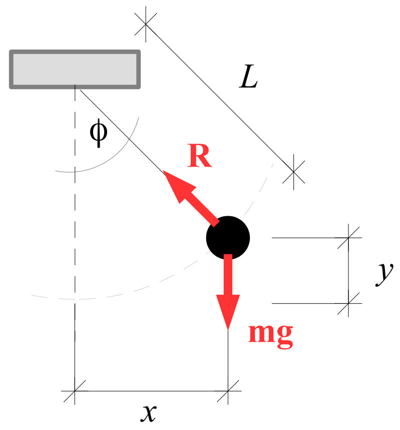
EXERCISE 1

Find an equation of motion of mass *m* placed at the end of a rigid pendulum of length *L* with the use of:

a) Newtonf's principles of motion
b) d'Alembert principle
c) Conservation of energy
d) Lagrange equations of the 2nd kind
e) Hamilton equations



SOLUTION:

The system has only a single degree of freedom.

AD a) – NEWTON'S 2nd PRINCIPLE OF MOTION

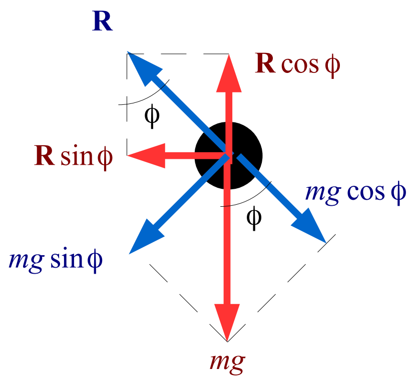
Geometry of the system gives us:

$$\begin{cases} x(t) = L\sin\phi(t) \\ y(t) = L(1-\cos\phi(t)) \end{cases}$$

Hence:

$$\begin{cases} \dot{x} = L\dot{\phi}\cos\phi \\ \dot{y} = L\dot{\phi}\sin\phi \end{cases}$$

$$\begin{cases} \ddot{x} = L\left[\ddot{\phi}\cos\phi-(\dot{\phi})^2\sin\phi\right] \\ \ddot{y} = L\left[\ddot{\phi}\sin\phi+(\dot{\phi})^2\cos\phi\right] \end{cases}$$



In the 2nd principle of motion we have to account for all forces acting on the body – these include **forces of reaction** $\mathbf{R}$. The motion is performed along a circle – this is an information that will be used to find the reaction in pendulum. It is not so simple – the reaction cannot simple equilibrate the gravity force, since zero resultant causes motion along a straight line. We know that in case of a motion along a circle the curvature of trajectory is caused by a centripetal force $\mathbf{F}_R$:

$$R = mg\cos\phi + F_R$$

We cannot, however, give a formula for the centripetal force, as the well known **formula:**

$$F_R = \frac{mv^2}{L}$$

is valid only in case of motion with constant angular velocity. In case of a pendulum this velocity changes with time. We may only write down that centripetal force is a certain function of time $F_R(t)$. We may decompose the forces acting on the mas into horizontal and vertical component obtaining:

$$\sum_i (F_{ix}+R_{ix}) = -mg\cos\phi\sin\phi - F_R\sin\phi$$
$$\sum_i (F_{iy}+R_{iy}) = -mg + mg\cos^2\phi + F_R\cos\phi$$

Equations of motion are determine daccording to the 2nd principle of motion:

$$\begin{cases} m\ddot{x} = -mg\cos\phi\sin\phi - F_R\sin\phi \\ m\ddot{y} = -mg + mg\cos^2\phi + F_R\cos\phi \end{cases}$$

Let's express horizontal and vertical accelerations by angular accelerations

$$\begin{cases} L\left[\ddot{\phi}\cos\phi - (\dot{\phi})^2\sin\phi\right] = -g\cos\phi\sin\phi - \frac{F_R}{m}\sin\phi \\ L\left[\ddot{\phi}\sin\phi + (\dot{\phi})^2\cos\phi\right] = -g + g\cos^2\phi + \frac{F_R}{m}\cos\phi \end{cases}$$

We may project both relations on a direction which is perpendicular to the axis of pendulum (tangent to trajectoryu) by multiplying the first equation with $\cos\phi$ and the second one with $\sin\phi$:

$$\begin{cases} \ddot{\phi}\cos^2\phi - (\dot{\phi})^2\cos\phi\sin\phi = -\frac{g}{L}\cos^2\phi\sin\phi - \frac{F_R}{mL}\sin\phi\cos\phi \\ \ddot{\phi}\sin^2\phi + (\dot{\phi})^2\cos\phi\sin\phi = -\frac{g}{L}\sin\phi + \frac{g}{L}\cos^2\phi\sin\phi + \frac{F_R}{mL}\sin\phi\cos\phi \end{cases}$$

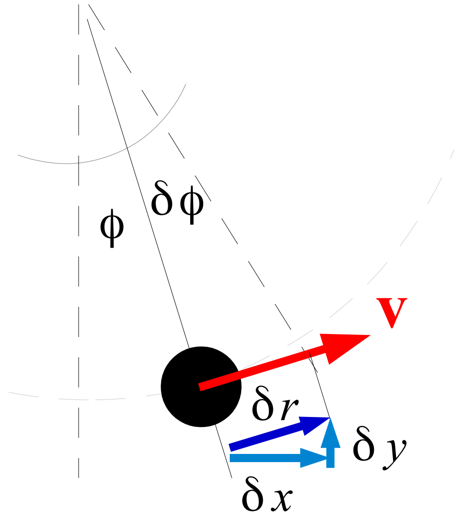
Summing the equations up:

$$\ddot{\phi}(\cos^2\phi + \sin^2\phi) = -\frac{g}{L}\sin\phi$$

After simple trigonomatric simplification we obtain:

$$\ddot{\phi} + \frac{g}{L}\sin\phi = 0$$

AD b) – D'ALEMBERT PRINCIPLE



We have to determine active forces and inerti order to find the inertial force we have tc acceleration vector. The use of geometry gives us.

$$\begin{cases} x(t) = L\sin\phi(t) \\ y(t) = L(1-\cos\phi(t)) \end{cases}$$

Hence:

$$\begin{cases} \dot{x} = L\dot{\phi}\cos\phi \\ \dot{y} = L\dot{\phi}\sin\phi \end{cases}$$

$$\begin{cases} \ddot{x} = L\left[\ddot{\phi}\cos\phi-(\dot{\phi})^2\sin\phi\right] \\ \ddot{y} = L\left[\ddot{\phi}\sin\phi+(\dot{\phi})^2\cos\phi\right] \end{cases}$$

Inertial forces are projected on horizontal and vertical direction:

$$\begin{cases} F_x = 0 \\ F_y = -mg \end{cases} \qquad \begin{cases} B_x = -m\ddot{x} = -mL\left[\ddot{\phi}\cos\phi-(\dot{\phi})^2\sin\phi\right] \\ B_y = -m\ddot{y} = -mL\left[\ddot{\phi}\sin\phi+(\dot{\phi})^2\cos\phi\right] \end{cases}$$

Permissible displacement is parallel to the vector of tangent velocity

$$\delta r = L\,\delta\phi$$

We project it on horizontal and vertical direction:

$$\begin{cases} \delta x = \delta r\cos\phi = L\cos\phi\,\delta\phi \\ \delta y = \delta r\sin\phi = L\sin\phi\,\delta\phi \end{cases}$$

Virtual work:

$$\delta L = (\mathbf{B}+\mathbf{F})\circ\delta\mathbf{r} = (B_x+F_x)\delta x+(B_y+F_y)\delta y = 0$$

$$\left[-mL\left(\ddot{\phi}\cos\phi-(\dot{\phi})^2\sin\phi\right)\right]\cdot(L\cos\phi\,\delta\phi) + \left[-mg-mL\left(\ddot{\phi}\sin\phi+(\dot{\phi})^2\cos\phi\right)\right]\cdot(L\sin\phi\,\delta\phi) = 0 \qquad \forall\,\delta\phi$$

$$\left[-mL^2\ddot{\phi}\cos^2\phi + mL^2(\dot{\phi})^2\sin\phi\cos\phi - mgL\sin\phi-mL^2\ddot{\phi}\sin^2\phi-mL^2(\dot{\phi})^2\sin\phi\cos\phi\right]\delta\phi=0 \qquad \forall\,\delta\phi$$

$$-mL^2\left[\ddot{\phi}(\sin^2\phi+\cos^2\phi) + \frac{g}{L}\sin\phi\right]\delta\phi=0 \qquad \forall\,\delta\phi$$

$$\ddot{\phi}+\frac{g}{L}\sin\phi = 0$$

AD c) – CONSERVATION OF ENERGY

A point mass at the end of pendulum has a linear velocity $\dot{r}$, which may be expressed by angular velocity as follows $\dot{r} = \dot{\phi} L$. **Kinetic energy** is equal:

$$E_k = \frac{1}{2} m (\dot{r})^2 = \frac{mL^2}{2} (\dot{\phi})^2$$

Potential energy: $E_p = mgy = mg L(1-\cos\phi)$

Total mechanical energy: $E = \frac{mL^2}{2}(\dot{\phi})^2 + mg L(1-\cos\phi)$

Law of conservation of energy provides us with an equation of motion. Remembering that angular displacement ϕ is a function of time we may write down:

$$\frac{\mathrm{d} E}{\mathrm{d} t} = m L^2 \dot{\phi} \ddot{\phi} + mg L \dot{\phi} \sin\phi = 0$$

what is equivalent to:

$$\ddot{\phi} + \frac{g}{L} \sin\phi = 0$$

AD d) – LAGRANGE EQUATION OF THE 2nd KIND

We chose a **generalized coordinate** being equal an angle of inclination of pendulum: $q = \phi$

Position vector: $\mathbf{r} = \begin{cases} x = L\sin q \\ y = L(1-\cos q) \end{cases}$

Velocity vector: $\dot{\mathbf{r}} = \begin{cases} \dot{x} = L\dot{q}\cos q \\ \dot{y} = L\dot{q}\sin q \end{cases}$

Kinetic energy: $E_k = \frac{1}{2} m (\dot{\mathbf{r}})^2 = \frac{mL^2(\dot{q})^2}{2}(\cos^2 q + \sin^2 q) = \frac{mL^2(\dot{q})^2}{2}$

The load is due to a single active force:

$$\mathbf{F} = \begin{cases} F_x = 0 \\ F_y = -mg \end{cases}$$

Generalized force: $Q = \frac{\partial \mathbf{r}}{\partial q} \circ \mathbf{F} = (L\cos q)\cdot 0 + (L\sin q)\cdot(-mg) = -mgL\sin q$

Lagrange equation of the 2nd kind:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E_k}{\partial \dot{q}}\right) - \frac{\partial E_k}{\partial q} = Q$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial}{\partial \dot{q}}\left(\frac{mL^2(\dot{q})^2}{2}\right)\right] - \frac{\partial}{\partial q}\left(\frac{mL^2(\dot{q})^2}{2}\right) = -mgL\sin q$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[mL^2\dot{q}\right] = -mgL\sin q$$

$$mL^2\ddot{q} = -mgL\sin q$$

Substituting $q = \phi$:

$$\ddot{\phi} + \frac{g}{L}\sin\phi = 0$$

Since the gravity is a conservative force it is possible to express the equations of motion in a simplfied form. We have to find a **potential energy**:

$$E_p = mgy = mg\,L(1-\cos q)$$

We define a **lagrangian**:

$$\mathscr{L} = E_k - E_p = \frac{mL^2(\dot{q})^2}{2} - mgL(1-\cos q)$$

Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathscr{L}}{\partial\dot{q}}\right) - \frac{\partial\mathscr{L}}{\partial q} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial}{\partial\dot{q}}\left(\frac{mL^2(\dot{q})^2}{2} - mgL(1-\cos q)\right)\right] - \frac{\partial}{\partial q}\left(\frac{mL^2(\dot{q})^2}{2} - mgL(1-\cos q)\right) = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(mL^2\dot{q}) - (-mgL\sin q) = 0$$

$$mL^2\ddot{q} + mgL\sin q = 0$$

Substituting $q = \phi$:

$$\ddot{\phi} + \frac{g}{L}\sin\phi = 0$$

AD e) – HAMILTON EQUATIONS

We chose a **generalized coordinate** being equal an angle of inclination of pendulum: $q = \phi$

Position vecotr: $\mathbf{r} = \begin{cases} x = L\sin q \\ y = L(1-\cos q) \end{cases}$

Velocity vector: $\dot{\mathbf{r}} = \begin{cases} \dot{x} = L\dot{q}\cos q \\ \dot{y} = L\dot{q}\sin q \end{cases}$

Kinetic energy: $E_k = \frac{1}{2}m(\dot{\mathbf{r}})^2 = \frac{mL^2(\dot{q})^2}{2}(\cos^2 q + \sin^2 q) = \frac{mL^2(\dot{q})^2}{2}$

Potential energy: $E_p = mgy = mg\,L(1-\cos q)$

Lagrangian: $L = E_k - E_p = \frac{mL^2(\dot{q})^2}{2} - mgL(1-\cos q)$

Generalized momentum: $p = \frac{\partial L}{\partial \dot{q}} = mL^2 \dot{q}$

Generalized velocity expressed by generalized momentum: $\dot{q} = \frac{p}{mL^2}$

Hamiltonian:

$$H = p\dot{q} - L = \frac{p^2}{mL^2} - \left[\frac{1}{2mL^2} p^2 - mgL(1-\cos q)\right] =$$
$$= \frac{p^2}{2mL^2} + mgL(1-\cos q)$$

Hamilton equations:

$$\begin{cases} \dot{p} = -\frac{\partial H}{\partial q} \\ \dot{q} = \frac{\partial H}{\partial p} \end{cases} \quad \Rightarrow \quad \begin{cases} \dot{p} = -mgL\sin q \\ \dot{q} = \frac{p}{mL^2} \end{cases}$$

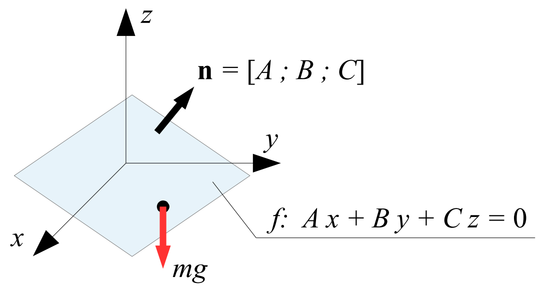
EXERCISE 2

Material point of mass m performs a motion in a uniform field of gravitational forces on a plane f given by equation:

$$f: \quad A\,x+B\,y+C\,z=0$$

Find the equations of motion with the use of:

a) Newton's 2nd principle of motion
b) d'Alembert principle
c) Lagrange equation of the 1st kind
d) Lagrange equation of the 2nd kind
e) Hamilton equation
f) Principle of least action



SOLUTION:

A point moving on the plane has two degrees of freedom.

