THE MOST IMPORTANT FORMULAE

ACTIONS ON VECTORS

Dot (scalar) product: $\mathbf{a} \circ \mathbf{b} = a_x b_x + a_y b_y + a_z b_z$

Vector (corss) product: $\mathbf{a} \times \mathbf{b} = \left[\begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} ; -\begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} ; \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \right]$

Mixed (triple) product: $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$

Projection on direction of a vector: $\mathbf{a}_n = \dfrac{\mathbf{a} \circ \mathbf{n}}{|\mathbf{n}|^2} \cdot \mathbf{n}$

Projection on a plane: $\mathbf{a}_\pi = \mathbf{a} - \dfrac{\mathbf{a} \circ \mathbf{n}}{\mathbf{n}^2} \cdot \mathbf{n}, \qquad \mathbf{n} \perp \pi$

MOMENT OF A FORCE AND PARAMETERS OF SYSTEM OF FORCES

Moment of a force (torque) about a point P: $\mathbf{M}_B = \mathbf{a} \times \vec{AB}$

Moment of a force (torque) about a straight line:

$$\mathbf{M}_p = \left(\mathbf{a} - \frac{\mathbf{a} \circ \vec{BC}}{|\vec{BC}|^2} \vec{BC} \right) \times \left(\vec{AC} - \frac{\vec{AC} \circ \vec{BC}}{|\vec{BC}|^2} \vec{BC} \right) = \frac{(\mathbf{a} \times \vec{AC}) \circ \vec{BC}}{|\vec{BC}|^2} \cdot \vec{BC} \qquad B, C \in p$$

Moment transport theorem for a single force: $\mathbf{M}_C = \mathbf{M}_B + \mathbf{a} \times \vec{BC}$

Sum of system of forces: $\mathbf{S} = \sum_{i=1}^{n} \mathbf{F}_i$

Sum of moments of a system of forces about P: $\mathbf{M}_P = \sum_{i=1}^{n} (\mathbf{F}_i \times \vec{P_i P})$

Moment transport theorem for system of forces: $\mathbf{M}_Q = \mathbf{M}_P + \mathbf{S} \times \vec{PQ}$

Parameter of the system: $K = \mathbf{S} \circ \mathbf{M}_P$

REDUCTION TO THE SIMPLEST EQUIVALENT SYSTEM

- $\mathbf{S} = \mathbf{0} \wedge \mathbf{M}_P = \mathbf{0} \Leftrightarrow K = 0$ $\Rightarrow$ **zero system in any point**
- $\mathbf{S} = \mathbf{0} \wedge \mathbf{M}_P \neq \mathbf{0} \Leftrightarrow K = 0$ $\Rightarrow$ **a couple in any point**
- $\mathbf{S} \neq \mathbf{0} \wedge K = 0 \Rightarrow (\mathbf{M}_P = \mathbf{0} \vee \mathbf{M}_P \perp \mathbf{S})$ $\Leftrightarrow$ **resultant in any point of a central axis**
- $K \neq 0$ $\Rightarrow$ **wrench in any point of a central axis**

Central axis equation: $\mathbf{r}(\lambda) = \vec{OP} + \dfrac{\mathbf{S} \times \mathbf{M}_P}{\mathbf{S}^2} + \lambda \mathbf{S}$

A moment of a couple in a wrench: $\mathbf{M}_Q = \dfrac{K}{|\mathbf{S}|^2} \cdot \mathbf{S}$

REDUCTION OF A SYSTEM OF PARALLEL FORCES

- A system of parallel forces is reduced either to a resultant, or a couple or a zero system
- A versor parallel to each force: $\mathbf{e}$
- Given system of forces: $\left\{\begin{pmatrix}\mathbf{F}_1\\A_1\end{pmatrix},\begin{pmatrix}\mathbf{F}_2\\A_2\end{pmatrix},\dots,\begin{pmatrix}\mathbf{F}_n\\A_n\end{pmatrix}\right\} \Rightarrow \mathbf{F}_i = F_i\,\mathbf{e}\ ,\ \ \mathbf{r}_i = \vec{OA_i}$
- Let's calculate: $S = \sum_{i=1}^{n} F_i\ ,\ \ \mathbf{M} = \sum_{i=1}^{n} F_i\,\mathbf{r}_i$
- **Sum of system**: $\mathbf{S} = S\,\mathbf{e}$
- **Sum of moments about any point B**: $\mathbf{M}_B = \mathbf{S}\times\vec{O^*B}$
- **Center of the system of parallel forces**: $\vec{OO^*} = \dfrac{\mathbf{M}}{S} = \dfrac{\sum_{i=1}^{n} F_i\,\mathbf{r}_i}{\sum_{i=1}^{n} F_i}$
- **Resultant** (if only sum $\mathbf{S}\neq\mathbf{0}$): $\mathbf{w} = \mathbf{S}$
- **Equation of the central axis:** $\mathbf{r}(\lambda) = \vec{OO^*} + \lambda\,\mathbf{S}$

REDUCTION OF A DISTRIBUTED LOAD TO A RESULTANT

- **Resultant** $S = \int_0^L q(x)\,\mathrm{d}x$
- **Moment about** $x=0$ $M = \int_0^L q(x)\cdot x\,\mathrm{d}x$
- **Position of resultant** $x_0 = \dfrac{M}{S}$

EXERCISE 1

There are vectors given: $\mathbf{a}=[1\;;\;2\;;\;3]$ $\quad\mathbf{b}=[2\;;\;1\;;\;-2]$ $\quad\mathbf{c}=[-4\;;\;2\;;\;-3]$ $\quad\mathbf{d}=[0\;;\;2\;;\;-1]$

Calculate:

a) $2\mathbf{a}+\mathbf{b}=?$ b) $3\mathbf{c}-2\mathbf{d}=?$
c) $\mathbf{a}\circ\mathbf{b}=?$ d) $\mathbf{b}\circ\mathbf{c}=?$ e) $\mathbf{c}\circ\mathbf{d}=?$
f) $\mathbf{a}\times\mathbf{b}=?$ g) $\mathbf{b}\times\mathbf{c}=?$ h) $\mathbf{c}\times\mathbf{d}=?$
i) $[\mathbf{a},\mathbf{b},\mathbf{c}]=?$ j) $[\mathbf{b},\mathbf{c},\mathbf{d}]=?$
k) zrzutować $\mathbf{a}$ na $\mathbf{b}$ l) zrzutować $\mathbf{c}$ na $\mathbf{d}$

SOLUTION:

a) $$2\mathbf{a}+\mathbf{b}=[2\cdot1+2\;;\;2\cdot2+1\;;\;2\cdot3-2]=[4\;;\;5\;;\;4]$$

b) $$3\mathbf{c}-2\mathbf{d}=[3\cdot(-4)-2\cdot0\;;\;3\cdot2-2\cdot2\;;\;3\cdot(-3)-2\cdot(-1)]=[-12\;;\;2\;;\;-7]$$

c) $$\mathbf{a}\circ\mathbf{b}=1\cdot2+2\cdot1+3\cdot(-2)=-2$$

d) $$\mathbf{b}\circ\mathbf{c}=2\cdot(-4)+1\cdot2+(-2)\cdot(-3)=0$$

e) $$\mathbf{c}\circ\mathbf{d}=(-4)\cdot0+2\cdot2+(-3)\cdot(-1)=7$$

f) $$\begin{array}{c}\times\begin{array}{c}{[1;2;3]}\\ {[2;1;-2]}\end{array}\\ \hline \left[\begin{vmatrix}2&3\\1&-2\end{vmatrix};-\begin{vmatrix}1&3\\2&-2\end{vmatrix};\begin{vmatrix}1&2\\2&1\end{vmatrix}\right]\end{array}=[2\cdot(-2)-1\cdot3\;;\;-1\cdot(-2)+2\cdot3\;;\;1\cdot1-2\cdot2]=[-7;8;-3]$$

g) $$\begin{array}{c}\times\begin{array}{c}{[2;1;-2]}\\ {[-4;2;-3]}\end{array}\\ \hline \left[\begin{vmatrix}1&-2\\2&-3\end{vmatrix};-\begin{vmatrix}2&-2\\-4&-3\end{vmatrix};\begin{vmatrix}2&1\\-4&2\end{vmatrix}\right]\end{array}=[1\cdot(-3)-2\cdot(-2)\;;\;-2\cdot(-3)+(-4)\cdot(-2)\;;\;2\cdot2-(-4)\cdot1]=[1;14;8]$$

h) $$\begin{array}{c}\times\begin{array}{c}{[-4;2;-3]}\\ {[0;2;-1]}\end{array}\\ \hline \left[\begin{vmatrix}2&-3\\2&-1\end{vmatrix};-\begin{vmatrix}-4&-3\\0&-1\end{vmatrix};\begin{vmatrix}-4&2\\0&2\end{vmatrix}\right]\end{array}=[2\cdot(-1)-2\cdot(-3)\;;\;-(-4)\cdot(-1)+0\cdot(-3)\;;\;(-4)\cdot2-0\cdot2]=[4;-4;-8]$$

i) $$[\mathbf{a},\mathbf{b},\mathbf{c}]=\begin{vmatrix}1&2&3\\2&1&-2\\-4&2&-3\end{vmatrix}=1\cdot1\cdot(-3)+2\cdot2\cdot3+(-4)\cdot2\cdot(-2)-3\cdot1\cdot(-4)-(-2)\cdot2\cdot1-(-3)\cdot2\cdot2=53$$

j) $$[\mathbf{b},\mathbf{c},\mathbf{d}]=\begin{vmatrix}2&1&-2\\-4&2&-3\\0&2&-1\end{vmatrix}=2\cdot2\cdot(-1)+(-4)\cdot2\cdot(-2)+0\cdot1\cdot(-3)-(-2)\cdot2\cdot0-(-3)\cdot2\cdot2-(-1)\cdot1\cdot(-4)=20$$

k) $$\frac{\mathbf{a}\circ\mathbf{b}}{\mathbf{b}^2}\cdot\mathbf{b}=\frac{1\cdot2+2\cdot1+3\cdot(-2)}{2^2+1^2+(-2)^2}\cdot[2;1;-2]=\frac{-2}{9}\cdot[2;1;-2]=\left[-\frac{4}{9};-\frac{2}{9};\frac{4}{9}\right]$$

l) $$\frac{\mathbf{c}\circ\mathbf{d}}{\mathbf{d}^2}\cdot\mathbf{d}=\frac{(-4)\cdot0+2\cdot2+(-3)\cdot(-1)}{0^2+2^2+(-1)^2}\cdot[0;2;-1]=\frac{7}{5}\cdot[0;2;-1]=\left[0;\frac{14}{5};-\frac{7}{5}\right]$$

EXERCISE 2

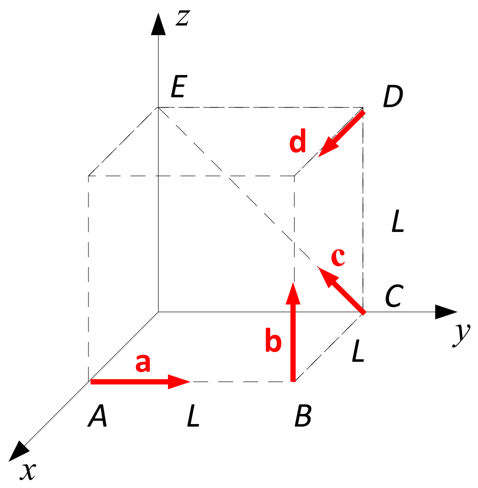
Find moments $M_D(\mathbf{a}), M_E(\mathbf{b}), M_B(\mathbf{c}), M_A(\mathbf{d}), \mathbf{M}_B(\mathbf{a})$

$|\mathbf{a}| = 2P$

$|\mathbf{b}| = P$

$|\mathbf{c}| = 3\sqrt{2}P$

$|\mathbf{d}| = 2P$



SOLUTION:

Moment $M_D(\mathbf{a})$:

$$\mathbf{a} = [0\ ;\ 2P\ ;\ 0]$$
$$\vec{AD} = [-L\ ;L\ ;L]$$
$$M_D(\mathbf{a}) = \mathbf{a} \times \vec{AD} = [2PL\ ;0\ ;2PL]$$

Moment $M_E(\mathbf{b})$:

$$\mathbf{b} = [0\ ;\ 0\ ;\ P]$$
$$\vec{BE} = [-L\ ;-L\ ;L]$$
$$M_E(\mathbf{b}) = \mathbf{b} \times \vec{BE} = [PL\ ;-PL\ ;0]$$

Moment $M_B(\mathbf{c})$:

Components of vector $\mathbf{c}$ will be found by first determining a unit vector which is parallel to $\mathbf{c}$ and then multiplying it by the length of $\mathbf{c}$:

$$\mathbf{e}_c = \frac{[0\ ;-L\ ;L]}{\sqrt{0^2+(-L)^2+L^2}} = \left[0\ ;-\frac{1}{\sqrt{2}}\ ;\frac{1}{\sqrt{2}}\right] \quad \Rightarrow \quad \mathbf{c} = |\mathbf{c}| \cdot \mathbf{e}_c = [0\ ;-3\ ;3]$$
$$\vec{CB} = [L\ ;0\ ;0]$$
$$M_B(\mathbf{c}) = \mathbf{c} \times \vec{CB} = [0\ ;3PL\ ;3PL]$$

Moment $M_A(\mathbf{d})$:

$$\mathbf{d} = [2P\ ;\ 0\ ;\ 0]$$
$$\vec{DA} = [L\ ;-L\ ;-L]$$
$$\mathbf{d} \times \vec{DA} = [0\ ;2PL\ ;-2PL]$$

Moment $M_B(\mathbf{a})$: Point B lies on a line of action of force $\mathbf{a}$, so the moment about it is 0.

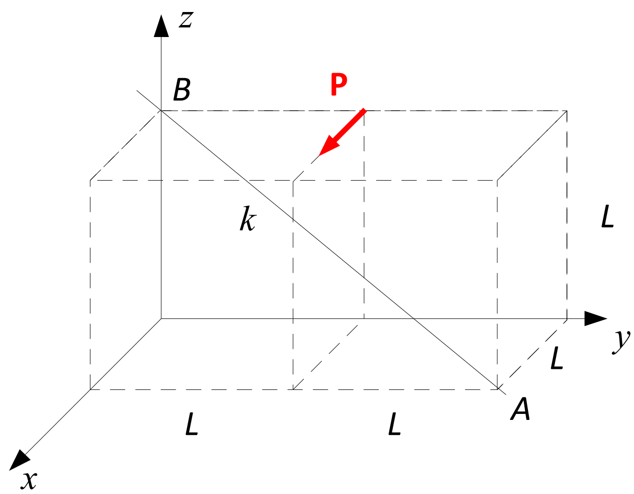
$$\mathbf{a} = [0\ ;\ 2P\ ;\ 0]$$
$$\vec{AB} = [0\ ;L\ ;0]$$
$$\mathbf{a} \times \vec{AB} = [0\ ;0\ ;0]$$

EXERCISE 3

Find a moment of force $\mathbf{P}$ about line k

a) with the use of definition

b) with the use of theorem



SOLUTION:

a) The use of definition

In order to make use of the definition of a moment about a line, we have to determine the plane which is perpendicular to this line.

