02. Geometrical characteristics of cross-sections of bars

According to assumptions of mechanics of bar structures, true three-dmensional deformable body will be modelled with a one-dimensional system, in which information concerning dimensions transverse to this distinguished dimension (axis of a bar) will be contained in a system of parameters characteristic for bar's cross-sections, and depending on the shape of this corss-section. Those quantities are: **surface area, location of centroid, statical moments, moments of inertia and deviation moment of inertia**. In English it is (to some extent) matter of interpretation whether those quantities should be called moments of **inertia** or moments of **area**. Strictly speaking "inertia" concerns mass and rotational motion, while "area" concerns purely geometric features of an object. It is, however, common to speak about "moments of inertia" concerning flexular stiffness of bent beam, which has nothing to do with mass and rotational inertia – it will be done so in this elaboration.

STATICAL MOMENT (FIRST MOMENT OF AREA) AND LOCATION OF CENTROID OF SHAPE

$A=\iint_A dx\,dy$ - **surface area** [m²]

$S_x=\iint_A y\,dx\,dy$ - **statical moment** (**first moment of area**) about plane XZ [m³]

$S_y=\iint_A x\,dx\,dy$ - **statical moment** (**first moment of area**) about plane YZ [m³]

Since we consider plane shapes lying on XY plane, statical moment about plane XZ or YZ will be often called simply statical moment about axes *x* and *y* respectively.

Location of centroid *O*: $x_O=\frac{S_y}{A} \qquad y_O=\frac{S_x}{A}$, stąd: $S_x=A\,y_O \qquad S_y=A\,x_O$

- **If the shape has a symmetry axis, then its centroid lies on this axis**
- **If the shape has more than one symmetry axis, then its centroid lies at intersection of those symmetry axes**
- **Statical moment about axes passing through centroid is equal 0**

MOMENT OF INERTIA (SECOND MOMENT OF AREA)

$I_x=\iint_A y^2\,dx\,dy$ - **moment of inertia** (**second moment of area**) about axis X [m⁴]

$I_y=\iint_A x^2\,dx\,dy$ - **moment of inertia** (**second moment of area**) about axis Y [m⁴]

$D_{xy}=\iint_A xy\,dx\,dy$ - **deviation moment of inertia** (**product of inertia**, **product moment of area**) about planes XZ i YZ [m⁴]

$I_0=\iint_A r^2\,dA=\iint_A (x^2+y^2)\,dx\,dy=I_x+I_y$ - **polar moment of inertia** (**polar moment of area**) [m⁴]

$i_x=\sqrt{\frac{I_x}{A}} \qquad i_y=\sqrt{\frac{I_y}{A}}$ - **radii of gyration** about axes *x* and *y* [m]

- **Moment of inertia is always positive.**
- **Deviation moment of inertia may be negative.**

CHANGE OF COORDINATE SYSTEM

The above quantities are defined with the use of coordinates of points of cross-section in a certain assumed coordinate system. Change of coordinate system results change in geometrical characteristics. Any change of coordinate system in planar case may be considered as a composition of two elementary transformation – **parallel translation** and **rotation**.

Geometrical characteristics in a coordinate system of axes moved parallel

In order to find moments of inertia about axes which are **parallel** but **translated** with respect to the given coordinate system we shall use **parallel axis theorem**, which is also called **Steiner-Huygens theorem**. Let's assume that we've already determined moments of inertia about axes *X, Y* containing centroid of cross-section. We are now looking for moments of inertia about axes *x, y* which are parallel to the original ones but displaced with a certain distance *d*. For I_X :

$$I_x = \iint_A (y+d)^2 \mathrm{d}A = \underbrace{\iint_A y^2 \mathrm{d}A}_{I_X} + d\underbrace{\iint_A y\,\mathrm{d}A}_{S_X=0} + d^2\underbrace{\iint_A \mathrm{d}A}_{A} = I_X + Ad^2$$