AD a) – NEWTON'S 2nd PRINCIPLE OF MOTION

In order to make use of the 2nd principle of motion it is necessary to determine all forces acting on a point – this includes also an unknown raction force that holds the point on the plane. This force – as a passive force that performs no work on displacement – will always be perpendicular to the plane, so it is parallel (proportional) to the gradient of this plane:

Reaction forces:
$$\mathbf{R}=\alpha\cdot\mathrm{grad}\,f=\alpha\left[\frac{\partial f}{\partial x}\,;\,\frac{\partial f}{\partial y}\,;\,\frac{\partial f}{\partial z}\right]=[\alpha A\,;\,\alpha B\,;\,\alpha C]$$

Active force due to gravity:
$$\mathbf{F}=[0\,;\,0\,;\,-mg]$$

Equations of motion are determined with the use of Newton's 2nd principle of motion:

$$m\,\ddot{\mathbf{r}}=\mathbf{F}+\mathbf{R} \quad \Leftrightarrow \quad \begin{cases} m\,\ddot{x}=\alpha\,A \\ m\,\ddot{y}=\alpha\,B \\ m\,\ddot{z}=\alpha\,C-mg \end{cases}$$

An unknown proportionality coefficient is found by double differentiation of the plane equation with respect to time and by expressing obtained derivatives by forces, according to the principle of motion written above:

$$A\,\ddot{x}+B\,\ddot{y}+C\,\ddot{z}=0 \quad \Rightarrow \quad \alpha\,A^2+\alpha\,B^2+\alpha\,C^2-Cmg=0 \quad \Rightarrow \quad \alpha=\frac{Cmg}{A^2+B^2+C^2}$$

The above equations allow us also to write down:

$$\ddot{z}=-\frac{A\,\ddot{x}+B\,\ddot{y}}{C}$$

Substituting both results to the equations of motions we obtain:

$$\begin{cases} \ddot{x} = \dfrac{CAg}{A^2+B^2+C^2} \\ \ddot{y} = \dfrac{CBg}{A^2+B^2+C^2} \\ -\dfrac{A\ddot{x}+B\ddot{y}}{C} = \dfrac{C^2 g}{A^2+B^2+C^2} - g \end{cases} \quad \Rightarrow \quad \begin{cases} \ddot{x} = \dfrac{CAg}{A^2+B^2+C^2} \\ \ddot{y} = \dfrac{CBg}{A^2+B^2+C^2} \\ A\ddot{x}+B\ddot{y} = \dfrac{C(A^2+B^2)g}{A^2+B^2+C^2} \end{cases}$$

We can notice that those equations are not idependent. Multiplying the first equation with *A* and the second one with *B* and summing them together we obtain the third one – it the first two are satisfied, then the third one is also satisfied. Equations of motion have thus the following form:

$$\begin{cases} \ddot{x} = \dfrac{CAg}{A^2+B^2+C^2} \\ \ddot{y} = \dfrac{CBg}{A^2+B^2+C^2} \end{cases}$$

AD b) – D'ALEMBERT PRINCIPLE

Coordinates (*x*,*y*) of a point on a plane are considered independent coordinates. Location vector is equal:

$$\mathbf{r} = \left[x \; ; \; y \; ; \; -\frac{Ax+By}{C} \right]$$

Active and inertial forces acting on a mass:

Active forces: $\mathbf{F} = [0 \; ; \; 0 \; ; \; -mg]$

Inertial forces: $\mathbf{B} = -m\ddot{\mathbf{r}} = \left[-m\ddot{x} \; ; \; -m\ddot{y} \; ; \; \dfrac{m}{C}(A\ddot{x}+B\ddot{y}) \right]$

Permissible displacement of a point is any displacement which lies in the plane *f*.

Permissible displacement: $\delta\mathbf{r} = [\delta x \; ; \; \delta y \; ; \; \delta z]$

We can calculate it as any vector which is perpendicular to the vector which is perpendicular to the plane, namely to the gradient of plane:

Vector perpendicular to *f*: $\mathbf{n} = \mathrm{grad}\, f = \left[\dfrac{\partial f}{\partial x} \; ; \; \dfrac{\partial f}{\partial y} \; ; \; \dfrac{\partial f}{\partial z} \right] = [A \; ; \; B \; ; \; C]$

Orthogonality condition: $\delta\mathbf{r} \circ \mathbf{n} = A\delta x + B\delta y + C\delta z = 0 \quad \Rightarrow \quad \delta z = -\dfrac{1}{C}(A\delta x + B\delta y)$

Permissible displacement: $\delta\mathbf{r} = \left[\delta x \; ; \; \delta y \; ; \; -\dfrac{1}{C}(A\delta x + B\delta y) \right]$

Virtual work:

$$\delta L = (\mathbf{F}+\mathbf{B})\circ\delta\mathbf{r} = \left[-m\ddot{x}\ ;\ -m\ddot{y}\ ;\ \frac{m}{C}(A\ddot{x}+B\ddot{y})-mg\right]\circ\left[\delta x\ ;\ \delta y\ ;\ -\frac{1}{C}(A\delta x+B\delta y)\right]=0\quad\forall\delta x,\delta y$$

$$\left[\ddot{x}+\frac{A}{C}\left(\frac{1}{C}(A\ddot{x}+B\ddot{y})\right)-g\right]\delta x+\left[\ddot{y}+\frac{B}{C}\left(\frac{1}{C}(A\ddot{x}+B\ddot{y})-g\right)\right]\delta y=0\quad\forall\delta x,\delta y$$

Equations of motion:

$$\begin{cases}\left(1+\dfrac{A^2}{C^2}\right)\ddot{x}+\dfrac{AB}{C^2}\ddot{y}=\dfrac{A}{C}g\\ \left(1+\dfrac{B^2}{C^2}\right)\ddot{y}+\dfrac{AB}{C^2}\ddot{x}=\dfrac{B}{C}g\end{cases}$$

Solving the above system with respect to $\ddot{x}$ and $\ddot{y}$ gives us:

$$\begin{cases}\ddot{x}=\dfrac{CAg}{A^2+B^2+C^2}\\ \ddot{y}=\dfrac{CBg}{A^2+B^2+C^2}\end{cases}$$

AD c) – LAGRANGE EQAUTIONS OF THE 1st KIND

In order to find the equations of motion with the use of Lagrange equations of the 1st kind we have to know all active forces and constraint equations:

Active forces: $\mathbf{F}=[0\ ;\ 0\ ;\ -mg]$
Constraints: $f(x,y,z)=Ax+By+Cz=0$

We have only a single constraint equation, so we introduce only a single Lagrange multiplier λ .

Lagrange equations of the 1st kind:

$$\begin{cases}m\ddot{x}=F_x+\lambda\dfrac{\partial f}{\partial x}\\ m\ddot{y}=F_y+\lambda\dfrac{\partial f}{\partial y}\\ m\ddot{z}=F_z+\lambda\dfrac{\partial f}{\partial z}\end{cases}\quad\Rightarrow\quad\begin{cases}\ddot{x}=\dfrac{A\lambda}{m}\\ \ddot{y}=\dfrac{B\lambda}{m}\\ \ddot{z}=-g+\dfrac{C\lambda}{m}\end{cases}$$

We have an additional **constraint equation** – after double differentiation with respect to time:

$$A\ddot{x}+B\ddot{y}+C\ddot{z}=0$$

Accounting for relation obtained from Lagrange equations:

$$\frac{1}{m}\left[A^2\lambda+B^2\lambda+C^2\lambda-Cmg\right]=0\quad\Rightarrow\quad\lambda=\frac{Cmg}{A^2+B^2+C^2}$$

Differentiated equation of constraints gives us also:

$$\ddot{z} = -\frac{A\ddot{x}+B\ddot{y}}{C}$$

Substituting these results into the equations of motion:

$$\begin{cases} \ddot{x} = \dfrac{CAg}{A^2+B^2+C^2} \\ \ddot{y} = \dfrac{CBg}{A^2+B^2+C^2} \\ -\dfrac{A\ddot{x}+B\ddot{y}}{C} = \dfrac{C^2 g}{A^2+B^2+C^2} - g \end{cases} \Rightarrow \begin{cases} \ddot{x} = \dfrac{CAg}{A^2+B^2+C^2} \\ \ddot{y} = \dfrac{CBg}{A^2+B^2+C^2} \\ A\ddot{x}+B\ddot{y} = \dfrac{C(A^2+B^2)g}{A^2+B^2+C^2} \end{cases}$$

We can notice that those equations are not idependent. Multiplying the first equation with A and the second one with B and summing them together we obtain the third one – it the first two are satisfied, then the third one is also satisfied. Equations of motion have thus the following form:

$$\begin{cases} \ddot{x} = \dfrac{CAg}{A^2+B^2+C^2} \\ \ddot{y} = \dfrac{CBg}{A^2+B^2+C^2} \end{cases}$$

AD d) – LAGRANGE EQUATIONS OF THE 2nd KIND

Generalized cordinates: $q_1 = x\,,\quad q_2 = y$

Position vector on plane *f*: $\mathbf{r} = \left[q_1\ ;\ q_2\ ;\ -\dfrac{Aq_1+Bq_2}{C} \right]$

Velocity vector: $\dot{\mathbf{r}} = \left[\dot{q}_1\ ;\ \dot{q}_2\ ;\ -\dfrac{A\dot{q}_1+B\dot{q}_2}{C} \right]$

Kinetic energy: $E_k = \dfrac{1}{2}m(\dot{\mathbf{r}})^2 = \dfrac{m}{2}\left[(\dot{q}_1)^2+(\dot{q}_2)^2+\dfrac{1}{C^2}(A\dot{q}_1+B\dot{q}_2)^2 \right]$

The motion is due to conservative forces:

Potential energy: $E_p = mgz = -\dfrac{mg}{C}(Aq_1+Bq_2)$

Lagnrangian: $\mathscr{L} = E_k - E_p = \dfrac{m}{2}\left[(\dot{q}_1)^2+(\dot{q}_2)^2+\dfrac{1}{C^2}(A\dot{q}_1+B\dot{q}_2)^2 \right] + \dfrac{mg}{C}(Aq_1+Bq_2)$

Lagrange eqautions of the 2nd kind:

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_1}\right)-\frac{\partial \mathscr{L}}{\partial q_1}=0 \\ \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_2}\right)-\frac{\partial \mathscr{L}}{\partial q_2}=0 \end{cases} \Rightarrow \begin{cases} \frac{d}{dt}\left[\frac{m}{2}\left(2\dot{q}_1+\frac{1}{C^2}(2A^2\dot{q}_1+2AB\dot{q}_2)\right)\right]-\frac{A}{C}mg=0 \\ \frac{d}{dt}\left[\frac{m}{2}\left(2\dot{q}_2+\frac{1}{C^2}(2AB\dot{q}_1+2B^2\dot{q}_2)\right)\right]-\frac{B}{C}mg=0 \end{cases} \Rightarrow$$

$$\begin{cases} \ddot{q}_1+\frac{1}{C^2}(A^2\ddot{q}_1+AB\ddot{q}_2)-\frac{A}{C}g=0 \\ \ddot{q}_2+\frac{1}{C^2}(AB\ddot{q}_1+B^2\ddot{q}_2)-\frac{B}{C}g=0 \end{cases} \Rightarrow \begin{cases} \left(1+\frac{A^2}{C^2}\right)\ddot{q}_1+\frac{AB}{C^2}\ddot{q}_2=\frac{A}{C}g \\ \left(1+\frac{B^2}{C^2}\right)\ddot{q}_2+\frac{AB}{C^2}\ddot{q}_1=\frac{B}{C}g \end{cases}$$

Substituting $q_1=x,\ q_2=y$ we obtain the same equations as previously.