A general equation of a plane in space:

$$\pi: \quad Ax+By+Cz+D=0$$

Vector $[A\,;\,B\,;\,C]$, the components of which are equal constant coefficients of this equation, is perpendicular to this plane (it is its gradient), so it will be obviously parallel to line k. This vector may be as follows:

$$\mathbf{k} = \vec{AB} = [-L\,;\,-2L\,;\,L]$$

Equation of a plane: $\quad \pi: \; -Lx-2Ly+Lz+D=0$

There is an infinite number of such planes. In particular we may chose a plane which contains the point to which force $\mathbf{P}$ is applied. Coordinates of this point are $P=(0\,;L\,;L)$ - substituting them in the equation of a plane we may find constant D:

$$-L\cdot 0-2L\cdot L+L\cdot L+D=0 \quad \Rightarrow \quad D=L^2$$

We will now find point C of intersection of plane and line k. An equation of a line passing through point A and being parallel to a vector $\mathbf{k}$ is as follows:

$$\mathbf{r} = \vec{OA}+\lambda\mathbf{k} \quad \Rightarrow \quad \begin{cases} x=L-\lambda L \\ y=2L-2\lambda L \\ z=0+\lambda L \end{cases}$$

Let's substitute it in the equation of plane:

$$-L\cdot(L-\lambda L)-2L\cdot(2L-2\lambda L)+L\cdot(\lambda L)+L^2=0 \quad \Rightarrow \quad \lambda=\frac{2}{3} \quad \Rightarrow \quad C: \begin{cases} x=\dfrac{L}{3} \\ y=\dfrac{2}{3}L \\ z=\dfrac{2}{3}L \end{cases}$$

We will now find a projection (on a plane) of a vector connecting a point of application of force with any point on the line – since the plane contains the point of application of force, the projection that we are looking for is simply the vector connecting point of application of force with a point of intersection of plane and line:

$$\vec{PC}=\left[\frac{L}{3}-0\ ;\ \frac{2}{3}L-L\ ;\ \frac{2}{3}L-L\right]=\left[\frac{L}{3}\ ;\ -\frac{L}{3}\ ;\ -\frac{L}{3}\right]$$

Projection of $\mathbf{P}$ on a vector normal to π :

$$\mathbf{P}_k=\frac{\mathbf{P}\circ\mathbf{k}}{\mathbf{k}^2}\cdot\mathbf{k}=\frac{P\cdot(-L)+0\cdot(-2L)+0\cdot L}{(-L)^2+(-2L)^2+L^2}[-L;-2L;L]=-\frac{PL}{6L^2}[-L;-2L;L]=\left[\frac{P}{6}\ ;\ \frac{P}{3}\ ;\ -\frac{P}{6}\right]$$

Projection of $\mathbf{P}$ on plane π : $\mathbf{P}_\pi=\mathbf{P}-\mathbf{P}_k=\left[\frac{5}{6}P\ ;\ -\frac{P}{3}\ ;\ \frac{P}{6}\right]$

Moment of $\mathbf{P}$ about line k : $\mathbf{M}_k(\mathbf{P})=\mathbf{P}_\pi\times\vec{PC}=\left[\frac{PL}{6}\ ;\ \frac{PL}{3}\ ;\ -\frac{PL}{6}\right]$

b) The use of theorem

At first, we determine the moment of a force about any point on the considered line and then we project the obtained vector on the direction of that line. Let's calculate a moment about e.g. point A:

Vector of a force:	$\mathbf{P}=[P\ ;\ 0\ ;\ 0]$
Vector connecting the point of application of force with a line:	$\vec{PA}=[L\ ;\ L\ ;\ -L]$
Moment of a force about A:	$\mathbf{M}_A(\mathbf{P})=[0\ ;\ PL\ ;\ PL]$

Projection of a moment on the direction of line:

$$\mathbf{M}_k(\mathbf{P})-\frac{\mathbf{M}_A(\mathbf{P})\circ\mathbf{k}}{\mathbf{k}^2}\cdot\mathbf{k}=\frac{0\cdot(-L)+PL\cdot(-2L)+PL\cdot L}{(-L)^2+(-2L)^2+L^2}[-L;-2L;L]=\frac{-PL^2}{6L^2}[-L;-2L;L]=$$
$$=\left[\frac{PL}{6}\ ;\ \frac{PL}{3}\ ;\ -\frac{PL}{6}\right]$$

EXERCISE 4

Find a couple corresponding with a vector of moment $\mathbf{M}=[2;-1;-2]$, in which one of the forces is applied in point $A=(1,2,-1)$.

SOLUTION:

A couple corresponding with a given vector of a moment lies in a plane which is perpendicular to that vector, so both vectors of couple $\mathbf{M}$. They may be chosen in any way. Let's $\mathbf{F}=[F_x;F_y;F_z]$ be the first vector of that couple:

$$\mathbf{F}\perp\mathbf{M} \quad \Rightarrow \quad \mathbf{F}\circ\mathbf{M}=0 \quad \Rightarrow \quad 2F_x-F_y-2F_z=0$$

The above equation is the only constraint in the choice of components of force $\mathbf{F}$. We may rewrite it in the following form:

$$F_y=2F_x-2F_z$$

Components F_x, F_z may be chosen arbitrary – we only cannot chose both of them equal 0, since we will obtain a zero vector. The easiest way is to chose one of them equal 0 and another one equal 1, then:

$$\mathbf{F}=[F_x;F_y;F_z]=[1\,;\,(2\cdot1-2\cdot0)\,;\,0]=[1;2;0]$$

The second force in the couple is an opposite vector: $\quad -\mathbf{F}=[-1;-2;0]$

We must now only find the point of application of the second force knowing that $\mathbf{F}$ is applied in $A=(1,2,-1)$. We know that a moment of a force about a point lying on its line of action is zero. We also know that a moment of a couple is independent of the choice of point about which it is calculated. Let's calculate a moment of $\mathbf{F}$ about point B which in yet unknown point of application of force $-\mathbf{F}$. This moment must be equal $\mathbf{M}=[2;-1;-2]$:

$$\begin{aligned}\mathbf{M}_B(\mathbf{F}) &= \mathbf{F}\times\vec{AB}=[1;2;0]\times[(x_B-x_A);(y_B-y_A);(z_B-z_A)]=\\ &=[1;2;0]\times[(x_B-1);(y_B-2);(z_B+1)]=[2z_B+2\,;\,-z_B-1\,;\,y_B-2x_B]\equiv[2;-1;-2]\end{aligned}$$

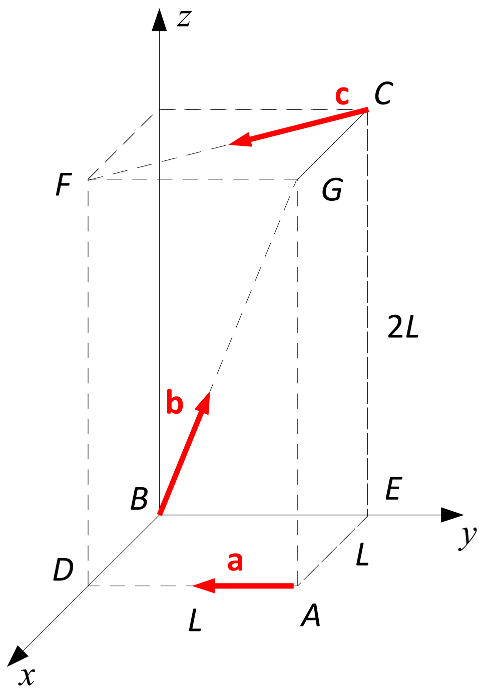
hence: $$\begin{cases}2z_B+2=2\\ -z_B-1=-1\\ y_B-2x_B=-2\end{cases} \quad\Rightarrow\quad \begin{cases}z_B=0\\ y_B=2x_B-2\end{cases}$$

Location of force $-\mathbf{F}$ may also be chosen with some freedom – it may be applied to any point on a line which is parallel to the line of action of $\mathbf{F}$ lying in distance $d=|\mathbf{M}|/|\mathbf{F}|$ from it. Let;s take e.g. $x_B=0$, then $y_B=-2$.

Finally:

A couple corresponding to a vector of moment $\mathbf{M}=[2;-1;-2]$ consists of force $\mathbf{F}=[1;2;0]$ applied in point $A=(1,2,-1)$ and force $-\mathbf{F}=[-1;-2;0]$ applied in point $B=(0,-2,0)$.

EXERCISE 5



Reduce the given system of forces in points D, E, F.

$|\mathbf{a}| = 2P$

$|\mathbf{b}| = 2\sqrt{6}P$

$|\mathbf{c}| = \sqrt{2}P$

SOLUTION:

Vectors of forces:

$$\mathbf{a} = [0\ ;\ -2P\ ;\ 0]$$

$$\mathbf{b} = 2\sqrt{6}P\cdot\frac{\vec{BG}}{|\vec{BG}|} = \frac{2\sqrt{6}P}{\sqrt{L^2+L^2+(2L)^2}}[L;L;2L] = \\ = [2P\ ;\ 2P\ ;\ 4P]$$

$$\mathbf{c} = \sqrt{2}P\cdot\frac{\vec{CF}}{|\vec{CF}|} = \frac{\sqrt{2}P}{\sqrt{L^2+(-L)^2+0^2}}[L;-L;0] = \\ = [P\ ;\ -P\ ;\ 0]$$

Sum of system: $\mathbf{S} = \mathbf{a}+\mathbf{b}+\mathbf{c} = [3P\ ;\ -P\ ;\ 4P]$

Reduction in point D:

$$\mathbf{M}_D(\mathbf{a}) = \mathbf{a}\times\vec{AD} = [0;-2P;0]\times[0;-L;0] = [0;0;0]$$
$$\mathbf{M}_D(\mathbf{b}) = \mathbf{b}\times\vec{BD} = [2P;2P;4P]\times[L;0;0] = [0;4PL;-2PL]$$
$$\mathbf{M}_D(\mathbf{c}) = \mathbf{c}\times\vec{CD} = [P;-P;0]\times[L;-L;-2L] = [2PL;2PL;0]$$

$$\mathbf{M}_D = [2PL;6PL;-2PL]$$

In point D the system is reduced to a vector $\mathbf{S} = [3P\ ;\ -P\ ;\ 4P]$ and to a couple of moment $\mathbf{M}_D = [2PL;6PL;-2PL]$.

Reduction in point E:

$$\mathbf{M}_E(\mathbf{a}) = \mathbf{a}\times\vec{AE} = [0;-2P;0]\times[-L;0;0] = [0;0;-2PL]$$
$$\mathbf{M}_E(\mathbf{b}) = \mathbf{b}\times\vec{BE} = [2P;2P;4P]\times[0;L;0] = [-4PL;0;2PL]$$
$$\mathbf{M}_E(\mathbf{c}) = \mathbf{c}\times\vec{CE} = [P;-P;0]\times[0;0;-2L] = [2PL;2PL;0]$$

$$\mathbf{M}_E = [-2PL;2PL;0]$$

The same result may be obtained with the use of **moment transport theorem**:

$$\mathbf{M}_E = \mathbf{M}_D+\mathbf{S}\times\vec{DE} = [2PL;6PL;-2PL]+[3P;-P;4P]\times[-L;L;0] = [-2PL;2PL;0]$$

In point E the system is reduced to a vector $\mathbf{W} = \mathbf{S} = [3P\ ;\ -P\ ;\ 4P]$ and to a couple of moment $\mathbf{M}_E = [-2PL;2PL;0]$.

Reduction in point F:

$$\mathbf{M}_F(\mathbf{a}) = \mathbf{a} \times \vec{AF} = [0;-2P;0] \times [0;-L;2L] = [-4PL;0;0]$$
$$\mathbf{M}_F(\mathbf{b}) = \mathbf{b} \times \vec{BF} = [2P;2P;4P] \times [L;0;2L] = [4PL;0;-2PL]$$
$$\mathbf{M}_F(\mathbf{c}) = \mathbf{c} \times \vec{CF} = [P;-P;0] \times [L;-L;0] = [0;0;0]$$

$$\mathbf{M}_F = [0;0;-2PL]$$

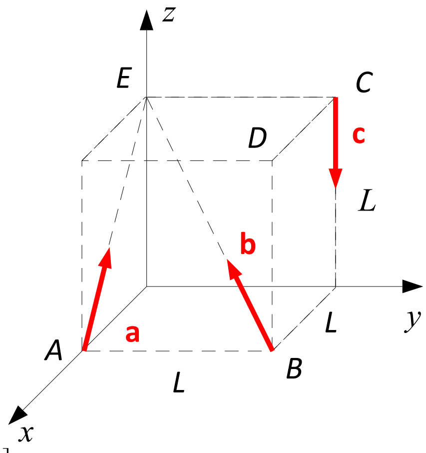
The same result may be obtained with the use of **moment transport theorem**:

$$\mathbf{M}_F = \mathbf{M}_D + \mathbf{S} \times \vec{DF} = [2PL;6PL;-2PL] + [3P;-P;4P] \times [0;0;2L] = [0;0;-2PL]$$

In point F the system is reduced to a vector $\mathbf{W} = \mathbf{S} = [3P\ ;-P\ ;\ 4P]$ and to a couple of moment $\mathbf{M}_F = [0;0\mathrm{a};-2PL]$.

EXERCISE 6

Reduce the given system of forces in point D and then reduce it to the simplest equivalent system.



$|\mathbf{a}| = \sqrt{2}\,P$
$|\mathbf{b}| = \sqrt{3}\,P$
$|\mathbf{c}| = 2\,P$

SOLUTION:

Vectors of forces:

$$\mathbf{a} = 2\sqrt{6}\,P\cdot\frac{\vec{AE}}{|\vec{AE}|} = \frac{\sqrt{2}\,P}{\sqrt{(-L)^2+0^2+L^2}}[-L\,;0\,;L] = [-P\,;\,0\,;\,P]$$

$$\mathbf{b} = \sqrt{2}\,P\cdot\frac{\vec{BE}}{|\vec{BE}|} = \frac{\sqrt{3}\,P}{\sqrt{(-L)^2+(-L)^2+L^2}}[-L\,;-L\,;L] = [-P\,;\,-P\,;\,P]$$

$$\mathbf{c} = [0\,;\,0\,;\,-2\,P]$$

Sum of the system: $\mathbf{S} = \mathbf{a}+\mathbf{b}+\mathbf{c} = [-2P\,;\,-P\,;\,0]$

Reduction in point D:

$$\mathbf{M}_D(\mathbf{a}) = \mathbf{a}\times\vec{AD} = [-P\,;0\,;P]\times[0\,;L\,;L] = [-PL\,;PL\,;-PL]$$
$$\mathbf{M}_D(\mathbf{b}) = \mathbf{b}\times\vec{BD} = [-P\,;-P\,;P]\times[0\,;0\,;L] = [-PL\,;PL\,;0]$$
$$\mathbf{M}_D(\mathbf{c}) = \mathbf{c}\times\vec{CD} = [0\,;0\,;-2\,P]\times[L\,;0\,;0] = [0\,;-2\,PL\,;0]$$

$$\mathbf{M}_D = [-2\,PL\,;0\,;-PL]$$

In point D the system is reduced to a force $\mathbf{W} = \mathbf{S} = [-2\,P\,;\,-P\,;\,0]$ and a couple of moment $\mathbf{M}_D = [-2\,PL\,;0\,;-PL]$.

Reduction to the simplest equivalent system:

Parameter of the system: $K = \mathbf{S}\circ\mathbf{M}_D = [-2\,P\,;-P\,;0]\circ[-2\,PL\,;0\,;-PL] = 4\,P^2\,L$

$K \neq 0 \quad \Rightarrow$ reduction to a **wrench**.

