcsin(u)\right]_0^1=\frac{3\pi}{16}\qquad\Rightarrow\qquad I_x=\frac{\pi ab^3}{4}$$

Similarly, for the second axis we obtain: $I_y=\dfrac{\pi a^3 b}{4}$

Deviation moment of inertia in considered coordinate system is equal: $D_{xy}=0$.

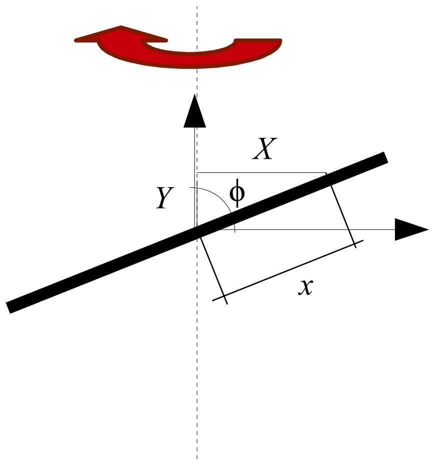
Moment of inertia about an axis containing centroid of an ellipse and inclined to its longer half-axis at angle α is equal:

$$I_k=I_x\cos^2\alpha+I_y\sin^2\alpha-2D_{xy}\sin\alpha\cos\alpha=\frac{\pi ab}{4}\left(b^2\cos^2\alpha+a^2\sin^2\alpha\right)$$
$$I_m=I_x\sin^2\alpha+I_y\cos^2\alpha+2D_{xy}\sin\alpha\cos\alpha=\frac{\pi ab}{4}\left(b^2\sin^2\alpha+a^2\cos^2\alpha\right)$$
$$D_{km}=\frac{I_y-I_x}{2}\sin 2\alpha-D_{xy}\cos 2\alpha=\frac{\pi ab}{8}\left(a^2-b^2\right)\sin 2\alpha$$



EXERCISE 8

Find the moment of inertia and deviation moment of inertia of a thin rod rotating about an axis containing its centroid and inclined to the axis of a rod at angle ϕ .



SOLUTION:

The term „thin rod" will be understood in such a way, that its transverse dimensions are so small, when compared to the longitudinal one, that we may assume that transverse coordinates of all points of a fixed cross-sections are the same, namely:

$$I_Y = \iiint_V \rho X^2 \mathrm{d}V = \rho A \int_{-L/2}^{L/2} X^2 \mathrm{d}x = \rho A \int_{-L/2}^{L/2} (x\sin\phi)^2 \mathrm{d}x =$$
$$= \rho A \sin^2\phi \left[\frac{x^3}{3}\right]_{-L/2}^{L/2} = \frac{1}{12}\rho A L^3 \sin^2\phi = \frac{mL^2}{12}\sin^2\phi = \frac{\mu L^3}{12}\sin^2\phi$$

where μ is linear density[kg/m]. In case of a rod rotating about an axis which is perpendicular to the axis of rod, namely for $\phi=90^\circ$:

$$I = \frac{mL^2}{12} = \frac{\mu L^3}{12}$$

A consequence of an assumption that transverse dimensions are small is that moment of inertia in case of rotation about rod's axis is zero $(\phi=0)$. In fact, a cylinder rotation about its own axis of symmetry has a non-zero moment of inertia, what is due to its non-zero transverse dimensions (see: exercise 3).

Deviation moment of inertia:

$$D_{XY} = \iiint_V \rho XY \mathrm{d}V = \rho A \int_{-L/2}^{L/2} XY \mathrm{d}x = \rho A \int_{-L/2}^{L/2} (x\sin\phi)(x\cos\phi) \mathrm{d}x =$$
$$= \rho A \sin\phi\cos\phi \left[\frac{x^3}{3}\right]_{-L/2}^{L/2} = \frac{mL^2}{12}\sin\phi\cos\phi = \frac{\mu L^3}{12}\sin\phi\cos\phi$$

EXERCISE 9

Find the moment of inertia of a thin rod rotating about an axis which is perpendicular to the rod's axis and contains on of its ends.

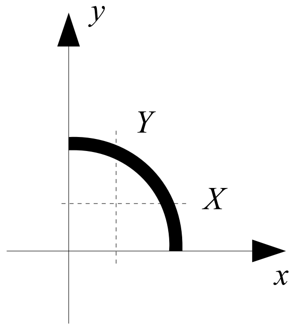
SOLUTION:

We find this quantity basing on previous solution and with the use of parallel axis theorem:

$$I' = I + m d^2 = \frac{mL^2}{12} + m\cdot\left(\frac{L}{2}\right)^2 = \frac{mL^2}{3}$$

EXERCISE 10

Find the centroid and central moments of inertia and deviation moments of inertia of a quarter of thin circular rod of mass *m*.



SOLUTION:

At first we will wind the moment of inertia about an axis *y* of a global coordinate system and then – having found the centroid – we will use the parallel axis theorem in order to find a central moment of inertia.

Total length of rod: $V = A\,L = \dfrac{A\pi r}{2}$ $\quad [V] = \mathrm{m}^3$

Material's density: $\rho = \dfrac{m}{V} = \dfrac{m}{A\,L} = \dfrac{2\,m}{A\pi r}$ $\quad [\rho] = \mathrm{kg/m^3}$

Material's linear density: $\mu = \rho\,A$ $\quad [\mu] = \mathrm{kg/m^3}$

Statical moment about plane (*y;z*) in global coordinate system:

$$S_y = \iiint_V \rho\,x\,\mathrm{d}V = \rho\,A \int_{\phi=0}^{\pi/2} r\,x\,\mathrm{d}\phi = \rho\,A\,r^2 \int_0^{\pi/2} \cos\phi\,\mathrm{d}\phi = \rho\,A\,r^2 = \mu\,r^2$$

Location of centroid: $x_O = \dfrac{S_y}{m} = \dfrac{2\,r}{\pi}$

Moment of inertia and deviation moment of inertia in global coordinate system:

$$I_y = \iiint_V \rho\,x^2\,\mathrm{d}V = \rho\,A \int_{\phi=0}^{\pi/2} r\,x^2\,\mathrm{d}\phi = \rho\,A\,r^3 \int_0^{\pi/2} \cos^2\phi\,\mathrm{d}\phi = \frac{\rho\,A\pi r^3}{4} = \mu\frac{\pi r^3}{4}$$

$$D_{xy} = \iiint_V \rho\,x\,y\,\mathrm{d}V = \rho\,A \int_{\phi=0}^{\pi/2} r\,x\,y\,\mathrm{d}\phi = \rho\,A\,r^3 \int_0^{\pi/2} \cos\phi\sin\phi\,\mathrm{d}\phi = \frac{\rho\,A\,r^3}{2} = \mu\frac{r^3}{2}$$

Central moment of inertia:

$$I_Y = I_y - m\,x_O^2 = \mu\frac{\pi r^3}{4} - \frac{\mu\pi r}{2}\cdot\left(\frac{2\,r}{\pi}\right)^2 = \mu\,r^2\left(\frac{\pi}{4} - \frac{2}{\pi}\right)$$

EXERCISE 11

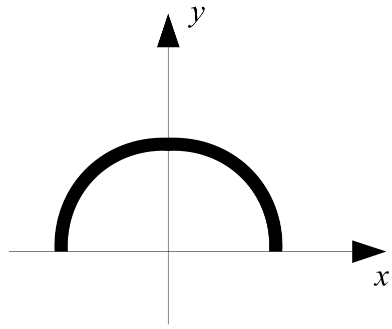
Find the moment of inertia and deviation moment of inertia of a half of thin circular rod.

SOLUTION:

According to the results obtained in previous exercise:

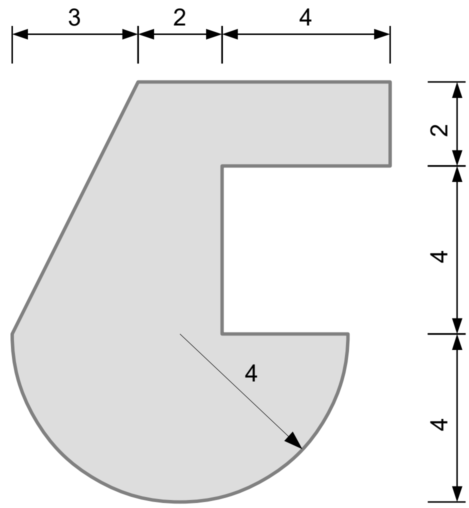
$$I_x = I_y = \mu\frac{\pi r^3}{2}$$

$$D_{xy} = 0$$



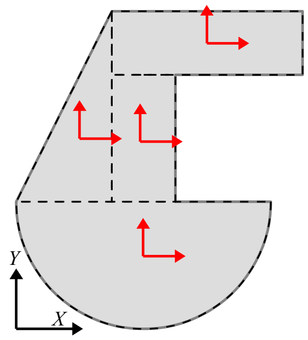
EXERCISE 12

Find the centroid of a shape presented in the picture and then find statical moment about a horizontal axis containing centroid from this part of the shape that lies above this line.



SOLUTION:

We divide the shape into a set of simple component shapes and we introduce a global coordinate system:



Surface area:

$$A = [2\cdot 6] + [2\cdot 4] + \left[\frac{1}{2}\cdot 3\cdot 6\right] + \left[\frac{\pi\cdot 4^2}{2}\right] = 54{,}133$$

Statical moments about chosen coordinate axes:

$$S_X = \left[2\cdot 6\cdot\left(8+\frac{2}{2}\right)\right] + \left[2\cdot 4\cdot\left(4+\frac{4}{2}\right)\right] + \left[\frac{1}{2}\cdot 3\cdot 6\cdot\left(4+\frac{6}{3}\right)\right] + \left[\frac{\pi\cdot 4^2}{2}\cdot\left(4-\frac{4}{3}\frac{4}{\pi}\right)\right] = 267{,}864$$

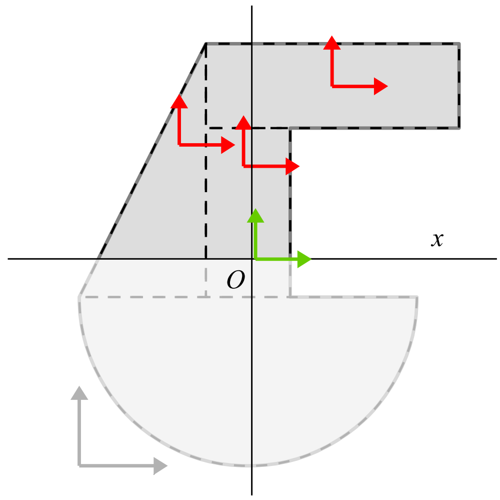
$$S_Y = \left[2\cdot 6\cdot\left(3+\frac{6}{2}\right)\right] + \left[2\cdot 4\cdot\left(3+\frac{2}{2}\right)\right] + \left[\frac{1}{2}\cdot 3\cdot 6\cdot\left(\frac{2}{3}\cdot 3\right)\right] + \left[\frac{\pi\cdot 4^2}{2}\cdot(4)\right] = 222{,}531$$