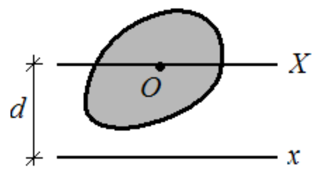
The second integral is equal 0 is due to fact that axis *X* passes through centroid, so statical moment about this axis must be zero. For daviation moment of inertia we have:

$$D_{xy} = \iint_A (x+d_x)(y+d_y)\mathrm{d}A = \underbrace{\iint_A xy\,\mathrm{d}A}_{D_{XY}} + d_x\underbrace{\iint_A y\,\mathrm{d}A}_{S_X=0} + d_y\underbrace{\iint_A x\,\mathrm{d}A}_{S_Y=0} + d_x d_y\underbrace{\iint_A \mathrm{d}A}_{A} = D_{XY} + A d_x d_y$$

Finally we may write down:

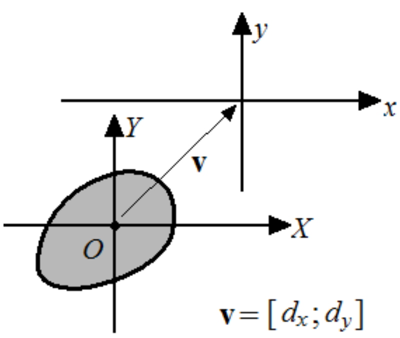
STEINER-HUYGENS THEOREM

Moment of inertia about any axis is equal to a moment of inertia about an axis which is parallel to it and pass through centroid of the shape amplified with a product of shape area and square the distance between those axes:



$$I_x = I_X + Ad^2$$

Deviation moment of inertia about two planes is equal to a deviation moment of inertia about planes which are parallel to them and pass through centroid of the shape amplified with a product of shape area and signed measures of distances between planes:



$$D_{xy} = D_{XY} + A d_x d_y$$

NOTE: d_x, d_y may be negative! While in case of moments of inertia it is not important as we square d anyway, in case of deviation moment of inertia accounting for a sign of d_x, d_y is necessary. d_x, d_y Are components of vector of translation of coordinate system – however it may be chosen arbitrary if this is a vector from the old coordinate system to the new one or in the opposite way.

Inverse theorems may also be used in order to find central moments of inertia while knowing the moments of inertia about certain parallel axes which do not pass through centroid:

$$I_X = I_x - Ad^2 \qquad D_{XY} = D_{xy} - A\,d_x d_y$$

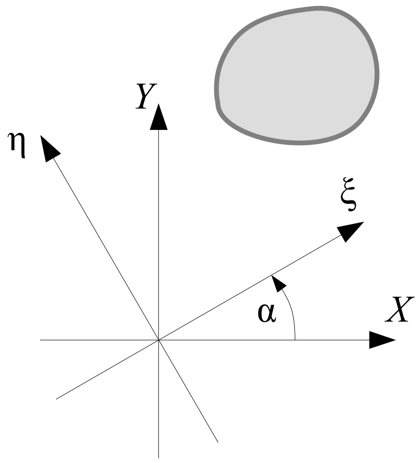
CONCLUSION: Since all quantities I_X, I_x, A, d^2 are positice, it yields that **among all parallel axes, the one which passes through centroid it the one, for which moment of inertia is the smallest**.

In order to find moments of inertia in two coordinate system, none of which contains centroid in its origin, we have to perform it in two steps, using the parallel axis twice – first: transform those values to the centroid and then transform them out of centroid to the target coordinate system.

Geometrical characteristics in rotated coordinate system

Knowing geometrical characteristics in a given coordinate system we may find moments of inertia in a **coordinate system rotated with angle α (counterclockwise) but with origin in the same point** in a following way:

New coordinates (ξ, η) obtained by rotation of original axes X, Y with angle α are equal:



$$\begin{cases} \xi = X\cos\alpha + Y\sin\alpha \\ \eta = -X\sin\alpha + Y\cos\alpha \end{cases}$$

New moment of inertia:

$$I_\xi = \iint_A \eta^2\,\mathrm{d}A = \iint_A (-X\sin\alpha + Y\cos\alpha)^2\,\mathrm{d}A =$$
$$= \sin^2\alpha \underbrace{\iint_A X^2\,\mathrm{d}A}_{I_Y} + \cos^2\alpha \underbrace{\iint_A Y^2\,\mathrm{d}A}_{I_X} - 2\sin\alpha\cos\alpha \underbrace{\iint_A XY\,\mathrm{d}A}_{D_{XY}} =$$
$$= I_X\cos^2\alpha + I_Y\sin^2\alpha - 2D_{XY}\sin\alpha\cos\alpha$$