Equation of the central axis

In case of reduction to resultant of to wrench, we know that one of the vectors of the reduced system is a vector equal to the vector of sum of the system, which is applied in any point of the central axis. The centra axis may be found in two ways::

- with the use of formula
- with the use of moment transport theorem

1) Central axis – the formula:

Position vector of the point of reduction: $\vec{OD}=[L\,;L\,;L]$

Vector connecting point D and the central axis:

$$\frac{\mathbf{S}\times\mathbf{M}_D}{\mathbf{S}^2}=\frac{[-2P\,;-P\,;0]\times[-2PL\,;0\,;-PL]}{[-2P\,;-P\,;0]\circ[-2P\,;-P\,;0]}=\frac{[P^2L\,;-2P^2L\,;-2P^2L]}{5P^2}=\left[\frac{1}{5}L\,;-\frac{2}{5}L\,;-\frac{2}{5}L\right]$$

Parametric equation of central axis:

$$\mathbf{r}=\vec{OD}+\frac{\mathbf{S}\times\mathbf{M}_D}{\mathbf{S}^2}+\lambda\mathbf{S}=[L\,;L\,;L]+\left[\frac{1}{5}L\,;-\frac{2}{5}L\,;-\frac{2}{5}L\right]+[-2\lambda P\,;-\lambda P\,;0]=$$
$$=\left[\frac{6}{5}L-2\lambda P\,;\frac{3}{5}L-\lambda P\,;\frac{3}{5}L\right]$$

Parameter λ may be chosen freely. In particular, instead of it, we may take $\mu=\lambda P$, in order to formally oust w force P (static quantity) from the definition of a line (geometric object). Finally, the equations of central axis may be written in the following form:

$$\mathbf{r}(\mu)=\begin{cases}x=\frac{6}{5}L-2\mu\\ y=\frac{3}{5}L-\mu\\ z=\frac{3}{5}L\end{cases}$$

2) Central axis – moment transport theorem:

Let's assume that there exists a certain $P=(x\,,y\,,z)$ belonging to the central axis. Knowing the sum of moments about point D, we may use the moment transport theorem to find this point. Let's determine the point connecting the old and the new center of moment by subtracting appropriate components:

$$\vec{DP}=[x-L\,;y-L\,;z-L]$$

Now, we shall use the moment transport theorem knowing that the moment about any point of central axis is equal:

$$\mathbf{M}_P=\frac{K}{\mathbf{S}^2}\cdot\mathbf{S}$$

The above formula is valid both in case of reduction to a wrench $(K\neq 0)$ and reduction to a resultant $(K=0)$ – then the vector of moment about a point of central axis is a zero vector.

$$\mathbf{M}_D+\mathbf{S}\times\vec{DP}=\mathbf{M}_P=\frac{K}{\mathbf{S}^2}\cdot\mathbf{S}$$

$$[-2PL\ ;\ 0\ ;\ -PL]+[-2P;-P;0]\times[x-L\ ;\ y-L\ ;\ z-L]=\left[-\frac{8}{5}PL\ ;\ -\frac{4}{5}PL\ ;\ 0\right]$$

$$[-2PL\ ;\ 0\ ;\ -PL]+[-P(z-L)\ ;\ 2P(x-L)\ ;\ -2P(y-L)+P(x-L)]=\left[-\frac{8}{5}PL\ ;\ -\frac{4}{5}PL\ ;\ 0\right]$$

$$[-Pz-PL\ ;\ 2Px-2PL\ ;\ Px-2Py]=\left[-\frac{8}{5}PL\ ;\ -\frac{4}{5}PL\ ;\ 0\right]$$

After moving all terms to the left side and dividing with P we obtain::

$$\begin{cases} -z+\frac{3}{5}L=0 \\ 2z-\frac{6}{5}L=0 \\ x-2y=0 \end{cases}$$

We've obtained a system of dependent eqautions – it can be easily noticed that the first and the second equation are equivalent. The above system is equivalent to the system of two equations:

$$\begin{cases} z-\frac{3}{5}L=0 \\ x-2y=0 \end{cases}$$

The above system describes a line which is an edge of intersection of two planes corresponding with those two equations. It can be noticed that the above result could be obtained from the parametric equations e.g. by substitution of $\mu=\frac{3}{5}L-y$ in an equation for coordinate x .

Moment in wrench

We have to calculate a moment about any point of the central axis. We can calculate if with the use of formula for projection of vector of moment on the direction of vector of sum or by calculation of a moment about any chosen point in the axis.

1) Moment in wrench – according to the formula:

$$\mathbf{M}_P=\frac{K}{\mathbf{S}^2}\cdot\mathbf{S}=\frac{4P^2L}{5P^2}[-2P;-P;0]=\left[-\frac{8}{5}PL\ ;\ -\frac{4}{5}PL\ ;\ 0\right]$$

2) Moment in wrench – calculated with the use of moment transport theorem

The same result could be obtained by calculation of sum of moments about a point on central axis. In order to do it we shall us a moment transport theorem. We will determine the vector connecting the point D with any point on central axis:

$$\vec{DP}=\vec{OP}-\vec{OD}=\left[\frac{6}{5}L-2\mu\ ;\ \frac{3}{5}L-\mu\ ;\ \frac{3}{5}L\right]-[L;L;L]=\left[\frac{1}{5}L-2\mu\ ;\ -\frac{2}{5}L-\mu\ ;\ -\frac{2}{5}L\right]$$

$$\mathbf{M}_P = \mathbf{M}_D + \mathbf{S}\times\vec{DP} = [-2\,PL\ ;\ 0\ ;\ -PL]+[-2\,P\,;-P\,;0]\times\left[\frac{1}{5}L-2\mu\ ;-\frac{2}{5}L-\mu\ ;\ -\frac{2}{5}L\right]=$$
$$=[-2\,PL\ ;\ 0\ ;\ -PL]+\left[\frac{2}{5}PL\,;-\frac{4}{5}PL\ ;\ PL\right]=\left[-\frac{8}{5}PL\,;-\frac{4}{5}PL\ ;\ 0\right]$$

Please note that the obtained result do not depend on parameter μ, despite the fact that it explicitly occurred in the position vector of a point of central axis. Since the central axis is parallel to the vector of sum, the results had to be just that. A vector connecting any center of moment with a center on central axis can always be decomposed into a sum of a constant part (the shortest connection to the central axis) and the one dependent on parameter μ - this second part is always parallel to the vector of sum. If the vector of sum is then multiplied (cross product) with such a vector, then – according to the distributivity of vector multiplication – it can be noticed that the part corresponding with the parameter-dependent part will always be 0 (both vectors are parallel). If we want to calculate the moment in wrench with the use of moment transport theorem, we could simply assume any value of the parameter, e.g. such for which the calculation is the simplest, e.g. $\mu=0$.

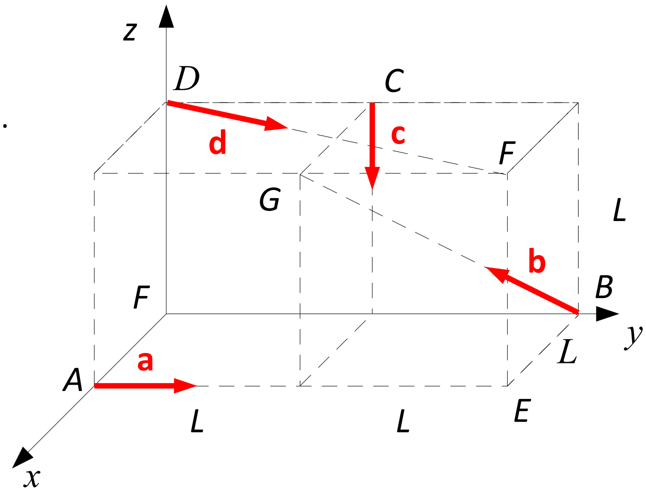
Finally, the result of reduction to the simplest equivalent system may be summarized as follows:

The given system of forces is reduced to the simplest equivalent system of a wrench consisting of a vector equal to the sum of the system $\mathbf{W}=\mathbf{S}=[-2\,P\ ;\ -P\ ;\ 0]$ applied to any point of the central axis given by equations:

$$\mathbf{r}(\mu)=\begin{cases} x=\frac{6}{5}L-2\mu \\ y=\frac{3}{5}L-\mu \\ z=\frac{3}{5}L \end{cases}$$

and of a couple of moment $\mathbf{M}_P=\left[-\frac{8}{5}PL\ ;\ -\frac{4}{5}PL\ ;\ 0\right]$, which is parallel to the vector $\mathbf{S}$.

EXERCISE 7



Reduce the given system of forces in point E
and then reduce it to the simplest equivalent system.

$|\mathbf{a}| = 3P$
$|\mathbf{b}| = \sqrt{3}P$
$|\mathbf{c}| = P$
$|\mathbf{d}| = 2\sqrt{5}P$

SOLUTION:

Vectors of forces:

$$\mathbf{a} = [0\ ;\ 3P\ ;\ 0]$$

$$\mathbf{b} = \sqrt{3}P \cdot \frac{\vec{BG}}{|\vec{BG}|} = \frac{\sqrt{3}P}{\sqrt{L^2+(-L)^2+L^2}}[L;-L;L] = [P\ ;-P\ ;\ P]$$

$$\mathbf{c} = [0\ ;\ 0\ ;-P]$$

$$\mathbf{d} = 2\sqrt{5}P \cdot \frac{\vec{DF}}{|\vec{DF}|} = \frac{2\sqrt{5}P}{\sqrt{L^2+(2L)^2+0^2}}[L;2L;0] = [2P\ ;\ 4P\ ;\ 0]$$

Sum of the system: $\mathbf{S} = \mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d} = [3P\ ;\ 6P\ ;\ 0]$

Reduction in point E:

$$\mathbf{M}_E(\mathbf{a}) = \mathbf{a}\times\vec{AE} = [0;3P;0]\times[0;2L;0] = [0;0;0]$$
$$\mathbf{M}_E(\mathbf{b}) = \mathbf{b}\times\vec{BE} = [P;-P;P]\times[L;0;0] = [0;PL;PL]$$
$$\mathbf{M}_E(\mathbf{c}) = \mathbf{c}\times\vec{CE} = [0;0;-P]\times[L;L;-L] = [PL;-PL;0]$$
$$\mathbf{M}_E(\mathbf{d}) = \mathbf{d}\times\vec{DE} = [2P;4P;0]\times[L;2L;-L] = [-4PL;2PL;0]$$

$$\mathbf{M}_E = [-3Pa;2Pa;Pa]$$

In point E the system is reduced to a vector $\mathbf{W}=\mathbf{S} = [6P\ ;\ 3P\ ;-P]$ equal the vector of sum and to a couple of moment $\mathbf{M}_E = [-3PL;2PL;PL]$

Reduction to the simplest equivalent system:

Parameter of the system: $K = \mathbf{M}_E \circ \mathbf{S} = [-3PL;2PL;PL]\circ[3P;6P;0] = 3P^2L$

$K \neq 0 \quad \Rightarrow$ reduction to a **wrench**.

Moment in wrench:

$$\mathbf{M}_P = \frac{K}{\mathbf{S}^2}\cdot\mathbf{S} = \frac{3P^2L}{45P^2}[3P;6P;0] = \left[\frac{1}{5}PL\ ;\ \frac{2}{5}PL\ ;\ 0\right]$$

Central axis:

Position vector of point of reduction: $\vec{OE} = [L\,;2L\,;0]$

Vector connecting point E with central axis:

$$\frac{\mathbf{S}\times\mathbf{M}_E}{\mathbf{S}^2} = \frac{[6P\,;3P\,;0]\times[-3PL\,;2PL\,;PL]}{[6P\,;3P\,;0]\circ[6P\,;3P\,;0]} = \frac{[6P^2L\,;-3P^2L\,;24P^2L]}{45P^2} =$$
$$=\left[\frac{2}{15}L\,;-\frac{1}{15}L\,;\,\frac{8}{15}L\right]$$

Parametric equations of central axis:

$$\mathbf{r} = \vec{OE}+\frac{\mathbf{S}\times\mathbf{M}_E}{\mathbf{S}^2}+\lambda\mathbf{S} = [L\,;2L\,;0]+\left[\frac{2}{15}L\,;\,-\frac{1}{15}L\,;\,\frac{8}{15}L\right]+[3\lambda P\,;6\lambda P\,;0] =$$
$$=\left[\frac{17}{15}L+3\mu\,;\,\frac{29}{15}L+6\mu\,;\,\frac{8}{15}L-\mu\right]$$

The given system of forces is reduced to the simplest equivalent system of wrench consisting of vector $\mathbf{W}=\mathbf{S}=[6P\,;\,3P\,;\,-P]$ equal to the vector of sum and applied in any point of the central axis given by equations:

$$\mathbf{r}(\mu)=\begin{cases} x=\dfrac{17}{15}L+3\mu \\ y=\dfrac{29}{15}L+6\mu \\ z=-\dfrac{8}{15}L \end{cases}$$

and of a couple of moment $\mathbf{M}_P=\left[\frac{1}{5}PL\,;\,\frac{2}{5}PL\,;\,0\right]$,which is parallel to the vector $\mathbf{S}$.

EXERCISE 8

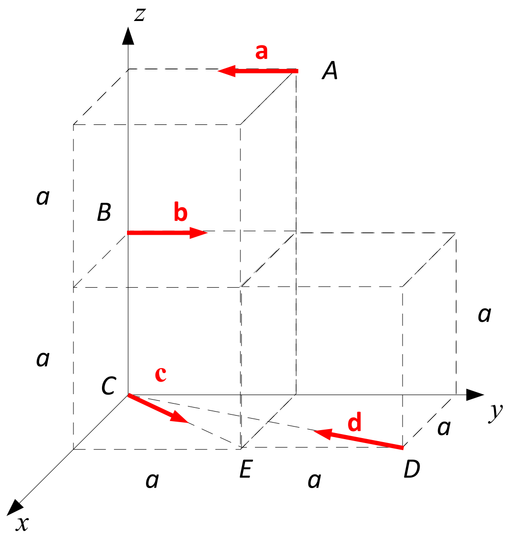
Reduce the given system of forces to the simplest equivalent system:

$|\mathbf{a}| = P$

$|\mathbf{b}| = 2P$

$|\mathbf{c}| = \sqrt{2}P$

$|\mathbf{d}| = \sqrt{5}P$



SOLUTION:

Vectors of forces:

$$\mathbf{a} = [0 \; ; \; -P \; ; \; 0]$$

$$\mathbf{b} = [0 \; ; \; 2P \; ; \; 0]$$

$$\mathbf{c} = \sqrt{2}P \cdot \frac{\vec{CE}}{|\vec{CE}|} = \frac{\sqrt{2}P}{\sqrt{L^2+L^2+0^2}}[L; L; 0] = [P \; ; \; P \; ; \; 0]$$

$$\mathbf{d} = \sqrt{5}P \cdot \frac{\vec{DC}}{|\vec{DC}|} = \frac{\sqrt{5}P}{\sqrt{(-L)^2+(-2L)^2+0^2}}[-L; -2L; 0] = [-P \; ; \; -2P \; ; \; 0]$$

Sum of the system: $\mathbf{S} = \mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d} = [0 \; ; \; 0 \; ; \; 0]$

The sum of the system is a zero vector so the system may reduce wither to a zero system or to a couple. This reduction is the same in all points and it is always a reduction to the simplest equivalent system.

In order to reduce the system to the simplest equivalent system we have to find the sum of moments of this system about any point – it will be most convenient to chose such a point that lies at intersection of lines of action of the greatest possible number of forces since the moment of all those forces about such a point is 0. Let's consider point C.

Reduction in point C:

$$\mathbf{M}_C(\mathbf{a}) = \mathbf{a} \times \vec{AC} = [0; -P; 0] \times [0; -L; -2L] = [2PL; 0; 0]$$
$$\mathbf{M}_C(\mathbf{b}) = \mathbf{b} \times \vec{BC} = [0; 2P; 0] \times [0; 0; -L] = [-2PL; 0; 0]$$
$$\mathbf{M}_E(\mathbf{c}) = \mathbf{0}$$
$$\mathbf{M}_E(\mathbf{d}) = \mathbf{0}$$

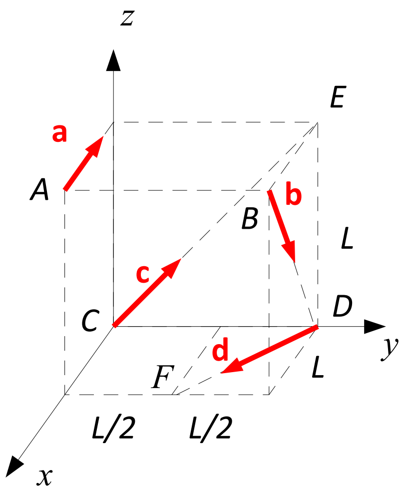
$$\mathbf{M}_E = [0; 0; 0]$$

Both sum and sum of moments are zero – the given system is reduced to a zero system in any point.