Location of centroid:

$$X_O = \frac{S_Y}{A} = 4{,}1108\ , \qquad Y_O = \frac{S_X}{A} = 4{,}9483$$

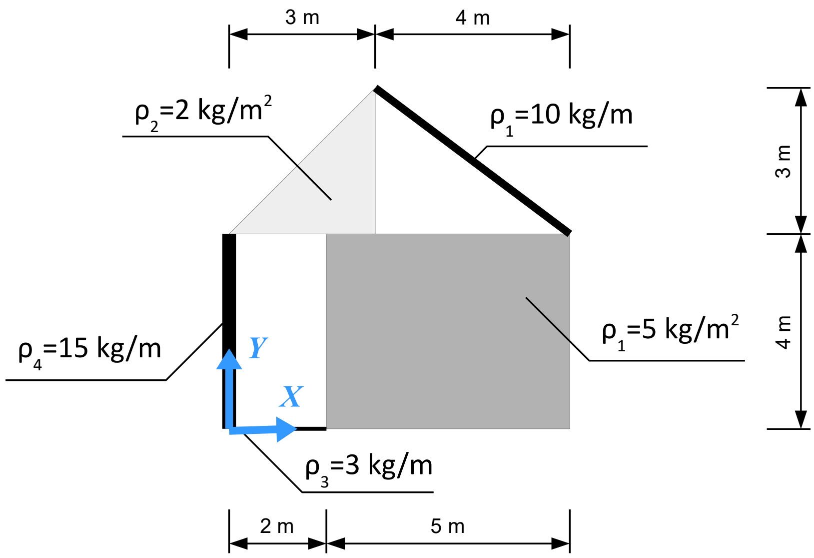
We place a horizontal line passing through centroid and calculate statical moment of a part of shape that lies above this line about this line.

$$S_x^{\uparrow} = \left[2\cdot 6\cdot(9-4{,}9483)\right] + \dots$$
$$\dots + \left[2\cdot(8-4{,}9483)\cdot\frac{(8-4{,}9483)}{2}\right] + \dots$$
$$\dots + \left[(10-4{,}9483)\cdot\left(\frac{3}{6}\cdot(10-4{,}9483)\right)\cdot\frac{(10-4{,}9483)}{3}\right] =$$
$$= 68{,}6768$$



EXERCISE 13

Find the centroid of a shape shown in the picture.



SOLUTION:

In case of materials of different density, their account in total mass (area), statical moments and moments of inertia requires multiplication of those geometric quantities by appropriate local density.

Total mass of the system: $m = 5\cdot[5\cdot 4]+2\cdot\left[\frac{1}{2}\cdot 3\cdot 3\right]+3\cdot[2]+15\cdot[4]+10\cdot\left[\sqrt{3^2+4^2}\right]=225\ \text{kg}$

Statical moments:

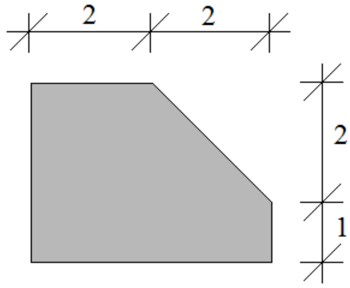
$$S_X = 5\cdot[20\cdot 2]+2\cdot\left[4{,}5\cdot\left(4+\frac{1}{3}\cdot 3\right)\right]+3\cdot[2\cdot 0]+15\cdot[4\cdot 2]+10\cdot\left[5\cdot\left(4+\frac{3}{2}\right)\right]=640\ \text{kg}\cdot\text{m}$$

$$S_Y = 5\cdot\left[20\cdot\left(2+\frac{5}{2}\right)\right]+2\cdot\left[4{,}5\cdot\left(\frac{2}{3}\cdot 3\right)\right]+3\cdot[2\cdot 1]+15\cdot[4\cdot 0]+10\cdot\left[5\cdot\left(3+\frac{4}{2}\right)\right]=724\ \text{kg}\cdot\text{m}$$

Location of centroid: $X_O = \dfrac{S_Y}{m} = 3{,}2178\ \text{m}$, $Y_O = \dfrac{S_X}{m} = 3{,}8444\ \text{m}$

EXERCISE 14

Find principal central moments of inertia and orientation of principal central axes of a shape shown in the figure. Then, find components of the tensor of moment of inertia in a central coordinate system inclined to horizontal and vertical axes at angle 60°.

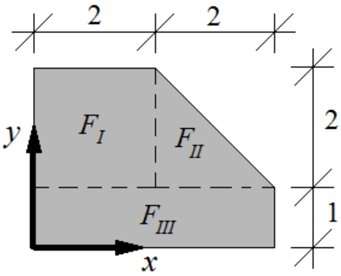


SOLUTION:

Outline of solution – discussed in a more detailed way in the next, more complicated exercise – is as follows:

1. Represent the shape as a sum or difference of possibly small number of smaller simple shapes (rectangles, right triangles, circles and their halves and quarters).
2. Calculate surface area of the whole shape and find statical moments about axes of chosen coordinate system – then, find coordinates of the centroid.
3. Determine geometric characteristics of each component shape in its own central coordinate system and sum up characteristics of all component shapes after having transformed it to the centroid of the whole shape with the use of parallel axis theorem.
4. Find principal central moments of inertia and orientation of axis of maximum moment of inertia.

Division of the shape and choice of global coordinate system:



Surface area of the shape:

$$A = A_I + A_{II} + A_{III} = [2\cdot 2] + \left[\frac{1}{2}\cdot 2\cdot 2\right] + [1\cdot 4] = 4+2+4=10$$

Statical moments of component shapes about axes *x* and *y* of chosen coordinate system are calculated as a product of their area and respectively *y* and *x* coordinate of their centroid. Centroid of rectangles lies in the middle of their width and height and centroid of right triangles lies at 1/3 of their width and height starting from right angle.:

$$S_x = \left[(2\cdot 2)\cdot\left(1+\frac{1}{2}\cdot 2\right)\right] + \left[\left(\frac{1}{2}\cdot 2\cdot 2\right)\cdot\left(1+\frac{1}{3}\cdot 2\right)\right] + \left[(1\cdot 4)\cdot\left(\frac{1}{2}\cdot 1\right)\right] = 13{,}333$$

$$S_y = \left[(2\cdot 2)\cdot\left(\frac{1}{2}\cdot 2\right)\right] + \left[\left(\frac{1}{2}\cdot 2\cdot 2\right)\cdot\left(2+\frac{1}{3}\cdot 2\right)\right] + \left[(1\cdot 4)\cdot\left(\frac{1}{2}\cdot 4\right)\right] = 17{,}333$$

Location of centroid: $x_O = \dfrac{S_y}{A} = 1{,}733$ $\qquad y_O = \dfrac{S_x}{A} = 1{,}333$

Moments of inertia and deviation moments of inertia for whole shape are calculated as a sum of appropriate moments of component shapes transformed to the centroid according to the parallel axis theorem.

$$I_X=\left[\frac{2\cdot 2^3}{12}+(2\cdot 2)\cdot\left(1+\frac{1}{2}\cdot 2-1{,}333\right)^2\right]+\left[\frac{2\cdot 2^3}{36}+\left(\frac{1}{2}\cdot 2\cdot 2\right)\cdot\left(1+\frac{1}{3}\cdot 2-1{,}333\right)^2\right]+\ldots$$
$$\ldots+\left[\frac{4\cdot 1^3}{12}+(4\cdot 1)\cdot\left(\frac{1}{2}\cdot 1-1{,}333\right)^2\right]=3{,}113+0{,}667+3{,}109=6{,}889$$

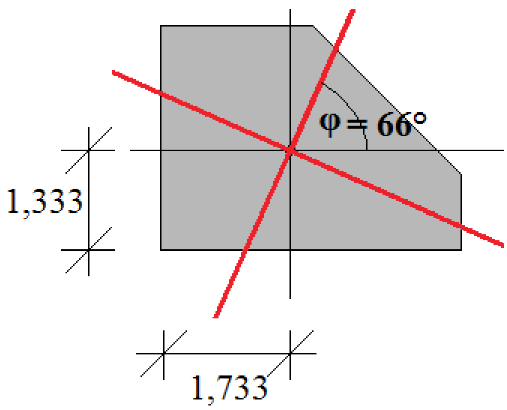
$$I_Y=\left[\frac{2^3\cdot 2}{12}+(2\cdot 2)\cdot\left(\frac{1}{2}\cdot 2-1{,}733\right)^2\right]+\left[\frac{2^3\cdot 2}{36}+\left(\frac{1}{2}\cdot 2\cdot 2\right)\cdot\left(2+\frac{1}{3}\cdot 2-1{,}733\right)^2\right]+\ldots$$
$$\ldots+\left[\frac{4^3\cdot 1}{12}+(4\cdot 1)\cdot\left(\frac{1}{2}\cdot 4-1{,}733\right)^2\right]=3{,}482+2{,}188+5{,}618=11{,}288$$

$$D_{XY}=\left[0+(2\cdot 2)\cdot\left(1+\frac{1}{2}\cdot 2-1{,}333\right)\left(\frac{1}{2}\cdot 2-1{,}733\right)\right]+\ldots$$
$$\ldots+\left[-\frac{2^2\cdot 2^2}{72}+\left(\frac{1}{2}\cdot 2\cdot 2\right)\cdot\left(1+\frac{1}{3}\cdot 2-1{,}333\right)\left(2+\frac{1}{3}\cdot 2-1{,}733\right)\right]+\ldots$$
$$\ldots+\left[0+(4\cdot 1)\cdot\left(\frac{1}{2}\cdot 1-1{,}333\right)\left(\frac{1}{2}\cdot 4-1{,}733\right)\right]=-1{,}956+0{,}401-0{,}890=-2{,}445$$

Principal central moments of inertia and orientation of the axis of maximum moment of inertia:

$$I_{max}=\frac{I_X+I_Y}{2}+\sqrt{\left(\frac{I_x-I_y}{2}\right)^2+D_{XY}^2}=12{,}377$$
$$I_{min}=\frac{I_X+I_Y}{2}-\sqrt{\left(\frac{I_x-I_y}{2}\right)^2+D_{XY}^2}=5{,}800$$

$$\varphi=\operatorname{arctg}\frac{D_{XY}}{I_y-I_{max}}=66^\circ$$



Components of tensor of moment of inertia about axes inclined to horizontal and vertical axis at angle 60° may be found with the use of following formulae:

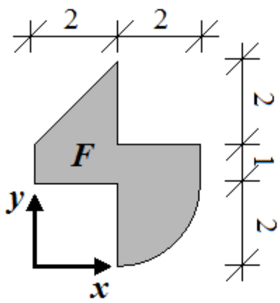
$$I_{x'}=I_X\cos^2\alpha+I_Y\sin^2\alpha-2D_{XY}\sin\alpha\cos\alpha$$
$$I_{y'}=I_X\sin^2\alpha+I_Y\cos^2\alpha+2D_{XY}\sin\alpha\cos\alpha$$
$$D_{x'y'}=\frac{I_X-I_\eta}{2}\sin2\alpha+D_{XY}\cos2\alpha$$

where:

$$\alpha=60^\circ\qquad\Rightarrow\qquad\sin\alpha=\frac{\sqrt{3}}{2}\quad\cos\alpha=\frac{1}{2}\quad\sin2\alpha=\frac{\sqrt{3}}{2}\quad\cos2\alpha=-\frac{1}{2}$$

$$I_{x'}=6{,}889\cdot\frac{1}{4}+11{,}288\cdot\frac{3}{4}-2\cdot(-2{,}445)\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{2}=12{,}306$$
$$I_{y'}=6{,}889\cdot\frac{3}{4}+11{,}288\cdot\frac{1}{4}+2\cdot(-2{,}445)\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{2}=5{,}871$$
$$D_{x'y'}=\frac{6{,}889-11{,}288}{2}\cdot\frac{\sqrt{3}}{2}+(-2{,}445)\cdot\left(-\frac{1}{2}\right)=-0{,}682$$

EXERCISE 15



Find principal central moments of inertia and orientation of principal axes of inertia in a shape shown in the figure. Then, find components of tensor of moment of inertia about axes inclined to the principal central axes of this shape at angle 45° .

