# Determining stress distribution with the use of the Airy stress function and FINITE DIFFERENCE METHOD

We consider **plane stress / strain state** with **no body forces**

## THEORETICAL INTRODUCTION

We define the **Airy stress function**:

$$F(x, y): \begin{cases} \dfrac{\partial^2 F}{\partial y^2} = \sigma_{xx} \\ \dfrac{\partial^2 F}{\partial z^2} = \sigma_{yy} \\ \dfrac{\partial^2 F}{\partial x \partial y} = -\sigma_{xy} = -\sigma_{yx} \end{cases}$$

Equilibrium equations are satisfied on the basis of the above definition. In order to obtain true solutions of the problem of linear theory of elasticity also strain compatibility conditions must be satisfied. For isotropic materials this requirement is equivalent to the statemet that **Airy stress function satisfied biharmonic equation**:

$$\nabla^4 F = 0 \qquad \text{where } \nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

Solution depends on:

- **Type of equation** (biharmonic equation)
- **Domain** (shape of membrane)
- **Boundary conditions** (load and supports on membrane's edge)

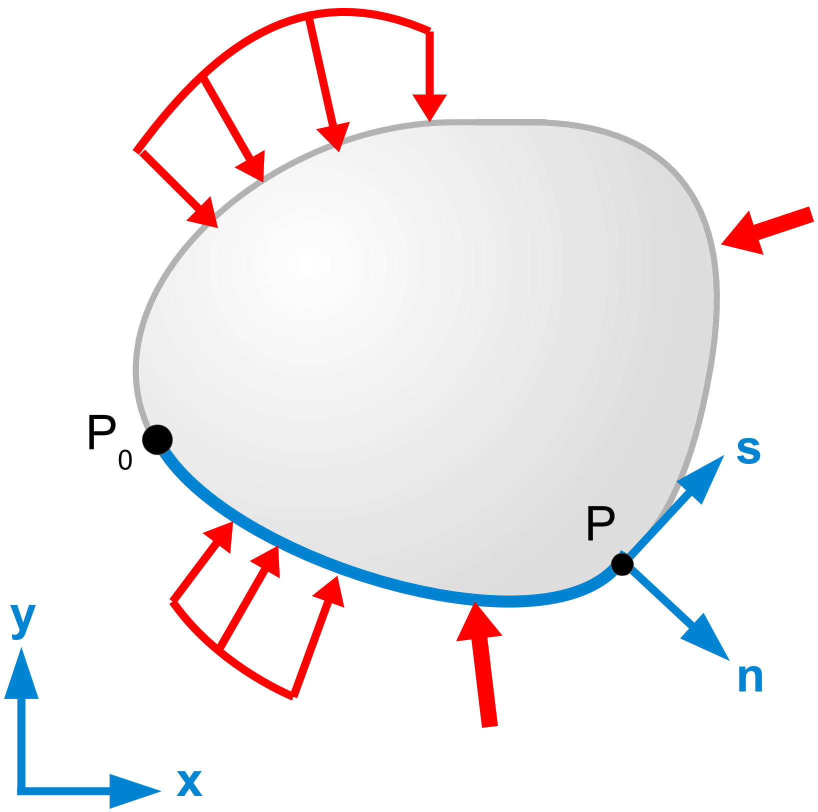
This is 4$^{th}$ order euqation – values of the function itself or its derivatives up to 3$^{rd}$ order are required to be given by boundary conditions in order to determine the solution uniquely.

We consider only the case when **boundary conditions are of only static type** (loads on boundary) – we consider no kinematics boundary conditions (displacements on boundary). **The membrane may deform freely**.

**Boundary conditions for Airy stress functions** may be determined as follows:

$$F\Big|_P = \frac{M\big|_P}{h} \qquad \frac{\partial F}{\partial n}\Big|_P = -\frac{Q_s\big|_P}{h}$$

## OUTLINE OF DETERMINING THE BOUNDARY CONDITIONS FOR AIRY STRESS FUNCTIONS ACCORDING TO THE BOUNDARY LOAD.

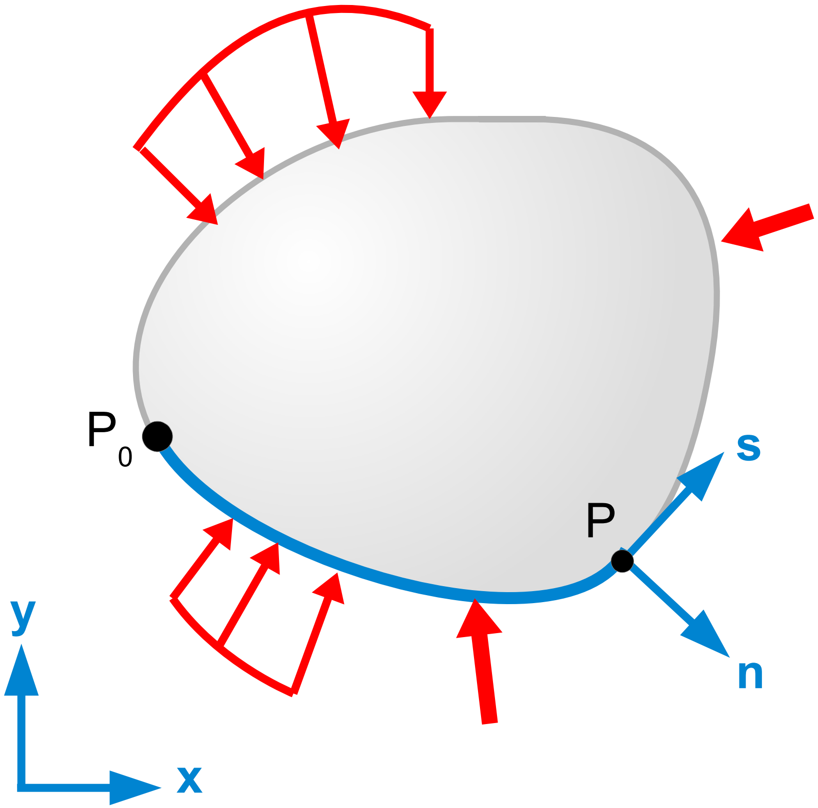



- **Point $P_0$** is chosen **arbitrary**. Its **location is fixed**.
- We chose **point P** then. **Its location varies**.
- In point P **local coordinate system (n,s)** is defined:
  - **n** – external normal direction
  - **s** – tangent direction oriented from $P_0$ to P

We determine:

$M|_P$ **Moment about P** from **all loads applied on the segment of boundary between point $P_0$ and P**. If the interior of membrane is to the left of the moving coordinate system, then **positive moment is counter-clockwise**.

$Q_s|_P$ **Sum of tangential forces** (forces which are parallel to the s axis of local coordinate systemtj) from **all loads applied on the segment of boundary between point $P_0$ and P**. **Positive tangential forces are considered oriented in the same way as axis s** in actual location of point P.




**OUTLINE OF DETERMINING THE BOUNDARY CONDITIONS FOR AIRY STRESS FUNCTIONS ACCORDING TO THE BOUNDARY LOAD.**

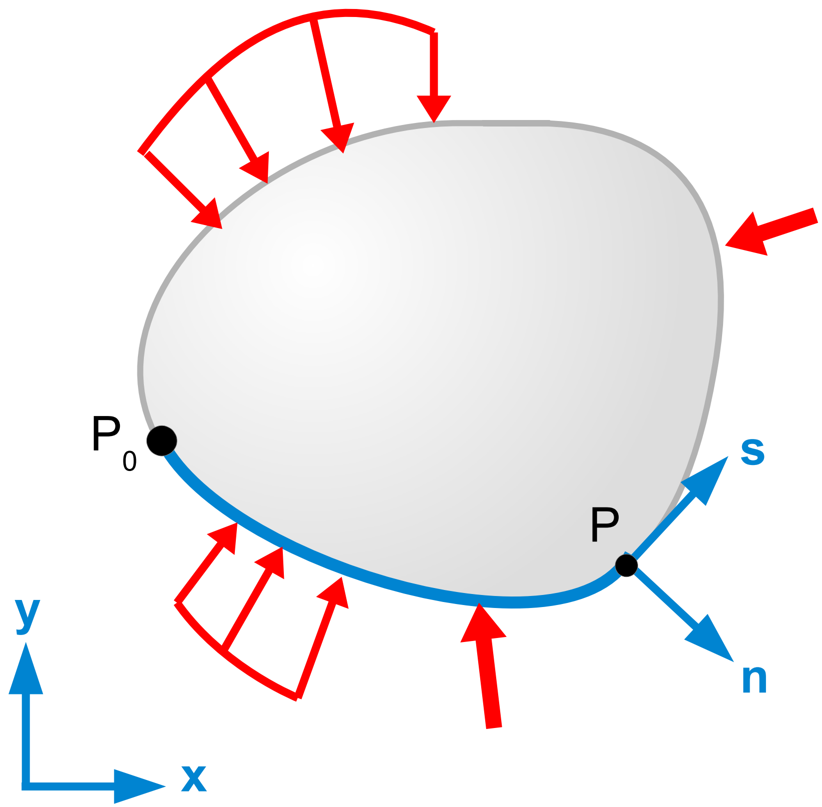
## BOUNDARY CONDITIONS

- **for values of Airy stress function** in point P:

$$F\big|_P = \frac{M\big|_P}{h}$$

- **for values of directional derivative of Airy stress function** along the direction of external normal in point P:

$$\left.\frac{\partial F}{\partial n}\right|_P = -\frac{Q_s\big|_P}{h}$$




## OUTLINE OF DETERMINING THE BOUNDARY CONDITIONS FOR AIRY STRESS FUNCTIONS ACCORDING TO THE BOUNDARY LOAD.

Directional derivative of Airy stress function along the direction of external normal in point P:

We find unit external normal vector in point P:

$$\mathbf{n}_P = \left[ n_x \; ; \; n_y \right]$$

Directional derivative:

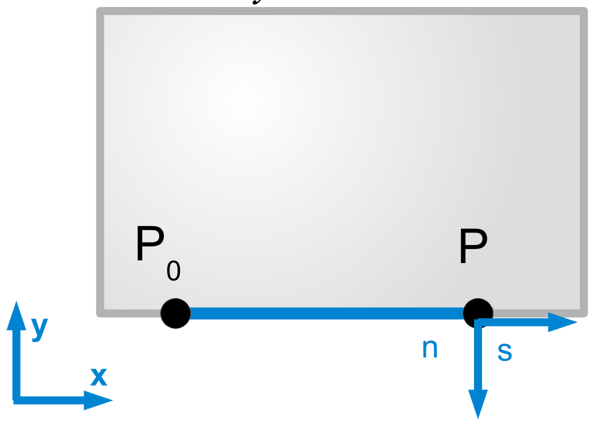
$$\left. \frac{\partial F}{\partial n} \right|_P = \operatorname{grad} F\big|_P \circ \mathbf{n}_P = \left[ \left. \frac{\partial F}{\partial x} \right|_P \; ; \; \left. \frac{\partial F}{\partial y} \right|_P \right] \circ \left[ n_x \; ; \; n_y \right] \Rightarrow$$

$$\boxed{\left. \frac{\partial F}{\partial n} \right|_P = n_x \left. \frac{\partial F}{\partial x} \right|_P + n_y \left. \frac{\partial F}{\partial y} \right|_P}$$

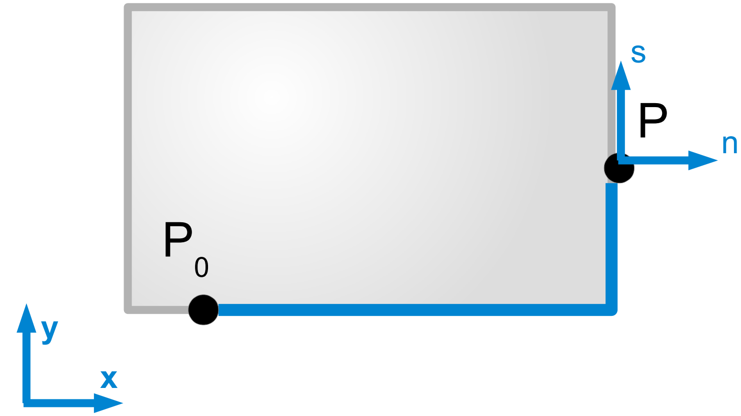
Directional derivative of Airy stress function along the direction of external normal in point P for a **rectangular domain**.

$$\left.\frac{\partial F}{\partial n}\right|_P = n_x \left.\frac{\partial F}{\partial x}\right|_P + n_y \left.\frac{\partial F}{\partial y}\right|_P$$

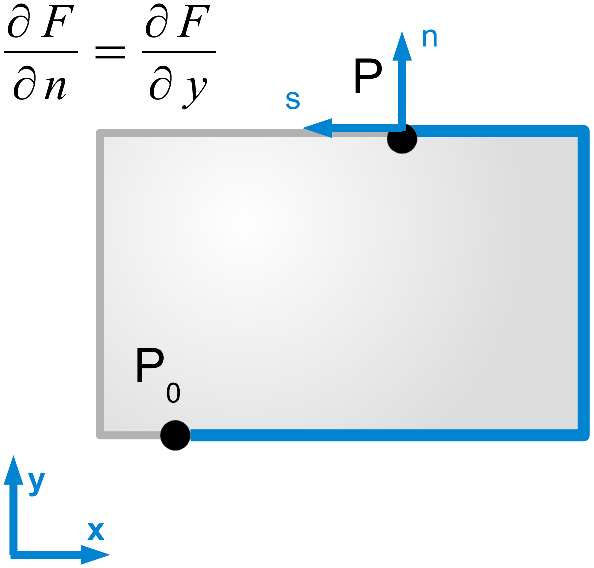
$$\frac{\partial F}{\partial n} = -\frac{\partial F}{\partial y}$$



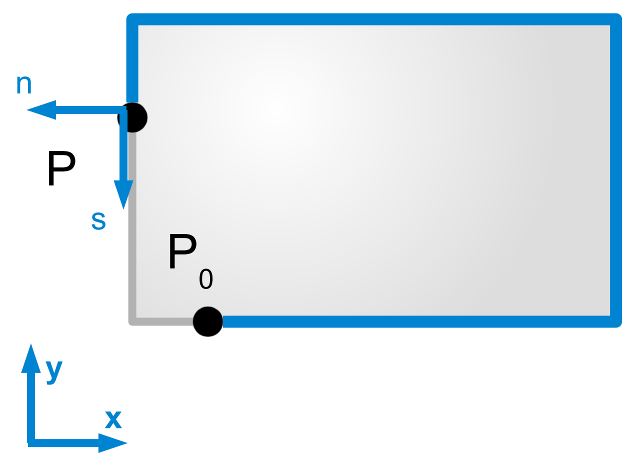

$$\frac{\partial F}{\partial n} = \frac{\partial F}{\partial x}$$




$$\frac{\partial F}{\partial n} = \frac{\partial F}{\partial y}$$




$$\frac{\partial F}{\partial n} = -\frac{\partial F}{\partial x}$$

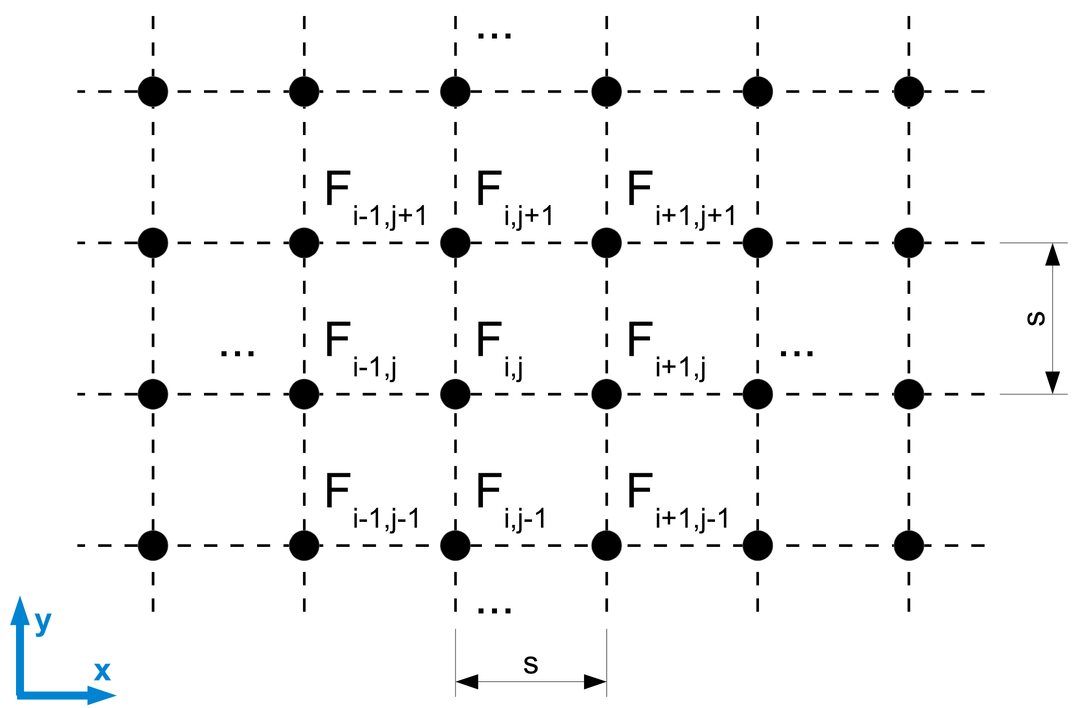


**FINITE DIFFERENCE METHOD** – numerical method allowing us to find approximate values of solutions of differential equations and systems of differential equations in a finite number of chosen points, so called **nodes of the FDM mesh**.

In each such node considered differential equation is a written down **approximating the derivatives occuring in that equation with difference quotient** for possibly small increments of independent variables Δx and Δy. Sine each quotient depends on the values of investigated solution in a linear way, as a result we obtain a **linear system of algebraic equations** for nodal values of that solution.

In the same way also boundary conditions are written down. Boundary conditions accounting for derivatives require introduction of fictious nodes outisde the considered domain – also in those node the values of function must be determined.

In the simplest variant of FDM, a rectangular mesh is assumed with equal incrementsj Δx = Δy = s.