Other moments may be found in a similar way. Finally we obtain **transformation formulae**:

$$I_\xi = I_X\cos^2\alpha + I_Y\sin^2\alpha - 2D_{XY}\sin\alpha\cos\alpha$$
$$I_\eta = I_X\sin^2\alpha + I_Y\cos^2\alpha + 2D_{XY}\sin\alpha\cos\alpha$$
$$D_{\xi\eta} = \frac{I_X - I_Y}{2}\sin 2\alpha + D_{XY}\cos 2\alpha$$

TENSOR OF MOMENT OF INERTIA– PRINCIPAL MOMENTS OF INERTIA OF CROSS-SECTION

We introduce a quantity called the **tensor of moment of inertia**: $\mathbf{I}=\begin{bmatrix} I_x & -D_{xy} \\ -D_{xy} & I_y \end{bmatrix}$

This tensor may be characterized by a system of scalars, the value of which do not change with the change of orientation of coordinate system – these are, so called, **tensor invariants**:

- **trace of tensor** $a = \mathrm{tr}(\mathbf{I}) = I_x + I_y$
- **determinant of tensor** $b = \det(\mathbf{I}) = I_x I_y - D_{xy}^2$
- **eigenvalues of tensor – principal moments of inertia**

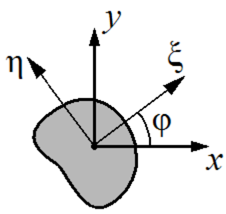
Principal moments of inertia – which are also the maximum and minimum possible values of moments of inertia among all orientations of coordinate system – **are the roots of** **secular equation**:

$$I^2 - a\,I + b = 0$$

$$I_{max/min} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + D_{xy}^2}$$

An angle between axis *x* and axis of maximum moment of inertia ξ is equal:

$$\mathrm{tg}\,\varphi = \frac{D_{xy}}{I_y - I_{max}} = \frac{I_x - I_{max}}{D_{xy}} = \frac{D_{xy}}{I_{min} - I_x} = \frac{I_{min} - I_y}{D_{xy}}$$



Axes of maximum and minimum moment of inertia are termed **principal axes** (directions) of tensor.

- **Principal axes of tensor of moment of inertia are always perpendicular.**
- **In the coordinate system od principal axes the tensor has a diagonal form with deviation moments of inertia equal 0.**
- **In a coordinate system inclined at angle 45° to principal axes deviation moments of inertia have extreme values (maximum or minimum).**
- **If the cross-section has a symmetry axis, then it is a principal axis of inertia and the second one is perpendicular to it.**
- **If the cross-section has more than two symmetry axes (circle, square, regular polygons for n>2, etc.) then any directions are principal directions of inertia.**

Geometrical characteristics of chosen plane shapes

x y – axes of certain chosen coordinate system
X Y – **central axes** – axes are parallel to *x* and *y,* but they pass thorugh centroid
ξ η – **preincipal central axes** – if it is not indicated otherwise, they coincide with axes *X Y*

I_x, I_y, D_z **- moments of inertia in a choses coordinate system (*x, y)***
I_X, I_Y, D_Z **- central moments of inertia**
I_{max}, I_{min}, I_0 **- principal central moments of inertia**

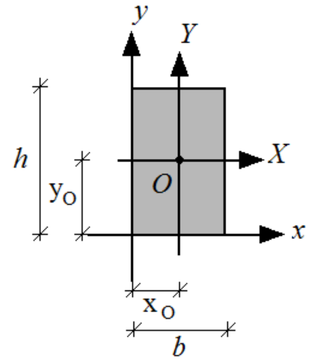
NOTE: If a shape (rectangle, right triangle, quarter of circle) is placed in the 2nd or the 4th quarter of chosen basic coordinate system (*x, y)*, axes of which coincide with edges of the shape, then deviation moments of inertia are signed in an opposite way than it is indicated below.