SOLUTION:

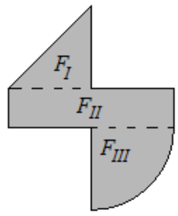
The shape may be represented as a sum of three shapes:

$F = F_I + F_{II} + F_{III}$

F_I - rectangle

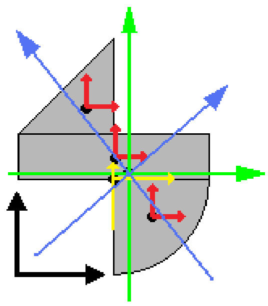
F_{II} - triangle

F_{III} - quarter of a circle



All exercises of this kind may be solved with the use of a simple otline (looking, however, complex at first glance). In order to use it we introduce a set of additional coordinate systems:

- ***global*** c.s. (**black**)
- ***local central*** c.s. of component shapes (**red**)
- for shapes being a sector of a circle:
  auxiliary *c.s.* in **the middle of circle** (**yellow**)
- ***central*** c.s. for the whole shape (**green**)
- ***principal central*** c.s. for the whole shape (**blue**)



Outline is as follows:

1. Calculate total surface area of the whole shape as a sum of areas of component shapes

   $A = \Sigma A_i$

2. Find centroids of all component shapes.

3. Find statical moments S_x, S_y for whole shape with respect to *global c.s.* (**black**) – each of them is a sum of statical moments of component shapes, which in turn are equal to a product of their area and appropriate coordinate of their centroid in *global c.s.* (**black**).

   $S_x = \Sigma S_{xi} \qquad S_{xi} = A_i \cdot \boldsymbol{y_{Oi}}$

   $S_y = \Sigma S_{yi} \qquad S_{yi} = A_i \cdot \boldsymbol{x_{Oi}}$

4. Find centroid of the whole shape in *global* (**black**) *c.s.*:

   $$\boldsymbol{x_O} = \frac{S_y}{A} \qquad \boldsymbol{y_O} = \frac{S_x}{A}$$

   In this point there is a *central c.s.* (**green**)*,* axes of which are parallel to the axes of *global c.s.* (**black**).

5. For each component shape find moments $\boldsymbol{I_{xi}}, \boldsymbol{I_{yi}}, \boldsymbol{D_{xyi}}$ in its *local central c.s.* (**red**). In case of shapes being a sector of a circle, first, find those moments in an *auxiliary c.s.* (**yellow**) and then, with the use of parallel axis theorem, find those values in *local central c.s.* (**red**) of the sector.

6. Find moments of the whole shape in its *central c.s.* (**green**) – they are sums of moments of component shapes $\boldsymbol{I_{Xi}}, \boldsymbol{I_{Yi}}, \boldsymbol{D_{XYi}}$ about *central c.s.* (**green**), calculated according to the

parallel axis theorem:

$$\boldsymbol{I}_X=\Sigma \boldsymbol{I}_{Xi} \qquad \boldsymbol{I}_{Xi}=\boldsymbol{I}_{xi}+A\cdot(\boldsymbol{y}_{Oi}-\boldsymbol{y}_O)^2$$
$$\boldsymbol{I}_Y=\Sigma \boldsymbol{I}_{Yi} \qquad \boldsymbol{I}_{Xi}=\boldsymbol{I}_{yi}+A\cdot(\boldsymbol{x}_{Oi}-\boldsymbol{x}_O)^2$$
$$\boldsymbol{D}_{XY}=\Sigma \boldsymbol{D}_{XYi} \qquad \boldsymbol{D}_{XYi}=\boldsymbol{D}_{xyi}+A\cdot(\boldsymbol{y}_{Oi}-\boldsymbol{y}_O)(\boldsymbol{x}_{Oi}-\boldsymbol{x}_O)$$

It is not allowed to change the sequence of subtraction of coordinates of centroids – in case of deviation moments of inertia change of this sequence in only one of the brackets lead to a mistake! NOTE: Some coordinates may be negative – the sign has to be accounted for in calculation.

7. We construct a tensor of moment of inertia and we find its eigenvalues and orientation of its principal axes – the axes of *principal central c.s.* (**blue**).

$$\boldsymbol{I}_{max}=\frac{\boldsymbol{I}_X+\boldsymbol{I}_Y}{2}+\sqrt{\left(\frac{\boldsymbol{I}_X-\boldsymbol{I}_Y}{2}\right)^2+\boldsymbol{D}_{XY}^2}, \quad \boldsymbol{I}_{min}=\frac{\boldsymbol{I}_X+\boldsymbol{I}_Y}{2}-\sqrt{\left(\frac{\boldsymbol{I}_X-\boldsymbol{I}_Y}{2}\right)^2+\boldsymbol{D}_{XY}^2}, \quad \operatorname{tg}\varphi=\frac{\boldsymbol{D}_{XY}}{\boldsymbol{I}_Y-\boldsymbol{I}_{max}}$$

Surface area of the whole shape:

$$A=\left[\frac{1}{2}\cdot 2\cdot 2\right]+[4\cdot 1]+\left[\frac{\pi\cdot 2^2}{4}\right]\approx 9{,}1416$$

Statical moments of the whole shape with respect to global c.s.:

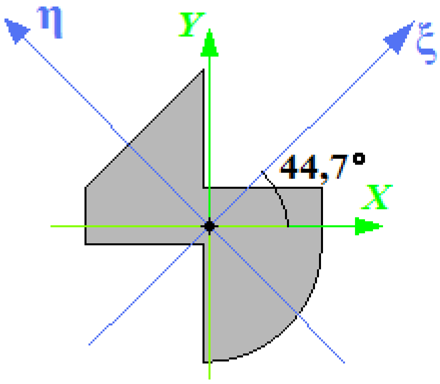
$$S_x=\left[\frac{1}{2}\cdot 2\cdot 2\cdot\left(3+\frac{1}{3}\cdot 2\right)\right]+\left[4\cdot 1\cdot\left(2+\frac{1}{2}\cdot 1\right)\right]+\left[\frac{\pi\cdot 2^2}{4}\cdot\left(2-\frac{4}{3}\frac{2}{\pi}\right)\right]\approx 20{,}9499$$
$$S_y=\left[\frac{1}{2}\cdot 2\cdot 2\cdot\left(\frac{2}{3}\cdot 2\right)\right]+[4\cdot 1\cdot 2]+\left[\frac{\pi\cdot 2^2}{4}\cdot\left(2+\frac{4}{3}\frac{2}{\pi}\right)\right]\approx 19{,}6165$$

Location of centroid of the whole shape: $x_O=\dfrac{S_y}{A}=2{,}1458 \qquad y_O=\dfrac{S_x}{A}=2{,}2917$

Central moments of inertia:

$$I_X=\left[\frac{2\cdot 2^3}{36}+\frac{1}{2}\cdot 2\cdot 2\cdot\left(3+\frac{1}{3}\cdot 2-2{,}2917\right)^2\right]+\left[\frac{4\cdot 1^3}{12}+4\cdot 1\cdot\left(2+\frac{1}{2}\cdot 1-2{,}2917\right)^2\right]+$$
$$+\left[\frac{\pi\cdot 2^4}{16}-\frac{\pi\cdot 2^2}{4}\cdot\left(\frac{4}{3}\frac{2}{\pi}\right)^2+\frac{\pi\cdot 2^2}{4}\cdot\left(2-\frac{4}{3}\frac{2}{\pi}-2{,}2917\right)^2\right]\approx 9{,}6970$$
$$I_Y=\left[\frac{2\cdot 2^3}{36}+\frac{1}{2}\cdot 2\cdot 2\cdot\left(\frac{2}{3}\cdot 2-2{,}1458\right)^2\right]+\left[\frac{1\cdot 4^3}{12}+4\cdot 1\cdot(2-2{,}1458)^2\right]+$$
$$+\left[\frac{\pi\cdot 2^4}{16}-\frac{\pi\cdot 2^2}{4}\cdot\left(\frac{4}{3}\frac{2}{\pi}\right)^2+\frac{\pi\cdot 2^2}{4}\cdot\left(2+\frac{4}{3}\frac{2}{\pi}-2{,}1458\right)^2\right]\approx 9{,}6138$$
$$D_{XY}=\left[+\frac{2^2\cdot 2^2}{72}+\frac{1}{2}\cdot 2\cdot 2\cdot\left(\frac{2}{3}\cdot 2-2{,}1458\right)\left(3+\frac{1}{3}\cdot 2-2{,}2917\right)\right]+$$
$$+\left[0+4\cdot 1\cdot(2-2{,}1458)\left(2+\frac{1}{2}\cdot 1-2{,}2917\right)\right]+$$
$$+\left[\frac{-2^4}{8}-\frac{\pi\cdot 2^2}{4}\left(\frac{4}{3}\frac{2}{\pi}\right)\left(-\frac{4}{3}\frac{2}{\pi}\right)+\frac{\pi\cdot 2^2}{4}\cdot\left(2+\frac{4}{3}\frac{2}{\pi}-2{,}1458\right)\left(2-\frac{4}{3}\frac{2}{\pi}-2{,}2917\right)\right]\approx -4{,}3889$$

Principal moments of inertia:



$$I_\xi = I_{max} = \frac{I_X + I_Y}{2} + \sqrt{\left(\frac{I_X - I_Y}{2}\right)^2 + D_{XY}^2} = 14{,}045$$

$$I_\eta = I_{min} = \frac{I_X + I_Y}{2} - \sqrt{\left(\frac{I_X - I_Y}{2}\right)^2 + D_{XY}^2} = 5{,}2663$$