Example: Partial derivative wrt x

- **definition:**

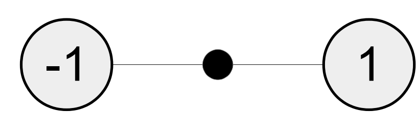
$$\frac{\partial F}{\partial x} = \lim_{s \to 0} \frac{F(x+s, y) - F(x-s, y)}{2s}$$

- **approximation:**

$$\left. \frac{\partial F}{\partial x} \right|_{i,j} \approx \frac{F_{i+1,j} - F_{i-1,j}}{2s}$$

- **symbol:**

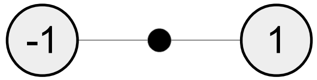
$$\frac{\partial}{\partial x} \approx \frac{1}{2s} \cdot$$



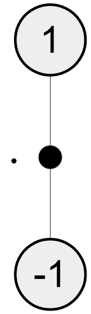
# CHOSEN FINITE DIFFERENCES

## $1^{st}$ order derivatives:

$$\frac{\partial}{\partial x} \approx \frac{1}{2s} \cdot$$

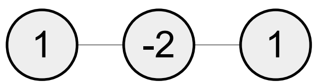



$$\frac{\partial}{\partial x} \approx \frac{1}{2s} \cdot$$

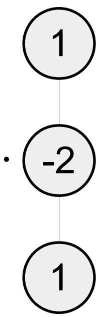



## $2^{nd}$ order derivatives:

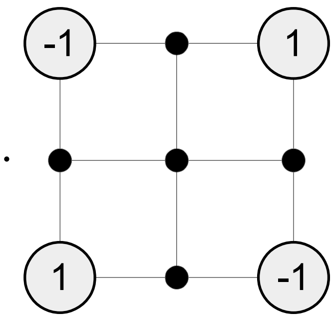
$$\frac{\partial^2}{\partial x^2} \approx \frac{1}{s^2} \cdot$$




$$\frac{\partial^2}{\partial y^2} \approx \frac{1}{s^2} \cdot$$

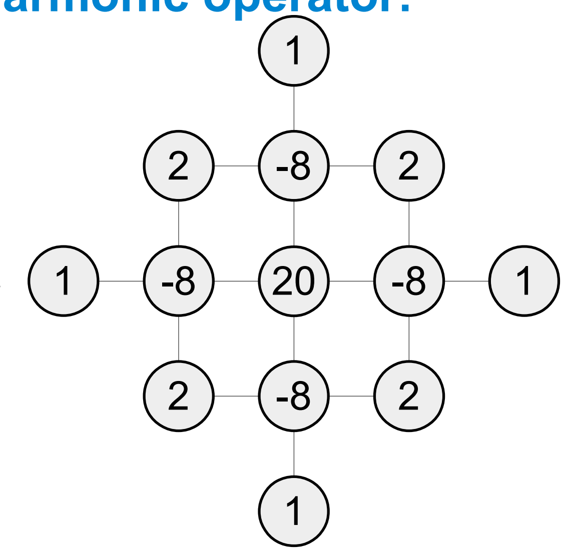



$$\frac{\partial^2}{\partial x \partial y} \approx \frac{1}{4s^2} \cdot$$




## $4^{th}$ order biharmonic operator:

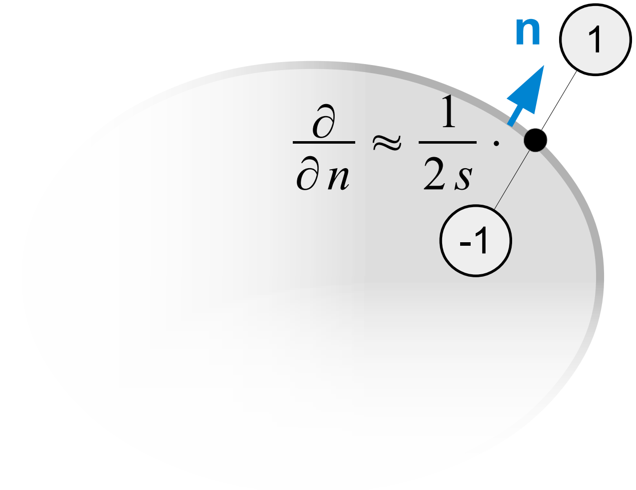
$$\nabla^4 \approx \frac{1}{s^4} \cdot$$



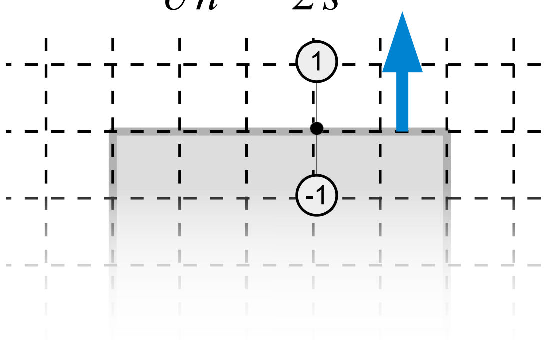
# CHOSEN FINITE DIFFERENCES

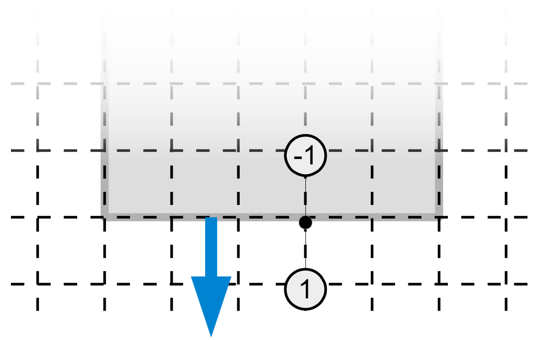
## Directional derivative along external normal:

- The node **outside** domain – multiplied by (1)
- The node **inside** domain – multiplied by (-1)

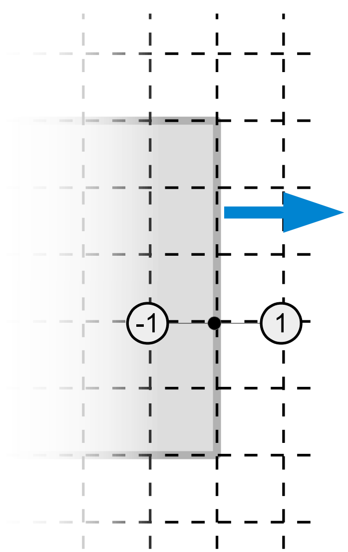



$$\frac{\partial}{\partial n} \approx \frac{1}{2s} \cdot$$



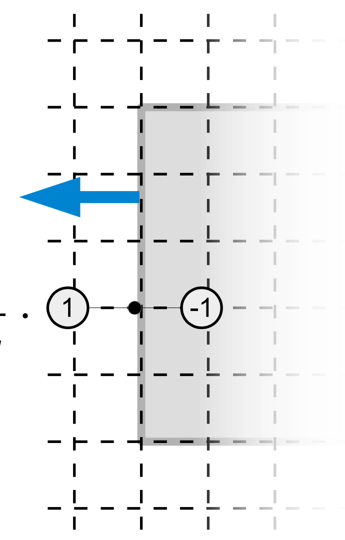




$$\frac{\partial}{\partial n} \approx \frac{1}{2s} \cdot$$



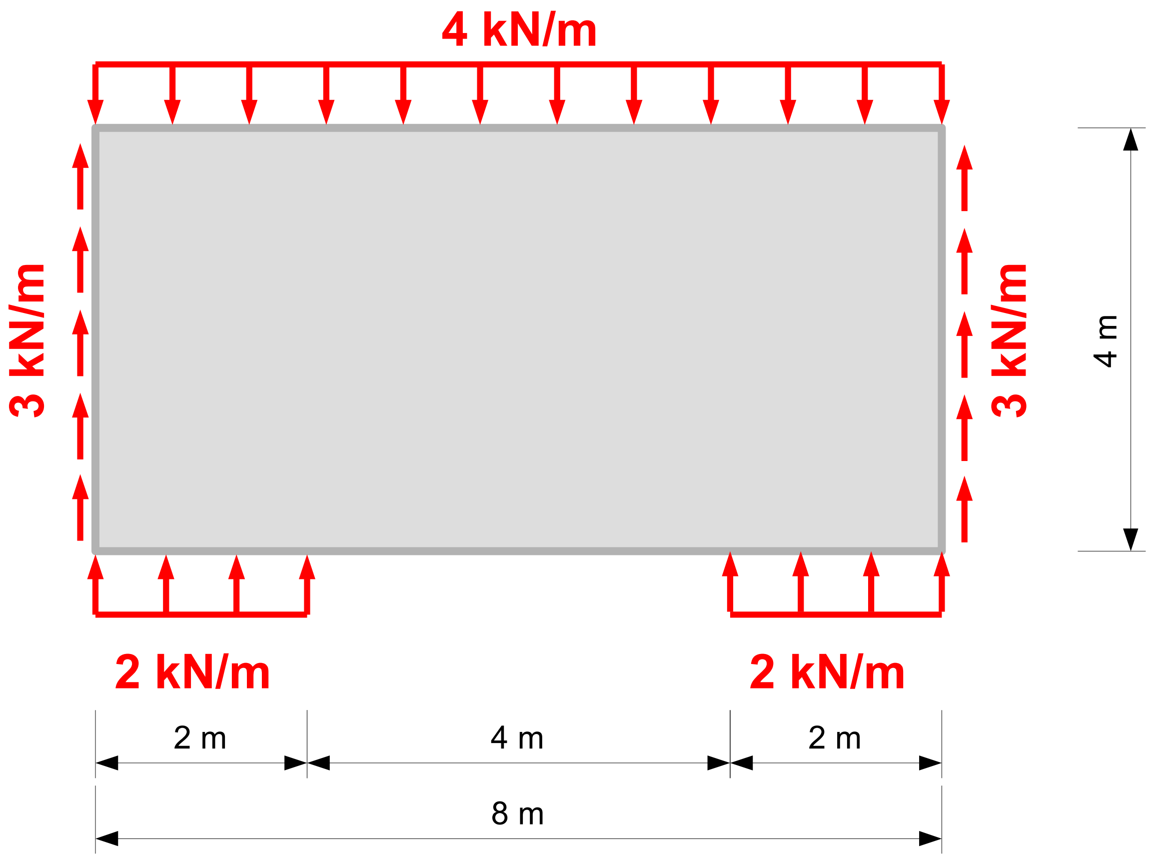

$$\cdot \frac{1}{2s} \approx \frac{\partial}{\partial n}$$

$$\frac{\partial}{\partial n} \approx \frac{1}{2s} \cdot$$



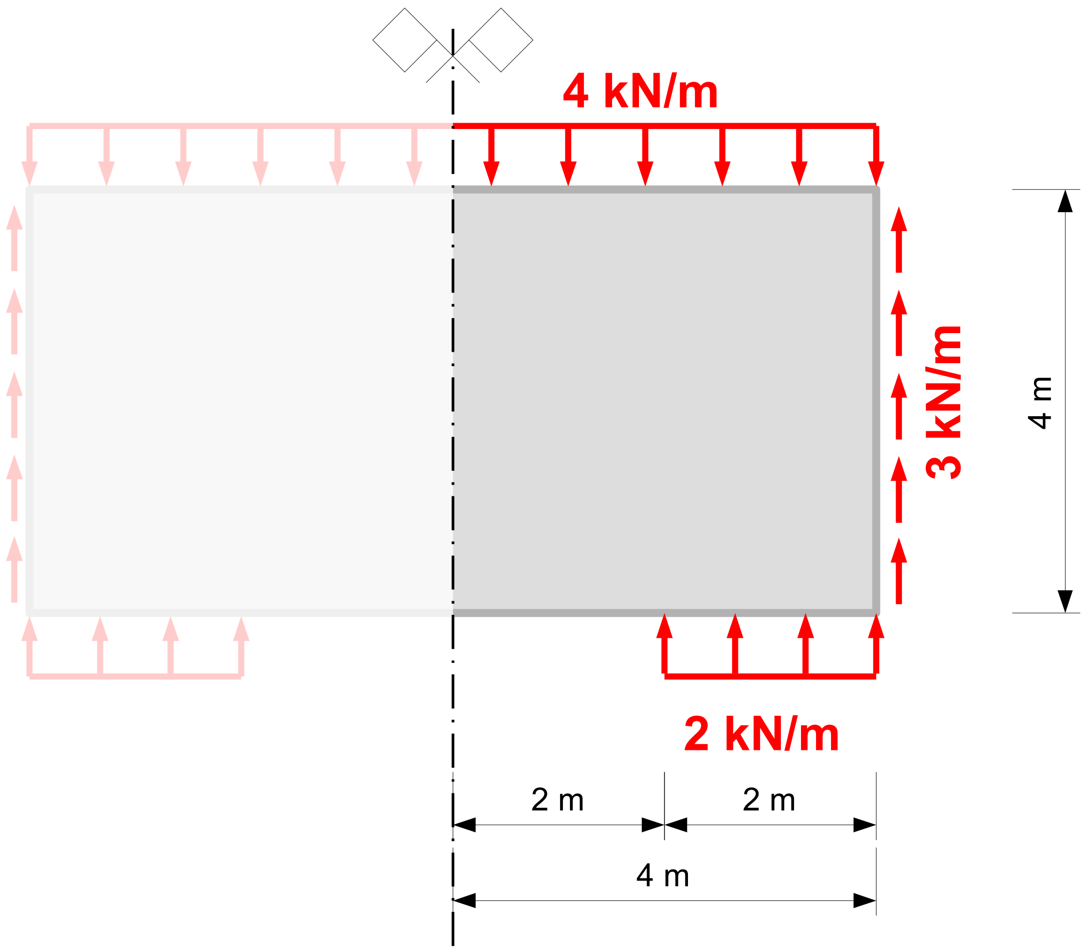
## EXAMPLE

Find the stress state in the middle of the membrane given in the figure below withthe use of Finite Difference Method. Assume thickness h=20cm and FDM grid space equal 1m.



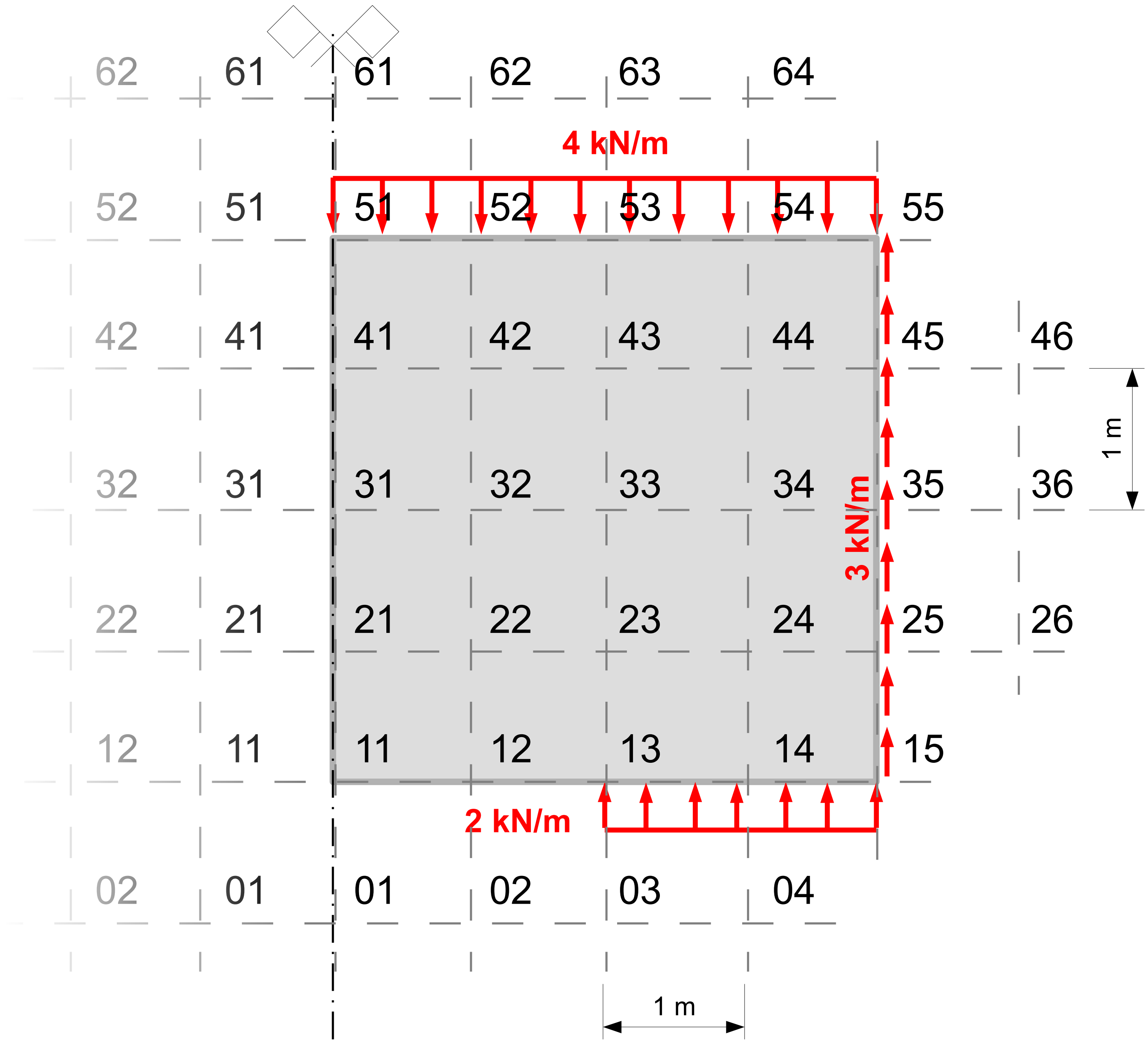
## EXAMPLE

The system is symmetric – we may consider only its half. Values of the Airy stress function for the second half are the same.



## EXAMPLE

Making use of the symmetry of the system the FDM mesh may be assumed as follows:

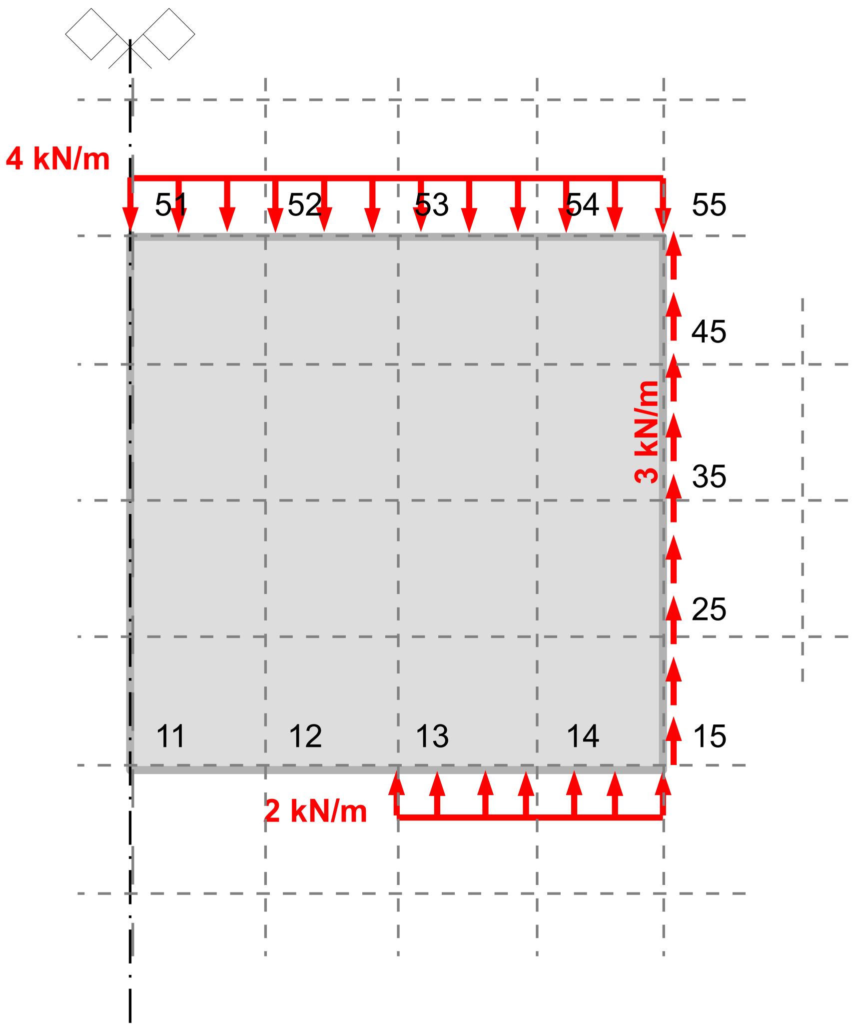


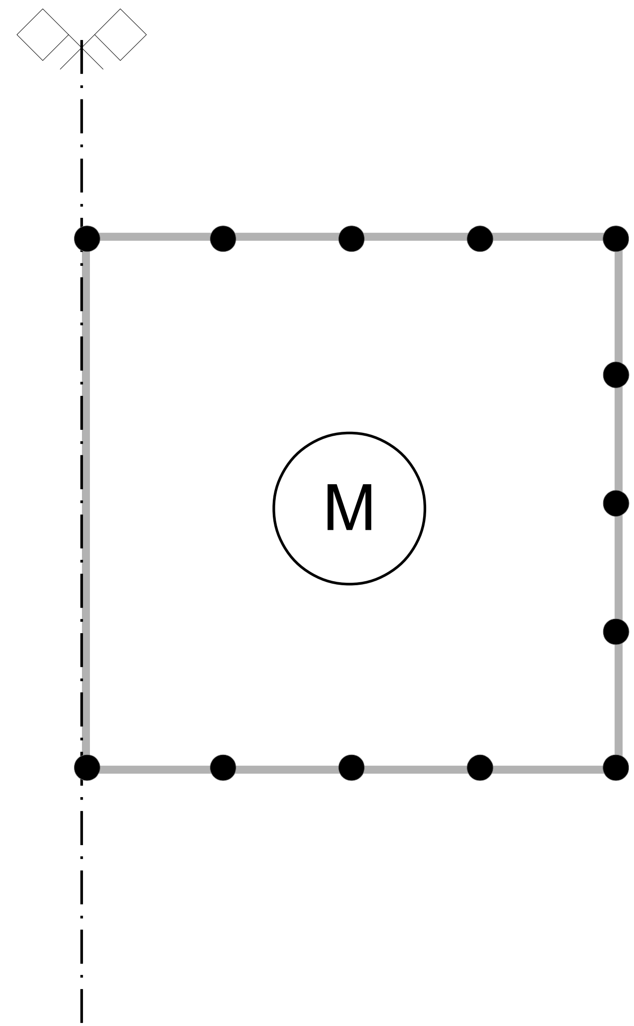
## EXAMPLE

We start with **boundary conditions** – accounting for them will reduce the number of unknown nodal values of Airy stress functions. Let's start with the **boundary condition for the value of the Airy stress function** itself.:

$$F|_P = \frac{M|_P}{h}$$

**We assume point $P_0$ in node 11.**



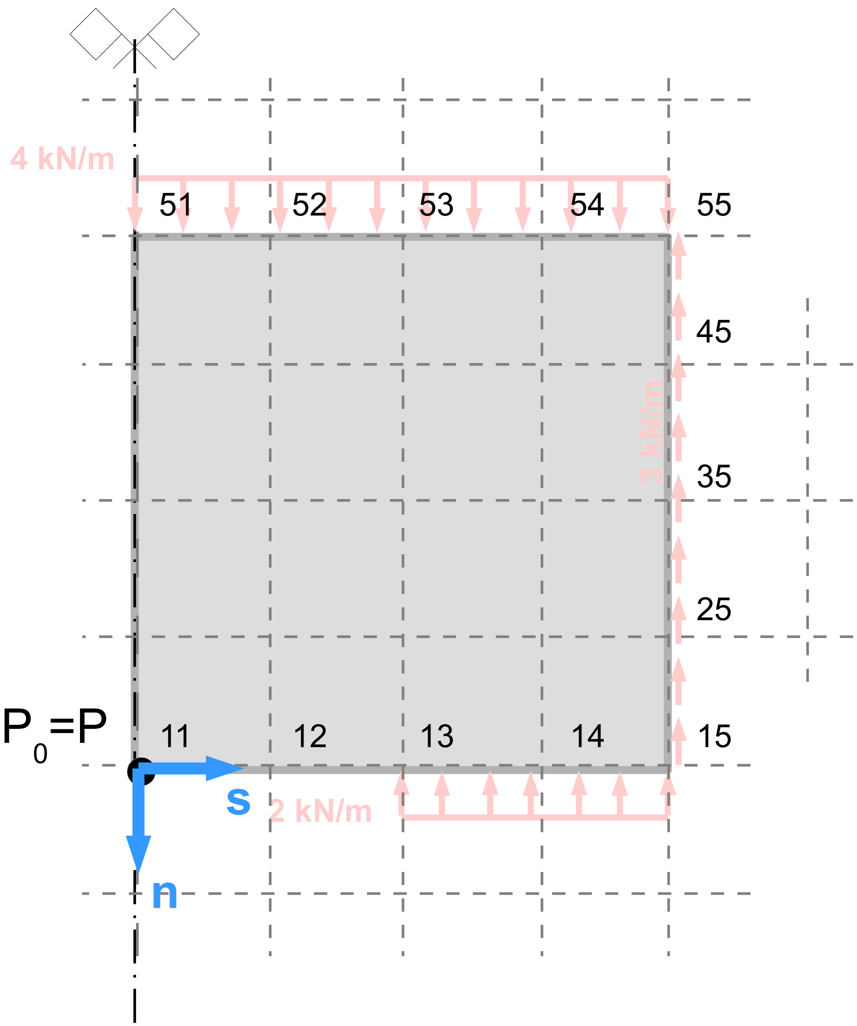
## EXAMPLE

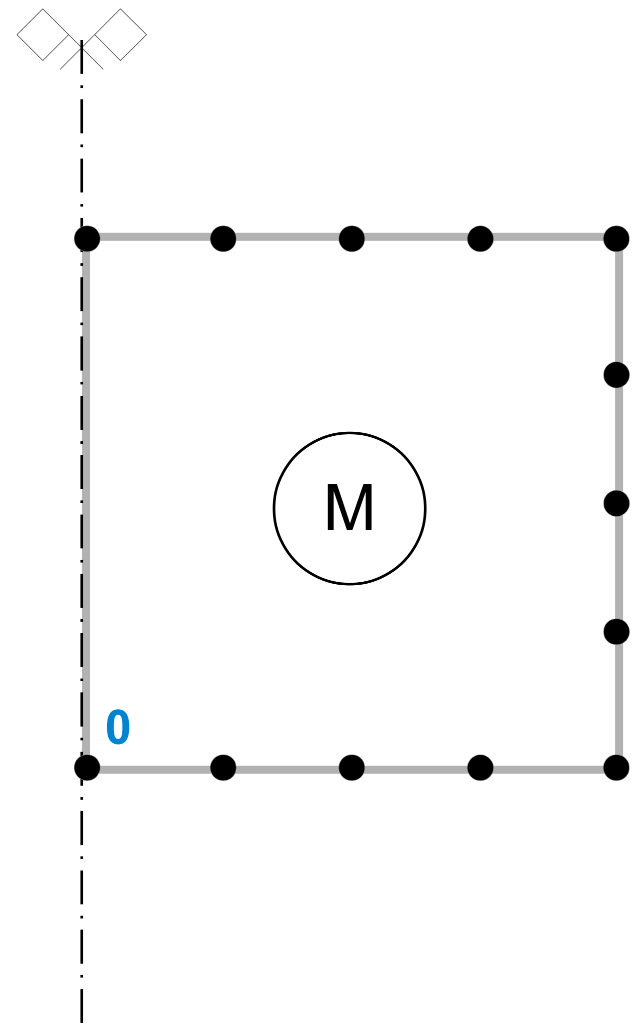
**We take point P in node 11.**

Moment about P (node 11) from load applied to the segment of the boundary between point $P_0$ (węzeł 11) and point P (węzeł 11):

$$M_{11} = 0 \text{ kNm}$$

For an assumed orientation of (n,s) coordinate system, **positive moment** is **counter-clockwise**.



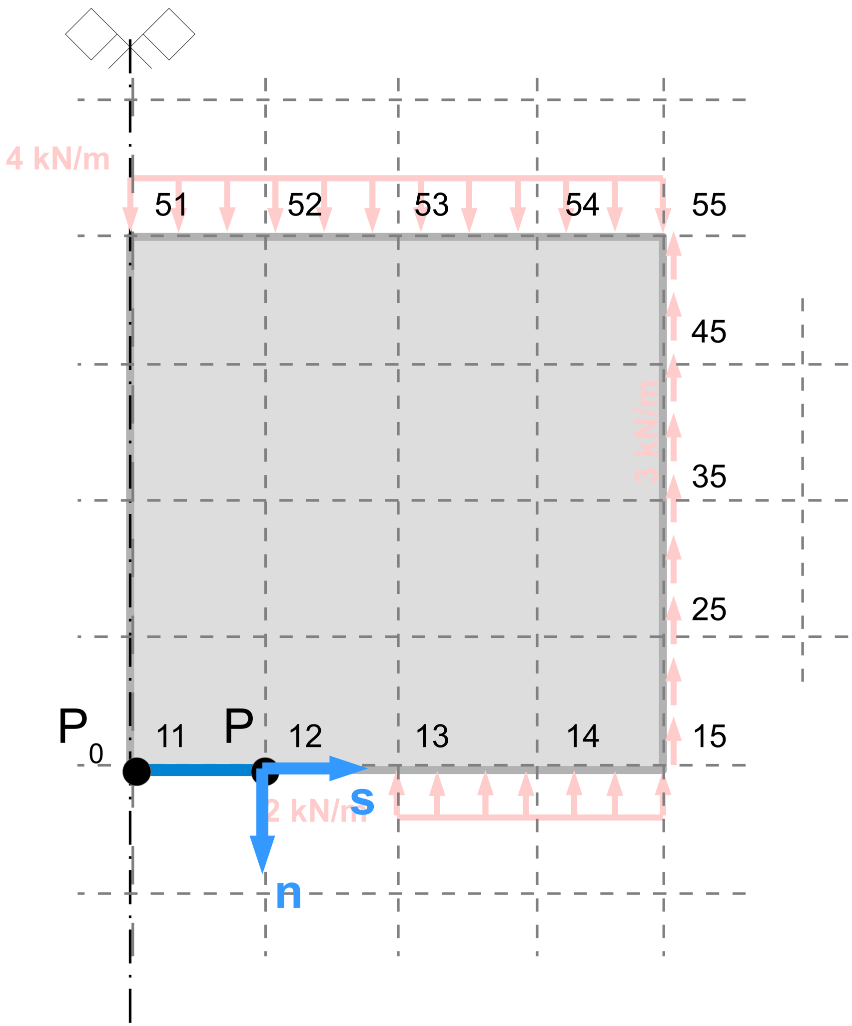



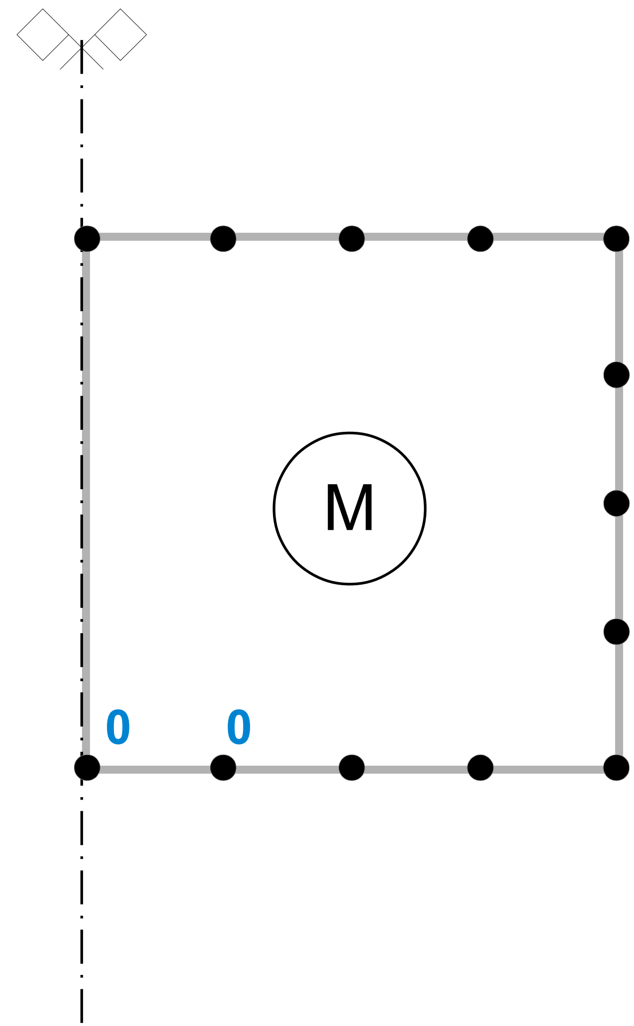
## EXAMPLE

**We take point P in node 12.**

Moment about P (node 12) from load applied to the segment of the boundary between point $P_0$ (węzeł 11) and point P (węzeł 12):

$$M_{12} = 0 \text{ kNm}$$



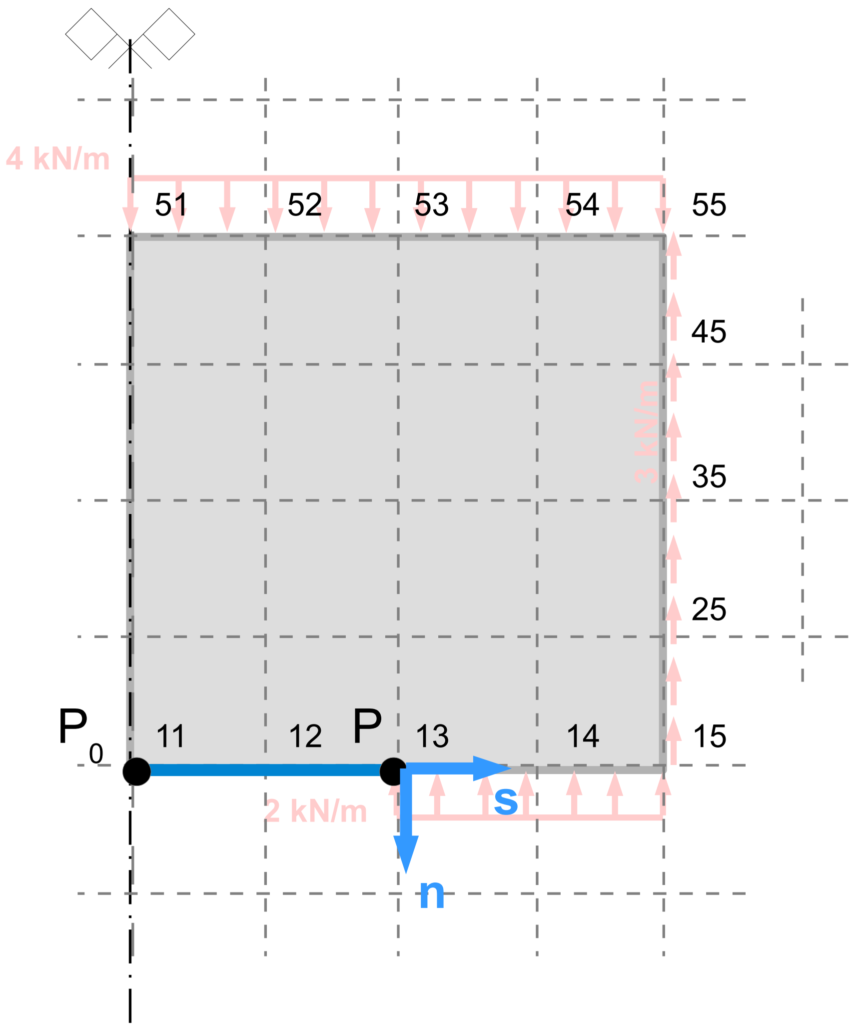



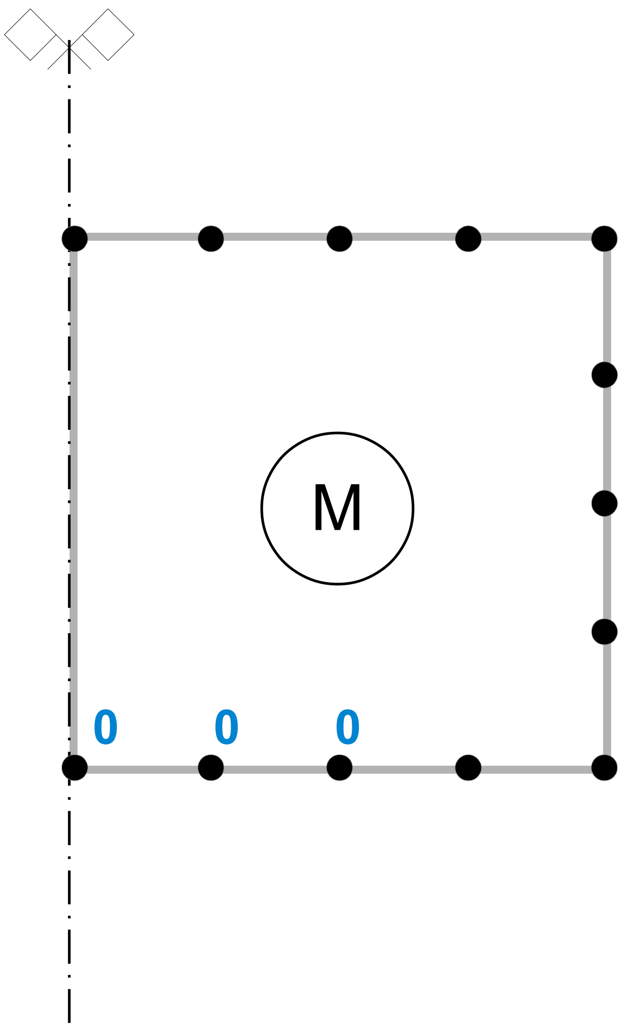
**EXAMPLE**

**We take point P in node 13.**

Moment about P (node 13) from load applied to the segment of the boundary between point $P_0$ (węzeł 11) and point P (węzeł 13):