Rectangle

$$A=bh \qquad D_z=\frac{b^2h^2}{4} \qquad I_0=\frac{bh}{12}(b^2+h^2)$$

$$x_O=\frac{b}{2} \qquad I_x=\frac{bh^3}{3} \qquad I_{max}=\frac{bh^3}{12}$$

$$y_O=\frac{h}{2} \qquad I_y=\frac{b^3h}{3} \qquad I_{min}=\frac{b^3h}{12}$$



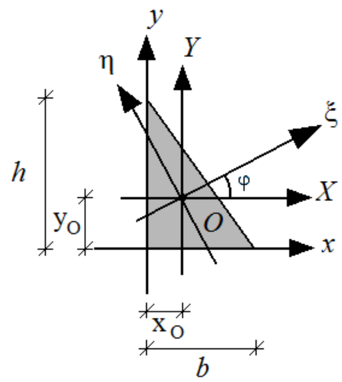
Triangle

$$A=\frac{1}{2}bh \qquad D_z=\frac{b^2h^2}{24} \qquad D_Z=\mp\frac{b^2h^2}{72} \qquad I_0=\frac{bh}{36}(b^2+h^2)$$

$$x_O=\frac{b}{3} \qquad I_x=\frac{bh^3}{12} \qquad I_X=\frac{bh^3}{36} \qquad I_{max}=\frac{bh}{72}\left[b^2+h^2+\sqrt{h^4-h^2b^2+b^4}\right]$$

$$y_O=\frac{h}{3} \qquad I_y=\frac{b^3h}{12} \qquad I_Y=\frac{b^3h}{36} \qquad I_{min}=\frac{bh}{72}\left[b^2+h^2-\sqrt{h^4-h^2b^2+b^4}\right]$$

$$\phi=\mathrm{arctg}\left(\frac{\pm bh}{h^2-b^2+\sqrt{h^4-h^2b^2+b^4}}\right)=\mathrm{arctg}\left(\mp\frac{h^2-b^2-\sqrt{h^4-h^2b^2+b^4}}{bh}\right)$$



NOTE: If the triangle is oriented in its central coordinate system (*X, Y*) in such a way, that trapezoids outlined be them are placed in the 2nd and the 4th quarter of this coordinate system, then deviation moment of inertia D_Z is negative – in they are placed in the 1st and the 3rd quarter, then it is positive.

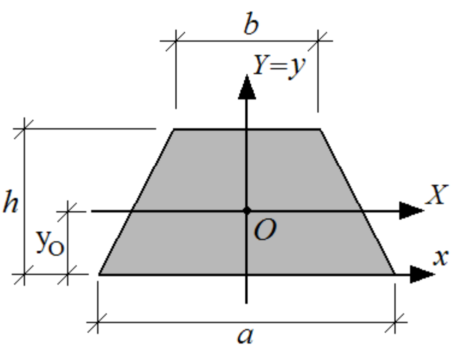
Isosceles trapezoid

$$A=\frac{(a+b)h}{2} \qquad D_z=0$$

$$x_O=0 \qquad I_x=\frac{(a+3b)h^3}{12} \qquad I_{max/min}=\frac{(a^2+4ab+b^2)h^3}{36(a+b)}$$

$$y_O=\frac{(a+2b)h}{3(a+b)} \qquad I_y=\frac{(a+b)(a^2+b^2)h}{48} \qquad I_{min/max}=\frac{(a+b)(a^2+b^2)h}{48}$$

$$I_0=\frac{4h^3(a^2+4ab+b^2)+3h(a^2+2a^3b+2a^2b^2+2ab^3+b^4)}{144(a+b)}$$

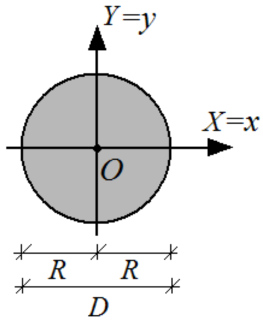


Circle

$$A=\pi R^2 \qquad I_0=\frac{\pi R^4}{2}=\frac{\pi D^4}{32}$$

$$x_O=0 \qquad I_{max}=\frac{\pi R^4}{4}=\frac{\pi D^4}{64}$$

$$y_O=0 \qquad I_{min}=\frac{\pi R^4}{4}=\frac{\pi D^4}{64}$$

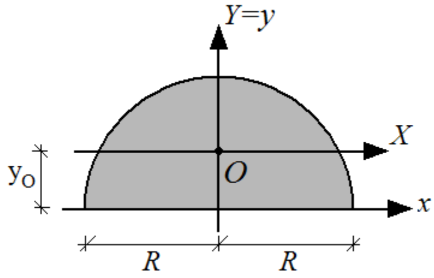


Half of a cricle

$$A=\frac{\pi R^2}{2} \qquad D_z=0 \qquad I_0=R^4\left(\frac{\pi}{4}-\frac{8}{9\pi}\right)$$