Orientation of axis of maximum moment of inertia:

$$\phi = \mathrm{arctg}\frac{D_{XY}}{I_Y - I_{max}} = \mathrm{arctg}(0{,}9903) = 44{,}728^\circ$$

Components of the tensor of moment of inertia in a coordinate system inclined at angle 45° to the principal central axes may be found with the use of following formulae:

$$\begin{aligned} I_{x'} &= I_\xi \cos^2\alpha + I_\eta \sin^2\alpha - 2 D_{\xi\eta} \sin\alpha\cos\alpha \\ I_{y'} &= I_\xi \sin^2\alpha + I_\eta \cos^2\alpha + 2 D_{\xi\eta} \sin\alpha\cos\alpha \\ D_{x'y'} &= \frac{I_\xi - I_\eta}{2}\sin 2\alpha + D_{\xi\eta}\cos 2\alpha \end{aligned}$$

where:

$$\alpha = 45^\circ \quad \Rightarrow \quad \begin{array}{ll} \sin\alpha = \frac{1}{\sqrt{2}} & \cos\alpha = \frac{1}{\sqrt{2}} \\ \sin 2\alpha = 1 & \cos 2\alpha = 0 \end{array}$$

$$I_{x'} = 13{,}9054 \cdot \frac{1}{2} + 5{,}4055 \cdot \frac{1}{2} - 2 \cdot 0 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = 9{,}6557$$

$$I_{y'} = 13{,}9054 \cdot \frac{1}{2} + 5{,}4055 \cdot \frac{1}{2} + 2 \cdot 0 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = 9{,}6557$$

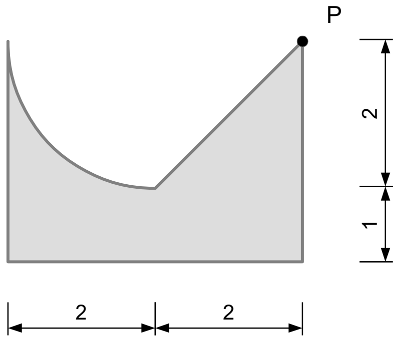
$$D_{x'y'} = \frac{13{,}9054 - 5{,}4055}{2} \cdot 1 + 0 \cdot 0 = 4{,}3894$$

Orientation at angle 45° with respect to principal axes is such an orientation for which:

- moments of inertia about both axes are both equal an arithmetic mean of principal moments of inertia.
- Deviation moment of inertia has an extreme (maximum or minimum) value

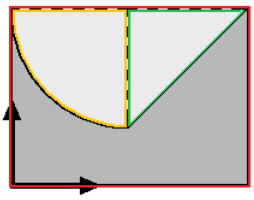
EXERCISE 16

Find principal central moments of inertia and orientation of principal central axes of inertia of a shape shown in the figure. Then, find moment of inertia about an axis passing through point P and inclined at angle 50° to the horizontal axis (conterclockwise)



SOLUTION:

Division of shape into component simple shapes and choice of global c.s.:



Surface area:

$$A = [4\cdot 3]_{pr} - \left[\frac{1}{2}\cdot 2\cdot 2\right]_{tr} - \left[\frac{\pi\cdot 2^2}{4}\right]_{ck} = 10-\pi = 6{,}8584$$

Statical moments and coordinates of centroid:

$$S_x = \left[4\cdot 3\cdot\frac{3}{2}\right]_{pr} - \left[\frac{1}{2}\cdot 2\cdot 2\cdot\left(1+\frac{2}{3}\cdot 2\right)\right]_{tr} - \left[\frac{\pi\cdot 2^2}{4}\cdot\left(3-\frac{4}{3}\frac{2}{\pi}\right)\right]_{ck} = 6{,}5752 \qquad y_O = \frac{S_x}{A} = 0{,}9587$$

$$S_y = [4\cdot 3\cdot 2]_{pr} - \left[\frac{1}{2}\cdot 2\cdot 2\cdot\left(2+\frac{1}{3}\cdot 2\right)\right]_{tr} - \left[\frac{\pi\cdot 2^2}{4}\cdot\left(2-\frac{4}{3}\frac{2}{\pi}\right)\right]_{ck} = 3{,}6165 \qquad x_O = \frac{S_y}{A} = 2{,}1944$$

Central moments of inertia:

$$I_X = \left[\frac{4\cdot 3^3}{12} + 4\cdot 3\cdot\left(\frac{3}{2}-0{,}9587\right)^2\right]_{pr} - \left[\frac{2\cdot 2^3}{36} + \frac{1}{2}\cdot 2\cdot 2\cdot\left(1+\frac{2}{3}\cdot 2-0{,}9587\right)^2\right]_{tr} - \left[\frac{\pi\cdot 2^4}{16} - \frac{\pi\cdot 2^2}{4}\cdot\left(\frac{4}{3}\frac{2}{\pi}\right)^2 + \frac{\pi\cdot 2^2}{4}\cdot\left(3-\frac{4}{3}\frac{2}{\pi}-0{,}9587\right)^2\right]_{ck} = 2{,}9425$$

$$I_Y = \left[\frac{4^3\cdot 3}{12} + 4\cdot 3\cdot(2-2{,}1944)^2\right]_{pr} - \left[\frac{2^3\cdot 2}{36} + \frac{1}{2}\cdot 2\cdot 2\cdot\left(2+\frac{1}{3}\cdot 2-2{,}1944\right)^2\right]_{tr} - \left[\frac{\pi\cdot 2^4}{16} - \frac{\pi\cdot 2^2}{4}\cdot\left(\frac{4}{3}\frac{2}{\pi}\right)^2 + \frac{\pi\cdot 2^2}{4}\cdot\left(2-\frac{4}{3}\frac{2}{\pi}-2{,}1944\right)^2\right]_{ck} = 11{,}2659$$

$$D_{XY} = \left[0+4\cdot 3\cdot\left(\frac{3}{2}-0{,}9587\right)(2-2{,}1944)\right]_{pr} - \left[\frac{2^2\cdot 2^2}{72} + \frac{1}{2}\cdot 2\cdot 2\cdot\left(1+\frac{2}{3}\cdot 2-0{,}9587\right)\left(2+\frac{1}{3}\cdot 2-2{,}1944\right)\right]_{tr} - \left[\frac{\pi\cdot 2^4}{16} - \frac{\pi\cdot 2^2}{4}\cdot\left(-\frac{4}{3}\frac{2}{\pi}\right)\left(-\frac{4}{3}\frac{2}{\pi}\right) + \frac{\pi\cdot 2^2}{4}\cdot\left(3-\frac{4}{3}\frac{2}{\pi}-0{,}9587\right)\left(2-\frac{4}{3}\frac{2}{\pi}-2{,}1944\right)\right]_{ck} = 1{,}3884$$

Central moments of inertia may be found also in a different way:

- Find components of the tensor of moment of inertia from each component shape with respect to an arbitrary chosen global coordinate system (with the use of parallel axis theorem)
- Find components of the tensor of moment of inertia in central coordinate system with the use of parallel axis theorem.

Moments of inertia with respect to axes of global coordinate system:

$$I_{X'}=\left[\frac{4\cdot 3^3}{12}+4\cdot 3\cdot\left(\frac{3}{2}\right)^2\right]-\left[\frac{2\cdot 2^3}{36}+\frac{1}{2}\cdot 2\cdot 2\cdot\left(1+\frac{2}{3}\cdot 2\right)^2\right]-\dots$$
$$\dots-\left[\frac{\pi\, 2^4}{16}-\frac{\pi\cdot 2^2}{4}\cdot\left(\frac{4}{3}\cdot\frac{2}{\pi}\right)^2+\frac{\pi\cdot 2^2}{4}\cdot\left(3-\frac{4}{3}\cdot\frac{2}{\pi}\right)^2\right]=9{,}251$$

$$I_{Y'}=\left[\frac{4^3\cdot 3}{12}+4\cdot 3\cdot\left(\frac{4}{2}\right)^2\right]-\left[\frac{2^3\cdot 2}{36}+\frac{1}{2}\cdot 2\cdot 2\cdot\left(2+\frac{1}{3}\cdot 2\right)^2\right]-\dots$$
$$\dots-\left[\frac{\pi\, 2^4}{16}-\frac{\pi\cdot 2^2}{4}\cdot\left(\frac{4}{3}\cdot\frac{2}{\pi}\right)^2+\frac{\pi\cdot 2^2}{4}\cdot\left(2-\frac{4}{3}\cdot\frac{2}{\pi}\right)^2\right]=44{,}2920$$

$$D_{X'Y'}=\left[0+4\cdot 3\cdot\left(\frac{4}{2}\right)\left(\frac{3}{2}\right)\right]-\left[\frac{2^2\cdot 2^2}{72}+\frac{1}{2}\cdot 2\cdot 2\cdot\left(2+\frac{1}{3}\cdot 2\right)\left(1+\frac{2}{3}\cdot 2\right)\right]-\dots$$
$$\dots-\left[\frac{2^4}{8}-\frac{\pi\cdot 2^2}{4}\cdot\left(\frac{4}{3}\cdot\frac{2}{\pi}\right)^2+\frac{\pi\cdot 2^2}{4}\cdot\left(2-\frac{4}{3}\cdot\frac{2}{\pi}\right)\left(3-\frac{4}{3}\cdot\frac{2}{\pi}\right)\right]=15{,}8171$$

Central moments of inertia:

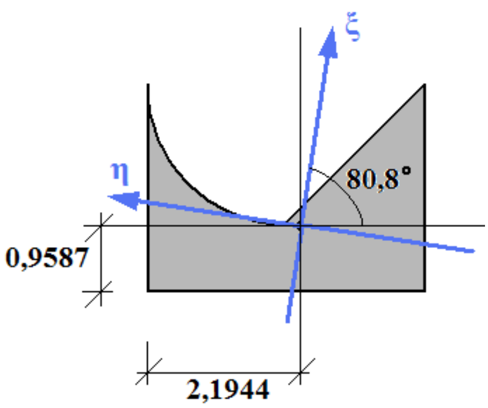
$$I_X=I_{X'}-A\,y_O^2=9{,}2507-6{,}8584\cdot(0{,}9587)^2=2{,}9471$$
$$I_Y=I_{Y'}-A\,x_O^2=44{,}2920-6{,}8584\cdot(2{,}1944)^2=11{,}2662$$
$$D_{XY}=D_{X'Y'}-A\,x_O\,y_O=15{,}8171-6{,}8584\cdot(2{,}1944\cdot 0{,}9587)=1{,}3886$$

Principal central moments of inertia:

$$I_\xi=I_{max}=\frac{I_X+I_Y}{2}+\sqrt{\left(\frac{I_X-I_Y}{2}\right)^2+D_{XY}^2}=11{,}4914$$
$$I_\eta=I_{min}=\frac{I_X+I_Y}{2}-\sqrt{\left(\frac{I_X-I_Y}{2}\right)^2+D_{XY}^2}=2{,}7170$$