$$M_{13} = 0 \text{ kNm}$$



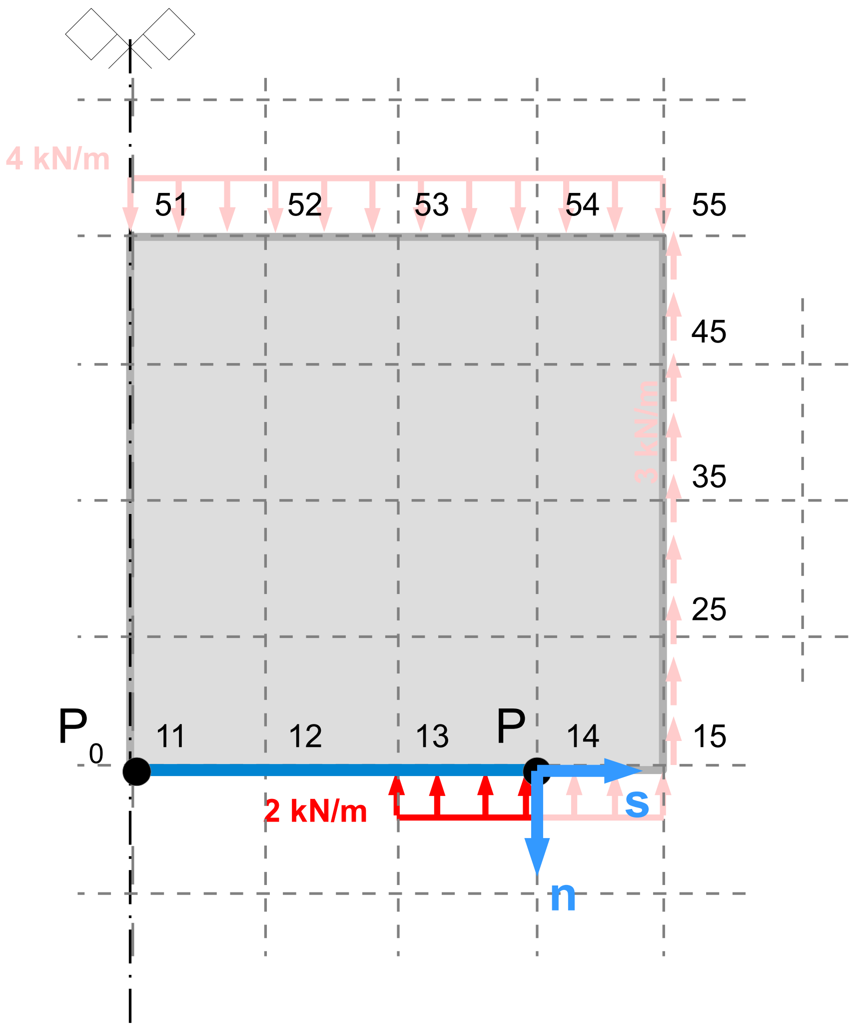



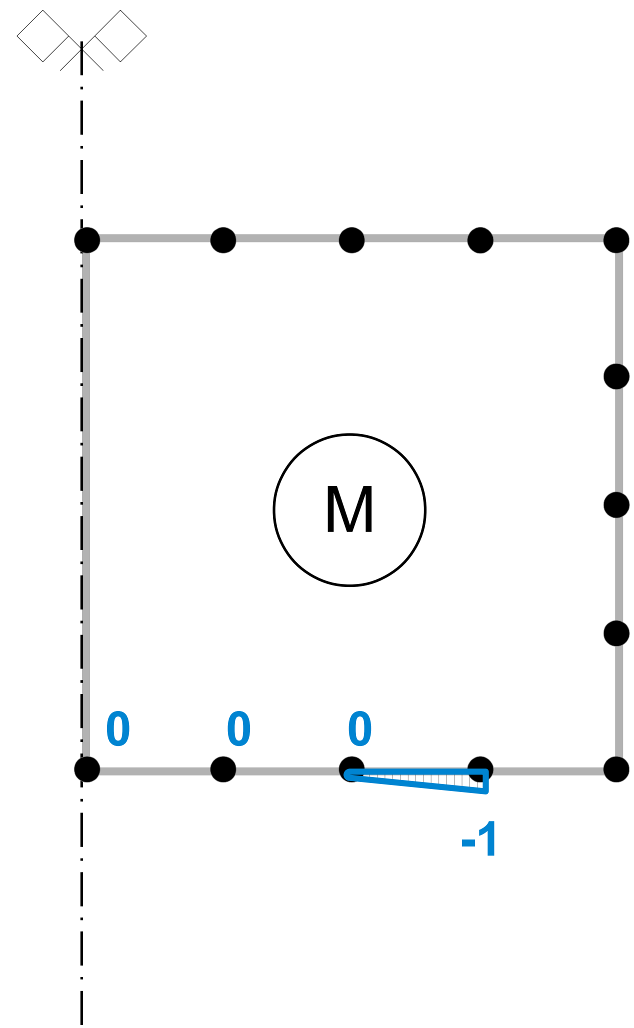
**EXAMPLE**

**We take point P in node 14.**

Moment about P (node 14) from load applied to the segment of the boundary between point $P_0$ (węzeł 11) and point P (węzeł 14):

$$M_{14} = -2\ \mathrm{kN/m^2} \cdot 1\,\mathrm{m} \cdot 0{,}5\,\mathrm{m} = -1\,\mathrm{kNm}$$



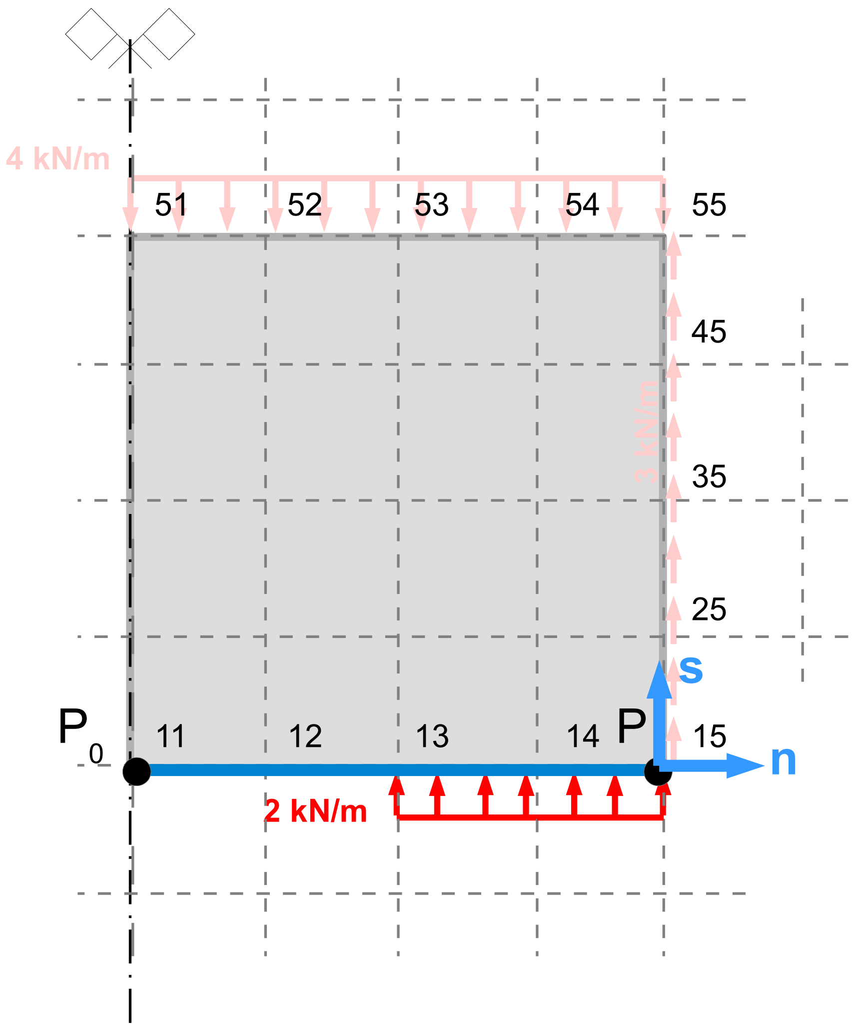



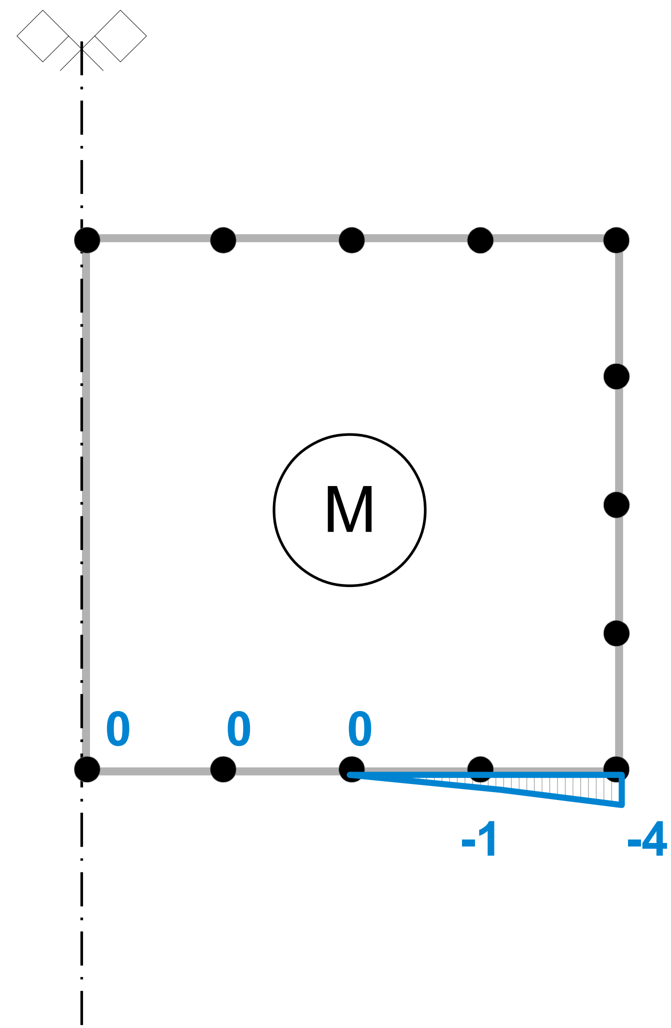
## EXAMPLE

**We take point P in node 15.**

Moment about P (node 15) from load applied to the segment of the boundary between point $P_0$ (węzeł 11) and point P (węzeł 15):

$$M_{15} = -2\ \mathrm{kN/m^2} \cdot 2\,\mathrm{m} \cdot 1\,\mathrm{m} = -4\,\mathrm{kNm}$$



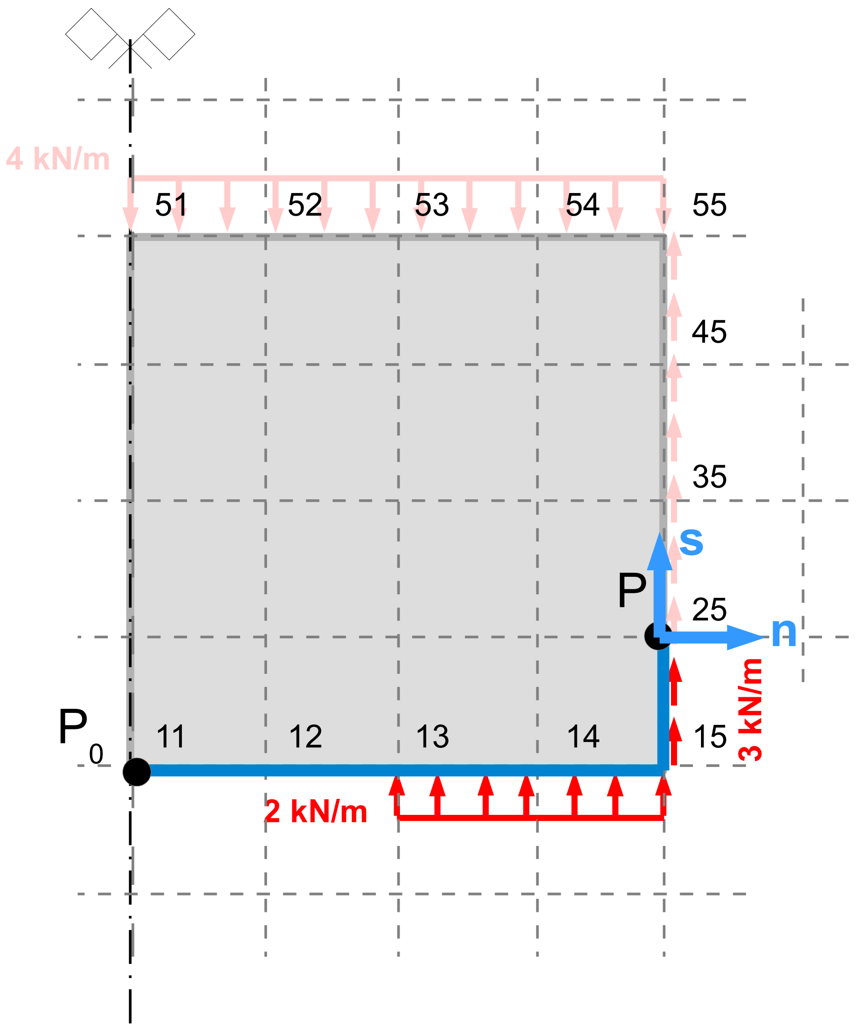



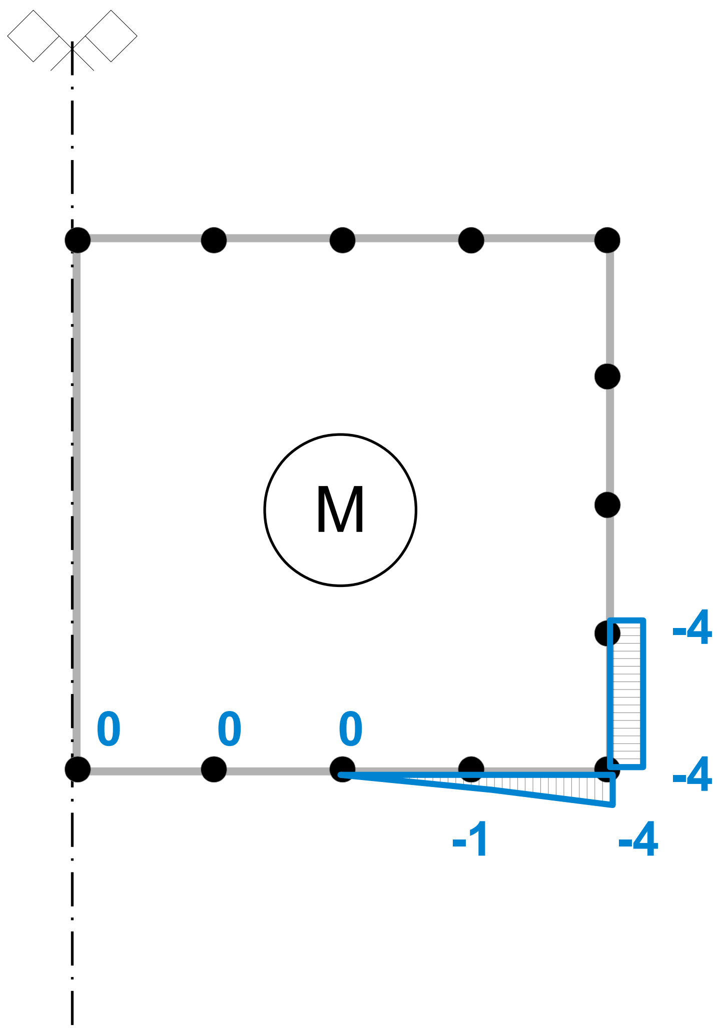
## EXAMPLE

**We take point P in node 25.**

Moment about P (node 25) from load applied to the segment of the boundary between point $P_0$ (węzeł 11) and point P (węzeł 25):

$$M_{25} = -2\ \text{kN/m}^2 \cdot 2\,\text{m} \cdot 1\,\text{m} = -4\,\text{kNm}$$



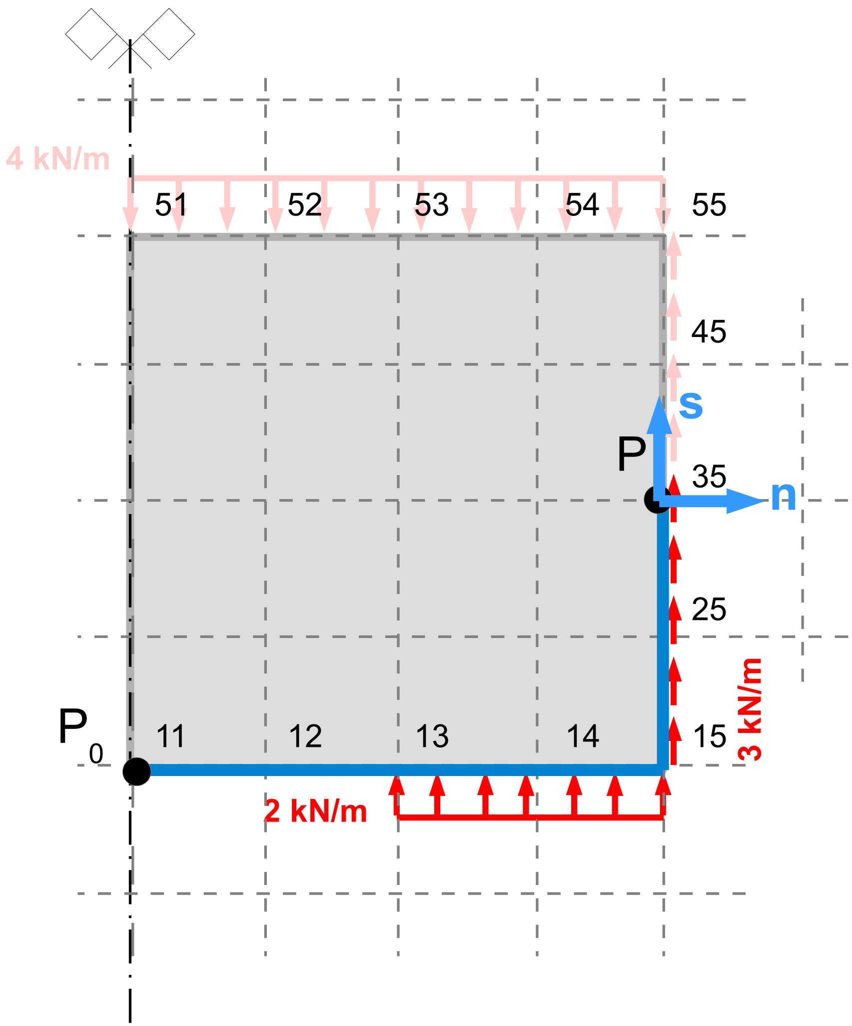



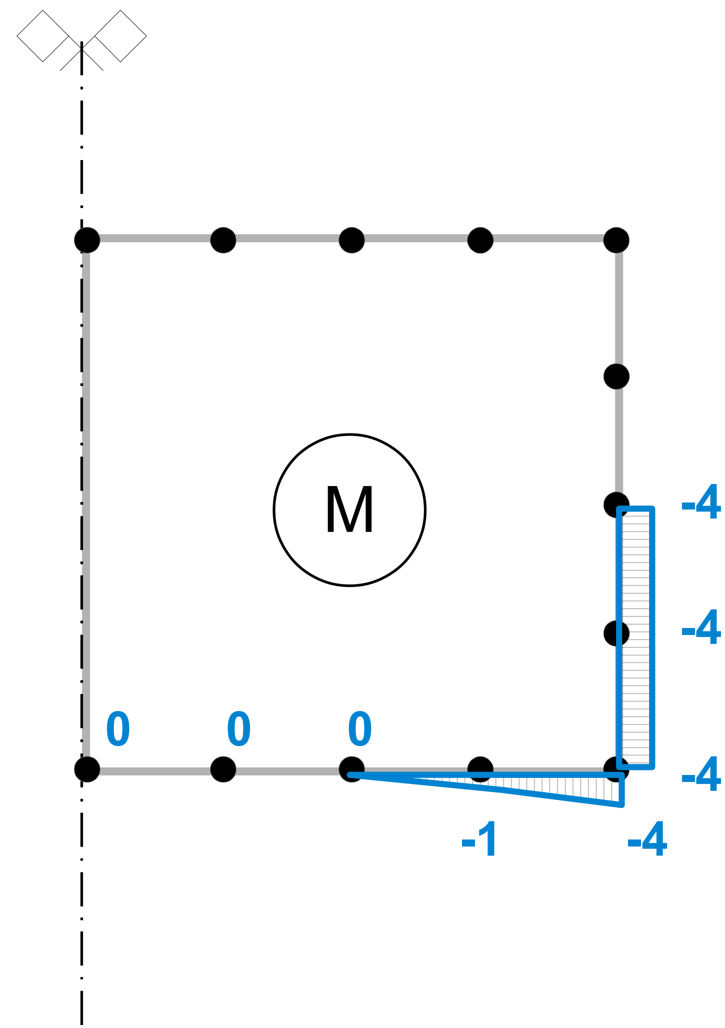
## EXAMPLE

**We take point P in node 35.**

Moment about P (node 35) from load applied to the segment of the boundary between point $P_0$ (węzeł 11) and point P (węzeł 35):

$$M_{35} = -2\ \mathrm{kN/m^2} \cdot 2\,\mathrm{m} \cdot 1\,\mathrm{m} = -4\,\mathrm{kNm}$$