$$x_O=0 \qquad I_x=\frac{\pi R^4}{8} \qquad I_{max}=I_Y=\frac{\pi R^4}{8}$$

$$y_O=\frac{4}{3}\frac{R}{\pi} \qquad I_y=\frac{\pi R^4}{8} \qquad I_{min}=I_X=R^4\left(\frac{\pi}{8}-\frac{8}{9\pi}\right)$$



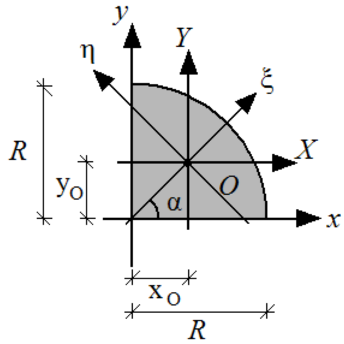
Quarter of a circle

$$A=\frac{\pi R^2}{4} \qquad D_z=\frac{R^4}{8} \qquad D_Z=R^4\left(\frac{1}{8}-\frac{4}{9\pi}\right) \qquad I_0=R^4\left(\frac{\pi}{8}-\frac{8}{9\pi}\right)$$

$$x_O=\frac{4}{3}\frac{R}{\pi} \qquad I_x=\frac{\pi R^4}{16} \qquad I_X=R^4\left(\frac{\pi}{16}-\frac{4}{9\pi}\right) \qquad I_{max}=R^4\frac{(\pi-2)}{16}$$

$$y_O=\frac{4}{3}\frac{R}{\pi} \qquad I_y=\frac{\pi R^4}{16} \qquad I_Y=R^4\left(\frac{\pi}{16}-\frac{4}{9\pi}\right) \qquad I_{min}=R^4\frac{(9\pi^2+18\pi-1}{144\pi}$$

$$\phi=45^\circ$$



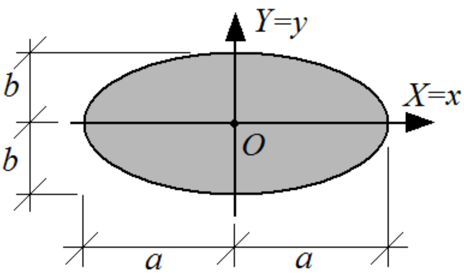
NOTE: D_Z and D_z are calculated for an orientation of the quarter as in the figure.

Ellipse

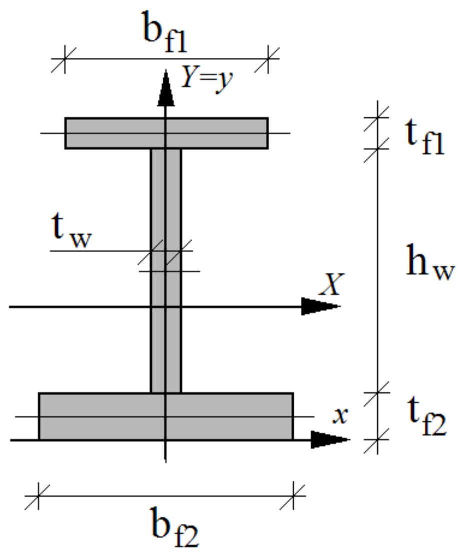
$$A=\pi a b \qquad I_0=\frac{\pi a b}{4}(a^2+b^2)$$

$$x_O=0 \qquad I_{max}=\frac{\pi a b^3}{4}$$

$$y_O=0 \qquad I_{min}=\frac{\pi a^3 b}{4}$$



Composite section – example of an I-section



Top flange:

$$A_{f1}=t_{f1}\cdot b_{f1} \qquad I_{x,f1}=\frac{b_{f1}t_{f1}^3}{12} \qquad I_{y,f1}=\frac{b_{f1}^3 t_{f1}}{12}$$

Bottom flange:

$$A_{f2}=t_{f2}\cdot b_{f2} \qquad I_{x,f2}=\frac{b_{f2}t_{f2}^3}{12} \qquad I_{y,f2}=\frac{b_{f2}^3 t_{f2}}{12}$$

Web:

$$A_w=t_w\cdot h_w \qquad I_{x,w}=\frac{t_w h_w^3}{12} \qquad I_{y,w}=\frac{b_w^3 h_w}{12}$$

One of principal axes coincides with symmetry axis *y*. The second one is perpendicular to it.