AD e) – HAMILTONA EQUATIONS

Generalized coordinates: $q_1=x,\quad q_2=y$

Position vector on plane *f*: $\mathbf{r}=\left[q_1\ ;\ q_2\ ;\ -\frac{Aq_1+Bq_2}{C}\right]$

Velocity vector: $\dot{\mathbf{r}}=\left[\dot{q}_1\ ;\ \dot{q}_2\ ;\ -\frac{A\dot{q}_1+B\dot{q}_2}{C}\right]$

Kinetic energy: $E_k=\frac{1}{2}m(\dot{\mathbf{r}})^2=\frac{m}{2}\left[(\dot{q}_1)^2+(\dot{q}_2)^2+\frac{1}{C^2}(A\dot{q}_1+B\dot{q}_2)^2\right]$

The motion is due to conservative forces:

Potential energy: $E_p=mgz=-\frac{mg}{C}(Aq_1+Bq_2)$

Lagrangian:

$$L=E_k-E_p=\frac{m}{2}\left[(\dot{q}_1)^2+(\dot{q}_2)^2+\frac{1}{C^2}(A\dot{q}_1+B\dot{q}_2)^2\right]+\frac{mg}{C}(Aq_1+Bq_2)=$$

$$=\frac{m}{2}\left[\left(1+\frac{A^2}{C^2}\right)\dot{q}_1^2+\left(1+\frac{B^2}{C^2}\right)\dot{q}_2^2+\frac{2AB}{C^2}\dot{q}_1\dot{q}_2\right]+\frac{mg}{C}(Aq_1+Bq_2)$$

Generalized momenta:

$$p_1=\frac{\partial L}{\partial \dot{q}_1}=m\left(1+\frac{A^2}{C^2}\right)\dot{q}_1+m\frac{AB}{C^2}\dot{q}_2$$

$$p_2=\frac{\partial L}{\partial \dot{q}_2}=m\left(1+\frac{B^2}{C^2}\right)\dot{q}_2+m\frac{AB}{C^2}\dot{q}_1$$

We determine the generalized velocities as functions of generalized momenta:

$$\dot{q}_1 = \frac{p_1(B^2+C^2)-p_2 AB}{m(A^2+B^2+C^2)} \qquad \dot{q}_2 = \frac{p_2(A^2+C^2)-p_1 AB}{m(A^2+B^2+C^2)}$$

Hamiltonian:

$$H(q_1,q_2,p_1,p_2) = \sum_j^2 p_j \cdot \dot{q}_j(p_1,p_2) - L(q_1,q_2,\dot{q}_1(p_1,p_2),\dot{q}_2(p_1,p_2)) = \\ = \frac{(B^2+C^2)p_1^2+(A^2+C^2)p_2^2-2ABp_1p_2}{2m(A^2+B^2+C^2)} - \frac{mg}{C}(Aq_1+Bq_2)$$

Equations of motion are provided by Hamilton equations:

$$\begin{cases} \dot{p}_1 = -\dfrac{\partial H}{\partial q_1} \\ \dot{p}_2 = -\dfrac{\partial H}{\partial q_2} \\ \dot{q}_1 = \dfrac{\partial H}{\partial p_1} \\ \dot{q}_2 = \dfrac{\partial H}{\partial p_2} \end{cases} \Rightarrow \begin{cases} \dot{p}_1 = mg \cdot \dfrac{A}{C} \\ \dot{p}_2 = mg \cdot \dfrac{B}{C} \\ \dot{q}_1 = \dfrac{(B^2+C^2)p_1 - AB\,p_2}{m(A^2+B^2+C^2)} \\ \dot{q}_2 = \dfrac{(A^2+C^2)p_2 - AB\,p_1}{m(A^2+B^2+C^2)} \end{cases}$$

AD f) – PRINCIPLE OF LEAST ACTION

The procedure of determining the Lagrangian is the same as in case of the Lagrange equations of the 2nd kind and as in case of Hamilton equations. We have thus:

$$L = \frac{m}{2}\left[\left(1+\frac{A^2}{C^2}\right)\dot{q}_1^2 + \left(1+\frac{B^2}{C^2}\right)\dot{q}_2^2 + \frac{2AB}{C^2}\dot{q}_1\dot{q}_2\right] + \frac{mg}{C}(Aq_1+Bq_2)$$

Integral of action:

$$S[q_1,q_2] = \int_{t_0}^{t}\left[\frac{m}{2}\left[\left(1+\frac{A^2}{C^2}\right)\dot{q}_1^2 + \left(1+\frac{B^2}{C^2}\right)\dot{q}_2^2 + \frac{2AB}{C^2}\dot{q}_1\dot{q}_2\right] + \frac{mg}{C}(Aq_1+Bq_2)\right]d\tau$$

Action will be have the least value when its variation will be equal 0. As the integrand depend on two functions (being independent variables of the functional) and on their first derivatives, we will calculate the linear increment of the functional δS (its variation) as a sum of variations connected with variability of each of those variables:

$$\delta S = \int_{t_0}^{0}\left[\frac{\partial L}{\partial q_1}\delta q_1 + \frac{\partial L}{\partial \dot{q}_1}\delta\dot{q}_1 + \frac{\partial L}{\partial q_2}\delta q_2 + \frac{\partial L}{\partial \dot{q}_2}\delta\dot{q}_2\right]d\tau$$

$$\delta S = \int_{t_0}^{0} \left[\left(\frac{A}{C} mg \right) \delta q_1 + \left[\left(1 + \frac{A^2}{C^2} \right) m \dot{q}_1 + \frac{mAB}{C^2} \dot{q}_2 \right] \delta \dot{q}_1 + \left(\frac{B}{C} mg \right) \delta q_2 + \left[\left(1 + \frac{B^2}{C^2} \right) m \dot{q}_2 + \frac{mAB}{C^2} \dot{q}_1 \right] \delta \dot{q}_2 \right] \mathrm{d}\tau$$

Integrals containing variations of functional with respect to generalized velocities will be integrated by part:

$$\int_{t_0}^{t} m \left(1 + \frac{A^2}{C^2} \right) \dot{q}_1 \delta \dot{q}_1 \mathrm{d}\tau = \begin{vmatrix} f = \dot{q}_1 & f' = \ddot{q}_1 \\ g' = \delta \dot{q}_1 & g = \delta q_1 \end{vmatrix} = m \left(1 + \frac{A^2}{C^2} \right) \left[\left(\dot{q}_1 \delta q_1 \right) \Big|_{t_0}^{t} - \int_{t_0}^{t} \ddot{q}_1 \delta q_1 \mathrm{d}\tau \right]$$

The term depending on boundary values is equal 0, since variations δq are always assumed to satisfy $\delta q(t_0) = \delta q(t) = 0$ – it is because we consider a variation of function between two points which are fixed:

$$\int_{t_0}^{t} m \left(1 + \frac{A^2}{C^2} \right) \dot{q}_1 \delta \dot{q}_1 \mathrm{d}\tau = - \int_{t_0}^{t} m \left(1 + \frac{A^2}{C^2} \right) \ddot{q}_1 \delta q_1 \mathrm{d}\tau$$

In the same manner other integrals are calculated:

$$\int_{t_0}^{t} m \left(1 + \frac{B^2}{C^2} \right) \dot{q}_2 \delta \dot{q}_2 \mathrm{d}\tau = - \int_{t_0}^{t} m \left(1 + \frac{B^2}{C^2} \right) \ddot{q}_2 \delta q_2 \mathrm{d}\tau$$
$$\int_{t_0}^{t} \frac{mAB}{C^2} \dot{q}_2 \delta \dot{q}_1 \mathrm{d}\tau = - \int_{t_0}^{t} \frac{mAB}{C^2} \ddot{q}_2 \delta q_1 \mathrm{d}\tau$$
$$\int_{t_0}^{t} \frac{mAB}{C^2} \dot{q}_1 \delta \dot{q}_2 \mathrm{d}\tau = - \int_{t_0}^{t} \frac{mAB}{C^2} \ddot{q}_1 \delta q_2 \mathrm{d}\tau$$

Let's substitue these results to the variation and put δq_1 and δq_2 outside brackets:

$$\delta S = \int_{t_0}^{0} \left[\left[\left(\frac{A}{C} mg \right) - \left(1 + \frac{A^2}{C^2} \right) m \ddot{q}_1 - \frac{mAB}{C^2} \ddot{q}_2 \right] \delta q_1 + \left[\left(\frac{B}{C} mg \right) - \left(1 + \frac{B^2}{C^2} \right) m \ddot{q}_2 - \frac{mAB}{C^2} \ddot{q}_1 \right] \delta q_2 \right] \mathrm{d}\tau$$

Principle of leas action states that $\delta S = 0$ for any δq_1 and δq_2 , hence:

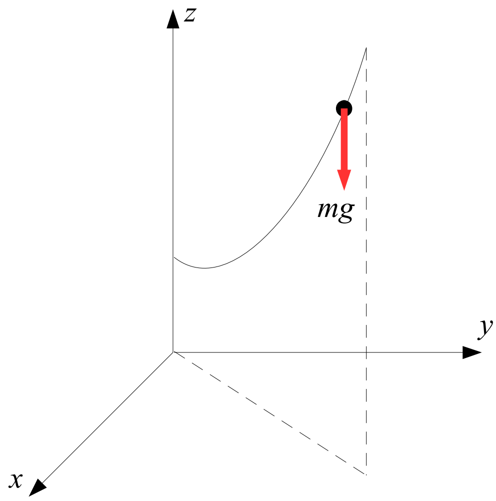
$$\begin{cases} \left(\frac{A}{C} mg \right) - \left(1 + \frac{A^2}{C^2} \right) m \ddot{q}_1 - \frac{mAB}{C^2} \ddot{q}_2 = 0 \\ \left(\frac{B}{C} mg \right) - \left(1 + \frac{B^2}{C^2} \right) m \ddot{q}_2 - \frac{mAB}{C^2} \ddot{q}_1 = 0 \end{cases},$$

what is equivalent to the results obtained from other methods.

ZADANIE 4

Find equations of motion of a point mass *m* moving along a curve given with parametric equations

$$K:\quad \begin{cases} x(\lambda)=\lambda \\ y(\lambda)=\lambda \\ z(\lambda)=3\lambda^2+1 \end{cases}$$



with the use of:

a) Newton's 2nd principle of motion
b) Conservation of energy
c) d'Alembert principle
d) Lagrange equations of the 1st kind
e) Lagrange equations of the 2nd kind
f) Principle of least action

SOLUTION:

A point performing a motion along a curve has only a single degree of freedom.

AD a) –NEWTON'S 2nd PRINCIPLE OF MOTION

In order to make use of the 2nd principle of motion it is necessary to know all forces acting on the system. We know the active force:

$$\mathbf{F}=[0\ ;\ 0\ ;\ -mg]$$

which is the force of gravity. We do not know, however, a passive force of reaction that holds the point on a given curve. We know that it is a passive force that performs no work on subsequent increments of displacement. Each increment of displacement will be a vector tangent to the trajectory, this is to the curve along which the point moves. It will be also parallel to the velocity vector – reaction must be then a vector which is perpendicular to the velocity vector in each moment of time.

Position vector is determined from the parametric equations of the curve:

$$\mathbf{r}(t)=\begin{cases} x(t)=\lambda(t) \\ y(t)=\lambda(t) \\ z(t)=3\lambda^2(t)+1 \end{cases}$$

Velocity and acceleration vectors are determined by differentiation of the position vector with respect to time – we must remember that parameter λ is also a function of time:

$$\dot{\mathbf{r}}=\begin{cases} \dot{x}(t)=\dot{\lambda} \\ \dot{y}(t)=\dot{\lambda} \\ \dot{z}(t)=6\lambda\dot{\lambda} \end{cases}\qquad\qquad \ddot{\mathbf{r}}=\begin{cases} \ddot{x}(t)=\ddot{\lambda} \\ \ddot{y}(t)=\ddot{\lambda} \\ \ddot{z}(t)=6(\dot{\lambda})^2+6\lambda\ddot{\lambda} \end{cases}$$

Unknown reaction force satisfies orthogonality condition: $\mathbf{R}\circ\dot{\mathbf{r}}=0 \quad \Rightarrow \quad R_x\dot{\lambda}+R_y\dot{\lambda}+6R_z\lambda\dot{\lambda}=0$

It reduces the number of unknown components of this force from 3 to 2. We may write:

$$\mathbf{R}=\left[\alpha\ ;\ \beta\ ;\ -\frac{\alpha+\beta}{6\lambda}\right]$$

where α,β are certain unknown parameters that we have to find. Let's write down the equations of motion according to the 2nd principle of motion:

$$m\ddot{\mathbf{r}}=\mathbf{F}+\mathbf{R} \quad \Leftrightarrow \quad \begin{cases} m\ddot{x}(t)=\alpha \\ m\ddot{y}(t)=\beta \\ m\ddot{z}(t)=-mg-\dfrac{\alpha+\beta}{6\lambda} \end{cases}$$

Comparing the acceleraions from the above equations to those obtained from differentiation of position vector gives us::

$$\begin{cases} \alpha=\ddot{\lambda}m \\ \beta=\ddot{\lambda}m \\ -g-\dfrac{\alpha+\beta}{6\lambda m}=6(\dot{\lambda})^2+6\lambda\ddot{\lambda} \end{cases}$$

Remembering that $\lambda=x$ we may easily find the values of components of reaction force $\alpha=\beta=m\ddot{x}(t)$. The function $x(t)$ will be determined with the use of the third equation which becomes an equation of motion:

$$-g-\frac{2m\ddot{x}}{6xm}=6(\dot{x})^2+6x\ddot{x}$$

$$-6gx-2\ddot{x}=36(\dot{x})^2x+36x^2\ddot{x}$$

Finally:

$$\ddot{x}(18x^2+1)+18(\dot{x})^2x+3gx=0$$

Equation of motion could be determined with the use of other functions, e.g. Equivalently by $y(t)$ or $\lambda(t)$.