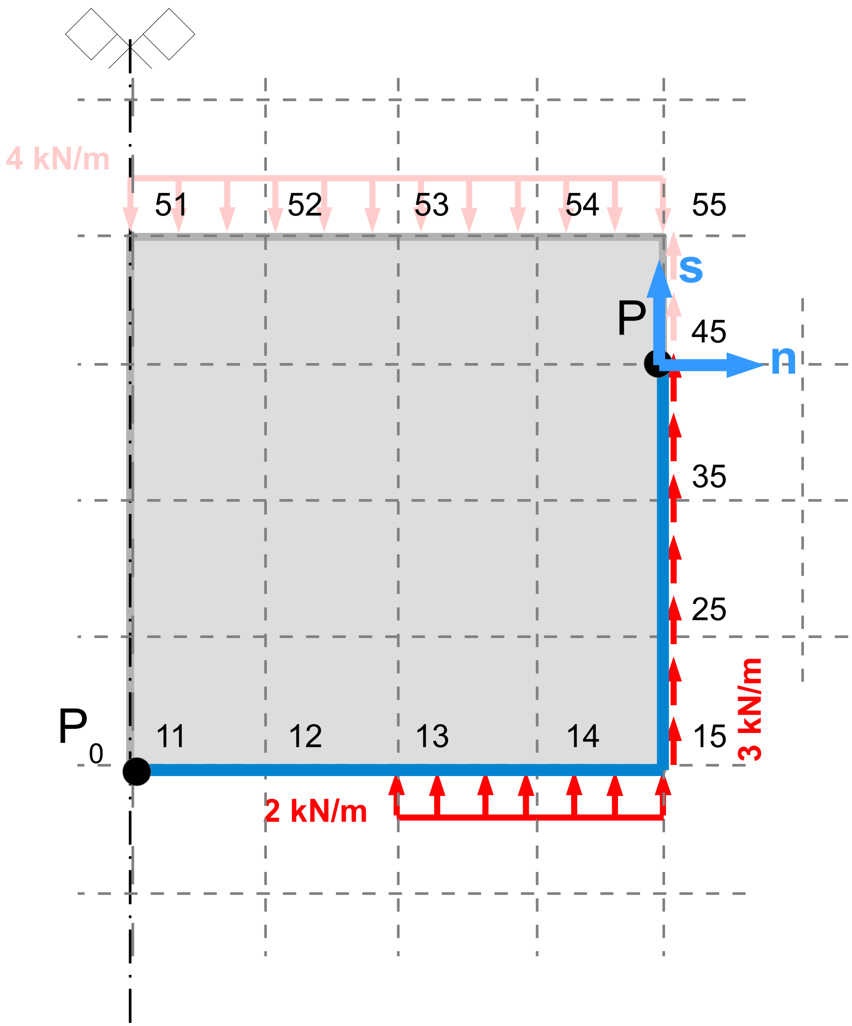



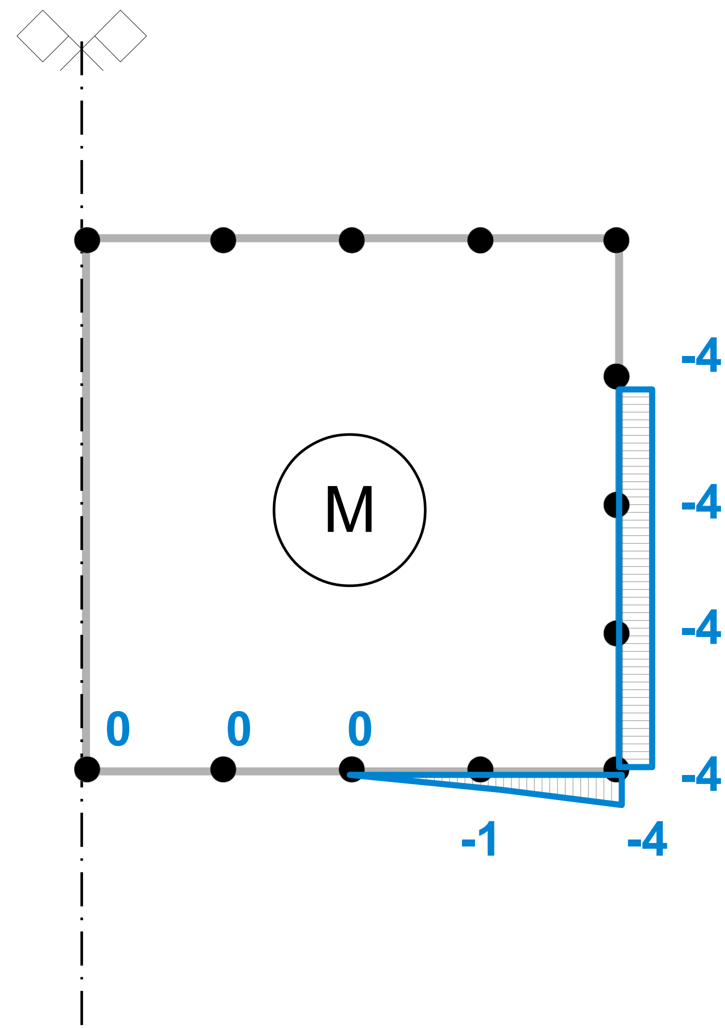
## EXAMPLE

**We take point P in node 45.**

Moment about P (node 45) from load applied to the segment of the boundary between point $P_0$ (węzeł 11) and point P (węzeł 45):

$$M_{45} = -2\ \mathrm{kN/m^2} \cdot 2\,\mathrm{m} \cdot 1\,\mathrm{m} = -4\,\mathrm{kNm}$$



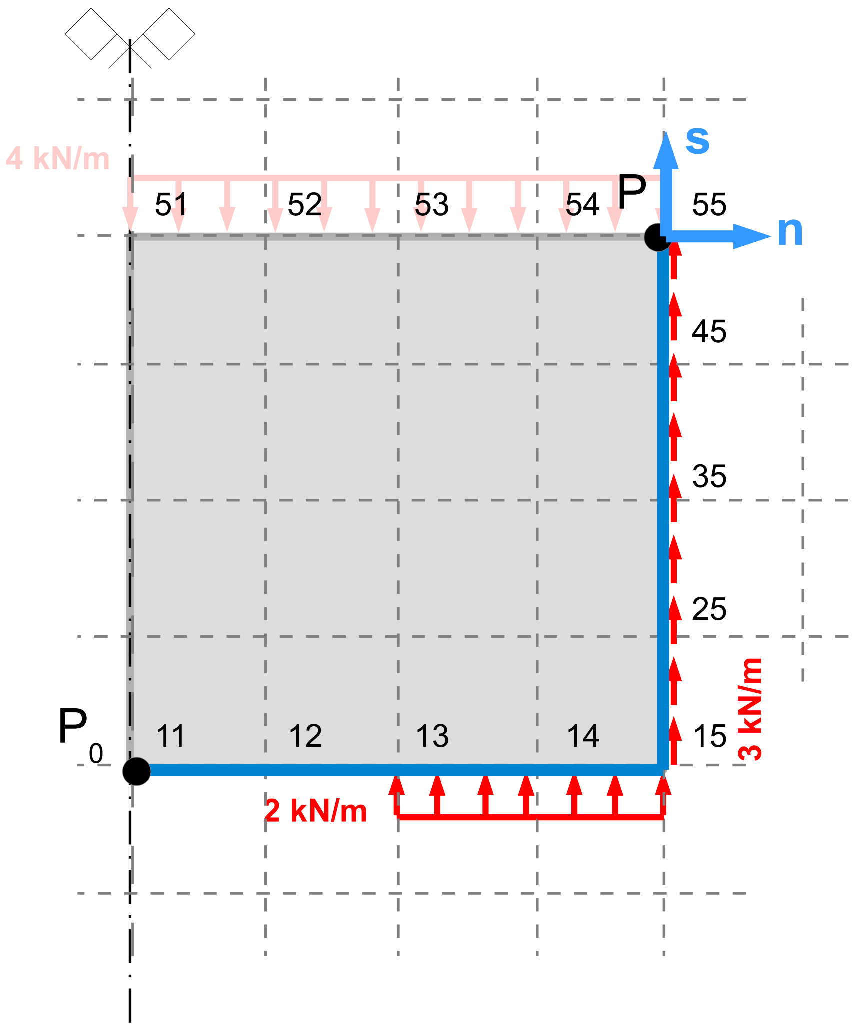



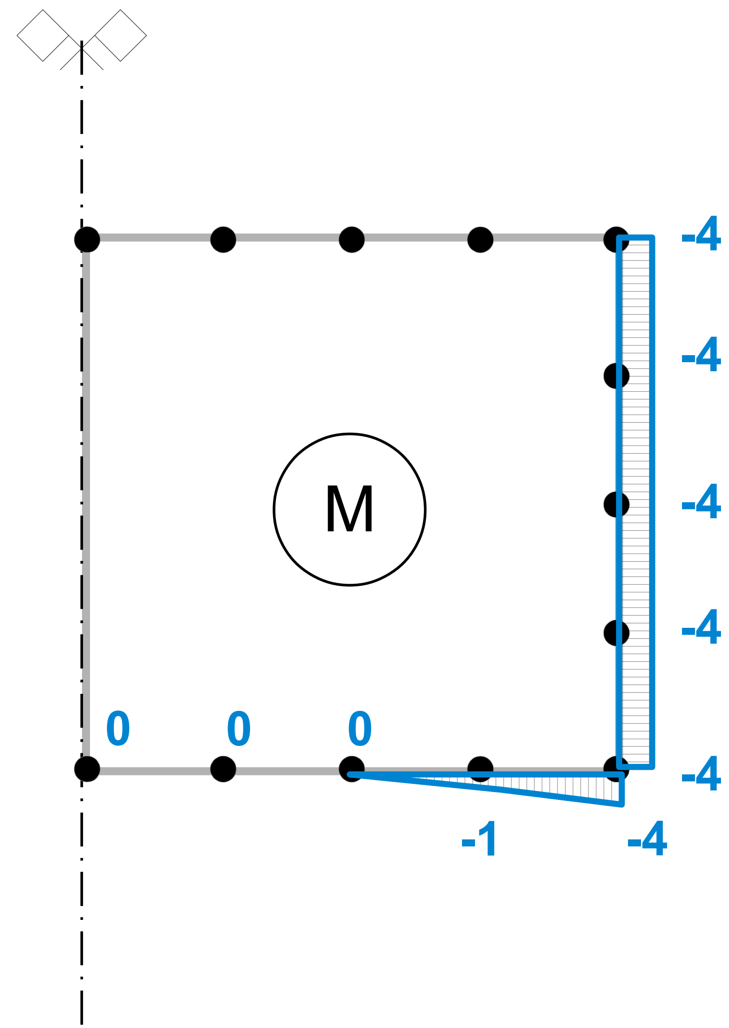
## EXAMPLE

**We take point P in node 55.**

Moment about P (node 55) from load applied to the segment of the boundary between point $P_0$ (węzeł 11) and point P (węzeł 55):

$$M_{55} = -2\ \mathrm{kN/m^2} \cdot 2\,\mathrm{m} \cdot 1\,\mathrm{m} = -4\,\mathrm{kNm}$$



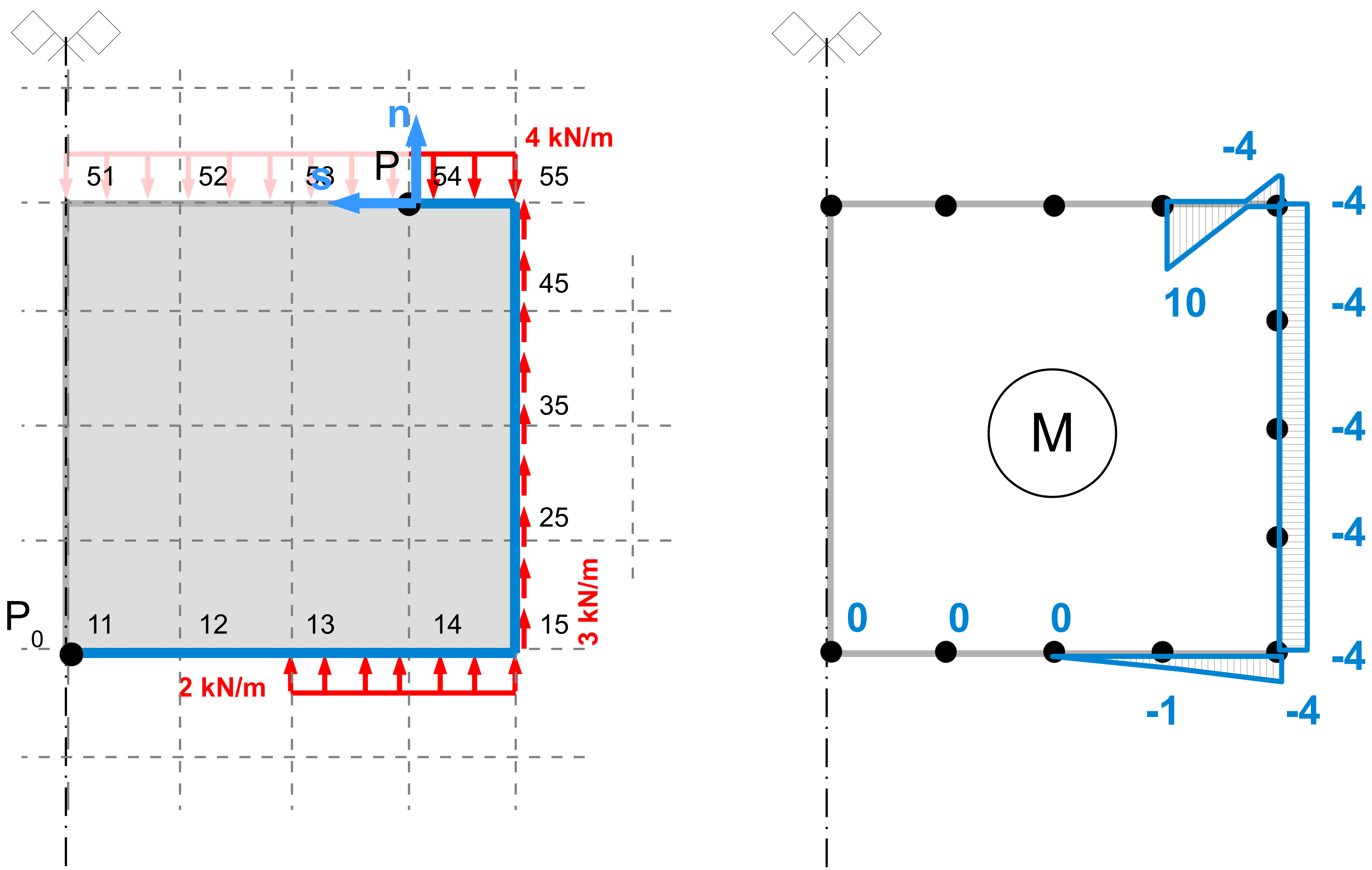



## EXAMPLE

**We take point P in node 54.**

Moment about P (node 54) from load applied to the segment of the boundary between point $P_0$ (węzeł 11) and point P (węzeł 54):

$$M_{54} = +\,3\cdot4\cdot1 \;-\; 4\cdot1\cdot0{,}5 = 10\,\mathrm{kNm}$$

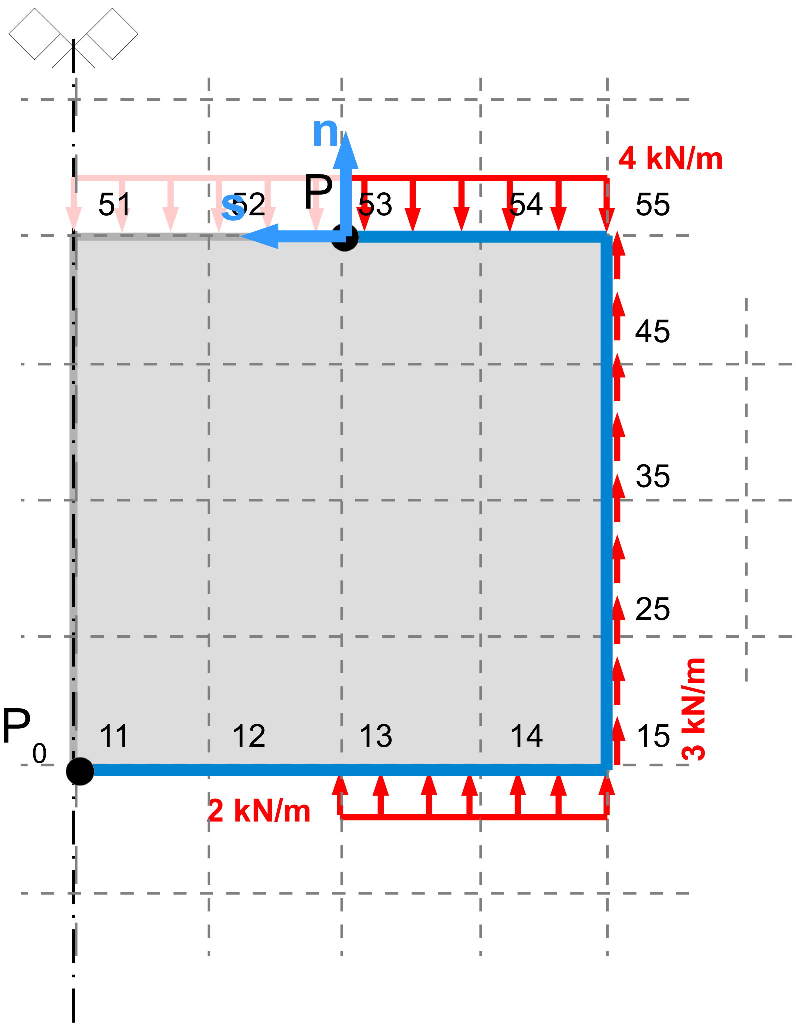


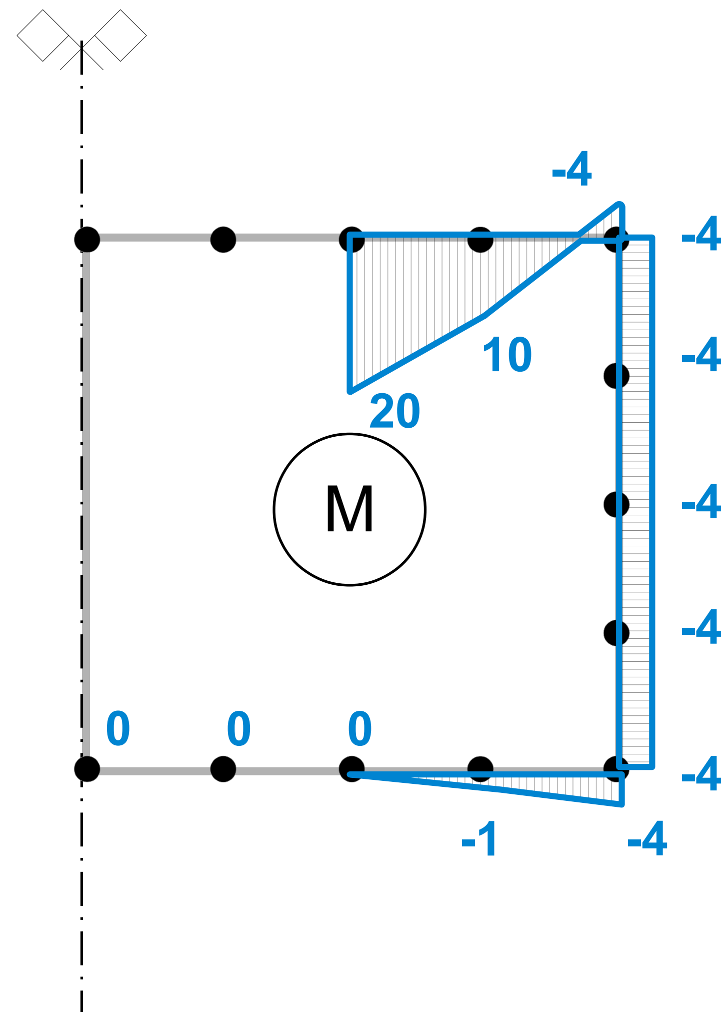
## EXAMPLE

**We take point P in node 53.**

Moment about P (node 53) from load applied to the segment of the boundary between point $P_0$ (węzeł 11) and point P (węzeł 53):

$$M_{53} = 2 \cdot 2 \cdot 1 + 3 \cdot 4 \cdot 2 - 4 \cdot 2 \cdot 1 = 20\,\mathrm{kNm}$$