EXERCISE 9

Reduce the given system of forces to the simplest equivalent system:

$|\mathbf{a}| = P$
$|\mathbf{b}| = \sqrt{2}\,P$
$|\mathbf{c}| = \sqrt{2}\,P$
$|\mathbf{d}| = \sqrt{5}\,P$



SOLUTION:

Vectors of forces:

$$\mathbf{a} = [-P\ ;\ 0\ ;\ 0]$$

$$\mathbf{b} = \sqrt{2}\,P\cdot\frac{\vec{BD}}{|\vec{BD}|} = \frac{\sqrt{2}\,P}{\sqrt{(-L)^2+0^2+(-L)^2}}[-L\,;0\,;-L] = [-P\ ;\ 0\ ;\ -P]$$

$$\mathbf{c} = \sqrt{2}\,P\cdot\frac{\vec{CE}}{|\vec{CE}|} = \frac{\sqrt{2}\,P}{\sqrt{0^2+L^2+L^2}}[0\,;L\,;L] = [0\ ;\ P\ ;\ P]$$

$$\mathbf{d} = \sqrt{5}\,P\cdot\frac{\vec{DF}}{|\vec{DF}|} = \frac{\sqrt{5}\,P}{\sqrt{L^2+(-L/2)^2+0^2}}[L\,;-L/2\,;0] = [2\,P\ ;\ -P\ ;\ 0]$$

Sum of the system: $\mathbf{S} = \mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d} = [0\ ;\ 0\ ;\ 0]$

The sum of the system is a zero vector so the system may reduce wither to a zero system or to a couple. This reduction is the same in all points and it is always a reduction to the simplest equivalent system.

Point D lies at intersection of lines of action of forces $\mathbf{b}$ and $\mathbf{d}$ - it is the point about which it will be the easiest to calculate the sum of moments:

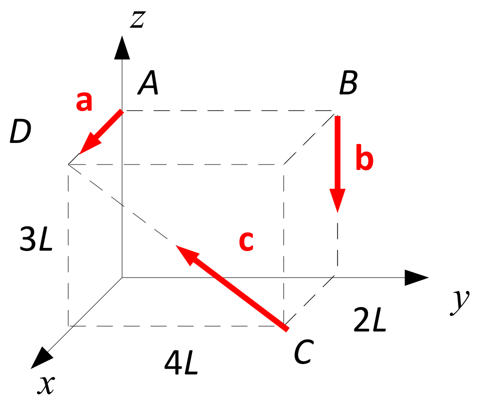
Reduction in point D:

$$\mathbf{M}_D(\mathbf{a}) = \mathbf{a}\times\vec{AD} = [-P\,;0\,;0]\times[-L\,;L\,;-L] = [0\,;-PL\,;-PL]$$
$$\mathbf{M}_D(\mathbf{b}) = \mathbf{0}$$
$$\mathbf{M}_D(\mathbf{c}) = \mathbf{c}\times\vec{CD} = [0\,;P\,;P]\times[0\,;L\,;0] = [-PL\,;0\,;0]$$
$$\mathbf{M}_D(\mathbf{d}) = \mathbf{0}$$

$$\mathbf{M}_E = [-PL\,;-PL\,;-PL]$$

The given system is reduced in any point to a couple of moment $\mathbf{M} = [-PL\,;-PL\,;-PL]$.

EXERCISE 10



Reduce the given system of forces to the simplest equivalent system:

$|\mathbf{a}| = P$
$|\mathbf{b}| = 2P$
$|\mathbf{c}| = 5P$

SOLUTION:

Vectors of forces:

$\mathbf{a} = [P\ ;\ 0\ ;\ 0]$
$\mathbf{b} = [0 ; 0 ; -2P]$
$$\mathbf{c} = 5P \cdot \frac{\vec{CD}}{|\vec{CD}|} = \frac{5P}{\sqrt{0^2+(-4L)^2+(3L)^2}}[0 ; -4L ; 3L] = [0\ ;\ -4P\ ;\ 3P]$$

Sum of the system: $\mathbf{S} = \mathbf{a}+\mathbf{b}+\mathbf{c} = [P\ ;\ -4P\ ;\ P]$

Reduction in point D:

$$\mathbf{M}_D(\mathbf{a}) = \mathbf{0}$$
$$\mathbf{M}_D(\mathbf{c}) = \mathbf{0}$$
$$\mathbf{M}_D(\mathbf{b}) = \mathbf{b} \times \vec{BD} = [0 ; 0 ; -2P] \times [2L ; -4L ; 0] = [-8PL ; -4PL ; 0]$$

$$\mathbf{M}_D = [-8PL ; -4PL ; 0]$$

Parameter of the system: $K = \mathbf{M}_D \circ \mathbf{S} = [-8PL ; -4PL ; 0] \circ [P ; -4P ; P] = 8P^2L$

$K \neq 0 \quad \Rightarrow$ Reduction to a **wrench**.

Reduction to the simplest equivalent system:

Position vector of point of reduction: $\vec{OD} = [2L\ ;\ 0\ ;\ 3L]$

Vector connecting point D with central axis:

$$\frac{\mathbf{S}\times\mathbf{M}_D}{\mathbf{S}^2} = \frac{[P ; -4P ; P] \times [-8PL ; -4PL ; 0]}{[P ; -4P ; P] \circ [P ; -4P ; P]} = \frac{[4P^2L ; -8P^2L ; -36P^2L]}{18P^2} =$$
$$= \left[\frac{2}{9}L ; -\frac{4}{9}L ; -2L\right]$$

Central axis:

$$\mathbf{r} = \vec{OD} + \frac{\mathbf{S}\times\mathbf{M}_D}{\mathbf{S}^2} + \lambda\mathbf{S} = [2L ; 0 ; 3L] + \left[\frac{2}{9}L\ ;\ -\frac{4}{9}L\ ;\ -2L\right] + [\lambda P ; -4\lambda P ; \lambda P] =$$
$$= \left[\frac{20}{9}L + \mu\ ;\ -\frac{4}{9}L - 4\mu\ ;\ L + \mu\right]$$

Moment in wrench:

$$\mathbf{M}_P = \frac{K}{\mathbf{S}^2}\cdot\mathbf{S} = \frac{8P^2L}{18P^2}[P\,;-4P\,;P] = \left[\frac{4}{9}PL\ ;-\frac{16}{9}PL\ ;\frac{4}{9}PL\right]$$

The given system of forces is reduced to the simplest equivalent system of wrench consisting of vector $\mathbf{W}=\mathbf{S}=[P\ ;\ -4P\ ;\ P]$ equal the vector of sum and applied in any point of central axis given by equations:

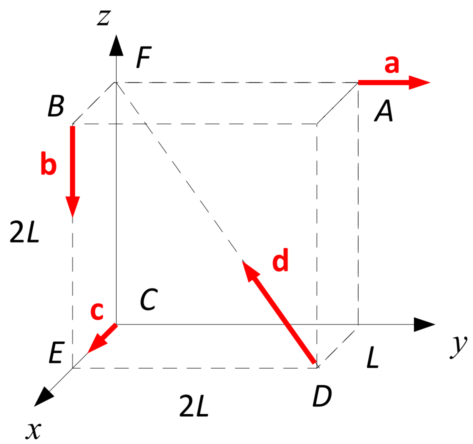
$$\mathbf{r}(\mu) = \begin{cases} x = \frac{20}{9}L+\mu \\ y = -\frac{4}{9}L-4\mu \\ z = L+\mu \end{cases}$$

and of a couple of moment $\mathbf{M}_P = \left[\frac{4}{9}PL\ ;-\frac{16}{9}PL\ ;\frac{4}{9}PL\right]$,which is parallel to vector $\mathbf{S}$.

EXERCISE 11

Reduce the given system of forces to the simplest equivalent system:

$|\mathbf{a}| = P$
$|\mathbf{b}| = 2P$
$|\mathbf{c}| = P$
$|\mathbf{d}| = 3P$



SOLUTION:

Vectors of forces:

$\mathbf{a} = [0\ ;\ P\ ;\ 0]$
$\mathbf{b} = [0;0;-2P]$
$\mathbf{c} = [P\,;0\,;0]$

$$\mathbf{d} = 3P\cdot\frac{\vec{DF}}{|\vec{DF}|} = \frac{3P}{\sqrt{(-a)^2+(-2a)^2+(2a)^2}}[-L;-2L;2L] = [-P\ ;\ -2P\ ;\ 2P]$$

Sum of the system: $\mathbf{S} = \mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d} = [0\ ;\ -P\ ;\ 0]$

Reduction in point E:

$$\mathbf{M}_E(\mathbf{a}) = \mathbf{a}\times\vec{AE} = [0\,;P\,;0]\times[L;-2L;-2L] = [-2PL;0\,;-PL]$$
$$\mathbf{M}_E(\mathbf{b}) = \mathbf{0}$$
$$\mathbf{M}_E(\mathbf{c}) = \mathbf{0}$$
$$\mathbf{M}_E(\mathbf{d}) = \mathbf{d}\times\vec{DE} = [-P\,;-2P\,;2P]\times[0;-2L;0] = [4PL;0;2PL]$$

$$\mathbf{M}_E = [2PL;0;PL]$$

Parameter of the system: $K = \mathbf{M}_E \circ \mathbf{S} = [2PL; 0; PL] \circ [0; -P; 0] = 0$

$K = 0 \quad \wedge \quad \mathbf{S} \neq \mathbf{0} \quad \Rightarrow$ reduction to a **resultant**.

Reduction to the simplest equivalent system:

Position vector of point of reduction: $\vec{OE} = [L; 0; 0]$

Vector connecting point E with central axis:

$$\frac{\mathbf{S} \times \mathbf{M}_E}{\mathbf{S}^2} = \frac{[0; -P; 0] \times [2PL; 0; PL]}{[0; -P; 0] \circ [0; -P; 0]} = \frac{[-P^2 L; 0; 2P^2 L]}{P^2} =$$
$$= [-L; 0; 2L]$$

Central axis:

$$\mathbf{r} = \vec{OE} + \frac{\mathbf{S} \times \mathbf{M}_E}{\mathbf{S}^2} + \lambda \mathbf{S} = [L; 0; 0] + [-L; 0; 2L] + [0; -\lambda P; 0] =$$
$$= [0; -\mu; 2L]$$

The given system of forces is reduced to the simplest equivalent system of resultant $\mathbf{W} = \mathbf{S} = [0; -P; 0]$ applied to any point of central axis given by equations:

$$\mathbf{r}(\mu) = \begin{cases} x = 0 \\ y = -\mu \\ z = 2L \end{cases}$$

Central axis is a horizontal line parallel to *y* axis (vector of sum is parallel to *y* axis), containing point $F = (0; 0; 2L)$ – coordinates of this point may be obtained by substituting $\mu = 0$. Let's see what would happen if the initial point of reduction was F. Then:

Reduction in point F:

$$\mathbf{M}_F(\mathbf{a}) = \mathbf{0}$$
$$\mathbf{M}_F(\mathbf{b}) = \mathbf{b} \times \vec{BF} = [0; 0; -2P] \times [-L; 0; 0] = [0; 2PL; 0]$$
$$\mathbf{M}_F(\mathbf{c}) = \mathbf{c} \times \vec{CF} = [P; 0; 0] \times [0; 0; 2L] = [0; -2PL; 0]$$
$$\mathbf{M}_F(\mathbf{d}) = \mathbf{0}$$

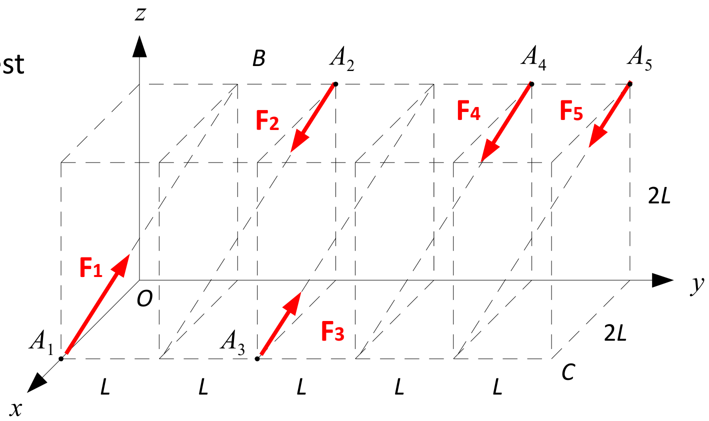
$$\mathbf{M}_F = [0; 0; 0]$$

From the fact that $\mathbf{M}_F = \mathbf{0}$ it is evident that $K = 0$ and $\mathbf{M}_F \times \mathbf{S} = \mathbf{0}$, so we may conclude that the system is reduced to a resultant applied to any point of central axis which contains point F and is parallel to vector **S**.

EXERCISE 12

Reduce the given system of forces to the simplest equivalent system:

$$|\mathbf{F_1}| = 6\,P$$
$$|\mathbf{F_2}| = 4\,P$$
$$|\mathbf{F_3}| = 3\,P$$
$$|\mathbf{F_4}| = 5\,P$$
$$|\mathbf{F_5}| = 3\,P$$



Find the sum of moments of this system about point C.

SOLUTION:

Position vectors of points of application of forces:

$\mathbf{r_1} = [2L;0;0]$
$\mathbf{r_2} = [0;2L;2L]$
$\mathbf{r_3} = [2L;2L;0]$
$\mathbf{r_4} = [0;4L;2L]$
$\mathbf{r_5} = [0;5L;2L]$

Let's assume a **versor**: $\mathbf{e} = \dfrac{\vec{A_1 B}}{|A_1 B|} = \dfrac{[-2L\,;\,L\,;\,2L]}{\sqrt{(-2L)^2+L^2+(2L)^2}} = \left[-\dfrac{2}{3}\,;\,\dfrac{1}{3}\,;\,\dfrac{2}{3}\right]$

$$\begin{array}{ll}
F_1=6P & F_1\mathbf{r_1} = 6P[2L;0;0] = [12PL;0;0] \\
F_2=-4P & F_2\mathbf{r_2} = -4P[0;2L;2L] = [0;-8PL;-8PL] \\
F_3=3P & F_3\mathbf{r_3} = 3P[2L;2L;0] = [6PL;6PL;0] \\
F_4=-5P & F_4\mathbf{r_4} = -5P[0;4L;2L] = [0;-20PL;-10PL] \\
F_5=-3P & F_5\mathbf{r_5} = -3P[0;5L;2L] = [0;-15PL;-6PL] \\
\hline
S=\sum_{i=1}^{5} F_i=-3P & \mathbf{M} = \sum_{i=1}^{5} F_i\mathbf{r_i}=[18PL;-37PL;-24PL]
\end{array}$$

The sum is not equal 0 – the system is reduced to a **resultant**.