AD b) – CONSERVATION OF ENERGY

Law of conservation of energy is always sufficient for determination of equation of motion of systems of one degree of freedom. In this case it is worth expressing all components of the position vector with the use of a single parameter:

$$x=\lambda \ \rightarrow \ \mathbf{r}=\begin{cases} x=x \\ y(x)=x \\ z(x)=3x^2+1 \end{cases}$$

Velocity vector: $\mathbf{r} = \begin{cases} \dot{x} = \dot{x} \\ \dot{y} = \dot{x} \\ \dot{z} = 6\dot{x}\,x \end{cases}$

Kinetic energy: $E_k = \dfrac{m(\dot{\mathbf{r}})^2}{2} = \dfrac{m}{2}\left[(\dot{x})^2 + (\dot{x})^2 + 36(\dot{x})^2 x^2\right]$

Potential energy: $E_p = m\,g\,z = m\,g\left[3x^2+1\right]$

Total mechanical energy: $E = E_k + E_p = \dfrac{m(\dot{x})^2}{2}\left[36x^2+2\right] + mg\left[3x^2+1\right]$

Conservation of energy:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(\dot{x})^2(18x^2+1) + g(3x^2+1)\right] = 0$$

$$2(\dot{x})\ddot{x}(18x+1) + 36(\dot{x})^3 x + 6\,g\,x\,\dot{x} = 0$$

Finally:

$$\ddot{x}(18x^2+1) + 18(\dot{x})^2 x + 3\,g\,x = 0$$

AD c) – D'ALEMBERT PRINCIPLE

Position vector of a point on a cruve is determined by parametric equations of the curve. Basing on it we can calculate vectors of velocity and acceleration assuming that the parameter i an unknown function of time:

$$\mathbf{r}(t) = \begin{cases} x(t) = \lambda(t) \\ y(t) = \lambda(t) \\ z(t) = 3\lambda^2(t)+1 \end{cases} \qquad \dot{\mathbf{r}} = \begin{cases} \dot{x}(t) = \dot{\lambda} \\ \dot{y}(t) = \dot{\lambda} \\ \dot{z}(t) = 6\lambda\dot{\lambda} \end{cases} \qquad \ddot{\mathbf{r}} = \begin{cases} \ddot{x}(t) = \ddot{\lambda} \\ \ddot{y}(t) = \ddot{\lambda} \\ \ddot{z}(t) = 6(\dot{\lambda})^2 + 6\lambda\ddot{\lambda} \end{cases}$$

Permissible displacement in every point must be tangent to the curve, so it will be parallel to the velocity vector. It is enough to preserve (in every point and moment) mutual proportions between components of the vector of permissible displacement to be the same as in velocity vector:

$$\delta\mathbf{r}(t) = \begin{cases} \delta x(t) = \delta r \\ \delta y(t) = \delta r \\ \delta z(t) = 6\lambda(t)\delta r \end{cases}$$

Active and inertial forces acting on the mass:

Active forces: $\mathbf{F} = [0\ ;\ 0\ ;\ -mg]$

Inertial forces: $\mathbf{B} = -m\ddot{\mathbf{r}} = [-m\ddot{\lambda}\ ;\ -m\ddot{\lambda}\ ;\ -6m(\dot{\lambda})^2 - 6m\lambda\ddot{\lambda}]$

Virtual work:

$$\delta L = (\mathbf{F}+\mathbf{B}) \circ \delta\mathbf{r} = (-m\ddot{\lambda})\cdot(\delta r) + (-m\ddot{\lambda})\cdot(\delta r) + (-mg - 6m(\dot{\lambda})^2 - 6m\lambda\ddot{\lambda})\cdot(6\lambda\delta r) =$$
$$= -m\left[\ddot{\lambda} + \ddot{\lambda} + 6g\lambda + 36\lambda(\dot{\lambda})^2 + 36\lambda^2\ddot{\lambda}\right]\delta r = -\,m\left[\ddot{\lambda}(36\lambda^2+2) + 6\lambda(6(\dot{\lambda})^2+g)\right]\delta r$$

d'Alembert principle:

$$\delta L = -m\left[\ddot{\lambda}(36\lambda^2+2)+6\lambda(6(\dot{\lambda})^2+g)\right]\delta r = 0 \qquad \forall\,\delta r$$
$$\ddot{\lambda}(36\lambda^2+2)+6\lambda(6(\dot{\lambda})^2+g) = 0$$

Substituting $x=\lambda$ we obtain the final form of the equation of motion:

$$\ddot{x}(18x^2+1)+18(\dot{x})^2x+3gx = 0$$

AD d) – LAGRANGE EQUATIONS OF THE 1st KIND

Active forces acting on mass: $\mathbf{F} = [0\ ;\ 0\ ;\ -mg]$

Constraint equations are determined from parametric equations of the curve:

$$K:\begin{cases} x(\lambda) = \lambda \\ y(\lambda) = \lambda \\ z(\lambda) = 3\lambda^2+1 \end{cases} \quad \Rightarrow \quad \begin{cases} f_1(x,y,z) = x-y=0 \\ f_2(x,y,z) = z-3x^2-1 = 0 \end{cases}$$

As there are two constraints, we have to introduce two **Lagrange multipliers** α,β .

Lagrange equation of the 1st kind:

$$\begin{cases} m\ddot{x} = F_x + \alpha\dfrac{\partial f_1}{\partial x}+\beta\dfrac{\partial f_2}{\partial x} \\ m\ddot{y} = F_y + \alpha\dfrac{\partial f_1}{\partial y}+\beta\dfrac{\partial f_2}{\partial y} \\ m\ddot{z} = F_z + \alpha\dfrac{\partial f_1}{\partial z}+\beta\dfrac{\partial f_2}{\partial z} \end{cases} \Rightarrow \begin{cases} m\ddot{x} = \alpha\cdot(1)+\beta\cdot(-6x) \\ m\ddot{y} = \alpha\cdot(-1)+\beta\cdot(0) \\ m\ddot{z} = -mg+\alpha\cdot(0)+\beta\cdot(1) \end{cases} \Rightarrow \begin{cases} m\ddot{x} = \alpha-6x\beta \\ m\ddot{y} = -\alpha \\ m\ddot{z} = -mg+\beta \end{cases}$$

There are 5 unknowns and three equations of motion – 2 required equations are provided by constraints. Double differentiation of those equations with respect to time gives us:

$$\begin{cases} \dot{x}-\dot{y}=0 \\ \dot{z}-6x\dot{x} = 0 \end{cases} \Rightarrow \begin{cases} \ddot{x}-\ddot{y}=0 \\ \ddot{z}-6(\dot{x})^2-6x\ddot{x} = 0 \end{cases}$$

The second equation of motion and the first constrain give us:

$$\alpha = -m\ddot{y} = -m\ddot{x}$$

The third equation of motion and the second constrain give us:

$$\beta = m\ddot{z}+mg = 6m((\dot{x})^2+x\ddot{x})+mg$$

Substituting to the first equation of motion results in:

$$m\ddot{x} = -m\ddot{x}-6\left(6mx((\dot{x})^2+x\ddot{x})+mg\right)$$

After some algebra:

$$2\,m\ddot{x} + 36\,m\,x(\dot{x})^2 + 36\,m\,x^2\,\ddot{x} + 6\,mg\,x = 0 \quad ,$$

what is equivalent to the result obtained from other methods:

$$\ddot{x}(18\,x^2+1)+18(\dot{x})^2\,x+3\,g\,x = 0$$

AD e) – LAGRANGE EQUATIONS OF THE 2nd KIND

The parameter of the curve is considered a **generalized coordinate** $q = \lambda$.

Position vector: $$\mathbf{r}(t) = \begin{cases} x = q \\ y = q \\ z = 3\,q^2+1 \end{cases}$$

Velocity vector: $$\dot{\mathbf{r}} = \begin{cases} \dot{x} = \dot{q} \\ \dot{y} = \dot{q} \\ \dot{z} = 6\,q\,\dot{q} \end{cases}$$

Kinetic energy: $$E_k = \frac{m(\dot{\mathbf{r}})^2}{2} = \frac{m}{2}\left[(\dot{q})^2+(\dot{q})^2+36\,q^2(\dot{q})^2\right] = \frac{m(\dot{q})^2}{2}\left[36\,q^2+2\right]$$

The motion is due to conservative forces:

Potential energy: $E_p = -V = mgz = mg(3\,q^2+1)$

Lagrangian: $\mathscr{L} = E_k - E_p = m(\dot{q})^2(18\,q^2+1) - mg(3\,q^2+1)$

Lagrange equations of the 2nd kind:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathscr{L}}{\partial\dot{q}}\right)-\frac{\partial\mathscr{L}}{\partial q} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[2\,m\dot{q}(18\,q^2+1)\right] - 36\,m(\dot{q})^2\,q + 6\,q\,mg = 0$$

$$2\,m\ddot{q}(18\,q^2+1)+72\,m(\dot{q})^2\,q - 36\,m(\dot{q})^2\,q + 6\,q\,mg = 0$$

Substituting $q = \lambda = x$ gives us:

$$\ddot{x}(18\,x^2+1) + 18(\dot{x})^2\,x + 3\,g\,x = 0$$

AD f) – ZASADA NAJMNIEJSZEGO DZIAŁANIA

Lagrangian is determined as in case of Lagrange equations of the 2nd kind

$$L = m(\dot{q})^2(18q^2+1) - mg(3q^2+1)$$

Integral of action:

$$S[q] = \int_{t_0}^{t} \left[m(\dot{q})^2(18q^2+1) - mg(3q^2+1)\right] \mathrm{d}\tau$$

The equation of motion will be found according to the principle of least action which states that integral of action must have minimum value. Action will be have the least value when its variation will be equal 0. As the integrand depend on unknown function and on its first derivative, we will calculate the linear increment of the functional δS (its variation) as a sum of variations connected with variability of each of those variables:

$$\delta S = \int_{t_0}^{0} \left[\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q}\right] \mathrm{d}\tau$$

$$\delta S[q] = \int_{t_0}^{t} \left[\left(36mq(\dot{q})^2 - 6mgq\right)\delta q + \left(2m\dot{q}(18q^2+1)\right)\delta \dot{q}\right] \mathrm{d}\tau$$

Knowing that $\delta \dot{q} = \frac{\mathrm{d}}{\mathrm{d}t}\delta q$, the integral which is a variation with respect to generalized velocity will be integrated by parts:

$$\int_{t_0}^{t} \left[\left(2m\dot{q}(18q^2+1)\right)\delta \dot{q}\right] \mathrm{d}\tau = \begin{vmatrix} f = 2m\dot{q}(18q^2+1) & f' = 2m\ddot{q}(18q^2+1) + 72mq(\dot{q})^2 \\ g' = \delta\dot{q} & g = \delta q \end{vmatrix} =$$

$$2m\dot{q}(18q^2+1)\delta q\Big|_{t_0}^{t} - \int_{t_0}^{t} \left[2m\ddot{q}(18q^2+1) + 72mq(\dot{q})^2\right]\delta q \, \mathrm{d}\tau$$

The term depending on boundary values is equal 0, since variations δq are always assumed to satisfy $\delta q(t_0) = \delta q(t) = 0$ – it is because we consider a variation of function between two points which are fixed:

$$\delta S[q] = \int_{t_0}^{t} \left[\left(36mq(\dot{q})^2 - 6mgq - 72mq(\dot{q})^2 - 2m\ddot{q}(18q^2+1)\right)\delta q\right] \mathrm{d}\tau =$$

$$= \int_{t_0}^{t} \left[\left(-36mq(\dot{q})^2 - 6mgq - 2m\ddot{q}(18q^2+1)\right)\delta q\right] \mathrm{d}\tau$$

Principle of leas action states that $\delta S = 0$ for any δq :

$$\delta S[q] = \int_{t_0}^{t} \left[\left(-36mq(\dot{q})^2 - 6mgq - 2m\ddot{q}(18q^2+1)\right)\delta q\right] \mathrm{d}\tau = 0 \qquad \forall \delta q$$

Hence:

$$18q(\dot{q})^2 + 3gq + \ddot{q}(18q^2+1) = 0$$

Substituting $q = x$ gives us finally:

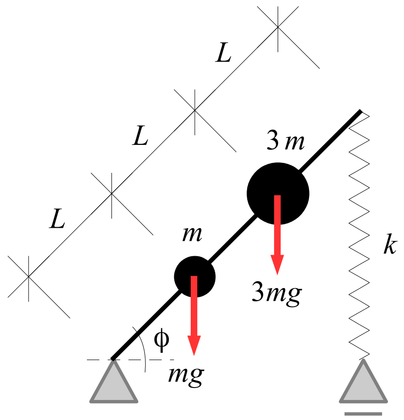
$$\ddot{x}(18x^2+1) + 18(\dot{x})^2 x + 3gx = 0$$

EXERCISE 5

Find the equations of motion of the system in the figue with the use of:

a) d'Alembert principle

b) Lagrange equations of the 2[nd] kind



SOLUTION:

The system has only a single degree of freedom.