Surface area: $A=\left[A_{f1}\right]+\left[A_{f2}\right]+\left[A_w\right]$

Statical moment about *x* axis: $S_x=\left[\frac{A_{f2}\cdot t_{f2}}{2}\right]+\left[A_w\cdot\left(t_{f2}+\frac{h_w}{2}\right)\right]+\left[A_{f2}\cdot\left(t_{f2}+h_w+\frac{t_{f1}}{2}\right)\right]$

Statical moment about *y* axis: $S_y=0$

Location of centroid: $x_O=\frac{S_y}{A}=0 \qquad y_O=\frac{S_x}{A}$

Principal central moments of inertia:

$$I_x=\left[I_{x,f1}+A_{f1}\cdot\left(y_O-\frac{t_{f2}}{2}\right)^2\right]+\left[I_{x,w}+A_w\cdot\left(y_O-t_{f2}-\frac{h_w}{2}\right)^2\right]+\left[I_{x,f2}+A_{f2}\cdot\left(y_O-t_{f2}-h_w-\frac{t_{f1}}{2}\right)^2\right]$$

$$I_y=\left[I_{y,f1}\right]+\left[I_{y,f2}\right]+\left[I_{y,w}\right]$$

MOMENTS OF INERTIA OF SYMMETRIC SHAPES

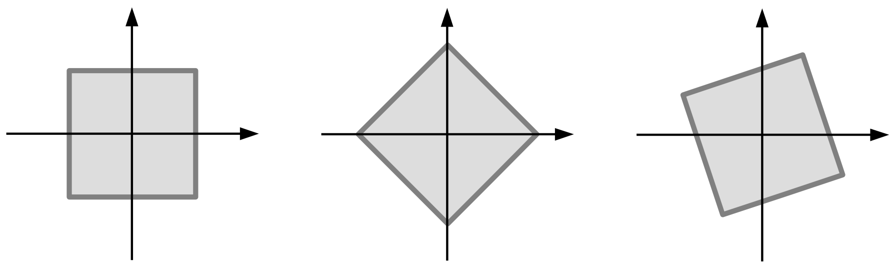
- If a plane shape has a **single symmetry axis**, then its **centroid lies on this axis** and **this axis is one of principal central axes of inertia** – the second one is **perpendicular** to it and they intersect in centroid.

- If a plane shape has a **two symmetry axes**, then its **centroid lies at intersection of those axes** and **both axes are principal central axes of inertia**

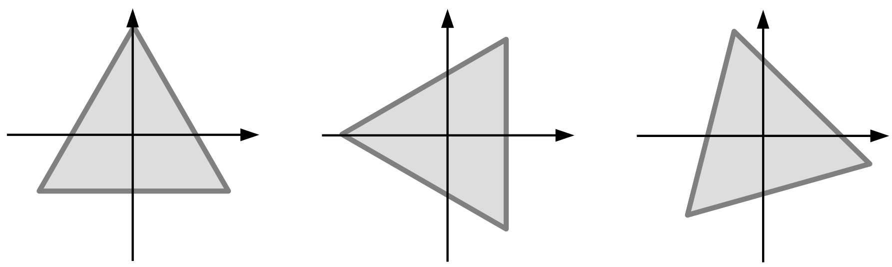
- If a plane shape has **more than two symmetry axes**, then its **centroid lies at intersection of those axes** and **any axis passing through centroid is a principal central axis of inertia**

If in certain coordinate system the shape has the same moments of inertia and zero deviation moment of inertia, then – according to transformation formulae for rotation of coordinate system – rotation of this shape (or system's axes) with any angle do not change value of moments of inertia. It concernes in perticular all **shapes that have more than two symmetry axes**.