Resultant: $\mathbf{W} = \mathbf{S} = S\,\mathbf{e} = -3P\left[-\dfrac{2}{3}\,;\,\dfrac{1}{3}\,;\,\dfrac{2}{3}\right] = [2P;-P;-2P]$

Center of the system: $\vec{OO^*} = \dfrac{\mathbf{M}}{S} = \dfrac{[18PL;-37PL;-24PL]}{-3P} = \left[-6L\,;\,\dfrac{37}{3}L\,;\,8L\right]$

Central axis:

$$\mathbf{r}(\lambda)=\overrightarrow{OO^*}+\lambda\mathbf{S}=\left[-6L\ ;\ \frac{37}{3}L\ ;\ 8L\right]+\lambda[2P;-P;-2P]=\begin{cases}x=-6L+2P\lambda\\ y=\frac{37}{3}L-P\lambda\\ z=8L-2P\lambda\end{cases}$$

The given system of forces is reduced to the simples equivalent system of a resultant $\mathbf{W}=\mathbf{S}=[2P;-P;-2P]$ applied to any point of central axis given by equations:

$$\mathbf{r}(\lambda)=\begin{cases}x=-6L+2P\lambda\\ y=\frac{37}{3}L-P\lambda\\ z=8L-2P\lambda\end{cases}$$

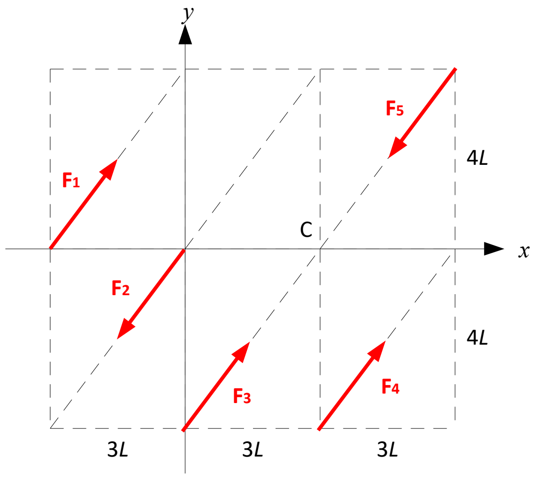
The sum of moments of the system about point C will be found basing on the momentr transport theorem and on the fact the sum of moments about center of the system is zero:

$$\mathbf{M}_C=\mathbf{S}\times\overrightarrow{O^*C}=[2P;-P;-2P]\times\left[2L-(-6L);5L-\frac{37}{3}L;0-8L\right]=$$
$$=\left[-\frac{20}{3}PL\ ;\ 0\ ;\ -\frac{20}{3}PL\right]$$

EXERCISE 13

Reduce the given system of forces to the simplest equivalent system:

$$|\mathbf{F}_1| = 5\,P$$
$$|\mathbf{F}_2| = 10\,P$$
$$|\mathbf{F}_3| = 5\,P$$
$$|\mathbf{F}_4| = 15\,P$$
$$|\mathbf{F}_5| = 20\,P$$



Draw the result in the picture.
Find the sum of moments about point C.

ROZWIĄZANIE:

Let's assume a **versor**: $\mathbf{e} = \dfrac{[3\,L\ ;\ 4\,L\ ;\ 0]}{\sqrt{(3\,L)^2+(4\,L)^2+0^2}} = \left[\dfrac{3}{5}\ ;\ \dfrac{4}{5}\ ;\ 0\right]$

$$\begin{array}{llll}
F_1 = 5\,P & F_1\mathbf{r}_1 = & 5\,P[-3\,L\,;0\,;0] & =[-15\,PL\,;0\,;0] \\
F_2 = -10\,P & F_2\mathbf{r}_2 = & -10\,P[0\,;0\,;0] & =[0\,;0\,;0] \\
F_3 = 5\,P & F_3\mathbf{r}_3 = & 5\,P[0\,;-4\,L\,;0] & =[0\,PL\,;-20\,PL\,;0] \\
F_4 = 15\,P & F_4\mathbf{r}_4 = & 15\,P[3\,L\,;-4\,L\,;0] & =[45\,PL\,;-60\,PL\,;0] \\
F_5 = -20\,P & F_5\mathbf{r}_5 = & -20\,P[6\,L\,;4\,L\,;0] & =[-120\,PL\,;-80\,PL\,;0] \\
\hline
S=\sum_{i=1}^{5} F_i = -5\,P & & \mathbf{M} = \sum_{i=1}^{5} F_i\mathbf{r}_i = [-90\,Pa\,;-160\,Pa\,;0] &
\end{array}$$

The sum is not equal 0 – the system is reduced to a **resultant**.

Resultant: $\mathbf{W} = \mathbf{S} = S\,\mathbf{e} = (-5\,P)\left[\dfrac{3}{5}\ ;\ \dfrac{4}{5}\ ;\ 0\right] = [-3\,P\,;-4\,P\,;0]$

Center of the system: $\vec{OO}^* = \dfrac{\mathbf{M}}{S} = \dfrac{[-90\,PL\,;-160\,PL\,;0]}{5\,P} = [18\,L\ ;\ 32\,L\ ;\ 0]$

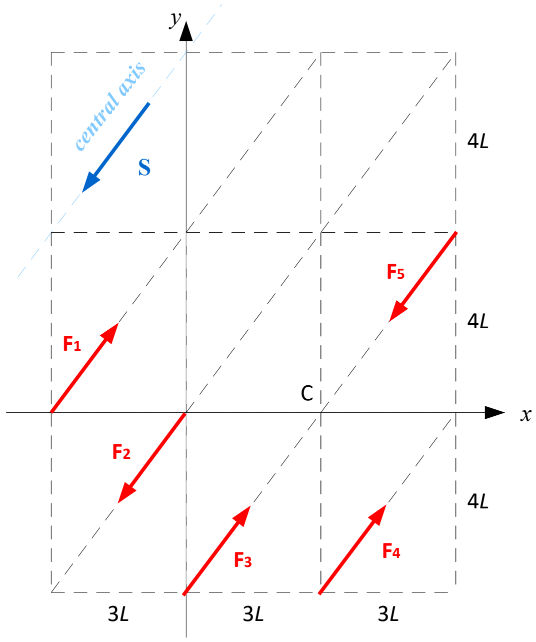
Central axis:

$$\mathbf{r}(\lambda) = \vec{OO}^* + \lambda\mathbf{S} = [18\,L\ ;\ 32\,L\ ;\ 0] + \lambda[3\,P\,;4\,P\,;0] = \begin{cases} x = 18\,L + 3\,P\lambda \\ y = 32\,L + 4\,P\lambda \\ z = 0 \end{cases}$$

A line on a plane may be rewritten in a simpler form by ousting parameter λ . Multiplying the first equation with (-4), the second one with 3 and adding them together gives us:

$$-4\,x + 3\,y = -72\,L - 12\,P\lambda + 96\,L + 12\,P\lambda \quad\Rightarrow\quad y = \frac{4}{3}x + 8\,L$$

The given system of forces is reduced to the simplest equivalent system of resultant $\mathbf{W} = \mathbf{S} = [-3P; -4P; 0]$ applied to any point of the central axis of equation: $y = \frac{4}{3}x + 8L$

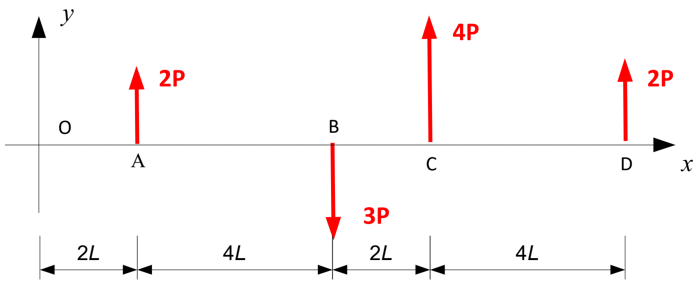


Sum of moments of the system about point C is equal:

$$\mathbf{M}_C = \mathbf{S} \times \overrightarrow{O^* C} = [-3P; -4P; 0] \times [3L - 18L; 0 - 32L; 0] = [0\ ;\ 0\ ;\ 36PL]$$

EXERCISE 14

Reduce the given system of forces to the simplest equivalent system:



SOLUTION:
Let's assume a **versor**: $\mathbf{e} = [0;1;0]$

$$
\begin{array}{ll}
F_1 = 2P & F_1\mathbf{r_1} = 2P[2L;0;0] = [4PL;0;0] \\
F_2 = -3P & F_2\mathbf{r_2} = -3P[6L;0;0] = [-18PL;0;0] \\
F_3 = 4P & F_3\mathbf{r_3} = 4P[8L;0;0] = [32PL;0;0] \\
F_4 = 2P & F_4\mathbf{r_4} = 2P[12L;0;0] = [24PL;0;0] \\
\hline
S = \sum_{i=1}^{5} F_i = 5P & \mathbf{M} = \sum_{i=1}^{5} F_i \mathbf{r}_i = [42PL;0;0]
\end{array}
$$

Sum in not equal 0 – the system is reduced to a **resultant**.

Resultant: $\mathbf{W} = \mathbf{S} = S\,\mathbf{e} = 5P[0;1;0] = [0;5P;0]$

Center of the system: $\vec{OO}^* = \dfrac{\mathbf{M}}{S} = \dfrac{[42PL;0;0]}{5P} = [8{,}4L\ ;\ 0\ ;\ 0]$

Central axis: $\mathbf{r}(\lambda) = \vec{OO}^* + \lambda \mathbf{S} = [8{,}4L\ ;\ 0\ ;\ 0] + \lambda[0;5P;0] = \begin{cases} x = 8{,}4 \\ y = 5\lambda P \\ z = 0 \end{cases}$

The above calculation suggest simpler way of solving the problem. Since all forces are vertical it is not necessary to describe them with the use of vectors as their direction is known anyway. Similarly, since they all lie in a single plane and they are all applied to the points os a single horizontal line, it is not necessary to describe the position of their points of application with vectors – it is enough to give appropriate value of x. Finally, since all vectors (forces and position vectors) lie in a single plane, then vectors of moments are all parallel, namely, the are perpendicular to the plane – it is enough to give this only non-zero component of those vectors.

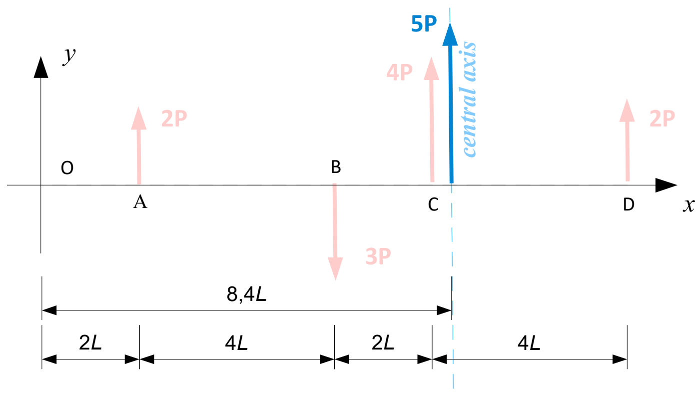
We can write:

$$S=\sum_{i=1}^{n}a_i=2P-3P+4P+2P=5P$$
$$M=\sum_{i=1}^{n}a_i r_i=(2P)\cdot(2L)+(-3P)\cdot(6L)+(4P)\cdot(8L)+(2P)\cdot(12L)=42Pa$$

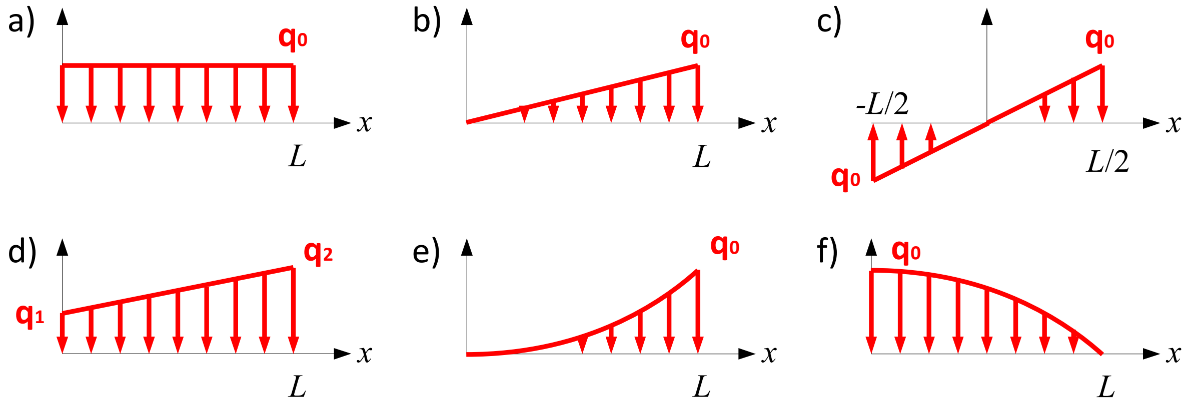
All vactors are vertical and lie in a single plane, so the central axis will also be vertical and lie in that plane. It is enough to find a single point belonging to it. In particular, such a point is the center of the system of parallel forces:

$$x_O=\frac{M}{S}=8{,}4\,L$$

The given system of forces is reduced to the simplest equivalent system of resultant $\mathbf{W}=\mathbf{S}=[0;5P;0]$ applied to any point of the central axis lying in plane (x,y) and given by equation $x=8{,}4L$.



EXERCISE 15
Reduce the distributed loads.



SOLUTION:

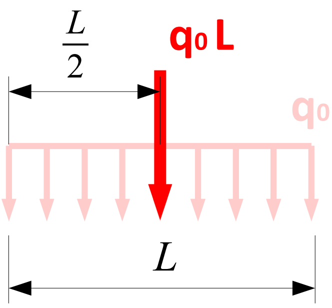
a) Rectangular (uniform) load

Load intensity: $q(x) = q_0$

Sum: $S = \int\limits_0^L q_0 \, \mathrm{d}x = [qx]_0^L = q_0 L$

Moment: $M = \int\limits_0^L q_0 x \, \mathrm{d}x = \left[\dfrac{q_0 x^2}{2} \right]_0^L = \dfrac{q_0 L^2}{2}$

Center: $x_O = \dfrac{M}{S} = \dfrac{L}{2}$



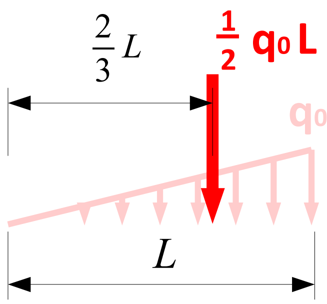
b) Triangular (hydrostatic) load

Load intensity: $q(x) = \dfrac{q_0}{L} x$

Sum: $S = \int\limits_0^L \dfrac{q_0}{L} x \, \mathrm{d}x = \left[\dfrac{q_0 x^2}{2L} \right]_0^L = \dfrac{q_0 L}{2}$

Moment: $M = \int\limits_0^L \dfrac{q_0}{L} x^2 \, \mathrm{d}x = \left[\dfrac{q_0 x^3}{3L} \right]_0^L = \dfrac{q_0 L^2}{3}$

Center: $x_O = \dfrac{M}{S} = \dfrac{2}{3} L$



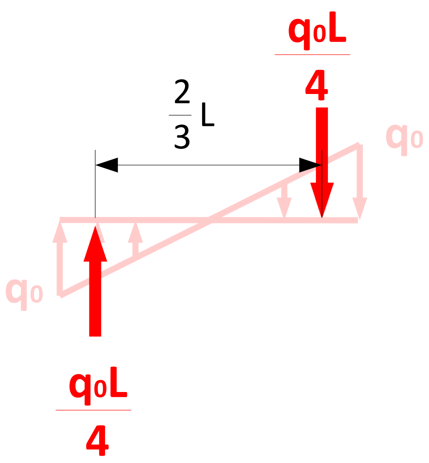
c) Antisymmetric linear load

Load intensity: $q(x) = \frac{2q_0}{L}x$

Sum: $S = \int_{-L/2}^{L/2} \frac{2q_0}{L} x\,\mathrm{d}x = \left[\frac{q_0 x^2}{L}\right]_{-L/2}^{L/2} = 0$

Moment: $M = \int_{-L/2}^{L/2} \frac{2q_0}{L} x^2\,\mathrm{d}x = \left[\frac{2q_0 x^3}{3L}\right]_{-L/2}^{L/2} = \frac{q_0 L^2}{6}$