AD a) – D'ALEMBERT PRINCIPLE

Geometry of the system allow us to determine the position vector of any point in which a force is applied:

$$\mathbf{r_1} = [L\cos\phi \;;\; L\sin\phi]$$
$$\mathbf{r_2} = [2L\cos\phi \;;\; 2L\sin\phi]$$
$$\mathbf{r_3} = [3L\cos\phi \;;\; 3L\sin\phi]$$

Velocity vectors may be determined by differentiation. In the point in which a mass is present, further differentiation enables us to determine an acceleration vector which will be used to determine the inertial forces. Remembering that ϕ is a function of time:

$$\dot{\mathbf{r}}_1 = L\dot{q}[-\sin\phi \;;\; \cos\phi] \qquad \ddot{\mathbf{r}}_1 = L\left[-\sin\phi\,\ddot\phi - \cos\phi\,(\dot\phi)^2 \;;\; \cos\phi\,\ddot\phi - \sin\phi\,(\dot\phi)^2\right]$$
$$\dot{\mathbf{r}}_2 = 2L\dot{q}[-\sin\phi \;;\; \cos\phi] \qquad \ddot{\mathbf{r}}_2 = 2L\left[-\sin\phi\,\ddot\phi - \cos\phi\,(\dot\phi)^2 \;;\; \cos\phi\,\ddot\phi - \sin\phi\,(\dot\phi)^2\right]$$
$$\dot{\mathbf{r}}_3 = 3L\dot{q}[-\sin\phi \;;\; \cos\phi]$$

Active and **inertial forces** acting on the system:

$$\mathbf{F_1} = [0 \;;\; -mg] \qquad \mathbf{B_1} = -m_1\ddot{\mathbf{r}}_1 = mL\left[\sin\phi\,\ddot\phi + \cos\phi\,(\dot\phi)^2 \;;\; -\cos\phi\,\ddot\phi + \sin\phi\,(\dot\phi)^2\right]$$
$$\mathbf{F_2} = [0 \;;\; -3mg] \qquad \mathbf{B_2} = -m_2\ddot{\mathbf{r}}_2 = 6mL\left[\sin\phi\,\ddot\phi + \cos\phi\,(\dot\phi)^2 \;;\; -\cos\phi\,\ddot\phi + \sin\phi\,(\dot\phi)^2\right]$$
$$\mathbf{F_3} = [0 \;;\; -k\,y_3] = [0 \;;\; -3kL\sin\phi] \qquad \mathbf{B_3} = [0 \;;\; 0]$$

In order to find the virtual work we need to find virtual displacements. Virtual displacements will be parallel to the linear velocity vectors. Velocity vectors may be expressed by angular velocity which will be the same for all point. Virtual displacements have to be decomposed into horizontal and vertical components as the active forces are described in a Cartesian coordinate system:

$$\delta r_1 = L\dot\phi, \quad \delta x_1 = -\delta r_1\sin\phi = -L\delta\phi\sin\phi, \quad \delta y_1 = \delta r_1\cos\phi = L\delta\phi\cos\phi$$
$$\delta r_2 = 2L\dot\phi, \quad \delta x_2 = -\delta r_2\sin\phi = -2L\delta\phi\sin\phi, \quad \delta y_2 = \delta r_2\cos\phi = 2L\delta\phi\cos\phi$$
$$\delta r_3 = 3L\dot\phi, \quad \delta x_3 = -\delta r_3\sin\phi = -3L\delta\phi\sin\phi, \quad \delta y_3 = \delta r_3\cos\phi = 3L\delta\phi\cos\phi$$

Virtual work:

$$\delta L = \sum_{i=1}^{3}(\mathbf{F}_i + \mathbf{B}_i)\circ\delta\mathbf{r}_i = \sum_{i=1}^{3}(F_{xi} + B_{xi})\cdot\delta x_i + (F_{yi} + B_{yi})\cdot\delta y_i =$$

$$= \left[mL(\sin\phi\ddot{\phi}+\cos\phi(\dot{\phi})^2)\right]\cdot(-L\delta\phi\sin\phi) + \left[mL(-\cos\phi\ddot{\phi}+\sin\phi(\dot{\phi})^2)-mg\right]\cdot(L\delta\phi\cos\phi) +$$
$$+\left[6mL\sin\phi\ddot{\phi}+\cos\phi(\dot{\phi})^2\right]\cdot(-2L\delta\phi\sin\phi)+\left[6mL(-\cos\phi\ddot{\phi}+\sin\phi(\dot{\phi})^2)-3mg\right]\cdot(2L\delta\phi\cos\phi)+$$
$$+[0]\cdot(-3L\delta\phi\sin\phi) + [-3kL\sin\phi]\cdot(3L\delta\phi\cos\phi) =$$

$$= \left[-13mL^2\ddot{\phi}(\sin^2\phi+\cos^2\phi)-7mg\,L\cos\phi-9kL^2\cos\phi\sin\phi\right]\delta\phi =$$

$$= -\left[13mL^2\ddot{\phi}+L\cos\phi[7mg + 9kL\sin\phi]\right]\delta\phi$$

d'Alemberta principle:

$$\delta L=0 \qquad \forall\delta r$$
$$-\left[13mL^2\ddot{\phi}+L\cos\phi[7mg + 9kL\sin\phi]\right]\delta\phi = 0 \qquad \forall\delta\phi$$

Finally:

$$13mL^2\ddot{\phi}+L\cos\phi[7mg + 9kL\sin\phi]=0$$

AD b) – LAGRANGE EQUATIONS OF THE 2nd KIND

The angle of rotation of a rod about the pinned support is considered the generalized coordinate.

Position vectors and velocity vectors of point in which a mass is present or in which a force is applied:

$$\mathbf{r}_1 = [L\cos q\ ;\ L\sin q] \qquad \dot{\mathbf{r}}_1 = [-L\dot{q}\sin q\ ;\ L\dot{q}\cos q]$$
$$\mathbf{r}_2 = [2L\cos q\ ;\ 2L\sin q] \qquad \dot{\mathbf{r}}_2 = [-2L\dot{q}\sin q\ ;\ 2L\dot{q}\cos q]$$
$$\mathbf{r}_3 = [3L\cos q\ ;\ 3L\sin q]$$

Kinetic energy:

$$E_k = \sum_{i=1}^{2}\frac{m_i(\dot{\mathbf{r}}_i)^2}{2} = \frac{m}{2}\left[L^2(\dot{q})^2(\sin^2 q+\cos^2 q)\right]+\frac{3m}{2}\left[4L^2(\dot{q})^2(\sin^2 q+\cos^2 q)\right] = \frac{13}{2}mL^2(\dot{q})^2$$

Derivatives of the position vectors of point of application of force with respect to generalized coordinate and vectors of active forces:

$$\frac{\partial\mathbf{r}_1}{\partial q} = [-L\sin q\ ;\ L\cos q] \qquad \mathbf{F}_1 = [0\ ;\ -mg]$$
$$\frac{\partial\mathbf{r}_2}{\partial q} = [-2L\sin q\ ;\ 2L\cos q] \qquad \mathbf{F}_2 = [0\ ;\ -3mg]$$
$$\frac{\partial\mathbf{r}_2}{\partial q} = [-3L\sin q\ ;\ 3L\cos q] \qquad \mathbf{F}_3 = [0\ ;\ -3kL\sin q]$$

Generalized force:

$$Q = \sum_{i=1}^{3}\frac{\partial\mathbf{r}_i}{\partial q}\circ\mathbf{F}_i = -mgL\cos q-6mgL\cos q-9kL^2\cos q\sin q =$$
$$= -L\cos q[9kL\sin q+7mg]$$

Lagrange equation of the 2nd kind:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E_k}{\partial \dot{q}}\right) - \frac{\partial E_k}{\partial q} = Q$$

Fianlly:

$$13\, mL^2 \ddot{q} + L\cos q\left[7\, mg + 9\, kL\sin q\right] = 0$$

AD b) – LAGRANGE EQUATIONS OF THE 2nd KIND FOR CONSERVATIVE FORCES

This task may be solved in a simplified for due to fact that all active forces are conservative. We will determine respective **potentials** for each of them. It concerns especially the gravity forces which – despite their common source – have different potentials, because masses on which those forces act are different:

$$V_1 = -m_1 g\, y_1 \quad \Rightarrow \quad \mathbf{F_1} = \mathrm{grad}\, V_1 = [0\ ;\ -m_1 g]$$
$$V_2 = -m_2 g\, y_2 \quad \Rightarrow \quad \mathbf{F_2} = \mathrm{grad}\, V_2 = [0\ ;\ -m_2 g]$$
$$V_3 = -\frac{k\, y_3^2}{2} \quad \Rightarrow \quad \mathbf{F_3} = \mathrm{grad}\, V_3 = [0\ ;\ -k y_3] = [0\ ;\ -3\, k\, L\sin q]$$

Potential energy needed to define the lagrangian is a sum of component potential energies for each element of the system, due to potential respective for it. Since – in general – each point may have a different *y* coordinate, they must be distinguished. Coordinate *y* may be expressed by generalized coordinate:

Potential energy:

$$E_p = m\, g\, L\sin q + (3\, m)\cdot g\cdot(2\, L)\sin q + \frac{k}{2}(3\, L\sin q)^2 =$$
$$= 7\, mgL\sin q + \frac{9\, kL^2}{2}\sin^2 q$$

Lagrangian:

$$\mathscr{L} = E_k - E_p = \frac{13}{2} m L^2 (\dot{q})^2 - 7\, mgL\sin q - \frac{9\, kL^2}{2}\sin^2 q$$

Lagrange equations of the 2nd kind:

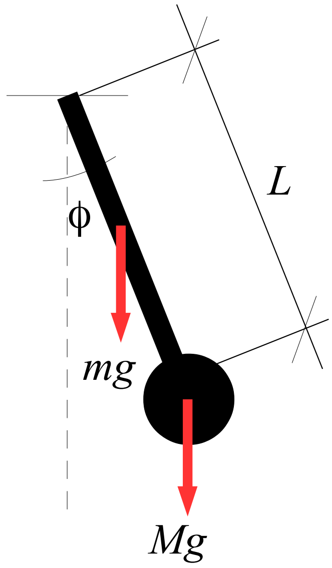
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}}\right) - \frac{\partial \mathscr{L}}{\partial q} = 0$$
$$13\, mL^2 \dot{q} + 7\, mgL\cos q + 9\, kL^2 \cos q \sin q$$

Finally:

$$13\, mL^2 \ddot{q} + L\cos q\left[7\, mg + 9\, kL\sin q\right] = 0$$

ZADANIE 6

Making use of hte Lagrange equations of the 2nd kind find the equation of motion of a pendulum consisting of a rigid bar of length *L* of finite mass *m* and finite moment of inertia. At the end of the bar there is a point mass *M* attached. The motion is due to gravity.



SOLUTION:

The system has only a single degree of freedom.
An angle of inclination of pendulum is considered a generalized coordinate

Position vectors of point to which forces are applied. In case of a bar it is its centroid.

$$\mathbf{r}_1 = \frac{L}{2}[\sin q \; ; \; 1-\cos q]$$
$$\mathbf{r}_2 = L[\sin q \; ; \; 1-\cos q]$$

As the motion is due to conservative forces it is not necessary to calculate derivatives of position vectors with respect to generalized coordinate. It is necessary, however, to find the kinetic energy. Kinetic energy of a point mass is equal kinetic energy of its translational motion – its velocity is thus needed:

$$\dot{\mathbf{r}}_2 = L\dot{q}[\cos q \; ; \; \sin q]$$

The bar performs only rotation about an axis containing one of its end. Its kinetic energy is the kinetic energy of rotational motion $\frac{1}{2}I\omega^2$ – we need to know the moment of inertia I and angular velocity ω . Since the angular displacement is our generalized coordinate we may write $\omega = \dot{q}$. Moment of inertia of a bar rotating about one of its ends is equal:

$$I = \frac{mL^2}{3}$$

Kinetic energy:

$$E_k = E_{k1}+E_{k2} = \frac{I\omega^2}{2} + \frac{M(\dot{\mathbf{r}}_2)^2}{2} = \frac{mL^2}{6}(\dot{q})^2 + \frac{ML^2(\dot{q})^2}{2}(\cos^2 q+\sin^2 q) = \frac{L^2(\dot{q})^2}{6}(m+3M)$$

Kinetic energy of a rotating bar may be calculated in such a way only when the center of rotation is constant in time. If it changes over time, the kinetic energy is calculated as a sum of energy of translational motion of its centroid and energy of rotational motion obut centroid, according to Koenigs theorem. Since the angular velocity is the same in both cases, independently of the choice of center of rotation, in such a situation we would have:

$$E_{k1} = \frac{m(\dot{\mathbf{r}}_1)^2}{2} + \frac{I_s\omega^2}{2} = \frac{1}{2}\frac{mL^2}{4}(\dot{q})^2(\sin^2 q+\cos^2 q) + \frac{1}{2}\frac{mL^2}{12}(\dot{q})^2 = \frac{mL^2}{6}(\dot{q})^2$$

It can be seen that both approaches are correct.