Orientation of axis of maximum moment of inertia:

$$\varphi=\mathrm{arctg}\frac{D_{XY}}{I_Y-I_{max}}=\mathrm{arctg}(6{,}1570)=80{,}775^\circ$$



Components of the tensor of moment of inertia about horizontal and vertical axis passing through point P may be found with the use of parallel axis theorem:

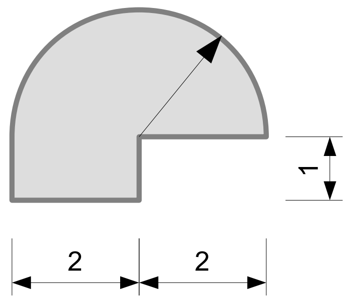
$$I_{x,P} = I_X + A\cdot(y_P - y_O)^2 = 2{,}9425 + 6{,}8584\cdot(3-0{,}9587)^2 = 31{,}5208$$
$$I_{y,P} = I_Y + A\cdot(x_P - x_O)^2 = 11{,}2659 + 6{,}8584\cdot(4-2{,}1944)^2 = 33{,}6256$$
$$D_{xy,P} = D_{XY} + A\cdot(x_P - x_O)(y_P - y_O) = 1{,}3884 + 6{,}8584\cdot(4-2{,}1944)\cdot(3-0{,}9587) = 26{,}6669$$

Moment of inertia about an axis passing through point P and inclined to the horizontal axis at angle 50°:

$$I_L = I_{x,P}\cos^2(50^\circ) + I_{y,P}\sin^2(50^\circ) - D_{xy,P}\sin(100^\circ) = 6{,}4942$$

EXERCISE 17

Find principal central moment of inertia and orientation of principal central axes of inertia of a shape shown in the figure. Then, find a moment of inertia about an axis passing through centroid and inclined to the horizontal axis at angle 30°.

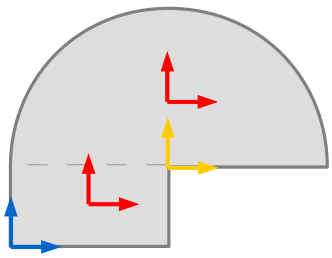


SOLUTION:

Surface area:

$$A = [2\cdot 1] + \left[\frac{\pi\cdot 2^2}{2}\right] \approx 8{,}2832$$

Statical moments:



$$S_x = \left[2\cdot 1\cdot\left(\frac{1}{2}\cdot 1\right)\right] + \left[\frac{\pi\cdot 2^2}{2}\cdot\left(1+\frac{4}{3}\frac{2}{\pi}\right)\right] \approx 12{,}6165$$
$$S_y = \left[2\cdot 1\cdot\left(\frac{1}{2}\cdot 2\right)\right] + \left[\frac{\pi\cdot 2^2}{2}\cdot 2\right] \approx 14{,}5664$$

Location of centroid: $x_O = \dfrac{S_y}{A} \approx 1{,}7585 \qquad y_O = \dfrac{S_x}{A} \approx 1{,}5231$

Moments of inertia:

$$I_X = \left[\frac{2\cdot 1^3}{12} + 1\cdot 2\cdot\left(\frac{1}{2}\cdot 1 - 1{,}5231\right)^2\right] + \left[\frac{\pi\cdot 2^4}{8} - \frac{\pi\cdot 2^2}{2}\cdot\left(\frac{4}{3}\frac{2}{\pi}\right)^2 + \frac{\pi\cdot 2^2}{2}\cdot\left(1+\frac{4}{3}\frac{2}{\pi} - 1{,}5231\right)^2\right] \approx 4{,}6829$$

NOTE: Y axis of an auxiliary c.s. (yellow) coincide with central axis of a half of circle – for this reason we don't have to use the parallel axis theorem in order to find I_Y :

$$I_Y=\left[\frac{1\cdot 2^3}{12}+1\cdot 2\cdot\left(\frac{1}{2}\cdot 2-1{,}7585\right)^2\right]+\left[\frac{\pi\cdot 2^4}{8}+\frac{\pi\cdot 2^2}{2}\cdot(2-1{,}7585)^2\right]\approx 8{,}4669$$

NOTE: Y axis of an auxiliary c.s. (yellow) coincide with central axis of a half of circle – for this reason, when calculating central deviation moment of inertia for hald of a circle, one component of a vector connecting the old coordinate system with the new one (yellow and red), which is present in the parallel axis theorem, is equal 0. Central deviation moment of inertia (in red c.s.) is thus the same as in the auxiliary c.s. (yellow) and we don't have to use the parallel axis theorem twice:

$$D_{XY}=\left[0+1\cdot 2\cdot\left(\frac{1}{2}\cdot 1-1{,}5231\right)\left(\frac{1}{2}\cdot 2-1{,}7585\right)\right]+\left[0+\frac{\pi\cdot 2^2}{2}\cdot\left(1+\frac{4}{3}\frac{2}{\pi}-1{,}5231\right)(2-1{,}7585)\right]\approx 2{,}0463$$

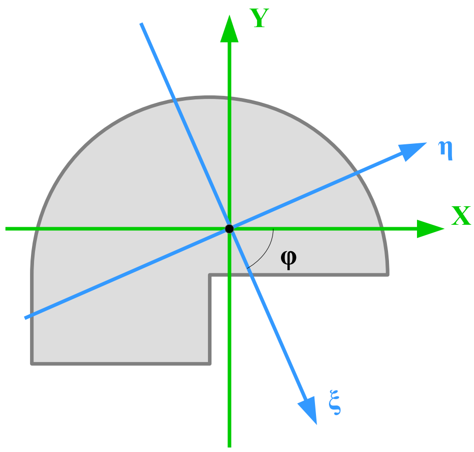
Principal moments of inertia:

$$I_\xi=I_{max}=\frac{I_X+I_Y}{2}+\sqrt{\left(\frac{I_X-I_Y}{2}\right)^2+D_{XY}^2}=9{,}3619$$

$$I_\eta=I_{min}=\frac{I_X+I_Y}{2}-\sqrt{\left(\frac{I_X-I_Y}{2}\right)^2+D_{XY}^2}=3{,}7880$$

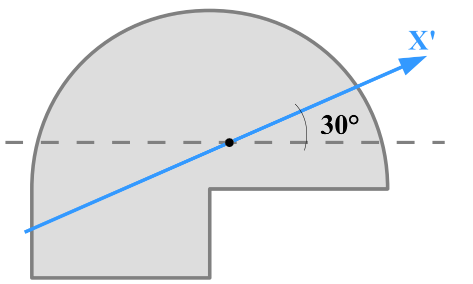
Orientation of an axis of maximum moment of inertia:

$$\phi=\mathrm{arctg}\frac{D_{XY}}{I_Y-I_{max}}=\mathrm{arctg}(-2{,}2866)=-66{,}3785^\circ$$



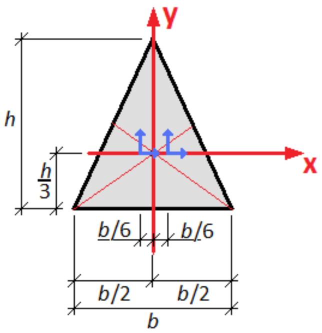
Moment of inertia about a central axis which is inclined to horizontal axis at angle 30° is equal:

$$I_{X'}=I_X\cos^2\alpha+I_Y\sin^2\alpha-2D_{XY}\sin\alpha\cos\alpha=$$
$$=4{,}6829\cdot\frac{3}{4}+8{,}4669\cdot\frac{1}{4}-2\cdot 2{,}0463\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{2}=3{,}857$$



EXERCISE 18

Find principal central moments of inertia for an isosceles triangle of base *b* and height *h*. Consider a special case of equilateral triangle.



SOLUTION:

Isosceles triangle has a symmetry axis, so it is one of principal central axes of inertia. The second one must be perpendicular to it and must pass through centroid which in this case is located on the symmetry axis at 1/3 of height of the triangle.

Principal central moments of inertia:

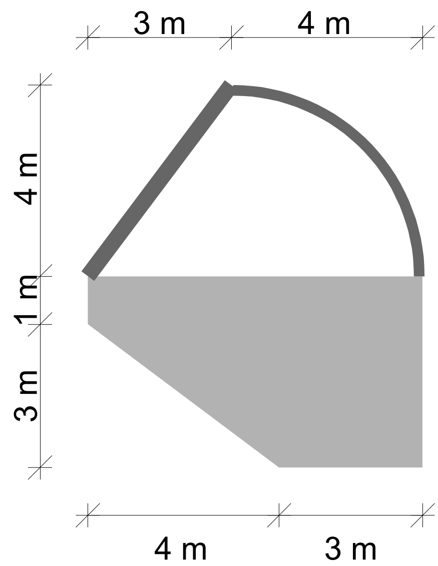
$$I_x=\left[\frac{(b/2)\cdot h^3}{36}\right]+\left[\frac{(b/2)\cdot h^3}{36}\right]=\frac{b\cdot h^3}{36}$$

$$I_y=\left[\frac{(b/2)^3\cdot h}{36}+\frac{1}{2}\cdot\frac{b}{2}\cdot h\cdot\left(\frac{1}{3}\cdot\frac{b}{2}\right)^2\right]+\left[\frac{(b/2)^3\cdot h}{36}+\frac{1}{2}\cdot\frac{b}{2}\cdot h\cdot\left(\frac{1}{3}\cdot\frac{b}{2}\right)^2\right]=\frac{b^3 h}{48}$$

For equilateral triangle we have: $b=a\,,\ h=\dfrac{a\sqrt{3}}{2}\quad\Rightarrow\quad I_x=I_y=\dfrac{a^4\sqrt{3}}{96}$

EXERCISE 19

Find principal central moments of inertia for a shape shown in the figure. Assume that surface density is $\rho=3\,\mathrm{kg/m^2}$ while linear density is $\mu=5\,\mathrm{kg/m^3}$.



SOLUTION:

Let's take a global coordinate system in the bottom-left corner of the picture.