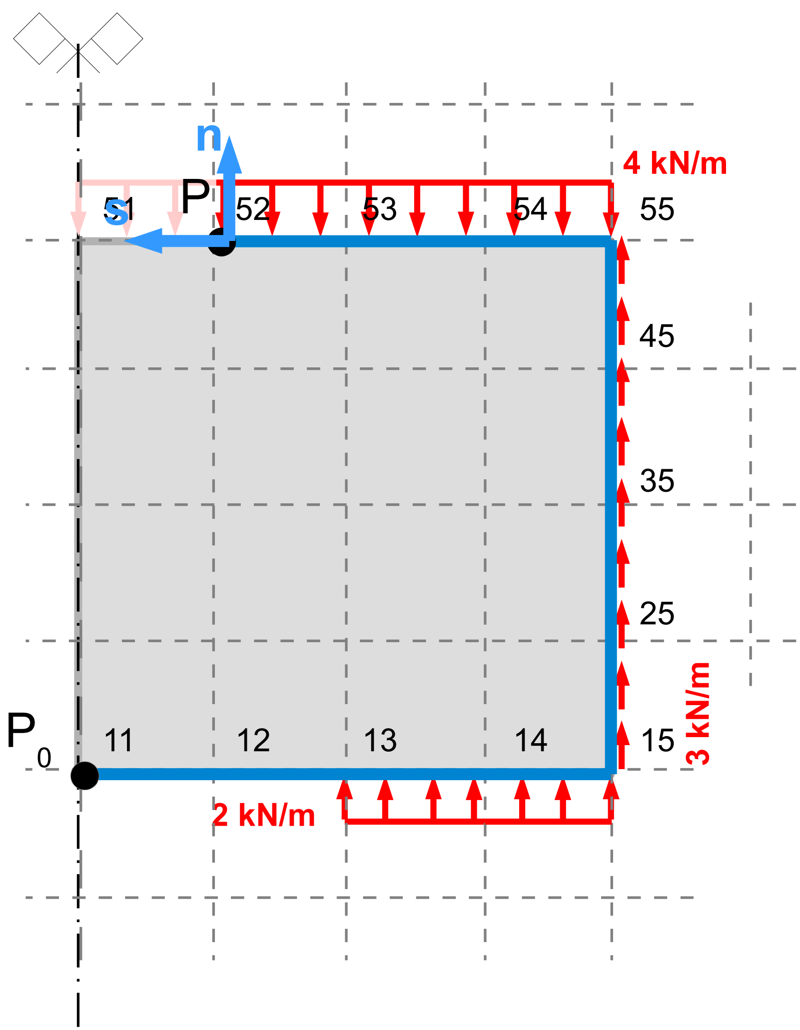



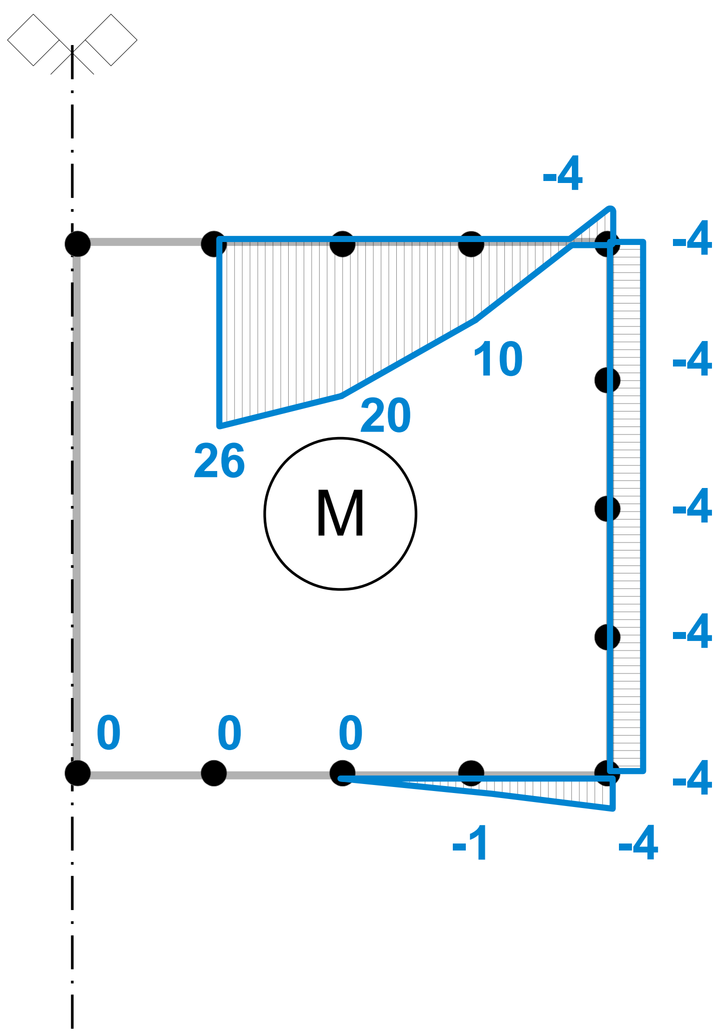
## EXAMPLE

**We take point P in node 52.**

Moment about P (node 52) from load applied to the segment of the boundary between point $P_0$ (węzeł 11) and point P (węzeł 52):

$$M_{52} = 2\cdot 2\cdot 2 + 3\cdot 4\cdot 3 - 4\cdot 3\cdot 1{,}5 = 26\,\text{kNm}$$



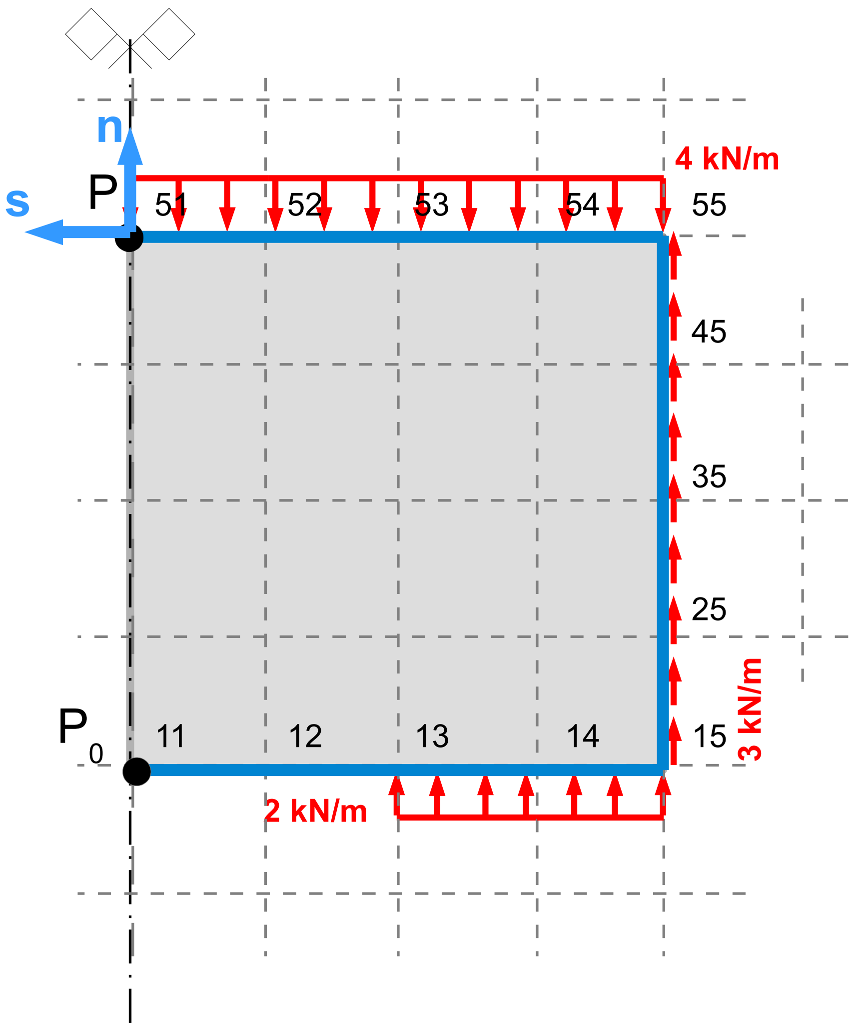



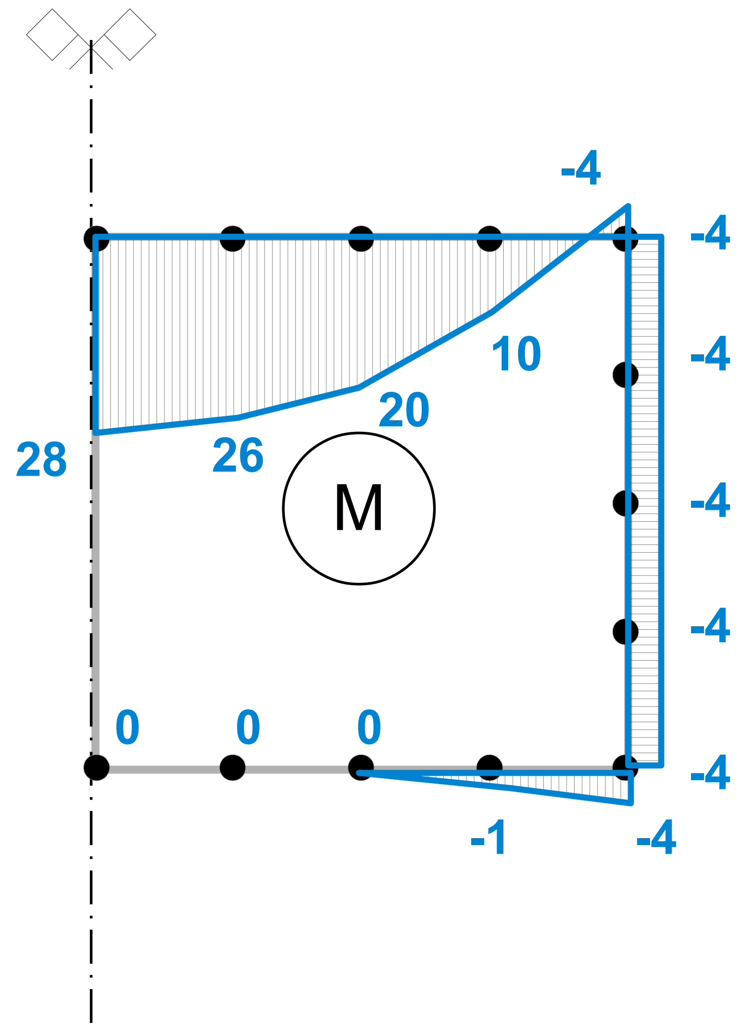
## EXAMPLE

**We take point P in node 51.**

Moment about P (node 51) from load applied to the segment of the boundary between point $P_0$ (węzeł 11) and point P (węzeł 51):

$$M_{51} = 2\cdot2\cdot3 + 3\cdot4\cdot4 - 4\cdot4\cdot2 = 28\,\text{kNm}$$



## EXAMPLE

### BOUNDARY CONDITIONS:

$$F|_P = \frac{M|_P}{h}$$

$$F_{11} = \frac{M_{11}}{h} = \frac{0\,\text{kNm}}{0{,}2\,\text{m}} = 0\ \text{kN}$$

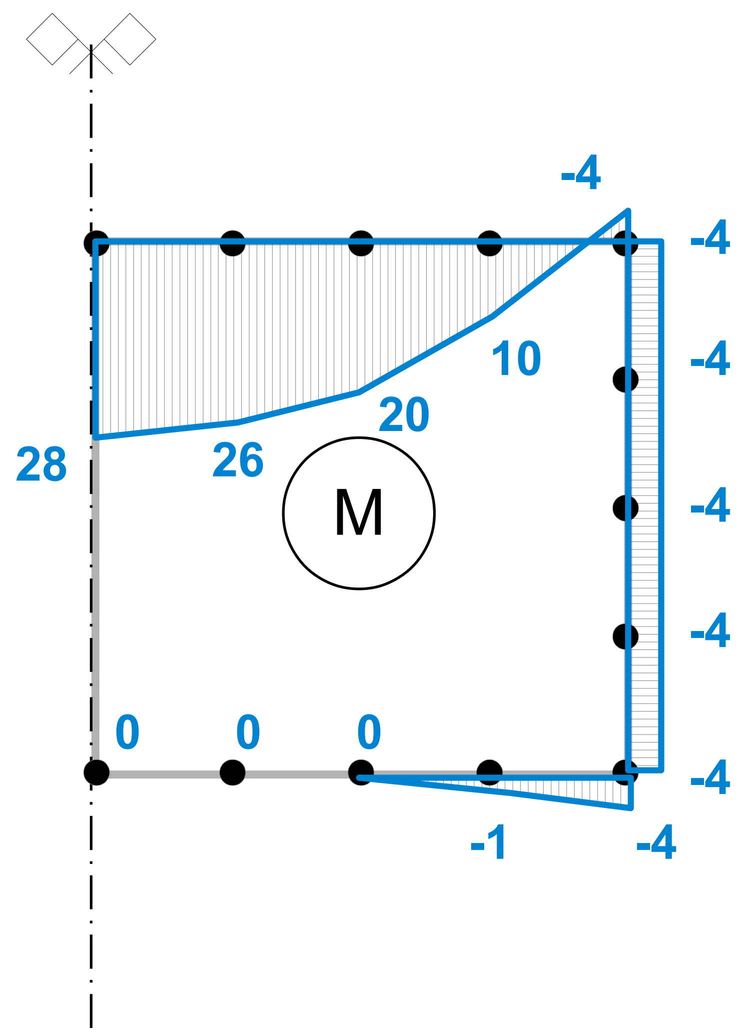
$$F_{12} = \frac{M_{12}}{h} = \frac{0\,\text{kNm}}{0{,}2\,\text{m}} = 0\ \text{kN}$$

...

$$F_{53} = \frac{M_{53}}{h} = \frac{20\,\text{kNm}}{0{,}2\,\text{m}} = 100\ \text{kN}$$

$$F_{52} = \frac{M_{52}}{h} = \frac{26\,\text{kNm}}{0{,}2\,\text{m}} = 130\ \text{kN}$$

$$F_{51} = \frac{M_{51}}{h} = \frac{28\,\text{kNm}}{0{,}2\,\text{m}} = 140\ \text{kN}$$



## EXAMPLE

### BOUNDARY CONDITIONS:

$$F|_P = \frac{M|_P}{h}$$

$$F_{11} = \frac{M_{11}}{h} = \frac{0\,\text{kNm}}{0{,}2\,\text{m}} = 0\ \text{kN}$$

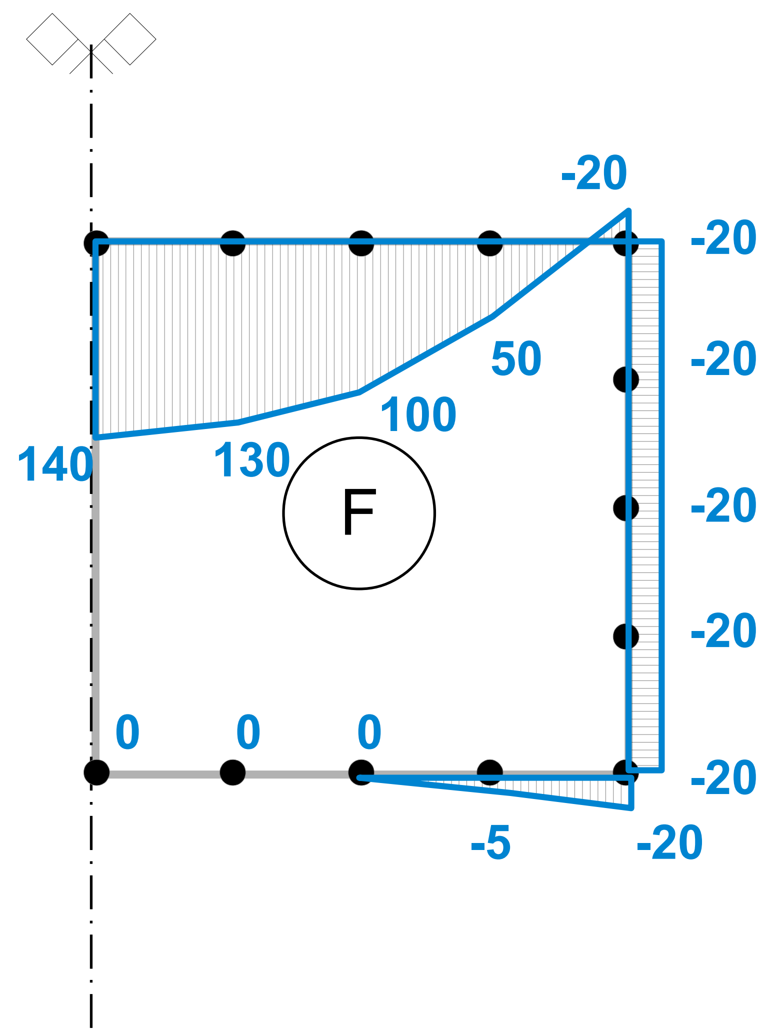
$$F_{12} = \frac{M_{12}}{h} = \frac{0\,\text{kNm}}{0{,}2\,\text{m}} = 0\ \text{kN}$$

...

$$F_{53} = \frac{M_{53}}{h} = \frac{20\,\text{kNm}}{0{,}2\,\text{m}} = 100\ \text{kN}$$

$$F_{52} = \frac{M_{52}}{h} = \frac{26\,\text{kNm}}{0{,}2\,\text{m}} = 130\ \text{kN}$$

$$F_{51} = \frac{M_{51}}{h} = \frac{28\,\text{kNm}}{0{,}2\,\text{m}} = 140\ \text{kN}$$