Potential energy will be a sum of potential energy of bar and of a point mass.

$$E_p = mgy_1 + Mgy_2 = \frac{1}{2} mgL(1-\cos q) + MgL(1-\cos q)$$

Lagrangian: $$\mathscr{L} = E_k - E_p = \frac{1}{6} L^2 (\dot{q})^2 (m+3M) - \frac{1}{2} mgL(1-\cos q) - MgL(1-\cos q)$$

Lagrange eqautions of the 2nd kind:

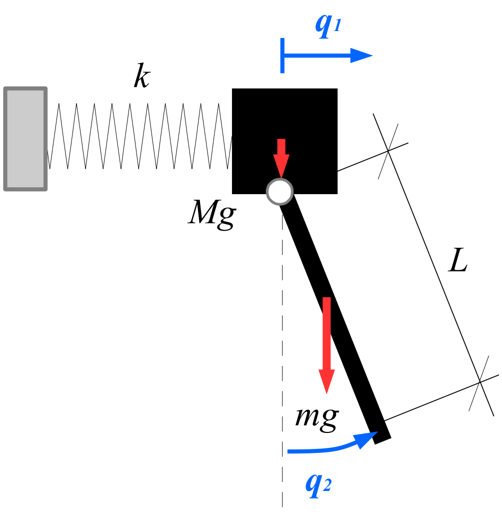
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}}\right) - \frac{\partial \mathscr{L}}{\partial q} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{3} L^2 \dot{q}(m+3M)\right) + \frac{1}{2} mgL \sin q + MgL \sin q = 0$$

Finally:

$$2L^2 \ddot{q}(m+3M) + (3m+6M) gL \sin q = 0$$

EXERCISE 7



Find the equations of motion for a system shown in the picture with the use of Lagrange equations of the 2nd kind and Hamilton equations.

SOLUTION:

1. LAGRANGE EQUATIONS OF THE 2nd KIND

Position vectors of point in which a force is applied:

$$\mathbf{r}_1 = [q_1 \ ; \ L] \qquad \mathbf{r}_2 = \left[q_1 + \frac{L}{2}\sin q_2 \ ; \ L - \frac{L}{2}\cos q_2\right]$$

Velocity vectors:

$$\dot{\mathbf{r}}_1 = [\dot{q}_1 \ ; \ 0] \qquad \dot{\mathbf{r}}_2 = \left[\dot{q}_1 + \frac{L\dot{q}_2}{2}\cos q_2 \ ; \ \frac{L\dot{q}_2}{2}\sin q_2\right]$$

Kinetic energy of a point mass wil be the energy of its translational motion:

$$E_{kI} = \frac{M(\dot{\mathbf{r}}_1)^2}{2} = \frac{M}{2}(\dot{q}_1)^2$$

Kinetic energy of a bar will be the sum of the kinetic energy of its translational and rotational motion. Translational motion of centroid is due to both motion of hinge and rotation about it. In case of rotational motion we have to account for moment of inertia respective for rotation about an axis passing through the centroid. The angle of rotation about centroid is the same as the chosen generalized coordinate q_2 , so the angular velocity $\omega = \dot{q}_2$:

$$E_{k2} = \frac{m(\dot{\mathbf{r}}_2)^2}{2} + \frac{I\omega^2}{2} = \frac{m}{2}\left[(\dot{q}_1)^2 + L\dot{q}_1\dot{q}_2\cos q_2 + \frac{L^2}{4}(\dot{q}_2)^2\right] + \frac{1}{2}\frac{mL^2}{12}(\dot{q}_2)^2 =$$
$$= \frac{m}{2}(\dot{q}_1)^2 + \frac{mL}{2}\dot{q}_1\dot{q}_2\cos q_2 + \frac{mL^2}{6}(\dot{q}_2)^2$$

Kinetic energy:

$$E_k = E_{kI} + E_{k2} = \frac{(m+M)}{2}(\dot{q}_1)^2 + \frac{mL}{2}\dot{q}_1\dot{q}_2\cos q_2 + \frac{mL^2}{6}(\dot{q}_2)^2$$

The motion is due to **conservative forces**– total **potential energy**:

$$E_p = -V_1 - V_2 - V_3 = \frac{kx_1^2}{2} + Mg\,y_1 + mgy_2 = \frac{k\,q_1^2}{2} + MgL + mg\left(L - \frac{L}{2}\cos q_2\right)$$

It is worth noting that assuming that gravitational potential has zero value in the most bottom position of the hinged bar is only a matter of agreement. Such a potential will give us always a positive potential energy if it was determined as $E_p = -V$. We know, however, that a conservative force field may be described with the use of an infinite number of potentials differing between each other only with a constant value. We could choose a much simpler description assuming that zero potential energy is at the level of hinge:

$$E_p = \frac{k\,q_1^2}{2} - mg\frac{L}{2}\cos q_2$$

We would – formally – deal with negative energy, what has no physical sense, however it would not have any significance from the point of view of determining the equations of motion because it is the variation of the energy (its dependence on generalized coordinates), not the value its self, what is important here – this additional term providing positive value of energy is anyway canceled by differentiation of the lagrangian with respect to generalized coordinates.

Lagrangian:

$$\mathscr{L} = E_k - E_p = \frac{(m+M)}{2}(\dot{q}_1)^2 + \frac{mL}{2}\dot{q}_1\dot{q}_2\cos q_2 + \frac{mL^2}{6}(\dot{q}_2)^2 - \frac{k\,q_1^2}{2} - MgL - mg\left(L - \frac{L}{2}\cos q_2\right)$$

Lagrange equations of the 2nd kind:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial \mathscr{L}}{\partial \dot{q}_1}\right) - \dfrac{\partial \mathscr{L}}{\partial q_1} = 0 \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial \mathscr{L}}{\partial \dot{q}_2}\right) - \dfrac{\partial \mathscr{L}}{\partial q_2} = 0 \end{cases}$$

After substitution:

$$\begin{cases} (m+M)\ddot{q}_1 + \dfrac{mL}{2}\left(\ddot{q}_2\cos q_2 - (\dot{q}_2)^2\sin q_2\right) + k\,q_1 = 0 \\ \dfrac{mL^2}{3}\ddot{q}_2 + \dfrac{mL}{2}(\ddot{q}_1\cos q_2 - \dot{q}_1\dot{q}_2\sin q_2) + \left(\dfrac{mL}{2}\dot{q}_1\dot{q}_2 - \dfrac{mgL}{2}\right)\sin q_2 = 0 \end{cases}$$

Finally:

$$\boxed{\begin{cases} (m+M)\ddot{q}_1 + \dfrac{mL}{2}\left(\ddot{q}_2\cos q_2 - (\dot{q}_2)^2\sin q_2\right) + k\,q_1 = 0 \\ \dfrac{mL^2}{3}\ddot{q}_2 + \dfrac{mL\,\ddot{q}_1}{2}\cos q_2 + \dfrac{mgL}{2}\sin q_2 = 0 \end{cases}}$$

2. HAMILTON EQUATIONS

Lagrangian is as above:

$$\mathscr{L} = E_k - E_p = \frac{(m+M)}{2}(\dot{q}_1)^2 + \frac{mL}{2}\dot{q}_1\dot{q}_2\cos q_2 + \frac{mL^2}{6}(\dot{q}_2)^2 - \frac{k q_1^2}{2} - MgL - mg\left(L - \frac{L}{2}\cos q_2\right)$$

Generalized momenta:

$$p_1 = \frac{\partial \mathscr{L}}{\partial \dot{q}_1} = (m+M)\dot{q}_1 + \frac{mL}{2}\dot{q}_2\cos q_2$$

$$p_2 = \frac{\partial \mathscr{L}}{\partial \dot{q}_2} = \frac{mL^2}{3}\dot{q}_2 + \frac{mL}{2}\dot{q}_1\cos q_2$$

We invert the above relations:

$$\dot{q}_1 = \frac{4Lp_1 - 6p_2\cos q_2}{L\left[4(M+m) - 3m\cos^2 q_2\right]}$$

$$\dot{q}_2 = \frac{12(m+M)p_2 - 6mLp_1\cos q_2}{mL^2\left[4(M+m) - 3m\cos^2 q_2\right]}$$

Hamiltonian:

$$H(\mathbf{q},\mathbf{p}) = \sum_i^s p_j \cdot \dot{q}_j(\mathbf{p}) - \mathscr{L}(\mathbf{q},\dot{\mathbf{q}}(\mathbf{p})) =$$

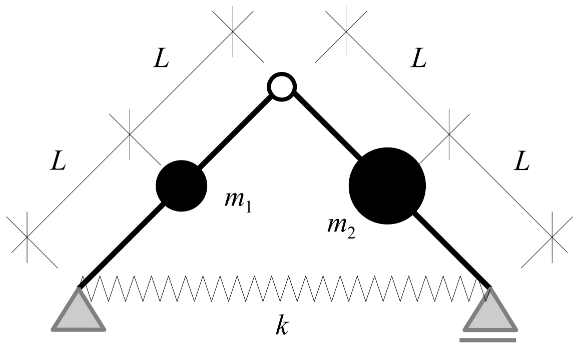
$$= \frac{2\left[m p_1^2 L^2 + 3(m+M)p_2^2 - 3m p_1 p_2 \cos q_2\right]}{mL^2\left[4(m+M) - 3m\cos^2 q_2\right]} + \frac{k q_1^2}{2} + mg\left(L - \frac{L}{2}\cos q_2\right)$$

Hamilton equations:

$$\begin{cases} \dot{q}_1 = \dfrac{\partial H}{\partial p_1} = \dfrac{4Lp_1 - 6p_2\cos q_2}{L\left[4(M+m) - 3m\cos^2 q_2\right]} \\ \dot{q}_2 = \dfrac{\partial H}{\partial p_2} = \dfrac{12(m+M)p_2 - 6mLp_1\cos q_2}{mL^2\left[4(M+m) - 3m\cos^2 q_2\right]} \\ \dot{p}_1 = -\dfrac{\partial H}{\partial q_1} = -kq_1 \\ \dot{p}_2 = -\dfrac{\partial H}{\partial q_2} = \dfrac{6\sin q_2\left[2Lp_1 - 3p_2\cos q_2\right]\left[mLp_1\cos q_2 - 2(m+M)p_2\right]}{L^2\left[4(m+M) - 3m\cos^2 q_2\right]^2} - \dfrac{mgL}{2}\sin q_2 \end{cases}$$

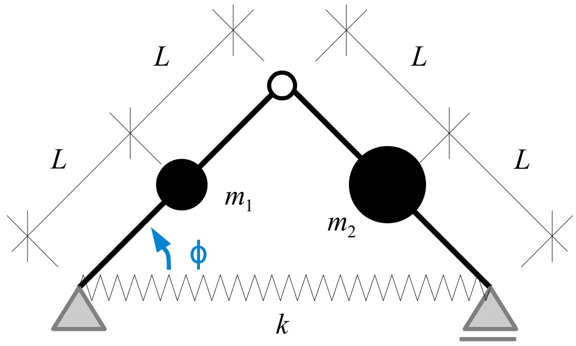
EXERCISE 8

Find eqautions of motion for a system in the picture with the use of Lagrange equations of the 2nd kind. The motion is due to gravity. The spring is in its natural state when both bars are inclined at angle 45°.



SOLUTION:

The system has only a single degree of freedom. An angle of inclination of bars with respect to horizon may be considered a generalized coordinate. We have to notice that both bars create an isosceles triangle.



Position vectors of points of application of forces:

$$\mathbf{r_1} = [L\cos\phi\ ;\ L\sin\phi]$$
$$\mathbf{r_2} = [3L\cos\phi\ ;\ L\sin\phi]$$
$$\mathbf{r_3} = [4L\cos\phi\ ;\ 0]$$

Velocities of masses:

$$\dot{\mathbf{r}}_1 = [-L\dot\phi\sin\phi\ ;\ L\dot\phi\cos\phi]$$
$$\dot{\mathbf{r}}_2 = [-3L\dot\phi\sin\phi\ ;\ L\dot\phi\cos\phi]$$

Kinetic energy:

$$E_k = \frac{m_1(\dot{\mathbf{r}}_1)^2}{2} + \frac{m_2(\dot{\mathbf{r}}_2)^2}{2} = \frac{L^2(\dot\phi)^2}{2}\left[m_1 + m_2(1+8\sin^2\phi)\right]$$

The motion is due to conservative forces. Total **potential energy** of the system:

$$E_p = m_1 g y_1 + m_2 g y_2 + \frac{k}{2}\left(x_3 - \frac{4L}{\sqrt{2}}\right)^2 = (m_1+m_2)gL\sin\phi + \frac{kL^2}{2}(4\cos\phi - 2\sqrt{2})^2$$

Lagrangian:

$$\mathscr{L} = E_k - E_p = \frac{L^2(\dot\phi)^2}{2}\left[m_1 + m_2(1+8\sin^2\phi)\right] - (m_1+m_2)gL\sin\phi - \frac{kL^2}{2}(4\cos\phi - 2\sqrt{2})^2$$

Lagrange equation of the 2nd kind:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathscr{L}}{\partial\dot\phi}\right) - \frac{\partial\mathscr{L}}{\partial\phi} = 0$$

Finally:

$$\boxed{L^2\ddot\phi\left[m_1+m_2(1+8\sin^2\phi)\right] + 8L^2(\dot\phi)^2 m_2\sin\phi\cos\phi + (m_1+m_2)gL\cos\phi - 4kL^2(4\cos\phi - 2\sqrt{2})\sin\phi = 0}$$

EXERCISE 9

Find equations of motions of a point mass *m* moving one a paraboloid of revolution:

$$f: \; 8x^2+8y^2-2z+6=0 \quad ,$$

due to gravitu. Make use of the Lagrange equations of the 2nd kind.