Shapes and their characteristics:

L.P.	SHAPE	DENSITY	AREA / LENGTH	x_O	y_O
1	Rectangle	3	4·7=28	3,5	2
2	Triangle (subtracted)	3	0,5·3·4=6	4 : 3 = 1,333	1
3	Straight rod	5	5	1,5	6
4	Circular rod	5	0,25·2·π·4=6,283	3+2·4:π=5,546	4+2·4:π=6,546

The straight rod is inclined to horizontal axis at angle: $\psi = \mathrm{arctg}\dfrac{4}{3} \approx 0{,}9273 \approx 53{,}13^\circ$

The straight rod is inclined to vertical axis at angle: $90^\circ - \psi \approx 36{,}87^\circ$

L.P.	I_x	I_y	D_{xy}
1	$\dfrac{7\cdot 4^3}{12} = 37{,}333$	$\dfrac{4\cdot 7^3}{12} = 114{,}333$	0
2	$\dfrac{4\cdot 3^3}{36} = 3$	$\dfrac{3\cdot 4^3}{36} = 5{,}333$	$-\dfrac{4^2\cdot 3^2}{72} = -2$
3	$\dfrac{5^3}{12}\sin^2(53{,}13^\circ) = 6{,}667$	$\dfrac{5^3}{12}\sin^2(36{,}87^\circ) = 3{,}750$	$\dfrac{+5^3}{12}\sin(53{,}13^\circ)\cos(53{,}13^\circ) = +5$
4	$\dfrac{\pi 4^3}{4} = 50{,}265$	$\dfrac{\pi 4^3}{4} = 50{,}265$	$+\dfrac{4^3}{2} = +32$

Total mass: $m = 3\cdot[28] - 3\cdot[6] + 5\cdot[5] + 5\cdot[6{,}283] = 122{,}415$

Statical moments:

$$S_x = 3\cdot[28\cdot 2] - 3\cdot[6\cdot 1] + 5\cdot[5\cdot 6] + 5\cdot[6{,}283\cdot 6{,}546] = 505{,}643$$
$$S_y = 3\cdot[28\cdot 3{,}5] - 3\cdot[6\cdot 1{,}333] + 5\cdot[5\cdot 1{,}5] + 5\cdot[6{,}283\cdot 5{,}546] = 481{,}734$$

Location of centroid: $x_O = 3{,}935 \qquad y_O = 4{,}131$

Moments of inertia:

$$I_x = 3\cdot\left[37{,}333 + 28\cdot(2-4{,}131)^2\right] - 3\cdot\left[3+6\cdot(1-4{,}131)^2\right] + 5\cdot\left[6{,}667+5\cdot(6-4{,}131)^2\right] +$$
$$+ 5\cdot\left[50{,}265 - 6{,}283\cdot\left(\frac{2\cdot 4}{\pi}\right)^2 + 6{,}283\cdot(6{,}546-4{,}131)^2\right] =$$
$$= 493{,}457 - 185{,}457 + 120{,}664 + 230{,}832 = 659{,}496$$

$$I_y = 3\cdot\left[114{,}333 + 28\cdot(3{,}5-3{,}935)^2\right] - 3\cdot\left[5{,}333+6\cdot(1{,}333-3{,}935)^2\right] +$$
$$+ 5\cdot\left[3{,}750 + 5\cdot(1{,}5-3{,}935)^2\right] + 5\cdot\left[50{,}265 - 6{,}283\cdot\left(\frac{2\cdot 4}{\pi}\right)^2 + 6{,}283\cdot(5{,}546-3{,}935)^2\right] =$$
$$= 358{,}893 - 137{,}866 + 166{,}981 + 129{,}145 = 517{,}153$$

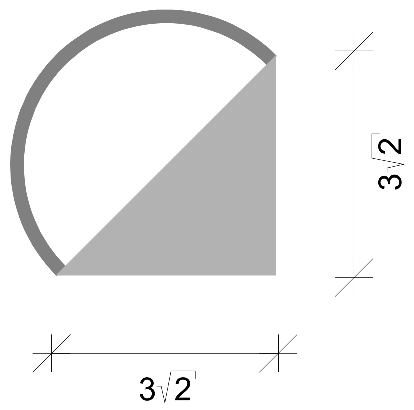
$$D_{xy} = 3\cdot\left[0 + 28\cdot(2-4{,}131)(3{,}5-3{,}935)\right] - 3\cdot\left[-2+6\cdot(1-4{,}131)(1{,}333-3{,}935)\right] +$$
$$+ 5\cdot\left[+5+5\cdot(6-4{,}131)(1{,}5-3{,}935)\right] +$$
$$+ 5\cdot\left[+32 - 6{,}283\cdot\left(\frac{2\cdot 4}{\pi}\right)\left(\frac{2\cdot 4}{\pi}\right) + 6{,}283\cdot(6{,}546-4{,}131)(5{,}546-3{,}935)\right] =$$
$$= 77{,}867 - 140{,}644 - 88{,}775 + 78{,}889 = -72{,}662$$

Principal moments of inertia and orientation of principal axes:

$$I_{max} = \frac{I_x + I_y}{2} + \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + D_{xy}^2} = 588{,}325 + 101{,}711 = 690{,}036$$
$$I_{min} = \frac{I_x + I_y}{2} - \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + D_{xy}^2} = 588{,}325 - 101{,}711 = 486{,}614$$
$$\phi = \mathrm{arctg}\frac{D_{xy}}{I_y - I_{max}} \approx 22{,}80^\circ$$

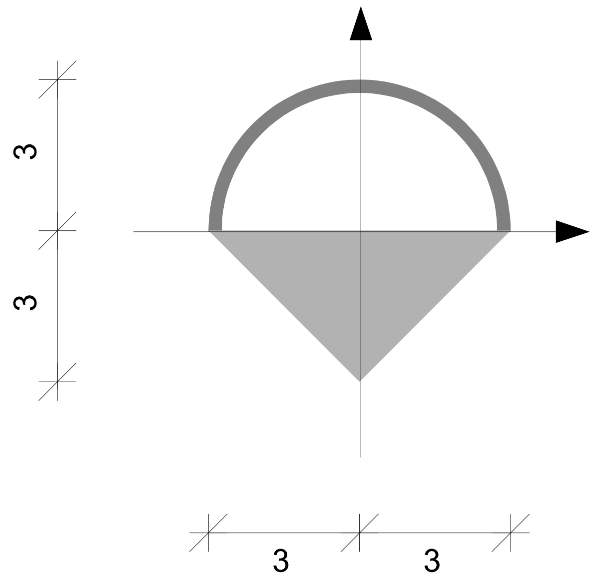
EXERCISE 20

Find principal central moments of inertia for a shape shown in the figure. Assume that surface density is $\rho = 1\ \mathrm{kg/m^2}$ and linear density is $\mu = 3\ \mathrm{kg/m^2}$.



SOLUTION:

The solution will simplify in great extent if we notice that considered shape has a symmetry axis – it is thus one of principal central axes. Let's rotate the whole shape in such a way that this axis was e.g. a vertical axis. Then, we will introduce an auxiliary horizontal axis about which we will calculate statical moment in order to find the centroid.



Surface area: $$A = 3\left[\frac{2\cdot\pi\cdot 3}{2}\right] + 2\cdot 1\cdot\left[\frac{1}{2}\cdot 3\cdot 3\right] = 37{,}27$$

Statical moment about auxiliary axis:

$$S_X = 3\cdot\left[\frac{2\cdot\pi\cdot 3}{2}\cdot\frac{2\cdot 3}{\pi}\right] + 2\cdot 1\cdot\left[\left(\frac{1}{2}\cdot 3\cdot 3\right)\cdot(-1)\right] = 45$$

Location of centroid with respect to the auxiliary axis: $$y = \frac{S_X}{A} = 1{,}20$$

Principal central moments of inertia:

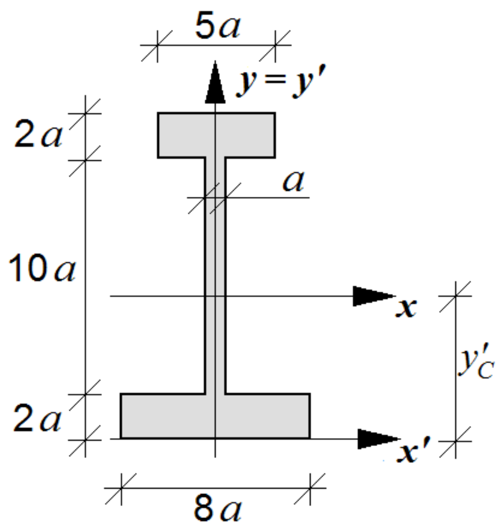
$$I_X = 3\cdot\left[\frac{\pi\cdot 3^3}{2} - 3\pi\cdot\left(\frac{2\cdot 3}{\pi}\right)^2 + 3\pi\cdot\left(\frac{2\cdot 3}{\pi} - 1{,}20\right)^2\right] + 2\cdot 1\cdot\left[\frac{3\cdot 3^3}{36} + \frac{1}{2}\cdot 3\cdot 3\cdot(-1 - 1{,}20)^2\right] = 86{,}41$$

$$I_Y = 3\cdot\left[\frac{\pi\cdot 3^3}{2}\right] + 2\cdot 1\cdot\left[\frac{3\cdot 3^3}{36} + \frac{1}{2}\cdot 3\cdot 3\cdot(1)^2\right] = 140{,}73$$

Since these are central principal axes the deviation moments of inertia are equal 0.

EXERCISE 21

Find principal central moments of inertia for an I-section shown in the figure.



SOLUTION:

Since the shape is symmetric, centroid must lie on symmetry axis which is also one of principal central axes of inertia – the second one must be perpendicular to it. It is enough to find an *y'* coordinate of centroid in any coordinate system (*x'* , *y'*), e.g. such a c.s., horizontal axis of which coincide with bottom edge of section. Having found location of centroid (by calculating statical moment about *x'*) we can calculate moments of inertia about principal central axes (passing through centroid) – in such a coordinate system deviation moments of inertia are equal 0.

Area of cross-section: $A=[8a\cdot 2a]+[10a\cdot a]+[2a\cdot 5a]=36a^2$

Statical moment about *x'* axis containing bottom edge of the section:

$$S_{x'}=[8a\cdot 2a\cdot a]+[10a\cdot a\cdot 7a]+[2a\cdot 5a\cdot 13a]=216a^3$$

Distance of a centroid from the bottom edge: $y_C{}'=\dfrac{S_{x'}}{A}=6a$

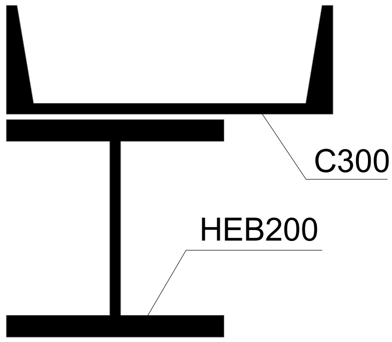
Principal central moments of inertia:

$$I_x=\left[\frac{8a\cdot(2a)^3}{12}+8a\cdot 2a\cdot(a-6a)^2\right]+\left[\frac{a\cdot(10a)^3}{12}+10a\cdot a\cdot(7a-6a)^2\right]+\dots$$
$$\dots+\left[\frac{5a\cdot(2a)^3}{12}+5a\cdot 2a\cdot(13a-6a)^2\right]=992a^4$$

$$I_y=\left[\frac{2a\cdot(8a)^3}{12}\right]+\left[\frac{10a\cdot a^3}{12}\right]+\left[\frac{2a\cdot(5a)^3}{12}\right]=107a^4$$

EXERCISE 22

Find principal central moments of inertia of a section built up of hot rolled section as it is shown in the figure.