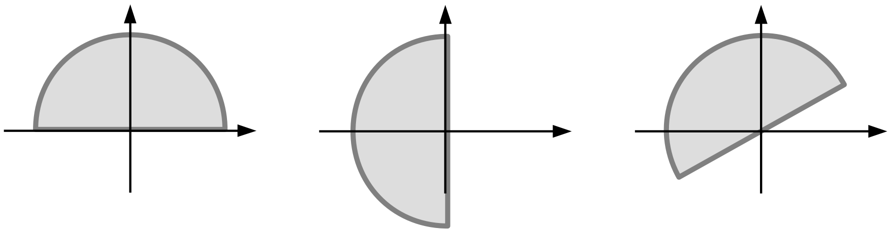
$$\begin{cases} I_X = I_Y = I \\ D_{XY} = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} I_\xi = I_\eta = I \\ D_{XY} = 0 \end{cases}$$



$$\mathbf{I} = \begin{bmatrix} \frac{a^4}{12} & 0 \\ 0 & \frac{a^4}{12} \end{bmatrix} \qquad \mathbf{I} = \begin{bmatrix} \frac{a^4}{12} & 0 \\ 0 & \frac{a^4}{12} \end{bmatrix} \qquad \mathbf{I} = \begin{bmatrix} \frac{a^4}{12} & 0 \\ 0 & \frac{a^4}{12} \end{bmatrix}$$

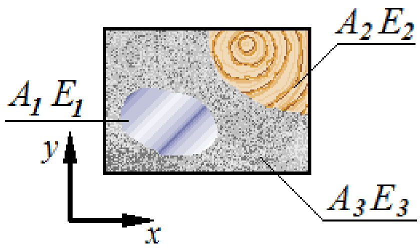


$$\mathbf{I} = \begin{bmatrix} \frac{a^4\sqrt{3}}{96} & 0 \\ 0 & \frac{a^4\sqrt{3}}{96} \end{bmatrix} \qquad \mathbf{I} = \begin{bmatrix} \frac{a^4\sqrt{3}}{96} & 0 \\ 0 & \frac{a^4\sqrt{3}}{96} \end{bmatrix} \qquad \mathbf{I} = \begin{bmatrix} \frac{a^4\sqrt{3}}{96} & 0 \\ 0 & \frac{a^4\sqrt{3}}{96} \end{bmatrix}$$



$$\mathbf{I} = \begin{bmatrix} \frac{\pi R^4}{8} & 0 \\ 0 & \frac{\pi R^4}{8} \end{bmatrix} \qquad \mathbf{I} = \begin{bmatrix} \frac{\pi R^4}{8} & 0 \\ 0 & \frac{\pi R^4}{8} \end{bmatrix} \qquad \mathbf{I} = \begin{bmatrix} \frac{\pi R^4}{8} & 0 \\ 0 & \frac{\pi R^4}{8} \end{bmatrix}$$

GEOMETRICAL CHARACTERISTICS OF COMPOSITE CROSS-SECTIONS



In case when a cross-section is built of few different materials that have different mechanical properties (e.g. steel, wood, concrete), then we determine **weighted geometrical characteristics**.

In case of problems of bending, we introduce a reference Young modulus E_0 , which is usually the smallest of Young moduli of all component materials $E_0 = \min(E_1, E_2, \ldots, E_n)$ and then for each material we calculate a fraction of its own Young modulus and the reference one:

$$\alpha_i = \frac{E_i}{E_0} \quad [-]$$

Weighted geometrical characteristics:

$A = \sum_i^n \alpha_i \iint_{A_i} dx\,dy$ - weighted area [m^2]

$S_x = \sum_i^n \alpha_i \iint_{A_i} y\,dx\,dy$ - weighted statical moment about plane XZ [m^3]

$S_y = \sum_i^n \alpha_i \iint_{A_i} x\,dx\,dy$ - weighted statical moment about plane YZ [m^3]

Location of centroid is determined as usual: $x_O = \frac{S_y}{A}$ $\qquad y_O = \frac{S_x}{A}$

$I_x = \sum_i^n \alpha_i \iint_{A_i} y^2\,dx\,dy$ - weighted moment of inertia about axis X [m^4]

$I_y = \sum_i^n \alpha_i \iint_{A_i} x^2\,dx\,dy$ - weighted moment of inertia about axis Y [m^4]

$D_{xy} = \sum_i^n \alpha_i \iint_{A_i} xy\,dx\,dy$ - weighted deviation moment of inertia about planes XZ i YZ [m^4]