SOLUTION:

A point moving on a surface in three-dimensional space has 2 degrees of freedom. Let's assume that generalized coordinates are the coordinates of Cartesian coordinate system:

$$q_1 = x \qquad q_2 = y$$

Position vector: $\mathbf{r} = [x, y, z] = [q_1 \; ; \; q_2 \; ; \; 4q_1^2+4q_2^2+3]$

Velocity vector: $\dot{\mathbf{r}} = [\dot{q}_1 \; ; \; \dot{q}_2 \; ; \; 8q_1\dot{q}_1+8q_2\dot{q}_2]$

Kinetic energy: $E_k = \dfrac{m(\dot{\mathbf{r}})^2}{2} = \dfrac{m}{2}\left((\dot{q}_1)^2+(\dot{q}_2)^2 + 64q_1^2(\dot{q}_1)^2+64q_2^2(\dot{q}_2)^2+128q_1q_2\dot{q}_1\dot{q}_2\right)$

Potential energy: $E_p = mgz = mg(4q_1^2+4q_2^2+3)$

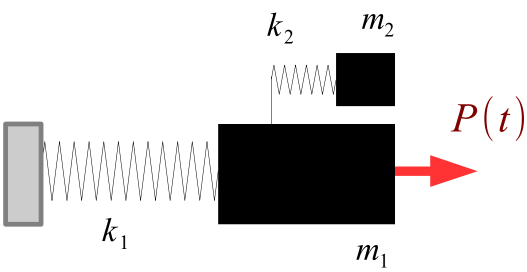
Lagrangian: $\mathscr{L} = E_k-E_p = \dfrac{m}{2}\left[(1+64q_1^2)(\dot{q}_1)^2+(1+64q_2^2)(\dot{q}_2)^2+128q_1q_2\dot{q}_1\dot{q}_2\right]-mg(4q_1^2+4q_2^2+3)$

Lagrange equations of the 2nd kind:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial\mathscr{L}}{\partial\dot{q}_1}\right)-\dfrac{\partial\mathscr{L}}{\partial q_1}=0 \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial\mathscr{L}}{\partial\dot{q}_2}\right)-\dfrac{\partial\mathscr{L}}{\partial q_2}=0 \end{cases}$$

$$\boxed{\begin{cases} m\left[64q_1(\dot{q}_1)^2+(1+64q_1^2)\ddot{q}_2+64q_1\left((\dot{q}_2)^2+q_2\ddot{q}_2\right)\right]+8mgq_1=0 \\ m\left[64q_2(\dot{q}_2)^2+(1+64q_2^2)\ddot{q}_1+64q_2\left((\dot{q}_1)^2+q_1\ddot{q}_1\right)\right]+8mgq_2=0 \end{cases}}$$

EXERCISE 10



a) Find equations of motion of a mass on elastic spring driven with external forceand equipped with a *tuned mass damper* (TMD) – make use of the Lagrange equations of the 2nd kind.

b) Assuming that the driving force is harmonic find such parameters of the damper that the vibration of mass m_1 reduced in greatest extent.

SOLUTION:

The system has 2 degrees of freedom. Let the generalized coordinates be: displacement of mass m_1 and relative displacement of mass m_2 with respect to mass m_1 . The problem is purely one-dimensional so instead of vector quantities it is enough to perform calculations of the only non-zero of their coordinates.

Position:

$$x_1 = q_1$$
$$x_2 = q_1 + q_2$$

Velocity:

$$\dot{x}_1 = \dot{q}_1$$
$$\dot{x}_2 = \dot{q}_1 + \dot{q}_2$$

Active forces:

$$F_1 = -k_1 q_1 + k_2 q_2 + P(t)$$
$$F_2 = -k_2 q_2$$

Kinetic energy:

$$E_k = \frac{1}{2} m_1 (\dot{x}_1)^2 + \frac{1}{2} m_2 (\dot{x}_2)^2 = \frac{1}{2}\Big[m_1(\dot{q}_1)^2 + m_2\big((\dot{q}_1)^2 + (\dot{q}_2)^2 + 2\dot{q}_1\dot{q}_2\big)\Big]$$

The presence of a driving non-conservative force requires calculation of generalized forces. Derivatives of position vectors with respect to generalized coordinated are required:

$$\frac{\partial x_1}{\partial q_1} = 1 \qquad \frac{\partial x_1}{\partial q_2} = 0$$
$$\frac{\partial x_2}{\partial q_1} = 1 \qquad \frac{\partial x_2}{\partial q_2} = 1$$

Generalized forces:

$$Q_1 = F_1 \cdot \frac{\partial x_1}{\partial q_1} + F_2 \cdot \frac{\partial x_2}{\partial q_1} = -k_1 q_1 + P(t)$$
$$Q_2 = F_1 \cdot \frac{\partial x_1}{\partial q_2} + F_2 \cdot \frac{\partial x_2}{\partial q_2} = -k_2 q_2$$

Lagrange equations of the 2nd kind:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial E_k}{\partial \dot{q}_1}\right) - \dfrac{\partial E_k}{\partial q_1} = Q_1 \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial E_k}{\partial \dot{q}_2}\right) - \dfrac{\partial E_k}{\partial q_2} = Q_2 \end{cases}$$

$$\boxed{\begin{cases} m_1 \ddot{q}_1 + m_2(\ddot{q}_1 + \ddot{q}_2) + k_1 q_1 = P(t) \\ m_2(\ddot{q}_1 + \ddot{q}_2) + k_2 q_2 = 0 \end{cases}}$$

The procedure of finding optimal parameters of the mass damper is called „tuning the damper”. We will solve this problem by solving the above equations of motion is case of **steady vibration driven by harmonic force**. Let's assume:

$$P(t) = P_0 \sin(\omega t)$$

The solution of the problem stated will be a particular solution which can be found by predicting its form according to the form on the inhomogeneous part of the euations. Let's assume:

$$q_1 = A\cos(\omega t) + B\sin(\omega t)$$
$$q_2 = C\cos(\omega t) + D\sin(\omega t)$$

Generalized velocities and accelerations:

$$\dot{q}_1 = -A\omega\sin(\omega t) + B\omega\cos(\omega t) \qquad \ddot{q}_1 = -A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t)$$
$$\dot{q}_2 = -C\omega\sin(\omega t) + D\omega\cos(\omega t) \qquad \ddot{q}_2 = -C\omega^2\cos(\omega t) - D\omega^2\sin(\omega t)$$

Let's substitute those results to the equations of motion:

$$\begin{cases} -\left[\left((m_1+m_2)-k_1\right)A + m_2\omega^2 C\right]\cos(\omega t) - \left[\left((m_1+m_2)-k_1\right)B + m_2\omega^2 D\right]\sin(\omega t) = P_0\sin(\omega t) \\ -\left[m_2\omega^2 A + (m_2\omega^2 - k_2)C\right]\cos(\omega t) - \left[m_2\omega^2 B + (m_2\omega^2 - k_2)D\right]\sin(\omega t) = 0 \end{cases}$$

Coefficients *A, B, C, D* may be found by imposind a requirement that left and right side of each equation are equal in every moment *t*. It will be so if and only if the coefficients by sine and cosine functions at both sides are the same – in particular, if there is no such function on one side, then the coefficient is equal 0. It gives us a system of equations:

$$\begin{cases} -\left((m_1+m_2)-k_1\right)A - m_2\omega^2 C = 0 \\ -\left((m_1+m_2)-k_1\right)B - m_2\omega^2 D = P_0 \\ -m_2\omega^2 A + (m_2\omega^2 - k_2)C = 0 \\ -m_2\omega^2 B + (m_2\omega^2 - k_2)D = 0 \end{cases} \Rightarrow \begin{cases} A = 0 \\ B = \dfrac{(k_2 - m_2\omega^2)P_0}{m_1 m_2\omega^4 - \left((k_1+k_2)m_2 + k_2 m_1\right)\omega^2 + k_1 k_2} \\ C = 0 \\ D = \dfrac{m_2\omega^2 P_0}{m_1 m_2\omega^4 - \left((k_1+k_2)m_2 + k_2 m_1\right)\omega^2 + k_1 k_2} \end{cases}$$

It can be notices that amplitude of vibration of mass m_1 will be equal *B*, while amplitude of vibration of mass m_2 will be equal *D*.We may now choose the parameters of the damper in such a way, that the amplitude of vibration of m_1 was 0 – it is sufficient to take:

$$k_2 = m_2\omega^2$$

Usually the damper is tuned in such a way, that the resonance vibrations is damped as it is the dominant component of most of vibration. It is assumed then that angular frequency of driving force is the same as the natural angular frequency of mass m_1 without damper:

$$\omega = \omega_0 = \sqrt{\frac{k_1}{m_1}}$$

Hence:

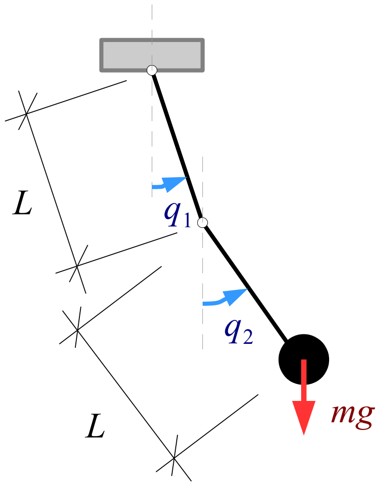
$$\frac{k_2}{m_2}=\frac{k_1}{m_1} \quad \Rightarrow \quad \omega_0^{(1)}=\omega_0^{(2)}$$

Damping will be most efficient when the natural angular frequency of the damper itself (relative to mass m_1) will be the same as natural angular frequency of mass m_1 without damper. The same approach – prediction of particular solution to the problem of harmonically driven vibration – shows that such a choice of damper's parameters minimizes vibration also in case when damping is accounted for. The solution is independent on the values of damping coefficients.

It must be noted, however, that the above solution concerns only the steady-state vibration – namely, when a sufficient time has elapsed and only under condition that the driving force acts harmonically all the time and its character (amplitude, frequency) do not change over time. It is obvious that oscillating loads (wind, tides, kinematic excitation due to earthquake, explosion etc.) in reality change significantly over time and they act for such a short period of time that it is never possible to deal with steady-state vibration. For these reasons the above estimations of parameters of the optimal tuned mass damper is only an approximation.

EXERCISE 11

Find equations of motion for a double pendulum with the use of Lagrange equations of the 2nd kind and Hamilton equations.