SOLUTION:

Section charts give us:

HEB200:

$b = 200 \text{ mm}$

$h = 200 \text{ mm}$

$A = 78{,}1 \text{ cm}^2$

$I_x = 5700 \text{ cm}^4$

$I_y = 2000 \text{ cm}^4$

C300:

$b = 100 \text{ mm}$

$h = 300 \text{ mm}$

$e = 27 \text{ mm}$

$A = 58{,}8 \text{ cm}^2$

$I_x = 8030 \text{ cm}^4$

$I_y = 493 \text{ cm}^4$

Surface area: $A = [78{,}1] + [58{,}8] = 136{,}9$

Statical moments:

$$S_x = [78{,}1\cdot 10] + [58{,}8\cdot(20+2{,}7)] = 2115{,}76$$
$$S_y = [78{,}1\cdot 10] + [58{,}8\cdot 15] = 1663$$

Location of centroid:

$$x = S_y / A = 12{,}15$$
$$y = S_z / A = 15{,}45$$

Moments of inertia:

$$I_x = \left[5700 + 78{,}1\cdot(10-15{,}45)^2\right] + \left[493 + 58{,}8\cdot(22{,}7-15{,}45)^2\right] = 11603{,}44$$
$$I_y = \left[2000 + 78{,}1\cdot(10-12{,}15)^2\right] + \left[8030 + 58{,}8\cdot(15-12{,}15)^2\right] = 10868{,}62$$
$$D_{xy} = \left[0 + 78{,}1(10-15{,}45)(10-12{,}15)\right] + \left[0 + 58{,}8(22{,}7-15{,}45)(15-12{,}15)\right] = 2130{,}09$$

Principal central moments of inertia:

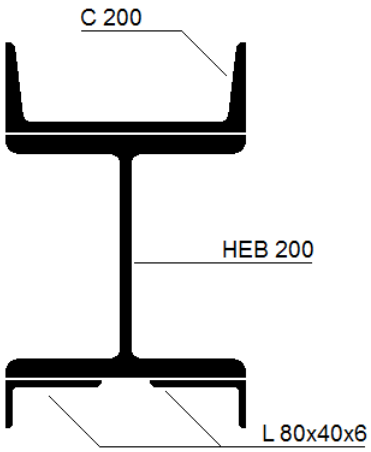
$$I_{max} = \frac{I_x + I_y}{2} + \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + D_{xy}^2} = 13397{,}57$$
$$I_{min} = \frac{I_x + I_y}{2} - \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + D_{xy}^2} = 9074{,}49$$

Orientation of principal central axes of inertia:

$$\phi = \operatorname{arctg} \frac{D_{xy}}{I_y - I_{max}} \approx -40{,}11^\circ$$

EXERCISE 23

Find principal central moments of inertia of a section built up of hot rolled section as it is shown in the figure.



SOLUTION:

Since the section is symmetric, centroid must lie on a symmetry axis, which is also one the principal central axes of inertia – the second one is perpendicular to it.

Characteristics of component sections:

WARNING*: When reading data from section charts it is necessary to check what is the orientation of axes of inertia in the charts that are used – it concernes especialle unequal angles (non-symmetric L-sections).*

	HEB 200	C 200	L 80x40x6
Surface area	$A=78{,}1\ \mathrm{cm}^2$	$A=32{,}2\,\mathrm{cm}^2$	$A=6{,}89\,\mathrm{cm}^2$
Moments of inertia	$I_x=5700\,\mathrm{cm}^4$	$I_x=148\,\mathrm{cm}^4$	$I_x=7{,}59\,\mathrm{cm}^4$
	$I_y=2000\,\mathrm{cm}^4$	$I_y=1910\,\mathrm{cm}^4$	$I_y=44{,}9\,\mathrm{cm}^4$

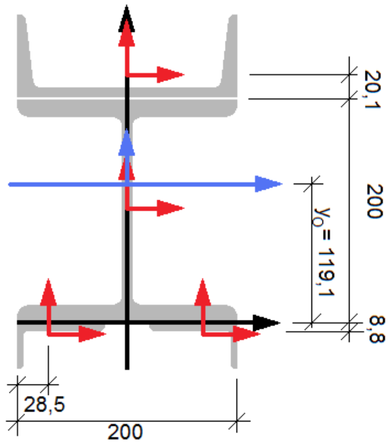
Location of centroids in chosen global coordinate system:

HEB 200 $\qquad x_O=0\,\mathrm{mm} \qquad y_O=\frac{1}{2}h_{HEB}=100\,\mathrm{mm}$

C 200 $\qquad x_O=0\,\mathrm{mm} \qquad y_O=h_{HEB}+e_C=220{,}1\ \mathrm{mm}$

L 80x40x6 $\qquad x_O=\pm\left(\frac{b_{HEB}}{2}-e_{Lx}\right)=\pm(200-28{,}5)=\pm171{,}5\,\mathrm{mm}$

$$y_O=-e_{Ly}=-8{,}8\,\mathrm{mm}$$



Area of cross-section:

$$A=78{,}1+32{,}2+2\cdot 6{,}89=124{,}08\quad[\mathrm{cm}^2]$$

Statical moment:

$$S_x=[78{,}1\cdot 10]+[32{,}2\cdot 22{,}01]+2\cdot[6{,}89\cdot(-0{,}88)]=1477{,}60\quad[\mathrm{cm}^3]$$

Location of centroid: $x_O=0 \qquad y_O=\frac{S_x}{A}=11{,}91\quad[\mathrm{cm}]$

Moments of inertia:

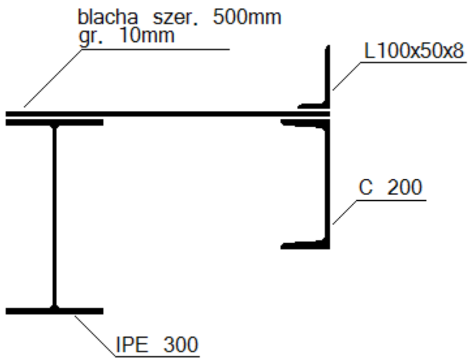
$$I_x=\left[5700+78{,}1\cdot(10-11{,}91)^2\right]+\left[148+32{,}2\cdot(22{,}01-11{,}91)^2\right]+\ldots$$

$$\ldots+2\cdot\left[7{,}59+6{,}89\cdot(-0{,}88-11{,}91)^2\right]=5984{,}92+3432{,}72+2269{,}37=11687{,}01\quad[\mathrm{cm}^4]$$

$$I_y=[2000]+[1910]+2\cdot\left[44{,}9+6{,}89\cdot(10-2{,}85)^2\right]=4704{,}27\quad[\mathrm{cm}^4]$$

EXERCISE 24

Find principal central moments of inertia and orientation of principal central axes of inertia of a non-symmetric section built up of hot rolled section as it is shown in the figure.



SOLUTION:

Geometrical characteristics:

<u>Economical I-section: IPE 300</u>

Height: $h_I = 300$ mm

Width: $b_I = 150$ mm

<u>C-section: C 200</u>

Height: $h_C = 200$ mm

Width: $b_C = 75$ mm

Distance of a centroid for external surface of web: $e_C = 2{,}01$ cm

<u>Sheet metal</u>

Width: $b_b = 500$ mm

Thickness: $t_b = 10$ mm

<u>Unequal angle: L100x50x8</u>

Length of long leg: $a_L = 100$ mm

Length of short leg: $b_L = 50$ mm

Distance of a centroid for external surface of long leg: $e_a = 1{,}13$ cm

Distance of a centroid for external surface of short leg: $e_b = 3{,}59$ cm

NOTE: It is common that for L-sections one cannot find in charts deviation moment of inertia in its central coordinate system of axes parallel to section's legs. This value can be found with the use of moments of inertia about those axes I_x, I_y and principal central moments of inertia I_{max}, I_{min}:

$$D_{xy} = \pm\sqrt{\left(\frac{I_x+I_y}{2} - I_{min}\right)^2 - \left(\frac{I_x-I_y}{2}\right)^2} = \pm\sqrt{\left(\frac{I_x+I_y}{2} - I_{max}\right)^2 - \left(\frac{I_x-I_y}{2}\right)^2}$$

Whether the sign „+" or „–" should be used, it is in a similar way as in case of triangle – if legs are placed in the 1st or 3rd quarter of central coordinate system of axes parallel to legs (namely corner of section is placed in 2nd or 4th quarter), then we take the „+" sign – otherwise, take „-".

Geometrical characteristics:

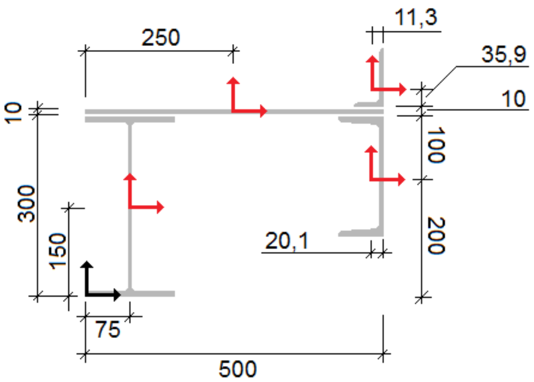
Section	A [cm^2]	I_x [cm^4]	I_y [cm^4]	D_{xy} [cm^4]
IPE 300	53,8	8360	604	0
C 200	32,2	1910	148	0
blacha	50	4,167	10417	0
L 100x50x8	11,5	116	19,5	26,5

Area of cross-section:

$$A = 53{,}8 + 32{,}2 + 50 + 11{,}5 = 147{,}5$$

Statical moments about axes of assumed global coordinate system:

$$S_x = [53{,}8 \cdot 15] + [32{,}2 \cdot 20] + [50 \cdot (30+0{,}5)] + [11{,}5 \cdot (30+1+3{,}59)] =$$
$$= 3373{,}785\,[\mathrm{cm}^3]$$
$$S_y = [53{,}8 \cdot 7{,}5] + [32{,}2 \cdot (50-2{,}01)] + [50 \cdot 25] + [11{,}5 \cdot (50-1{,}13)] =$$
$$= 3760{,}783\,[\mathrm{cm}^3]$$



Location of centroid:

$$x_O = \frac{S_y}{A} = 25{,}497\,\mathrm{cm} \qquad y_O = \frac{S_x}{A} = 22{,}873\,\mathrm{cm}$$