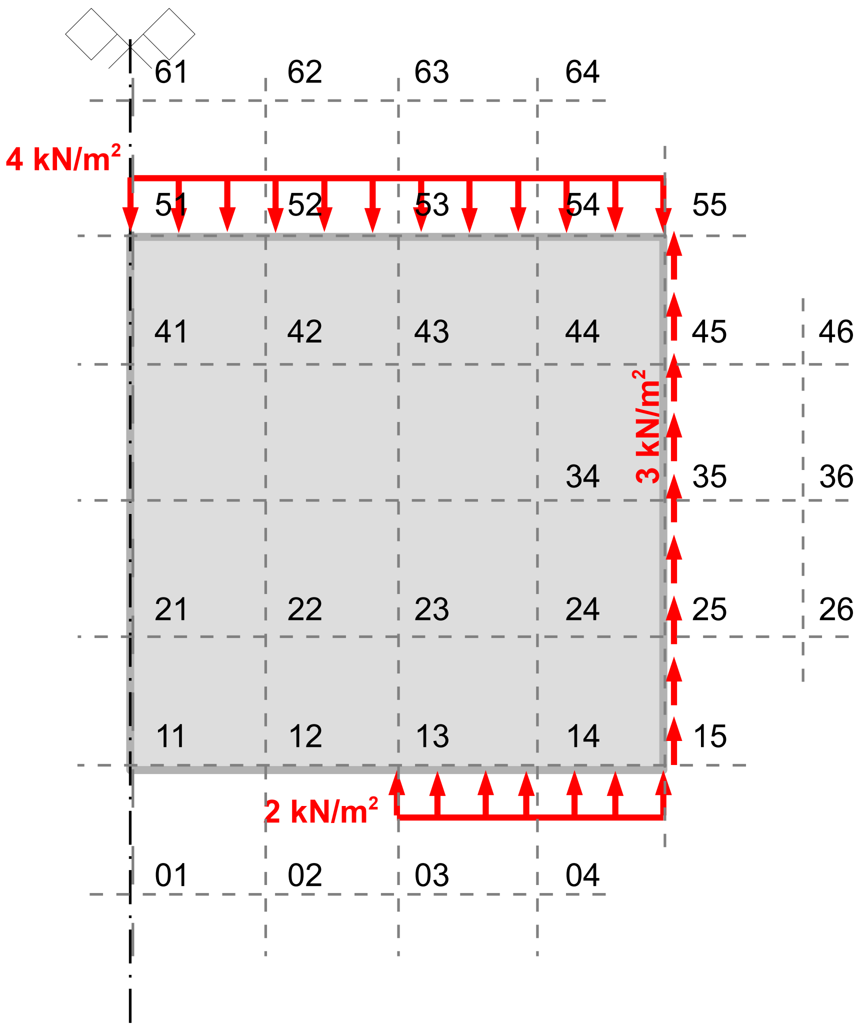


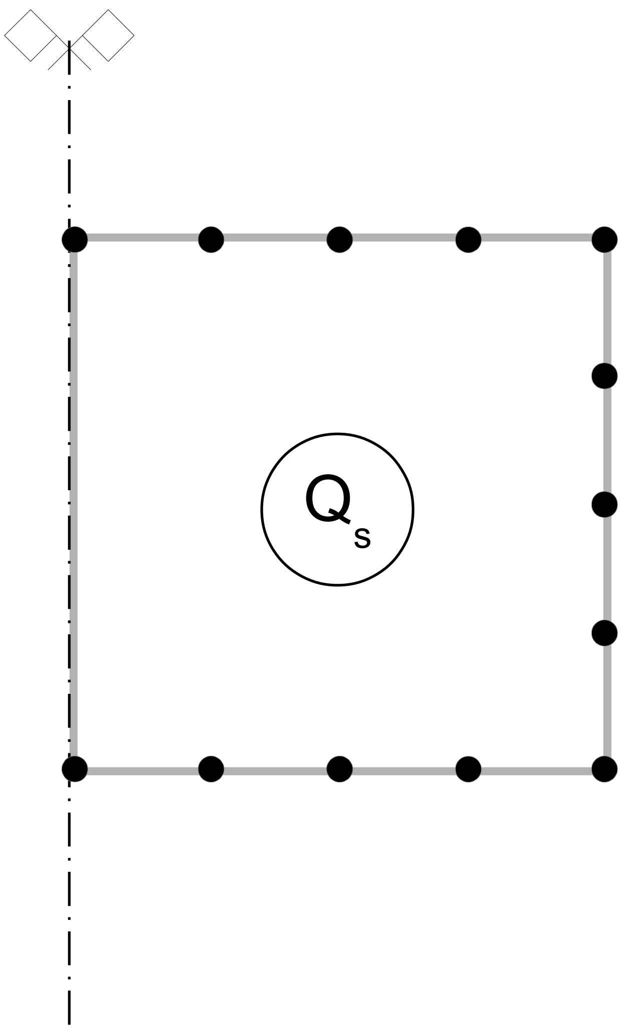
## EXAMPLE

Let's write down the boundary conditions for **directional derivatives of the Airy stress function along external normal**:

$$\left.\frac{\partial F}{\partial n}\right|_P = -\frac{Q_s|_P}{h}$$

**Point $P_0$ is assumed in the same place as it was done when calculating moments, namely in node 11.**



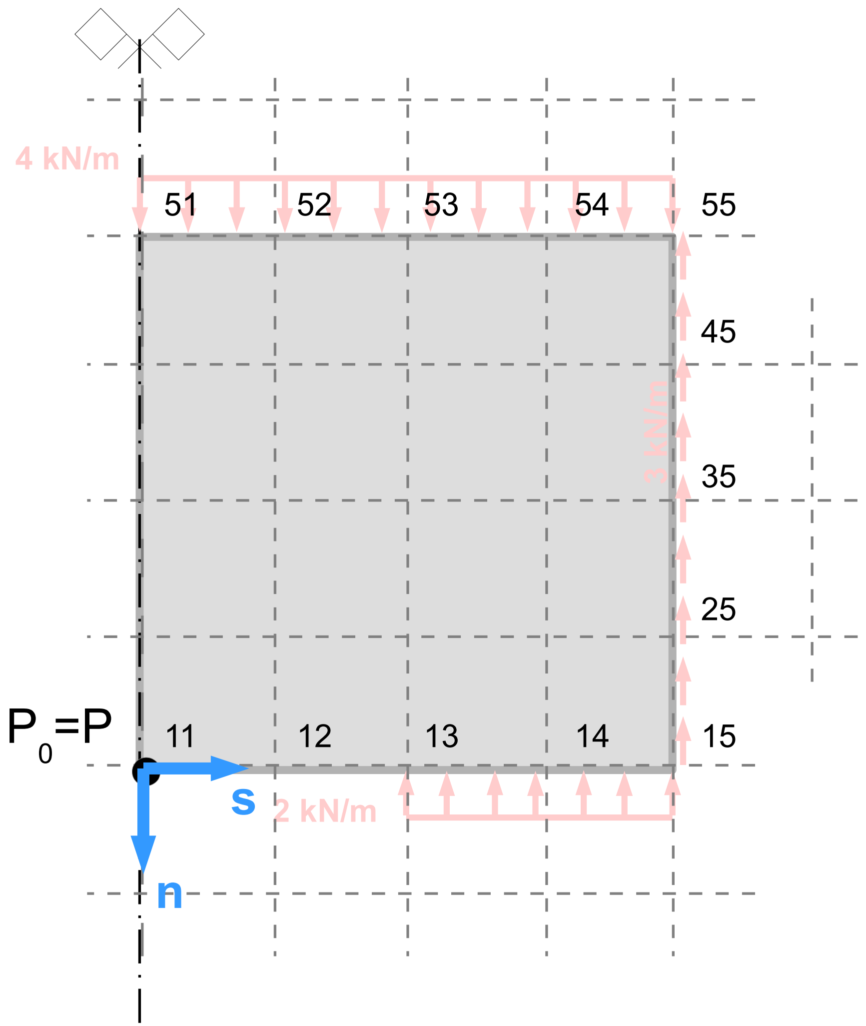



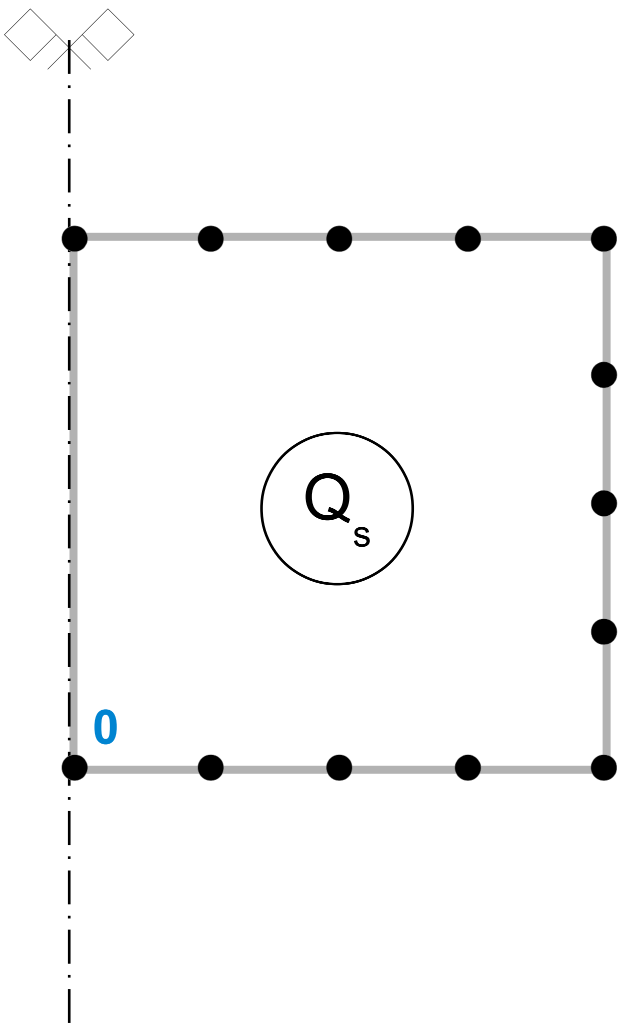
## EXAMPLE

**We take point P in node 11.**

Sum of all forces parallel to axis s applied to the segment of the boundary between point $P_0$ (węzeł 11) and point P (węzeł 11):

$$Q_{s,11} = 0 \text{ kN}$$



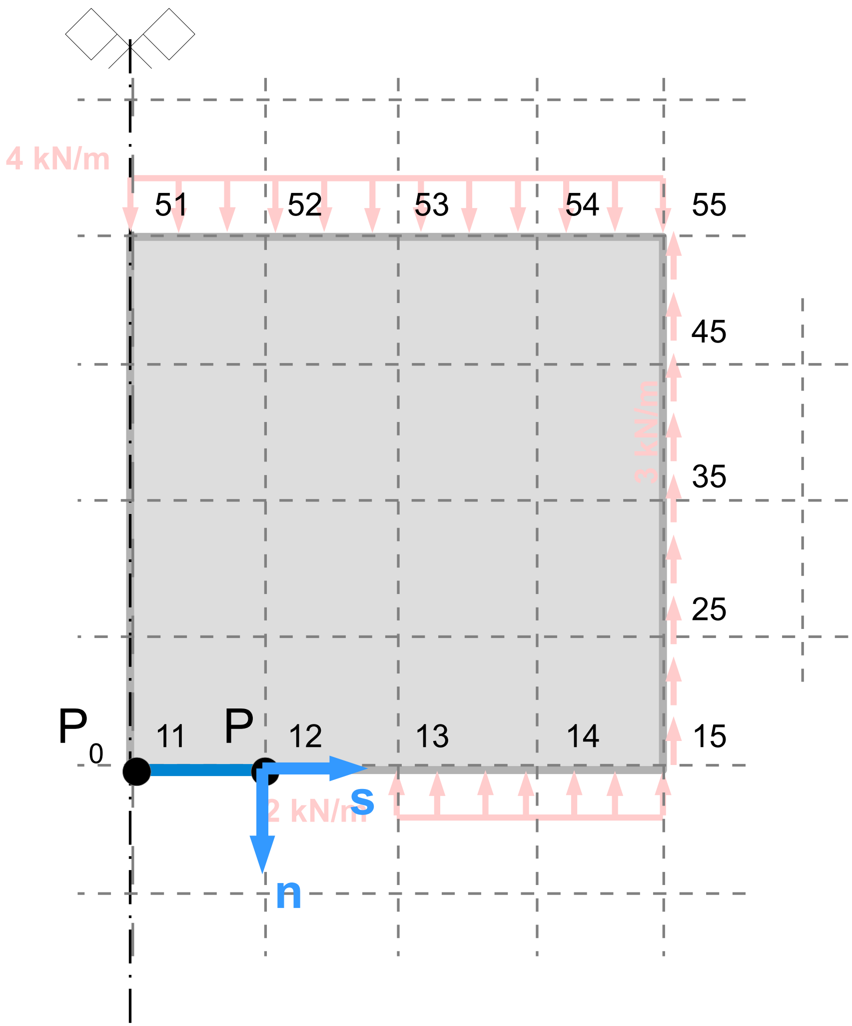



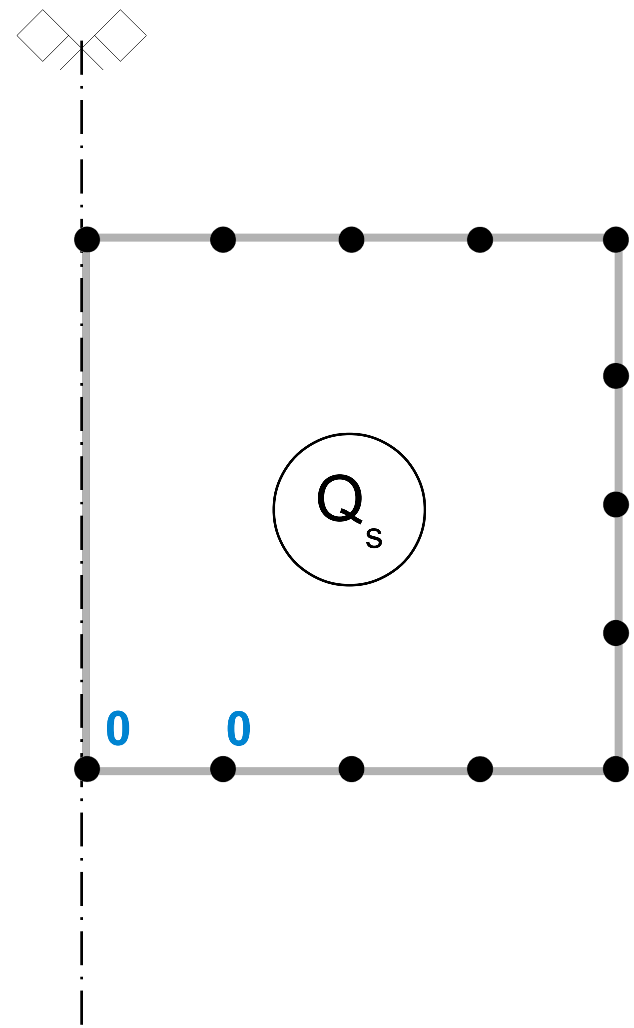
## EXAMPLE

**We take point P in node 12.**

Sum of all forces parallel to axis s applied to the segment of the boundary between point $P_0$ (węzeł 11) and point P (węzeł 12):

$$Q_{s,12} = 0 \text{ kN}$$



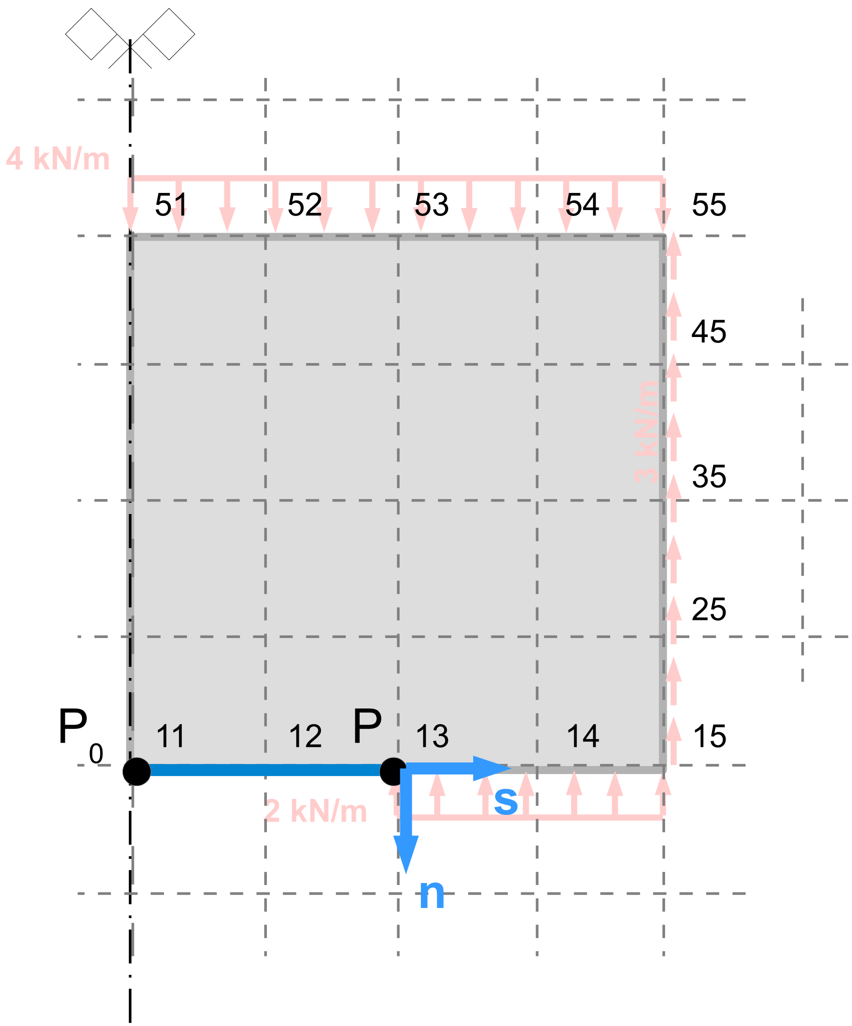



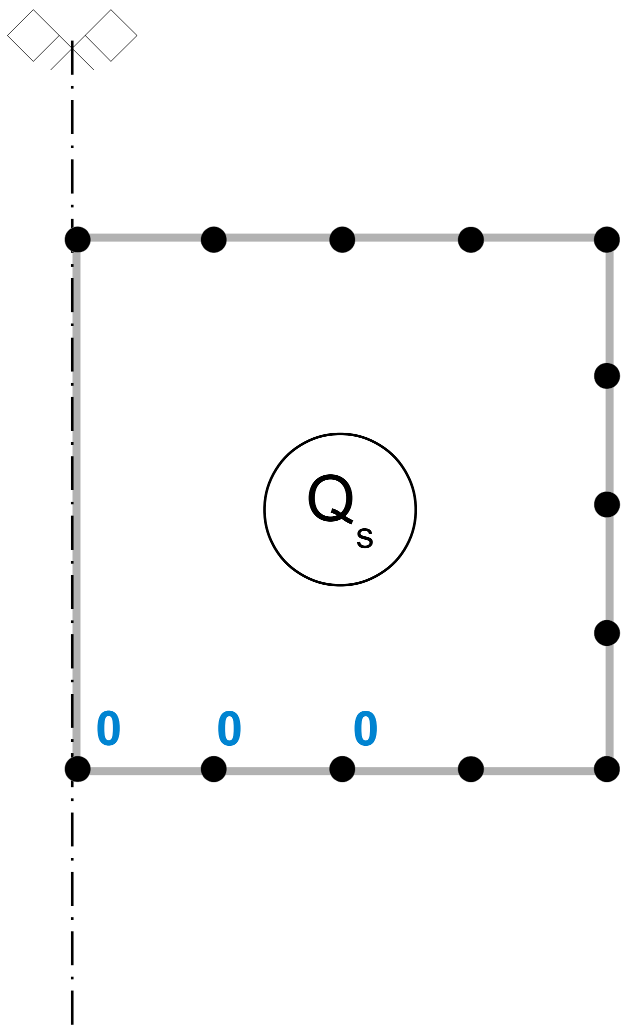
## EXAMPLE

**We take point P in node 13.**

Sum of all forces parallel to axis s applied to the segment of the boundary between point $P_0$ (węzeł 11) and point P (węzeł 13):