SOLUTION:

1. LAGRANGE EQUATIONS OF THE 2nd KIND

The system has two degrees of freedom. Following **generalized coordinates** are chosen:

1. Angle of declination of the first pendulum from vertical direction q_1
2. Angle of declination of the second pendulum from vertical direction q_2

Location of mass: $\mathbf{r} = [L\sin q_1 + L\sin q_2 \; ; \; 2L - L\cos q_1 - L\cos q_2]$

Velocity: $\dot{\mathbf{r}} = L[\dot{q}_1 \cos q_1 + \dot{q}_2 \cos q_2 \; ; \; \dot{q}_1 \sin q_1 + \dot{q}_2 \sin q_2]$

Kinetic energy:

$$E_k = \frac{m(\dot{\mathbf{r}})^2}{2} = \frac{mL^2}{2}\left[(\dot{q}_1)^2 + (\dot{q}_2)^2 + 2\dot{q}_1\dot{q}_2(\cos q_1 \cos q_2 + \sin q_1 \sin q_2)\right] =$$
$$= \frac{mL^2}{2}\left[(\dot{q}_1)^2 + (\dot{q}_2)^2 + 2\dot{q}_1\dot{q}_2 \cos(q_2 - q_1)\right]$$

Potential energy: $E_p = mgy = mgL(2 - \cos q_1 - \cos q_2)$

Lagrangian: $\mathscr{L} = E_k - E_p = \frac{mL^2}{2}\left[(\dot{q}_1)^2 + (\dot{q}_2)^2 + 2\dot{q}_1\dot{q}_2\cos(q_2 - q_1)\right] + mgL(\cos q_1 + \cos q_2 - 2)$

Lagrange equations of the 2nd kind:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_1}\right) - \frac{\partial \mathscr{L}}{\partial q_1} = 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_2}\right) - \frac{\partial \mathscr{L}}{\partial q_2} = 0 \end{cases}$$

$$\begin{cases} \ddot{q}_1 + \ddot{q}_2 \cos(q_2 - q_1) - (\dot{q}_2)^2 \sin(q_2 - q_1) + \frac{g}{L}\sin q_1 = 0 \\ \ddot{q}_2 + \ddot{q}_1 \cos(q_2 - q_1) + (\dot{q}_1)^2 \sin(q_2 - q_1) + \frac{g}{L}\sin q_2 = 0 \end{cases}$$

2. HAMILTONA EQUATIONS

The system has two degrees of freedom. Following **generalized coordinates** are chosen:

1. Angle of declination of the first pendulum from vertical direction q_1
2. Angle of declination of the second pendulum from vertical direction q_2

Location of mass: $\mathbf{r} = [L\sin q_1 + L\sin q_2 \; ; \; 2L - L\cos q_1 - L\cos q_2]$

Velocity: $\dot{\mathbf{r}} = L[\dot{q}_1\cos q_1 + \dot{q}_2\cos q_2 \; ; \; \dot{q}_1\sin q_1 + \dot{q}_2\sin q_2]$

Kinetic energy:

$$E_k = \frac{m(\dot{\mathbf{r}})^2}{2} = \frac{mL^2}{2}\left[(\dot{q}_1)^2 + (\dot{q}_2)^2 + 2\dot{q}_1\dot{q}_2(\cos q_1\cos q_2 + \sin q_1\sin q_2)\right] =$$
$$= \frac{mL^2}{2}\left[(\dot{q}_1)^2 + (\dot{q}_2)^2 + 2\dot{q}_1\dot{q}_2\cos(q_2 - q_1)\right]$$

Potential energy: $E_p = mgy = mgL(2 - \cos q_1 - \cos q_2)$

Lagrangian: $\mathscr{L} = E_k - E_p = \frac{mL^2}{2}\left[(\dot{q}_1)^2 + (\dot{q}_2)^2 + 2\dot{q}_1\dot{q}_2\cos(q_2 - q_1)\right] + mgL(\cos q_1 + \cos q_2 - 2)$

Generalized momenta:

$$p_1 = \frac{\partial \mathscr{L}}{\partial \dot{q}_1} = mL^2[\dot{q}_1 + \dot{q}_2\cos(q_2 - q_1)]$$
$$p_2 = \frac{\partial \mathscr{L}}{\partial \dot{q}_2} = mL^2[\dot{q}_2 + \dot{q}_1\cos(q_2 - q_1)]$$

Inverse of the above relations: $\dot{q}_1 = \frac{p_1 - \cos(q_2 - q_1)p_2}{\sin^2(q_2 - q_1)mL^2}$ $\quad \dot{q}_2 = \frac{p_2 - \cos(q_2 - q_1)p_1}{\sin^2(q_2 - q_1)mL^2}$

Hamiltonian:

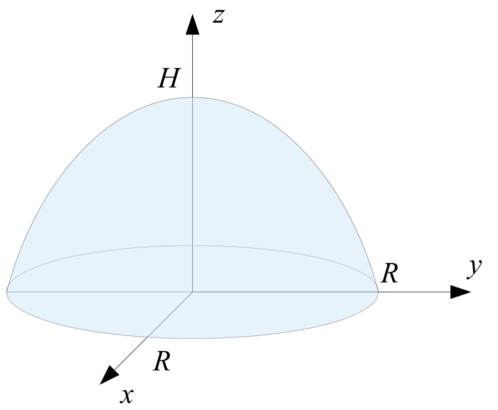
$$H(\mathbf{q},\mathbf{p}) = \sum_i^s p_j\cdot\dot{q}_j(\mathbf{p}) - L(\mathbf{q},\dot{\mathbf{q}}(\mathbf{p})) = \frac{p_1^2 + p_2^2 - 2p_1p_2\cos(q_2 - q_1)}{2\sin^2(q_2 - q_1)mL^2} - mgL(\cos q_1 + \cos q_2 - 2)$$

Hamilton equations:

$$\begin{cases} \dot{q}_1 = \dfrac{\partial H}{\partial p_1} \\ \dot{q}_2 = \dfrac{\partial H}{\partial p_1} \\ \dot{p}_1 = -\dfrac{\partial H}{\partial q_1} \\ \dot{p}_2 = -\dfrac{\partial H}{\partial q_2} \end{cases} \Rightarrow \begin{cases} \dot{q}_1 = \dfrac{p_1 - \cos(q_2 - q_1)p_2}{\sin^2(q_2 - q_1)mL^2} \\ \dot{q}_2 = \dfrac{p_2 - \cos(q_2 - q_1)p_1}{\sin^2(q_2 - q_1)mL^2} \\ \dot{p}_1 = \dfrac{p_1p_2(1 + \cos^2(q_2 - q_1)) - (p_1^2 + p_2^2)\cos(q_2 - q_1)}{mL^2\sin^3(q_2 - q_1)} - mgL\sin(q_1) \\ \dot{p}_2 = -\dfrac{p_1p_2(1 + \cos^2(q_2 - q_1)) - (p_1^2 + p_2^2)\cos(q_2 - q_1)}{mL^2\sin^3(q_2 - q_1)} - mgL\sin(q_2) \end{cases}$$

EXERCISE 12

Find equations of motion of a point mass *m* moving on a paraboloid of revolution due to gravity.



SOLUTION:

LAGRANGE EQUATIONS OF THE 2nd KIND

Let's determine the equation of the surface. It is symmetric with respect to planes XZ and YZ, so it will be given by an equation of general form:

$$z = a\cdot(x^2+y^2)+b$$

To of paraboloid: $H = a\cdot(0+0)+b = b \quad \Rightarrow \quad b = H$

Base of paraboloid: $0 = a\cdot(R^2+0)+H = aR^2+H \quad \Rightarrow \quad a = -\dfrac{H}{R^2}$ (the same for *x* = 0, *y* = *R*)

Finally:

$$z = H\left[1-\frac{x^2+y^2}{R^2}\right]$$

A point moving on a surface in a three-dimensional space has 2 degrees of freedom. Since the paraboloid is axis-symmetric, it will convenient to choose the cylindrical coordinates as generalized coordinates:

$$q_1 = r = \sqrt{x^2+y^2} \qquad q_2 = \phi = \operatorname{atan}\frac{y}{x} \qquad \Rightarrow \begin{cases} x = r\cos\phi \\ y = r\sin\phi \\ z = z \end{cases} .$$

For point on paraboloid:

$$\begin{cases} x = r\cos\phi \\ y = r\sin\phi \\ z = H\left(1-\dfrac{r^2}{R^2}\right) \end{cases}$$

Location vector: $$\mathbf{r} = [x, y, z] = \left[q_1\cdot\cos q_2 ;\ q_1\cdot\sin q_2 ;\ H\left(1-\frac{q_1^2}{R^2}\right)\right]$$

Velocity vector: $$\dot{\mathbf{r}} = \left[\dot{q}_1\cos q_2 - q_1\dot{q}_2\sin q_2 ;\ \dot{q}_1\sin q_2 + q_1\dot{q}_2\cos q_2 ;\ -\frac{2H}{R^2}q_1\dot{q}_1\right]$$

Kinetic energy: $$E_k = \frac{m(\dot{\mathbf{r}})^2}{2} = \frac{m}{2}\left[(\dot{q}_1)^2 + q_1^2(\dot{q}_2)^2 + \frac{4H^2}{R^4}q_1^2(\dot{q}_1)^2\right]$$

Potential energy: $$E_p = mgz = mgH\left(1-\frac{q_1^2}{R^2}\right)$$

Lagrangian: $$\mathscr{L} = E_k - E_p = \frac{m}{2}\left[(\dot{q}_1)^2 + q_1^2(\dot{q}_2)^2 + \frac{4H^2}{R^4}q_1^2(\dot{q}_1)^2\right] + mgH\left(\frac{q_1^2}{R^2}-1\right)$$

Lagrange equations of the 2nd kind:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial\mathscr{L}}{\partial\dot{q}_1}\right) - \dfrac{\partial\mathscr{L}}{\partial q_1} = 0 \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial\mathscr{L}}{\partial\dot{q}_2}\right) - \dfrac{\partial\mathscr{L}}{\partial q_2} = 0 \end{cases}$$

$$\frac{\partial\mathscr{L}}{\partial q_1} = m q_1\left[\frac{4H^2}{R^4}(\dot{q}_1)^2 + (\dot{q}_2)^2 + \frac{2gH}{R^2}\right] \qquad \frac{\partial\mathscr{L}}{\partial q_2} = 0$$

$$\frac{\partial\mathscr{L}}{\partial\dot{q}_1} = m\dot{q}_1\left[1+\frac{4H^2}{R^4}q_1^2\right] \qquad \frac{\partial\mathscr{L}}{\partial\dot{q}_2} = m q_1^2\dot{q}_2$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathscr{L}}{\partial\dot{q}_1}\right) = m\ddot{q}_1\left[1+\frac{4H^2}{R^4}q_1^2\right] + \frac{8mH^2}{R^4}q_1(\dot{q}_1)^2 \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathscr{L}}{\partial\dot{q}_2}\right) = m\left[2q_1\dot{q}_1\dot{q}_2 + q_1^2\ddot{q}_2\right]$$

Finaly:

$$\begin{cases} \ddot{q}_1\left[1+\dfrac{4H^2}{R^4}q_1^2\right] + q_1\left[\dfrac{4H^2}{R^4}(\dot{q}_1)^2 - (\dot{q}_2)^2 - \dfrac{2gH}{R^2}\right] = 0 \\ 2q_1\dot{q}_1\dot{q}_2 + q_1^2\ddot{q}_2 = 0 \end{cases}$$

D'ALEMBERT PRICIPLE

The system has only 2 degrees of freedom. Among 3 variables x, y, z describing position of a point, we will choose x, y as the independent ones. The third coordinate may be found with the use of the constraint equation (equation of a surface on which the point may only move), as previously:

$$f: \quad z - H\left[1-\frac{x^2+y^2}{R^2}\right] = 0$$

Position vector $\qquad \mathbf{r} = [x; y; z] = \left[x; y; H\left(1-\frac{x^2+y^2}{R^2}\right)\right]$

Active and inertial forced:

- **active forced:** $\qquad \mathbf{F} = [0; 0; -mg]$
- **inertial forces:** $\quad \mathbf{B} = -m\ddot{\mathbf{r}} = \left[-m\ddot{x}; -m\ddot{y}; \frac{2Hm}{R^2}\left[(\dot{x})^2+x\ddot{x}+(\dot{y})^2+y\ddot{y}\right]\right]$

Virtual displacement vector has a general form: $\qquad \delta\mathbf{r} = [\delta x ; \delta y ; \delta z]$

It is any vector which is tangent to the surface, this is, which is perpendicular to the gradient of the surface:

$$\mathbf{n} = \text{grad}\, f = \left[\frac{\partial f}{\partial x} ; \frac{\partial f}{\partial y} ; \frac{\partial f}{\partial z}\right] = \left[\frac{2H}{R^2}x ; \frac{2H}{R^2}y ; 1\right]$$

Orthogonality condition:

$$\delta\mathbf{r}\circ\mathbf{n} = \frac{2H}{R^2}x\cdot\delta x + \frac{2H}{R^2}y\cdot\delta y + 1\cdot\delta z = 0 \qquad \Rightarrow \qquad \delta z = -\frac{2H}{R^2}(x\cdot\delta x + y\cdot\delta y)$$

$$\delta\mathbf{r} = \left[\delta x ; \delta y ; -\frac{2H}{R^2}(x\cdot\delta x + y\cdot\delta y)\right]$$

Virtual work:

$$\delta L = (\mathbf{F}+\mathbf{B})\circ\delta\mathbf{r} = \left[-m\ddot{x}; -m\ddot{y}; \frac{2Hm}{R^2}\left[(\dot{x})^2+x\ddot{x}+(\dot{y})^2+y\ddot{y}\right]-mg\right]\circ\left[\delta x ; \delta y ; -\frac{2H}{R^2}(x\cdot\delta x + y\cdot\delta y)\right] =$$

$$= \left[-m\ddot{x} - \frac{4H^2m}{R^4}x\left[(\dot{x})^2+x\ddot{x}+(\dot{y})^2+y\ddot{y}\right]+\frac{2mgH}{R^2}x\right]\cdot\delta x +$$

$$+ \left[-m\ddot{y} - \frac{4H^2m}{R4}y\left[(\dot{x})^2+x\ddot{x}+(\dot{y})^2+y\ddot{y}\right]+\frac{2mgH}{R^2}y\right]\cdot\delta y$$

d'Alembert principle:

$$\delta L = 0 \quad \forall\, \delta x, \delta y \ ,$$

hence the **equations of motion** are:

$$\begin{cases} -m\ddot{x} - \frac{4H^2m}{R^4}x\left[(\dot{x})^2+x\ddot{x}+(\dot{y})^2+y\ddot{y}\right]+\frac{2mgH}{R^2}x = 0 \\ -m\ddot{y} - \frac{4H^2m}{R^4}y\left[(\dot{x})^2+x\ddot{x}+(\dot{y})^2+y\ddot{y}\right]+\frac{2mgH}{R^2}y = 0 \end{cases}$$