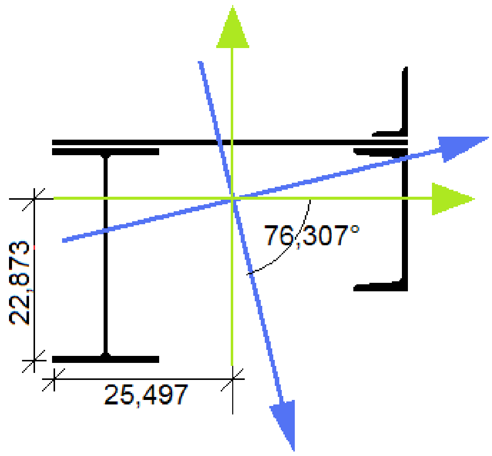
Moments of inertia:

$$I_x = [8360 + 53{,}8 \cdot (15-22{,}873)^2] + [1910 + 32{,}2 \cdot (20-22{,}873)^2] + [4{,}167 + 50 \cdot (30+0{,}5-22{,}873)^2] +$$
$$+ [116 + 11{,}5 \cdot (30+1+3{,}59-22{,}873)^2] = 11694{,}746 + 2175{,}783 + 2912{,}723 + 1694{,}813 =$$
$$= 18478{,}065\,[\mathrm{cm}^4]$$
$$I_y = [604 + 53{,}8 \cdot (7{,}5-25{,}497)^2] + [148 + 32{,}2 \cdot (50-2{,}01-25{,}497)^2] + [10417 + 50 \cdot (25-25{,}497)^2] +$$
$$+ [19{,}5 + 11{,}5 \cdot (50-1{,}13-25{,}497)^2] = 18029{,}390 + 16439{,}109 + 10429{,}350 + 6301{,}917 =$$
$$= 51199{,}766\,[\mathrm{cm}^4]$$

$$D_{xy} = [0 + 53{,}8 \cdot (15-22{,}873) \cdot (7{,}5-25{,}497)] + [0 + 32{,}2 \cdot (20-22{,}873) \cdot (50-2{,}01-25{,}497)] +$$
$$+ [0 + 50 \cdot (30+0{,}5-22{,}873) \cdot (25-25{,}497)] + [26{,}5 + 11{,}5 \cdot (30+1+3{,}59-22{,}873) \cdot (50-1{,}13-25{,}497)]$$
$$= 7622{,}942 - 2080{,}841 - 189{,}531 + 3122{,}567 = 8475{,}137\,[\mathrm{cm}^4]$$

Principal central moments of inertia:

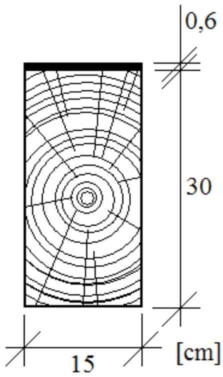
$$I_{max} = \frac{Ix + I_y}{2} + \sqrt{\left(\frac{I_x - I - y}{2}\right)^2 + D_{xy}^2} = 53264{,}587\ \mathrm{cm}^4$$
$$I_{min} = \frac{Ix + I_y}{2} - \sqrt{\left(\frac{I_x - I - y}{2}\right)^2 + D_{xy}^2} = 16413{,}244\ \mathrm{cm}^4$$



Orientation of principal central axes of inertia:

$$\varphi = \mathrm{arctg}\frac{D_{xy}}{I_y - I_{max}} = \mathrm{arctg}(-4{,}104) = -76{,}307^{\circ}$$

EXERCISE 25



Find weighted geometrical characteristics of a timber section strengthened with a sheet metal as it is shown in the figure. Young modulus of wood $E_d = 10\ \text{GPa}$,Young modulus of steel $E_s = 205\ \text{GPa}$.

SOLUTION:

Reference Young modulus: $E_0 = E_d = 10\ \text{GPa}$

Weight coefficient for steel: $\alpha = \dfrac{E_s}{E_0} = 20{,}5$

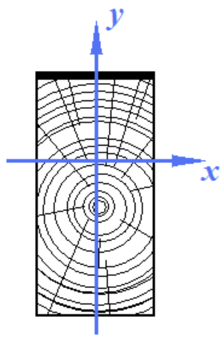
The section is symmetric – symmetry axis is one of principal central axes of inertia.

Weighted area:

$$A = A_d + \alpha A_s = [15\cdot 30] + 20{,}5\cdot[15\cdot 0{,}6] = 634{,}5\ [\text{cm}^2]$$

Weighted statical moment about bottom edge:

$$S_{x'} = S_{x'd} + \alpha S_{x's} = \\ = \left[15\cdot 30\cdot\frac{30}{2}\right] + 20{,}5\left[15\cdot 0{,}6\cdot\left(30+\frac{0{,}6}{2}\right)\right] = 12340{,}35\quad [\text{cm}^3]$$



Location of weighted centroid:

$$y'_C = \frac{S_{x'}}{A} = 19{,}45\quad [\text{cm}]$$

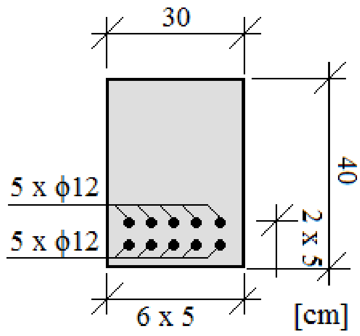
Weighted moments of inertia:

$$I_x = I_{xd} + \alpha I_{xs} = \\ = \left[\frac{15\cdot 30^3}{12} + 15\cdot 30\cdot(15-19{,}45)^2\right] + 20{,}5\left[\frac{15\cdot 0{,}6^3}{12} + 15\cdot 0{,}6\cdot(30{,}3-19{,}45)^2\right] = 64386{,}47\quad [\text{cm}^4]$$

$$I_y = I_{yd} + \alpha I_{ys} = \left[\frac{15^3\cdot 30}{12}\right] + 20{,}5\left[\frac{15^3\cdot 0{,}6}{12}\right] = 11896{,}88\quad [\text{cm}^4]$$

EXERCISE 26

Find weighted geometrical characteristics of a reinforced-concrete (RC) section as it is shown in the figure. Young modulus of concrete $E_c = 32\ \mathrm{GPa}$, Young modulus of reinforcement steel $E_s = 210\ \mathrm{GPa}$. Rebar diameter $\phi = 12\ \mathrm{mm}$. Rebar span $a = 5\,\mathrm{cm}$. Concrete cover: $c = 5\,\mathrm{cm} - 0{,}5\,\phi = 4{,}4\,\mathrm{cm}$. Compare the solution with results obtained for purely concrete section.



SOLUTION:

Characteristics of purely concrete section:

$$A = 30 \cdot 40 = 1200 \quad [\mathrm{cm}^2]$$

$$I_x = \frac{30 \cdot 40^3}{12} = 160000 \quad [\mathrm{cm}^4] \qquad I_y = \frac{30^3 \cdot 40}{12} = 90000 \quad [\mathrm{cm}^4]$$

Weighted characteristics of composite (reinforced-concrete) section :

Reference Young modulus: $E_0 = E_c = 32\ \mathrm{GPa}$

Weight coefficient for steel: $\alpha = \dfrac{E_s}{E_0} = 6{,}563$

The section is symmetric – symmetry axis is one of principal central axes of inertia.

Weighted area:

$$A = A_c + \alpha A_s = [30 \cdot 40] + 6{,}563 \cdot \left[10 \cdot \frac{\pi \cdot 1{,}2^2}{4}\right] = 1274{,}226\ [\mathrm{cm}^2]$$

Weighted statical moment about bottom edge:

$$S_{x'} = S_{x'c} + \alpha S_{x's} = \left[30 \cdot 40 \cdot \frac{40}{2}\right] + 6{,}563\left[5 \cdot \frac{\pi \cdot 1{,}2^2}{4} \cdot 5 + 5 \cdot \frac{\pi \cdot 1{,}2^2}{4} \cdot 10\right] = 24556{,}693 \quad [\mathrm{cm}^3]$$

Location of weighted centroid: $y'_C = \dfrac{S_{x'}}{A} = 19{,}272 \quad [\mathrm{cm}]$

Weighted moments of inertia:

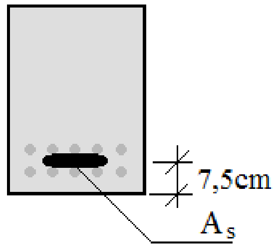
$$I_x = I_{xc} + \alpha I_{xs} = \left[\frac{30 \cdot 40^3}{12} + 30 \cdot 40 \cdot (20 - 19{,}272)^2\right] +$$
$$+ 6{,}563\left[10 \cdot \frac{\pi \cdot 1.2^4}{64} + 5 \cdot \frac{\pi \cdot 1{,}2^2}{4} \cdot (5 - 19{,}272)^2 + 5 \cdot \frac{\pi \cdot 1{,}2^2}{4} \cdot (10 - 19{,}272)^2\right] = 171392{,}653 \quad [\mathrm{cm}^4]$$

$$I_y = I_{yc} + \alpha I_{ys} = \left[\frac{30^3 \cdot 40}{12}\right] + 6{,}563\left[10 \cdot \frac{\pi \cdot 1{,}2^4}{64} + 4 \cdot \frac{\pi \cdot 1{,}2^2}{4} \cdot 5^2 + 4 \cdot \frac{\pi \cdot 1{,}2^2}{4} \cdot 10^2\right] = 93717{,}969 \quad [\mathrm{cm}^4]$$

In practical calculations it is common to replace all rebars with a single point inclusion of an area equal to the total area of all rebars and placed in the centroid of system of rebars. We do not state what is the shape of this inclusion and in calculation of moments of inertia in the application of parallel axis theorem we disregard the first part (connected with moment of inertia of the shape in its own central axes) as a negligibly small value.

So we may replace the system of rebars with a single area placed on a symmetry axis in the distance 7,5 cm from the bottom edge and its surface area is equal:

$$A_s = 10 \cdot \frac{\pi \cdot 1{,}2^2}{4} = 11{,}310 \quad [\mathrm{cm}^2]$$



Weighted area:

$$A = A_c + \alpha A_s = [30 \cdot 40] + 6{,}563 \cdot \left[10 \cdot \frac{\pi \cdot 1{,}2^2}{4}\right] = 1274{,}226 \ [\mathrm{cm}^2]$$

Weighted statical moment:

$$S_{x'} = S_{x'c} + \alpha S_{x's} = \left[30 \cdot 40 \cdot \frac{40}{2}\right] + 6{,}563 [11{,}310 \cdot 7{,}5] = 24556{,}706 \quad [\mathrm{cm}^3]$$

Location of weighted centroid::

$$y'_C = \frac{S_{x'}}{A} = 19{,}272 \quad [\mathrm{cm}]$$

Weighted moments of inertia:

$$I_x = I_{xc} + \alpha I_{xs} =$$

$$\left[\frac{30 \cdot 40^3}{12} + 30 \cdot 40 \cdot (20 - 19{,}272)^2\right] + 6{,}563 \cdot \left[0 + 11{,}310 (7{,}5 - 19{,}272)^2\right] = 170922{,}431 \quad [\mathrm{cm}^4]$$

$$I_y = I_{yc} + \alpha I_{ys} = \left[\frac{30^3 \cdot 40}{12}\right] + 6{,}563 \cdot [0] = 90000 \quad [\mathrm{cm}^4]$$

Results obtained in this way do not differ much from those obtained by strict calculation, especially concerning this direction of bending to which the section is dedicated (I_x).