Sum is equal 0, while moment is not equal 0. The given system of forces is reduced to a couple of moment $\frac{q_0 L^2}{6}$. This couple may be determined arbitrary – in particular, each triangular part may be replaced with resultant:



$$M = P \cdot d = \frac{q_0 L}{4} \cdot \frac{2}{3} L = \frac{q_0 L^2}{6}$$

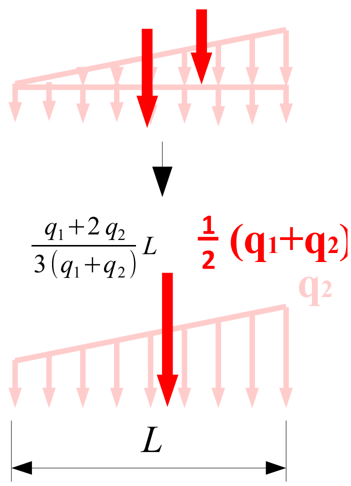
d) Trapezoidal load

Resultant of this load may be found by treating the load as a superposition of uniform lod of intensity q_1 and of triangular load of intensity $q_2 - q_1$ according to the picture:

Sum: $S = q_1 L + \frac{1}{2}(q_2 - q_1)L = \frac{(q_1+q_2)L}{2}$

Moment: $M = \frac{q_1 L^2}{2} + \frac{(q_2-q_1)L^2}{3} = \frac{(q_1+2q_2)L^2}{6}$

Center: $x_O = \frac{M}{S} = \frac{q_1+2q_2}{3(q_1+q_2)}L$



e) Parabolic load I

Intensity is given by a certain quadratic function $q(x) = Ax^2 + Bx + C$, parameters of which may be determined according to the picture:

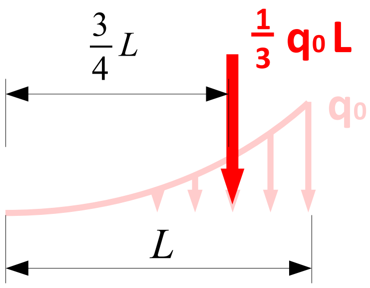
$$\begin{cases} q(0)=0 \\ q'(0)=0 \\ q(L)=q_0 \end{cases} \Rightarrow \begin{cases} C=0 \\ B=0 \\ AL^2+BL+C=q_0 \end{cases} \Rightarrow \begin{cases} A=\dfrac{q_0}{L^2} \\ B=0 \\ C=0 \end{cases}$$

Load intensity: $q(x) = \dfrac{q_0}{L^2}x^2$

Sum: $S = \displaystyle\int_0^L \frac{q_0}{L^2}x^2\,\mathrm{d}x = \left[\frac{q_0 x^3}{3L^2}\right]_0^L = \frac{q_0 L}{3}$

Moment: $M = \displaystyle\int_0^L \frac{q_0}{L^2}x^3\,\mathrm{d}x = \left[\frac{q_0 x^4}{4L^2}\right]_0^L = \frac{q_0 L^2}{4}$

Center: $x_O = \dfrac{M}{S} = \dfrac{3}{4}L$



f) Parabolic load II

Intensity is given by a certain quadratic function $q(x) = Ax^2 + Bx + C$, parameters of which may be determined according to the picture:

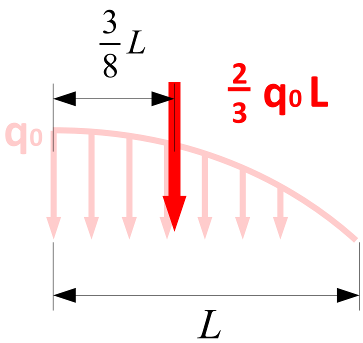
$$\begin{cases} q(0)=q_0 \\ q'(0)=0 \\ q(L)=0 \end{cases} \Rightarrow \begin{cases} C=q_0 \\ B=0 \\ AL^2+BL+C=0 \end{cases} \Rightarrow \begin{cases} A=-\dfrac{q_0}{L^2} \\ B=0 \\ C=q_0 \end{cases}$$

Load intensity: $q(x) = -\dfrac{q_0}{L^2}x^2 + q_0$

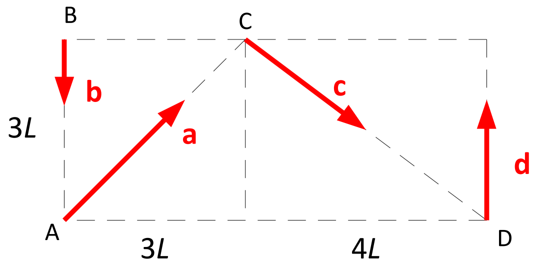
Sum:
$$S = \int_0^L -\frac{q_0}{L^2}x^2 + q_0\,\mathrm{d}x = \left[q_0 x - \frac{q_0 x^3}{3L^2}\right]_0^L = $$
$$= q_0 L - \frac{q_0 L}{3} = \frac{2}{3}q_0 L$$

Moment:
$$M = \int_0^L \left(-\frac{q_0}{L^2}x^2 + q_0\right)x\,\mathrm{d}x = \left[\frac{q_0 x^2}{2} - \frac{q_0 x^4}{4L^2}\right]_0^L =$$
$$= \frac{q_0 L^2}{2} - \frac{q_0 L^2}{4} = \frac{q_0 L^2}{4}$$

Center: $x_O = \dfrac{M}{S} = \dfrac{3}{8}L$



EXERCISE 16
Find the sum of moment of the given system of forces about points A and D



$$|\mathbf{a}| = P\sqrt{2}$$
$$|\mathbf{b}| = 2P$$
$$|\mathbf{c}| = 5P$$
$$|\mathbf{d}| = P$$

SOLUTION:

Vectors: $\mathbf{a} = [P; P; 0]$ $\mathbf{b} = [0; -2P; 0]$ $\mathbf{c} = [4P; -3P; 0]$ $\mathbf{d} = [0; P; 0]$

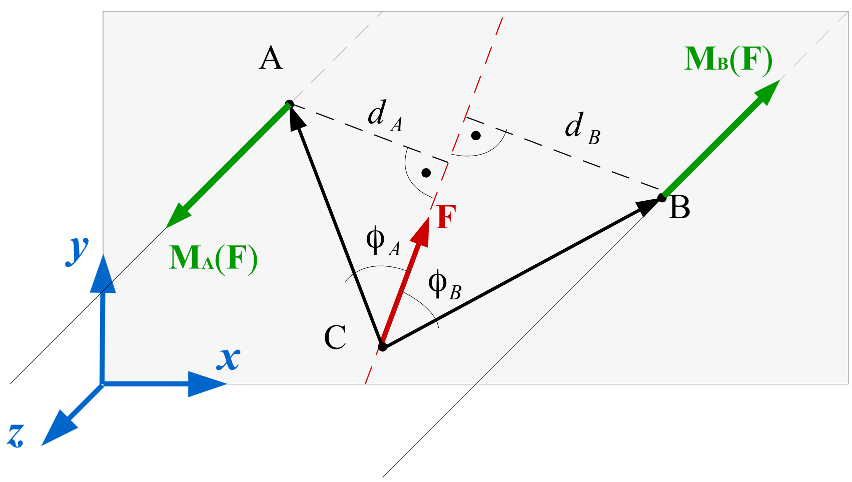
Sum of moments about point A. Moment of the forces $\mathbf{a}$ and $\mathbf{b}$ is zero. Each other force may be translated along their lines of action, so forces $\mathbf{c}$ and $\mathbf{d}$ may be translated to point D and added together:

$$\mathbf{M}_A = (\mathbf{c}+\mathbf{d}) \times \vec{DA} = [4P; -2P; 0] \times [-7L; 0; 0] = [0, 0, -14PL]$$

Sum of moments about point D. Moment of the forces $\mathbf{c}$ and $\mathbf{d}$ is zero. Each other force may be translated along their lines of action, so forces $\mathbf{a}$ and $\mathbf{b}$ may be translated to point A and added together:

$$\mathbf{M}_D = (\mathbf{a}+\mathbf{b}) \times \vec{AD} = [P; -P; 0] \times [7L; 0; 0] = [0, 0, 7PL]$$

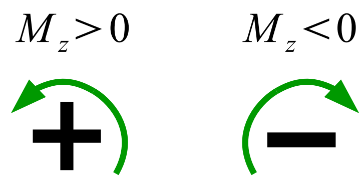
A moment of a plane system of forces is a vector which is perpendicular to the plane in which all forces are applied and lie. According to the below example, forces in plane (x, y) wil always have moments parallel to z axis. The z -axis components is equal (with + or - sign) their length, which may calculated according to the definition of the cross product:



$$\mathbf{F} = [F_x; F_y; 0]$$
$$\mathbf{r} = [r_x; r_y; 0]$$
$$\mathbf{M} = \mathbf{F} \times \mathbf{r} = [0; 0; M_z]$$

$$|M_z| = |\mathbf{F}| \cdot |\mathbf{r}| \cdot \sin\phi = |\mathbf{F}| \cdot d$$

The value of moment of a force about a point is equal **the magnitude of the force** multiplied by **the distance of the center of moment from the line of action** of that force. The sign in according to the right-hand rule. A **positive** moment is the one, for which the force "rotates" about the center of moment **counterclockwise**.



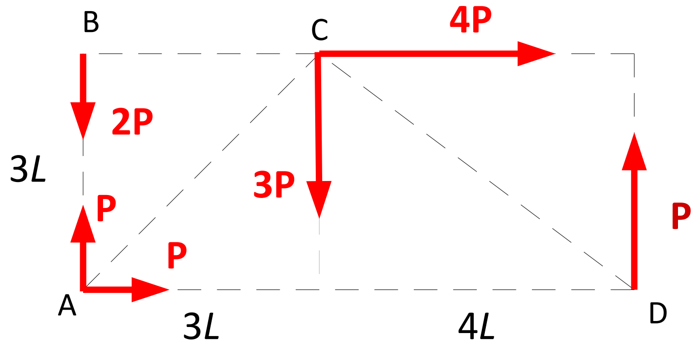
Our task may be solved now in a simpler way now:

- Decompose all forces into **horizontal and vertical component**.
- Value of the moment is the value of the force times the distance of the center from the line of action of that force (arm):

moment = force x arm

- When calculating the moment the **distance of the center from the lines of action (arm)** is found as follows**:**
  - **distance from vertical line (force)** is measured **horizontally**
  - **distance from horizontal line (force)** is measured **vertically**

- **Positive moment** corresponds with **counterclockwise rotation**
- **Negative moment** corresponds with **clockwise rotation**



Sum of moments about point A:

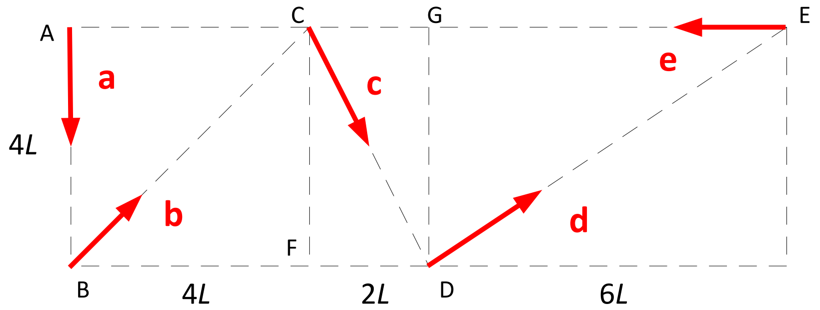
$$M_A = \underbrace{P\cdot 0 + P\cdot 0}_{\mathbf{M(a)}} + \underbrace{2P\cdot 0}_{\mathbf{M(b)}} \underbrace{-\,3P\cdot 3L - 4P\cdot 3L}_{\mathbf{M(c)}} \underbrace{+\,P\cdot 7L}_{\mathbf{M(d)}} =$$
$$= -9PL - 12PL + 7PL = -14PL$$

Sum of moments about point D:

$$M_D = \underbrace{-\,P\cdot 7L + P\cdot 0}_{\mathbf{M(a)}} \underbrace{+\,2P\cdot 7L}_{\mathbf{M(b)}} \underbrace{+\,3P\cdot 4L - 4P\cdot 3L}_{\mathbf{M(c)}} + \underbrace{P\cdot 0}_{\mathbf{M(d)}} =$$
$$= -7PL + 14PL + 12PL - 12PL = 7PL$$

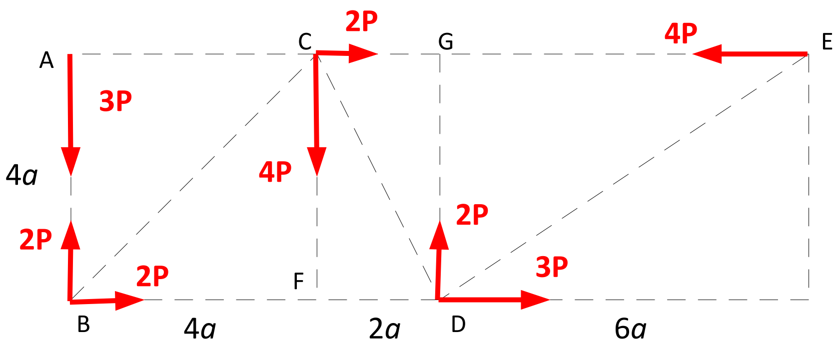
EXERCISE 17

Find the sum of moments of the given system of forces about points F and G



$$|\mathbf{a}| = 3P$$
$$|\mathbf{b}| = 2\sqrt{2}P$$
$$|\mathbf{c}| = 2\sqrt{5}P$$
$$|\mathbf{d}| = \sqrt{13}P$$
$$|\mathbf{e}| = 4P$$

SOLUTION:



$$\mathbf{a}=[0;-3P]$$
$$\mathbf{b}=[2P;2P]$$
$$\mathbf{c}=[2P;-4P]$$
$$\mathbf{d}=[3P;2P]$$
$$\mathbf{e}=[-4P;0]$$

Sum of moments about point F:

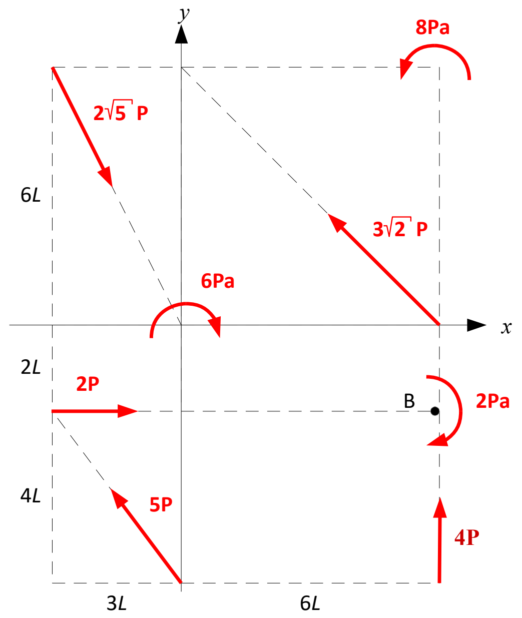
$$M_F = \underbrace{+3P\cdot 4L}_{\mathbf{M}(\mathbf{a})} \underbrace{-2P\cdot 4L + 2P\cdot 0}_{\mathbf{M}(\mathbf{b})} \underbrace{+4P\cdot 0 - 2P\cdot 4L}_{\mathbf{M}(\mathbf{c})} \underbrace{+2P\cdot 2L + 3P\cdot 0}_{\mathbf{M}(\mathbf{d})} \underbrace{+4P\cdot 4L}_{\mathbf{M}(\mathbf{e})} = 16PL$$

Sum of moments about point G:

$$M_G = \underbrace{+3P\cdot 6L}_{\mathbf{M}(\mathbf{a})} \underbrace{-2P\cdot 6L + 2P\cdot 4L}_{\mathbf{M}(\mathbf{b})} \underbrace{+4P\cdot 2L - 2P\cdot 0}_{\mathbf{M}(\mathbf{c})} \underbrace{+2P\cdot 0 + 3P\cdot 4L}_{\mathbf{M}(\mathbf{d})} \underbrace{+4P\cdot 0}_{\mathbf{M}(\mathbf{e})} = 34PL$$

EXERCISE 18

Reduce the given system of forces in point B, and then reduce it to the simplest equivalent system.



SOLUTION:
All forces are decomposed into horizontal and vertical components.

Sum of forces along x axis:

$$\Sigma F_x = 2P - 3P + 2P - 3P = -2P$$

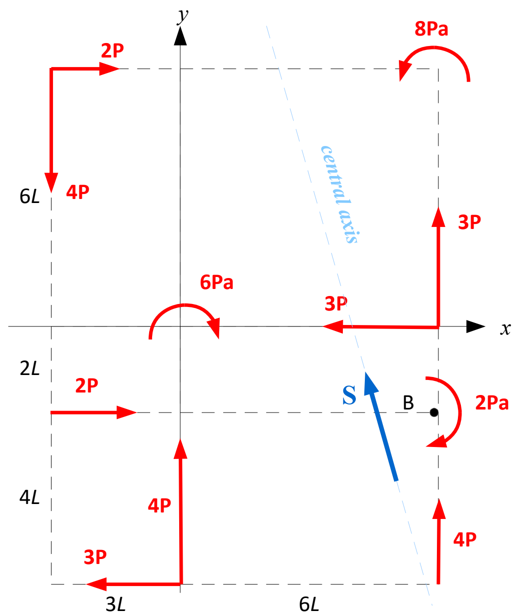
Sum of forces along y axis:

$$\Sigma F_y = -4P + 3P + 4P + 4P = 7P$$

Sum of moments about B:

$$\Sigma M_O = -2P\cdot 8a + 4P\cdot 9a + 8Pa - 6Pa + 3P\cdot 2a + \\ -2Pa - 3P\cdot 4a - 4P\cdot 6a = -10Pa$$

The system is reduced in point B to a vector equal to vector of sum $\mathbf{W} = \mathbf{S} = [-2P;7P;0]$ and to a couple of moment $\mathbf{M}_B = [0;0;-10Pa]$.



Plane system of forces may reduce only to zero system, to a couple or a resultant. Sum of the system is unequal 0, so the system is reduced to resultant. The central axis will be parallel to the vector of sum. We have to find a point $P(x,y,0)$ belonging to central axis. Moment about any such point is 0. We will use the moment transport theorem. Vector connecting the old center with the new one will be found by subtracting appropriate coordinates of its end and beginning:

$$\begin{aligned} P &= (x;y;0) \\ B &= (6L;-2L;0) \end{aligned} \quad \Rightarrow \quad \vec{BP} = [x-6L\,;\; y-(-2L)\,;\; 0]$$

$$\mathbf{0} = \mathbf{M}_P = \mathbf{M}_B + \mathbf{S}\times\vec{BP} \quad \Rightarrow \quad [0;0;0] = [0;0;-10PL] + [-2P;7P;0]\times[x-6L;y+2L;0]$$

$$[0;0;0] = [0;0;-10PL] + [0;0;-2P(y+2L) - 7P(x-6L)]$$

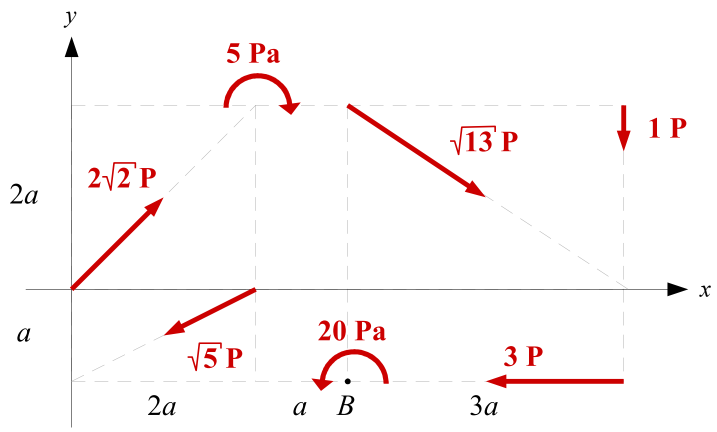
$$[0;0;0] = [0;0;-10PL - 2Py - 7Px - 4PL + 42PL]$$

$$28PL - 2Py - 7Px = 0 \quad \Rightarrow \quad y = 14L - \frac{7}{2}x$$

The given system of forces is reduced to the simplest equivalent form of resultant $\mathbf{W} = \mathbf{S} = [-2P;7P;0]$ applied to any point of central axis of equation: $y = 14a - \frac{7}{2}x$.

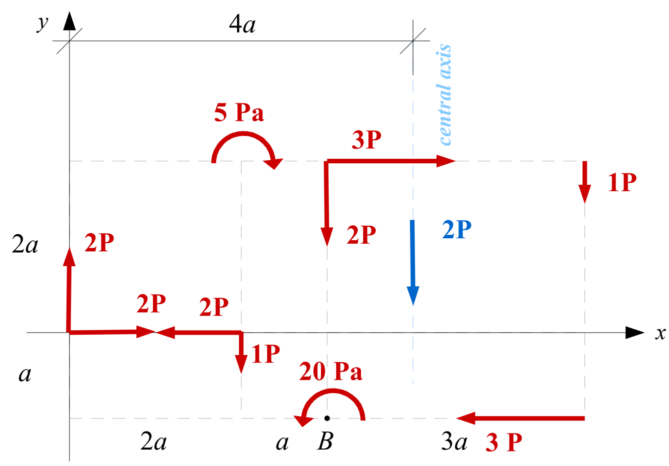
EXERCISE 19

Reduce the given system of forces in point B, and then reduce it to the simplest equivalent system.



SOLUTION:

We decompose all forces into horizontal and vertical components and calculate sum of the system and sum of moments about B. **REMARK**: Point moment is equivalent to a couple and it gives always the same value of moment – also about a point of its application.



Sum of forces along x axis:

$$\Sigma F_x = 3P + 2P - 2P - 3P = 0$$

Sum of forces along y axis:

$$\Sigma F_y = 2P - P - 2P - P = -2P$$

Sum of moments about B:

$$\Sigma M_B = -5Pa + 0 - 3P \cdot 3a - P \cdot 3a - 2P \cdot 3a - 2P \cdot a + 2P \cdot a + P \cdot a + 20Pa + 0 = -2Pa$$

The system is reduced in point B to a vector equal to vector of sum $\mathbf{W} = \mathbf{S} = [0; -2P; 0]$ and to a couple of moment $\mathbf{M} = [0; 0; -2Pa]$.

Sum of the system is not equal 9, so the system is reduced to a resultant. Location of resultant is found from the condition for the moment about points of central axis to be equal 0. We make use of the moment transport theorem:

$$\mathbf{0} = \mathbf{M}_P = \mathbf{M}_B + \mathbf{S} \times \vec{BP} \quad \Rightarrow \quad [0;0;0] = [0;0;-2Pa] + [0;-2P;0] \times [x-3a; y+a; 0]$$

$$[0;0;0] = [0;0;-2Pa] + [0;0;2Px - 6Pa]$$

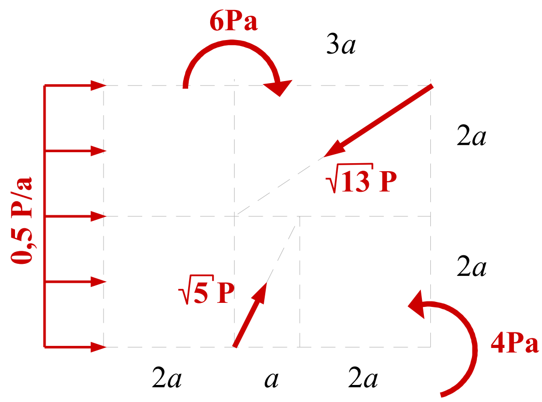
$$[0;0;0] = [0;0;2Px - 8Pa]$$

$$2Px - 8Pa = 0 \quad \Rightarrow \quad x = 4a$$

The given system of forces is reduced to the simplest equivalent form of resultant $\mathbf{W} = \mathbf{S} = [0; -2P; 0]$ applied to any point of central axis of equation $x = 4a$.

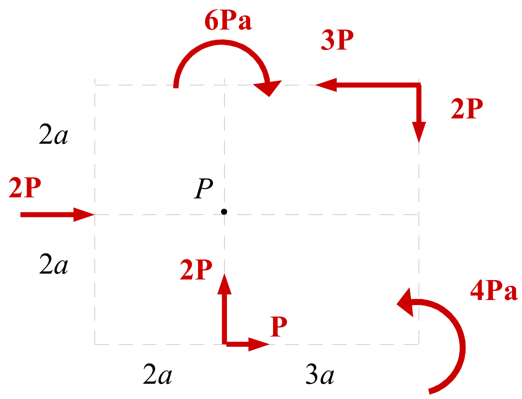
EXERCISE 20

Reduce the given system of forces to the simplest equivalent system.



SOLUTION:

We decompose all forces into horizontal and vertical components and the distributed load is replaced with a resultant.



Sum of forces along x axis:

$$\Sigma F_x = 2P + P - 3P = 0$$

Sum of forces along y axis:

$$\Sigma F_y = 2P - 2P = 0$$

The sum of the system is equal 0 – the system is reduced either to zero system or to a couple.

When reducing to the simplest equivalent system we may calculate the sum of moment about any points. Let's chose such a point for which the greatest possible number of forces gives zero moment. Let it be point P as in the picture.

Suma momentów względem punktu P:

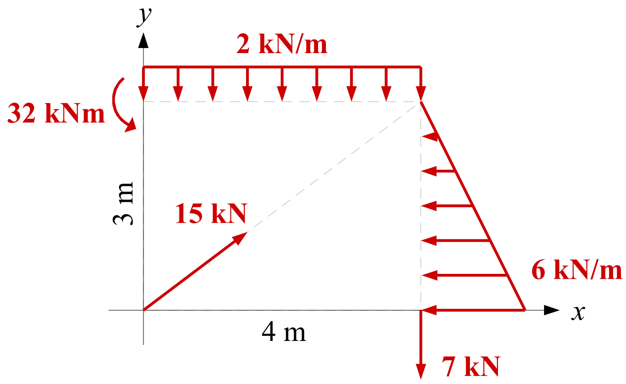
$$\Sigma M_P = -6Pa + 4Pa + P\cdot 2a = 0$$

Both sum and sum of moments are equal 0.

The given system of forces is reduced to the simplest equivalent system of zero system in any point.

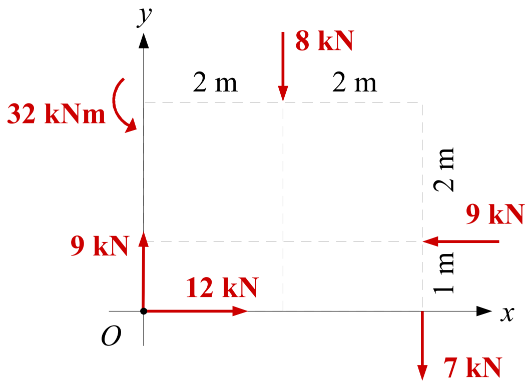
EXERCISE 21

Reduce the given system of forces to the simplest equivalent system and show the results of reduction.



SOLUTION:

- Oblique forces are decomposed into horizontal and vertical components.
- Distributed loads are replaced with their resultants



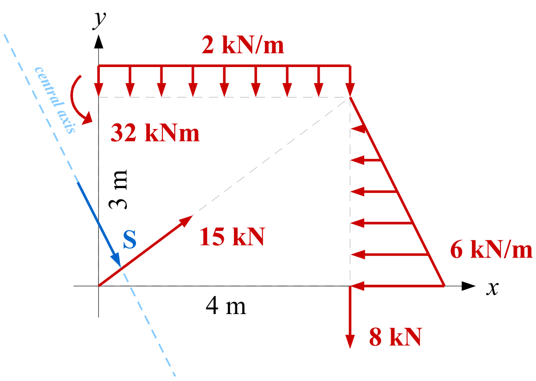
Let's reduce the system in the origin of chosen coordinate system:

$$\Sigma F_x = 12-9 = 3 \quad [\mathrm{kN}]$$
$$\Sigma F_y = -8+9-7 = -6 \quad [\mathrm{kN}]$$
$$\Sigma M_O = 32-8\cdot 2+9\cdot 1-7\cdot 4 = -3 \quad [\mathrm{kNm}]$$

Central axis:

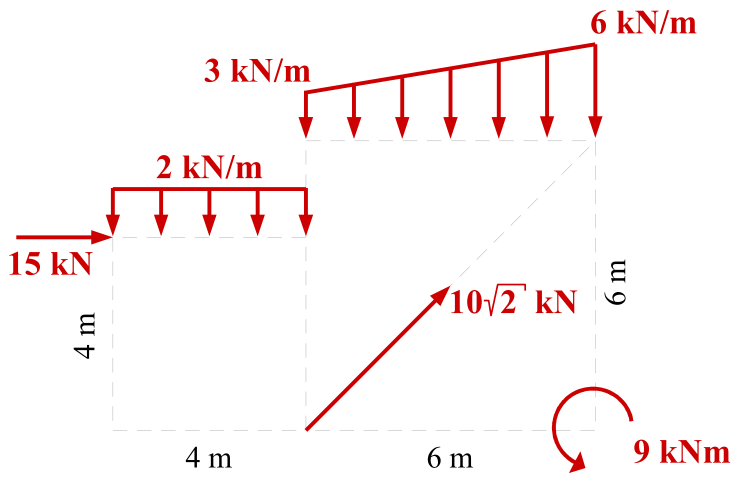
$$\mathbf{0} = \mathbf{M}_O + \mathbf{S}\times\vec{OP} = [0;0;-3] + [3;-6;0]\times[x;y;0] \quad [\mathrm{kNm}]$$
$$[0;0;0] = [0;0;-3] + [0;0;3y+6x]$$
$$0 = 3y+6x-3 \quad \Rightarrow \quad y = -2x+1$$

The given system of forces is reduced to the simplest equivalent form of resultant $\mathbf{W} = \mathbf{S} = [3\,\mathrm{kN}; -6\,\mathrm{kN}; 0]$ applied to any point of central axis of equation $y=-2x+1$



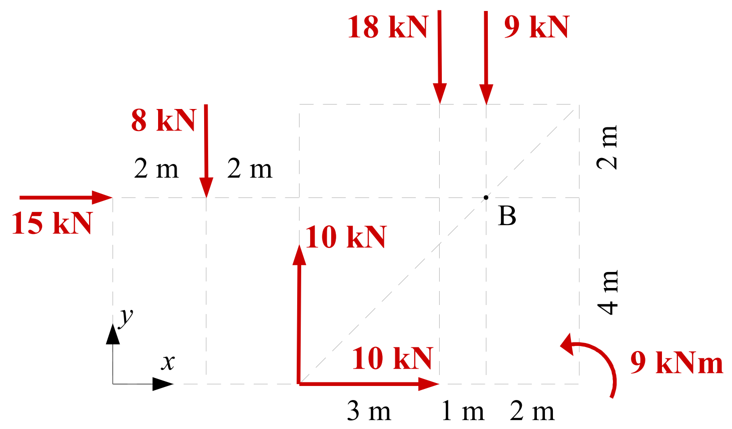
EXERCISE 22

Reduce the given system of forces to the simplest equivalent system and show the results of reduction.



SOLUTION:

- Oblique forces are decomposed into horizontal and vertical components.
- Trapezoidal load is decomposed into a rectangular and triangular load.
- Distributed loads are replaced with their resultants



It will be most convenient to reduce the system in point B – it lies at intersection of lines of action of three forces – horizontal point force, resultant of triangular load and oblique force (thus, we neglect the contribution of both its components).

$$\Sigma F_x = 15 + 10 = 25 \quad [\mathrm{kN}]$$

$$\Sigma F_y = -8 + 10 - 18 - 9 = -25 \quad [\mathrm{kN}]$$

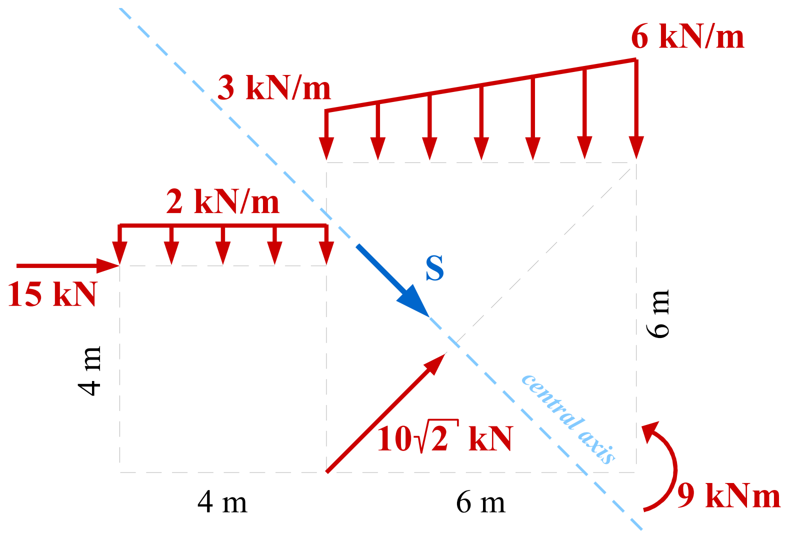
$$\Sigma M_B = 8 \cdot 6 + 18 \cdot 1 + 9 = 75 \quad [\mathrm{kNm}]$$

Central axis:

$$\mathbf{0}=\mathbf{M}_B+\mathbf{S}\times\vec{BP}=[0;0;75]+[25;-25;0]\times[x-8;y-4;0]\ \ [\mathrm{kNm}]$$

$$[0;0;0]=[0;0;75]+[0;0;25y+25x-300]$$

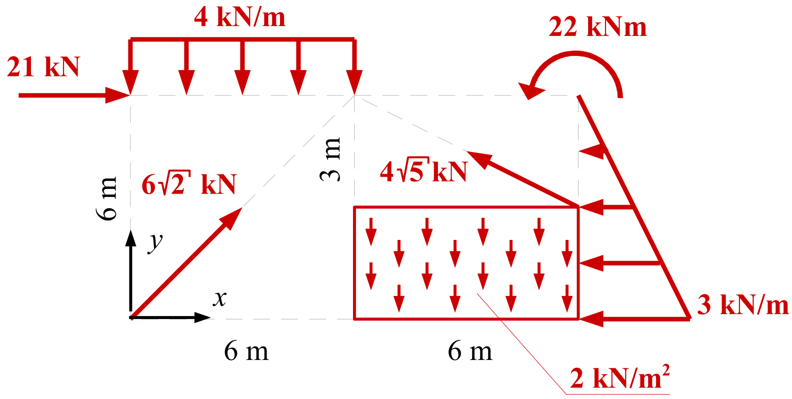
$$0=25y+25x-225 \quad\Rightarrow\quad y=-x+9$$



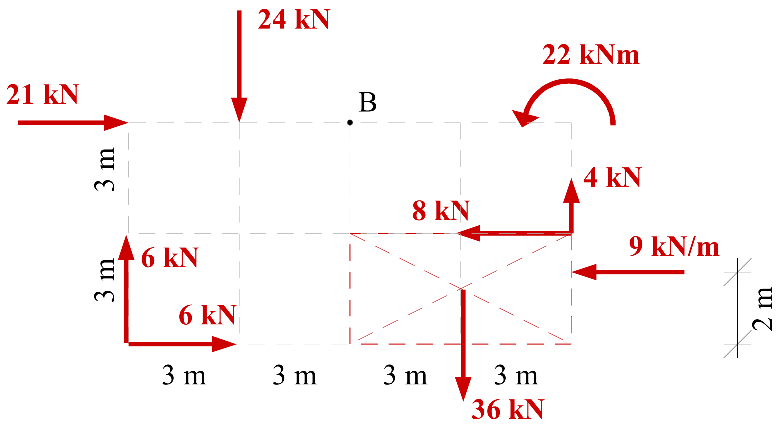
The given system of forces is reduced to the simplest equivalent form of resultant $\mathbf{W}=\mathbf{S}=[25\ \mathrm{kN}\ ;\ -25\ \mathrm{kN}\ ;\ 0]$ applied to any point of central axis of equation $y=-x+9\,[\mathrm{m}]$.

EXERCISE 23

Reduce the given system of forces to the simplest equivalent system.



SOLUTION:



$$\Sigma F_x = 21+6-8-9 = 10 \quad [\mathrm{kN}]$$

$$\Sigma F_y = 6-24-36+4 = -50 \quad [\mathrm{kN}]$$

The moment will be calculated about point B – it lies at intersection of lines of action of all point forces:

$$\Sigma M_B = 24\cdot 3+22-36\cdot 3-9\cdot 4 = -50 \quad [\mathrm{kNm}]$$

Central axis:

$$\mathbf{0} = \mathbf{M}_B+\mathbf{S}\times\vec{BP} = [0;0;-50] + [10;-50;0]\times[x-6;y-6;0] \quad [\mathrm{kNm}]$$

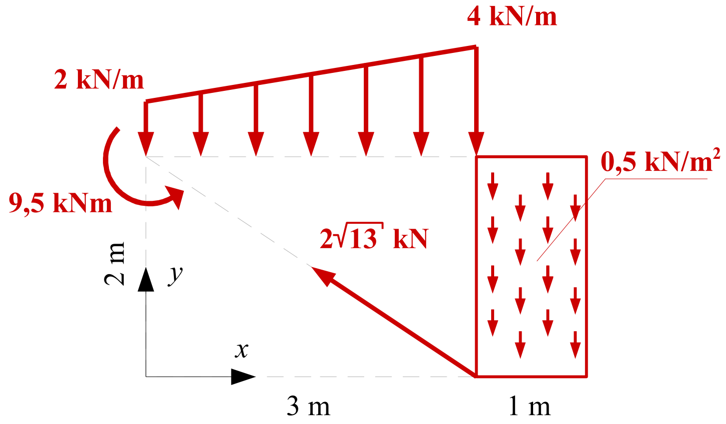
$$[0;0;0]= [0;0;-50] + [0;0;10y+50x-360]$$

$$0 = 10y+50x-410 \qquad \Rightarrow \qquad y = -5x+41$$

The given system of forces is reduced to the simplest equivalent form of resultant $\mathbf{W} = \mathbf{S}=[10\ \mathrm{kN}\ ;\ -50\ \mathrm{kN}\ ;\ 0]$ applied to any point of central axis of equation $y = -5x+41\ [\mathrm{m}]$.

EXERCISE 24

Reduce the given system of forces to the simplest equivalent system.

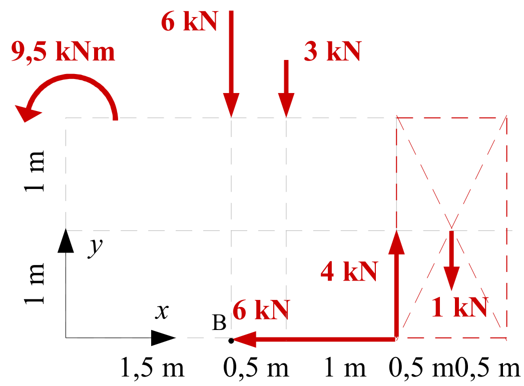


SOLUTION:

$$\Sigma F_x = -6 \quad [\mathrm{kN}]$$

$$\Sigma F_y = -6-3+4-1 = -6 \quad [\mathrm{kN}]$$

$$\Sigma M_B = 9{,}5-3\cdot 0{,}5+4\cdot 1{,}5-1\cdot 2 = 12 \quad [\mathrm{kNm}]$$



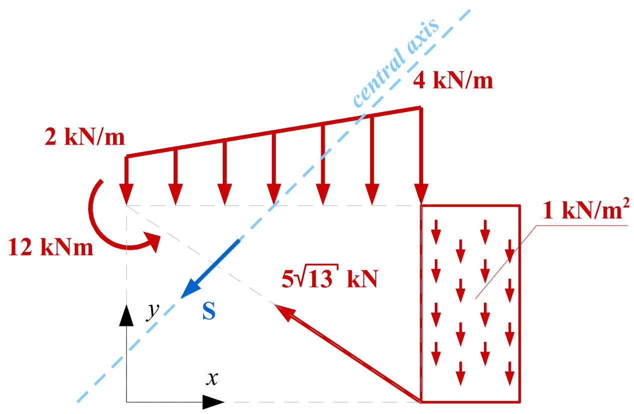
Central axis:

$$\mathbf{0} = \mathbf{M}_B + \mathbf{S}\times\vec{BP} = [0;0;12] + [-6;-6;0]\times[x-1{,}5;y-0;0] \quad [\mathrm{kNm}]$$

$$[0;0;0] = [0;0;12] + [0;0;-6y+6x-9]$$

$$0 = -6y+6x+3 \quad \Rightarrow \quad y = x+0{,}5$$

The given system of forces is reduced to the simplest equivalent form of resultant $\mathbf{W} = \mathbf{S} = [-6\ \mathrm{kN}\ ;-6\ \mathrm{kN}\ ;0]$ applied to any point of central axis of equation $y = x+0{,}5\ [\mathrm{m}]$.

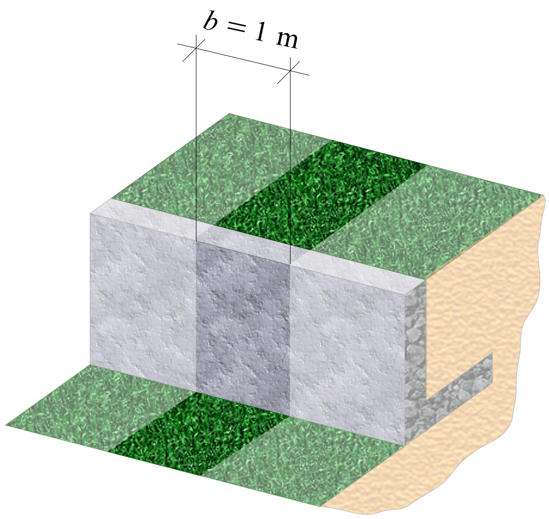


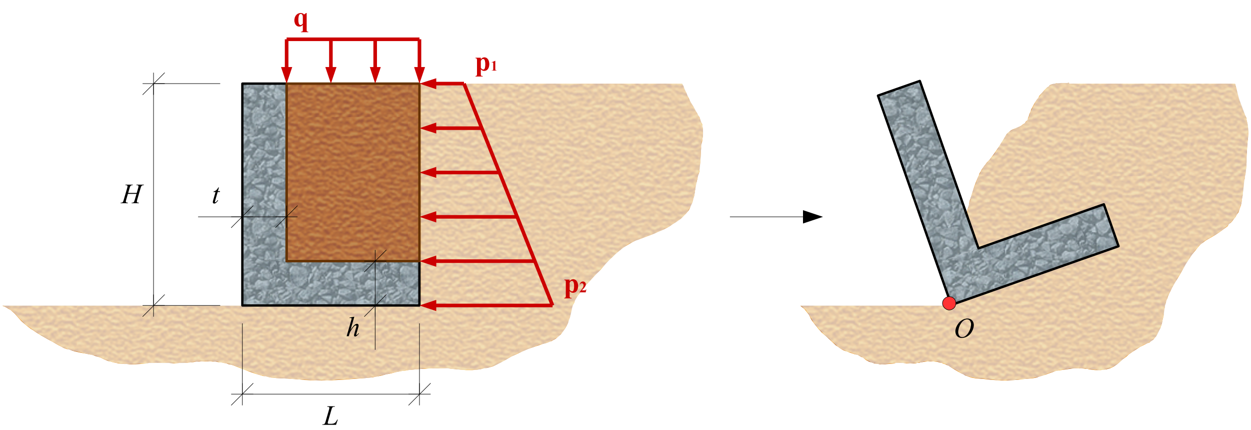
EXERCISE 25

Check the stability (against rotation) of a retaining wall for the data as below:

- height of retaining wall $H=4{,}5\ \mathrm{m}$
- thickness of retaining wall $t=0{,}5\ \mathrm{m}$
- depth of retaining wall $L=4\ \mathrm{m}$
- thickness of slab $h=0{,}5\ \mathrm{m}$
- Reinforced-concrete self-weight $\gamma_s = 25\ \mathrm{kN/m^3}$
- Soil self-weight $\gamma_g = 20\ \mathrm{kN/m^3}$
- Ground load $q = 5\ \mathrm{kN/m^2}$
- Angle of internal friction of soil $\phi=30^\circ$
- Soil pressure: $p(z) = \mathrm{tg}^2\left(45^\circ - \frac{\phi}{2}\right)\cdot\left[q+\gamma_g z\right]$

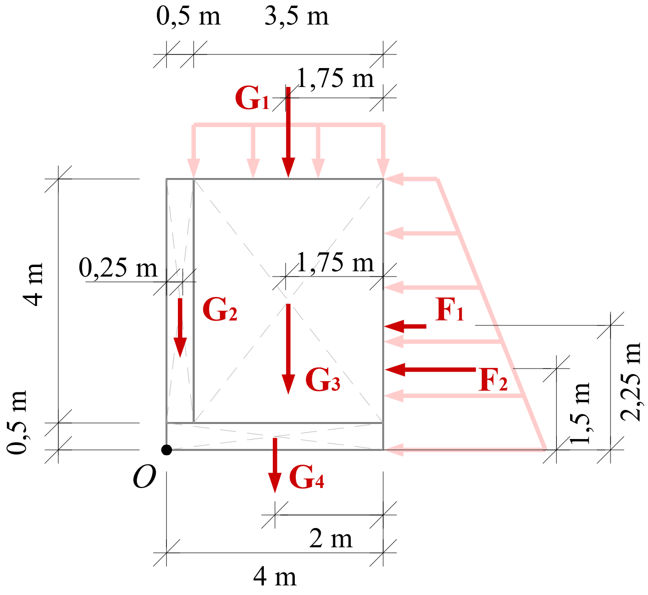
Calculation is performed for a section of a wall of width $b = 1\,\mathrm{m}$. Stability is verified in such a way that in an assumed center of rotation of the wall so called **overturning moment** (moments of forces overturning the wall – soil pressure) and **resisting moment** (moment of all forces acting in an opposite way – self-weight and soil load) are compared. If the resisting moment is greater then it is assumed that the wall won't be overturned. **Center of rotation** is assumed in point O as in the picture.





SOLUTION:

Static diagram is as in the picture:



- Resultants of weight of wall:

$$G_2 = \gamma_s \cdot (H-h) \cdot t \cdot b = 50 \text{ kN}$$
$$G_4 = \gamma_s \cdot L \cdot h \cdot b = 50 \text{ kN}$$

- Resultant of weight of soil:

$$G_3 = \gamma_g \cdot (H-h) \cdot (L-t) \cdot b = 280 \text{ kN}$$

- Resultant of load on soil:

$$G_1 = q \cdot (L-t) \cdot b = 17{,}5 \text{ kN}$$

- Soil pressure: $p_1 = p(0) = 1{,}667 \text{ kN/m}^2$
  $p_2 = p(H) = 31{,}667 \text{ kN/m}^2$

- Resultant of uniform part of soil pressure:

$$F_1 = p_1 \cdot H \cdot b = 7{,}5 \text{ kN}$$

- Resultant of hydrostatic part of soil pressure:

$$F_2 = \frac{1}{2}(p_2 - p_1) \cdot H \cdot b = 67{,}5 \text{ kN}$$

- **Resisting moment**

$$M_u = -(G_1 + G_3) \cdot \left(t + \frac{L-t}{2}\right) - G_1 \cdot \frac{t}{2} - G_4 \cdot \frac{L}{2} = -781{,}875 \text{ kNm}$$

- **Overturning moment**

$$M_o = F_1 \cdot \frac{H}{2} + F_2 \cdot \frac{H}{3} = 118{,}125 \text{ kNm}$$

$|M_u| > |M_o|$ **- resisting moment is greater than the overturning moment. The retaining wall won't be overturned.**