$$Q_{s,13} = 0 \text{ kN}$$



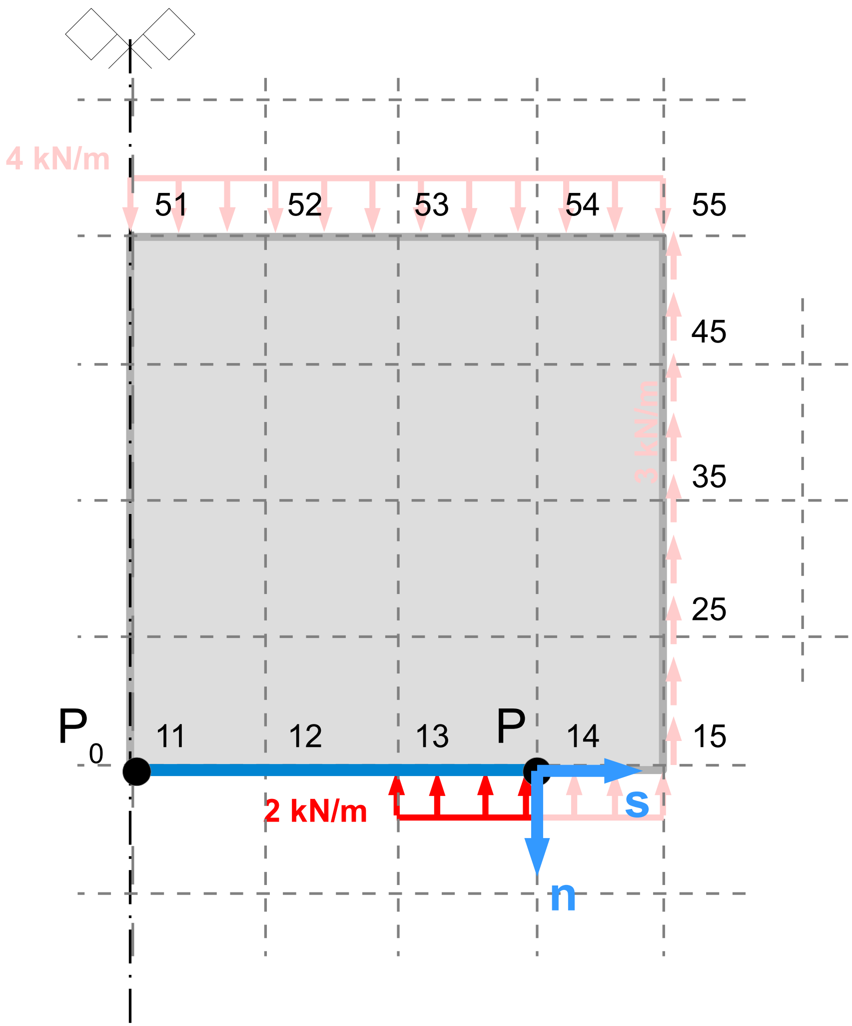
## EXAMPLE

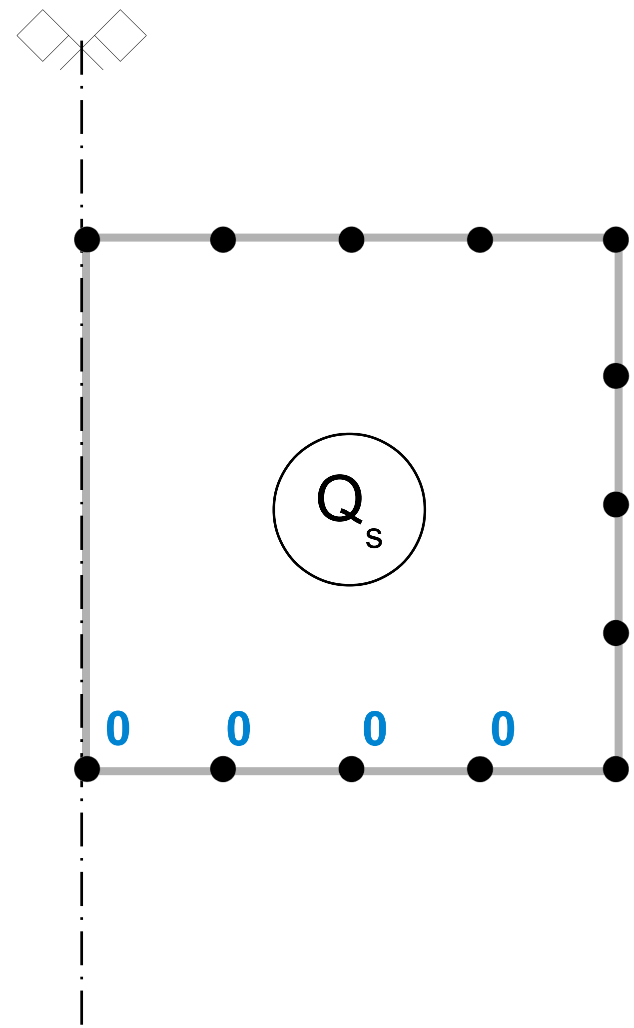
**We take point P in node 14.**

Sum of all forces parallel to axis s applied to the segment of the boundary between point $P_0$ (węzeł 11) and point P (węzeł 14):

$$Q_{s,14} = 0 \text{ kN}$$

(the load is parallel to n axis – **there is no load along s axis**)



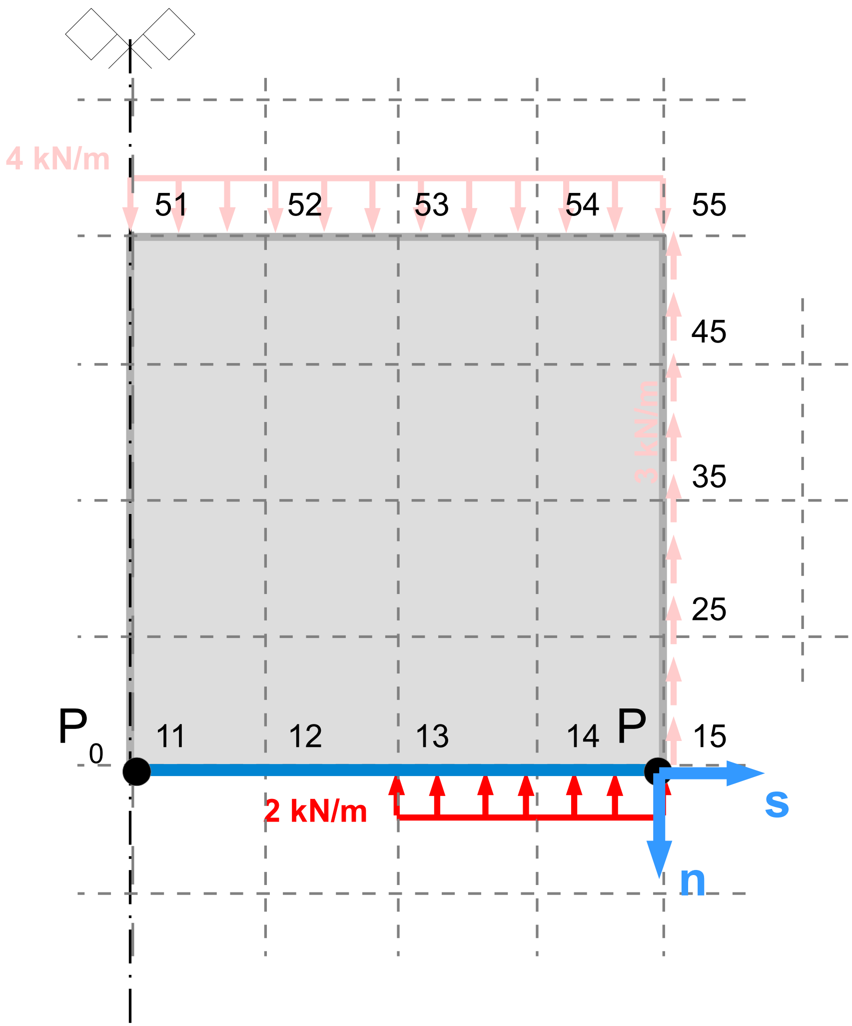
## EXAMPLE

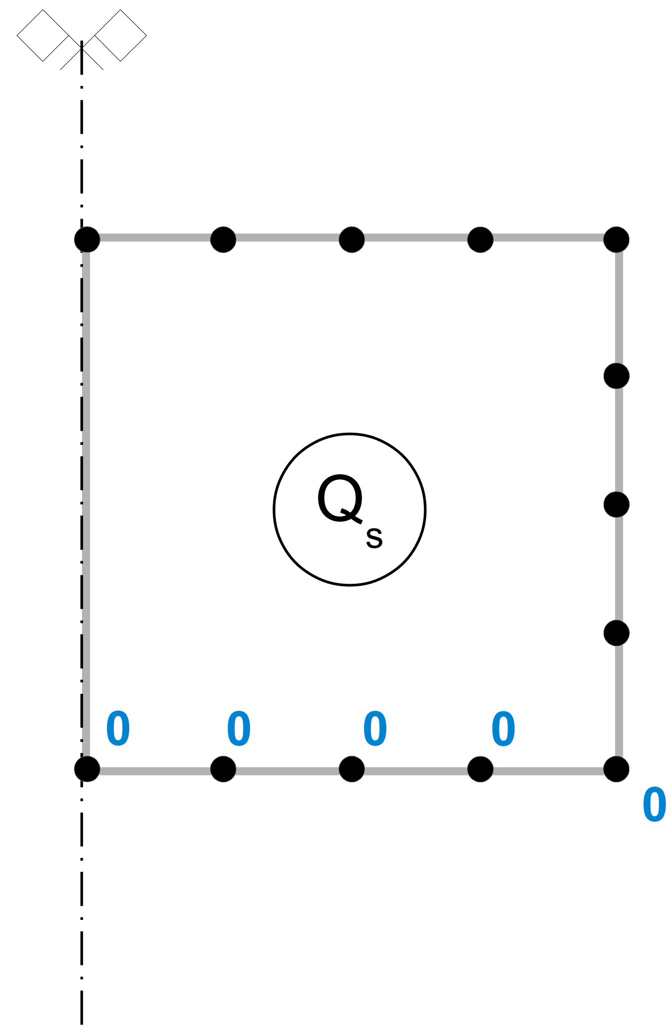
**We take point P in node 15.**

Sum of all forces parallel to axis s applied to the segment of the boundary between point $P_0$ (węzeł 11) and point P (węzeł 15):

$$Q^-_{s,15} = 0 \text{ kN}$$

(this is **one-sided left value**)



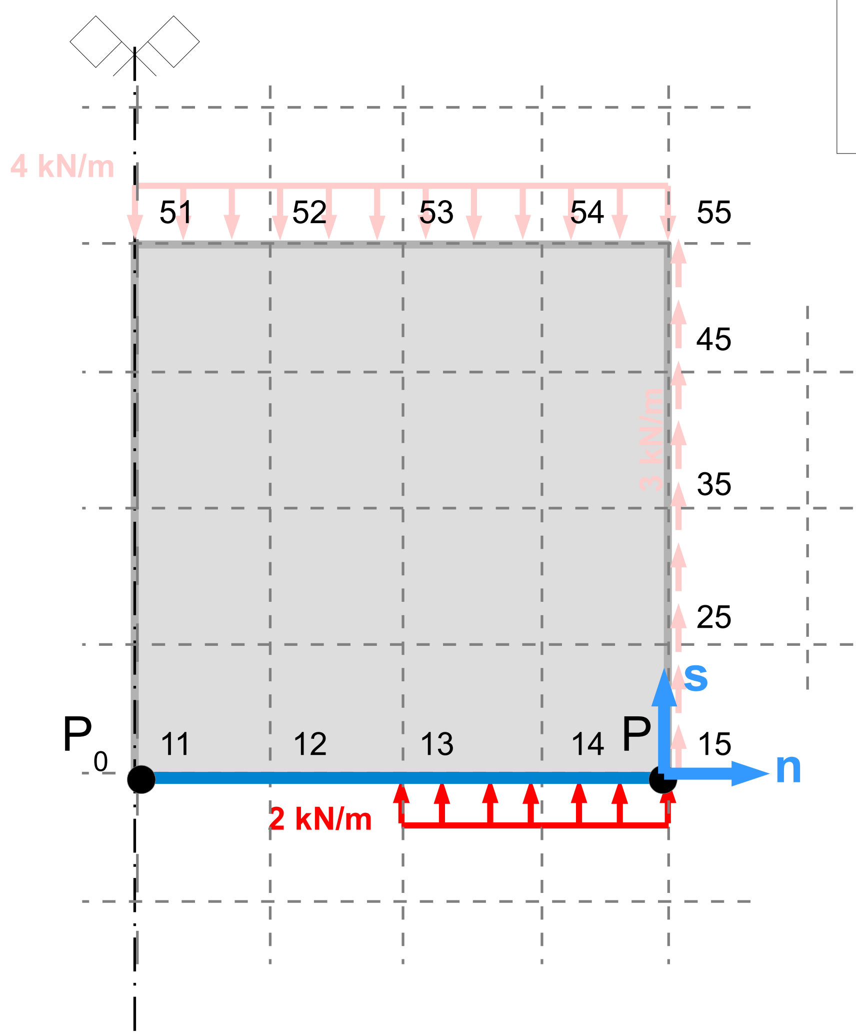
## EXAMPLE

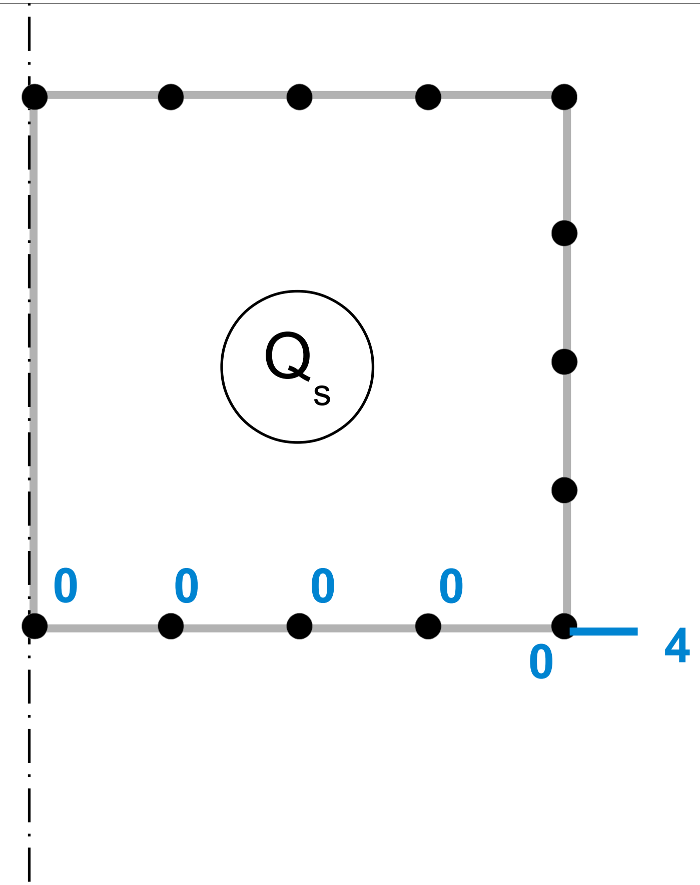
**We take point P in node 15.**

Sum of all forces parallel to axis s applied to the segment of the boundary between point $P_0$ (węzeł 11) and point P (węzeł 15):

$$Q^{+}_{s,15} = 2\,\mathrm{kN/m}\cdot 2\,\mathrm{m} = 4\,\mathrm{kN}$$

(this is **one-sided right value** - after passing to the vertical segment of the boundary, local coordinate system rotates and tangent axis s is now vertical. Now we can see load which is parallel to s. It is oriented in the same way, so it is considered positive. In fact, we don't need the value of this force in a corner point, because for such point the boundary condition for derivative is not written down.)



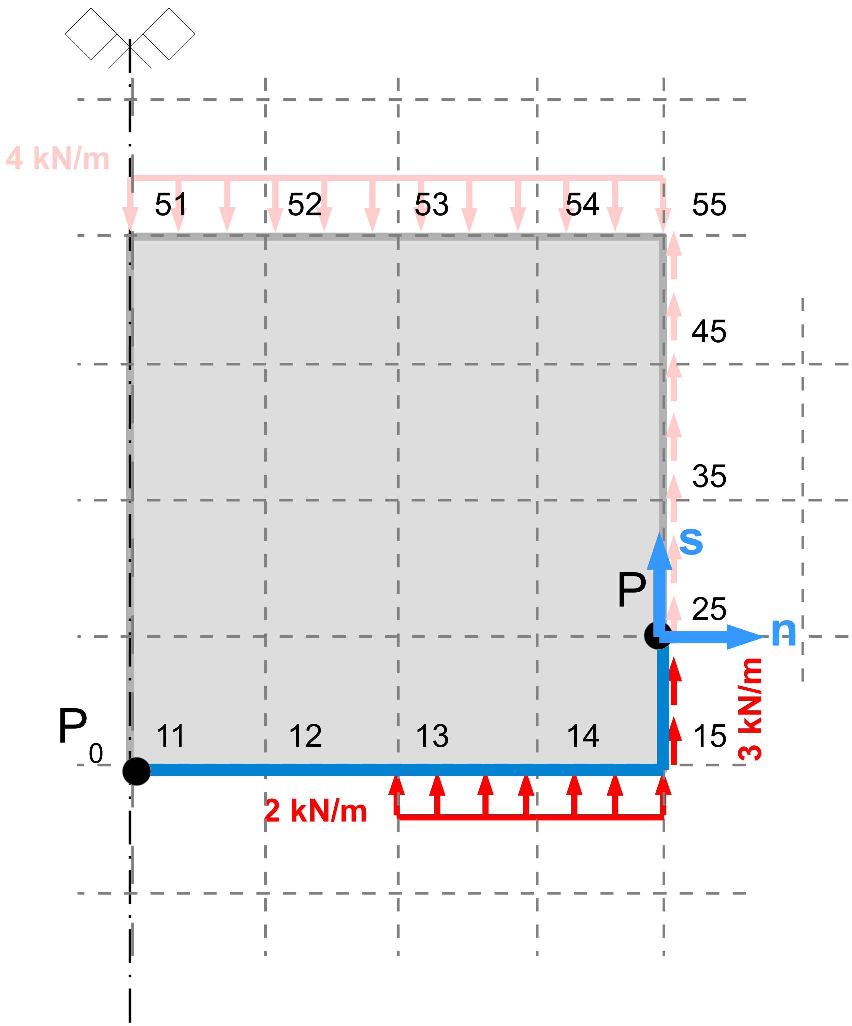



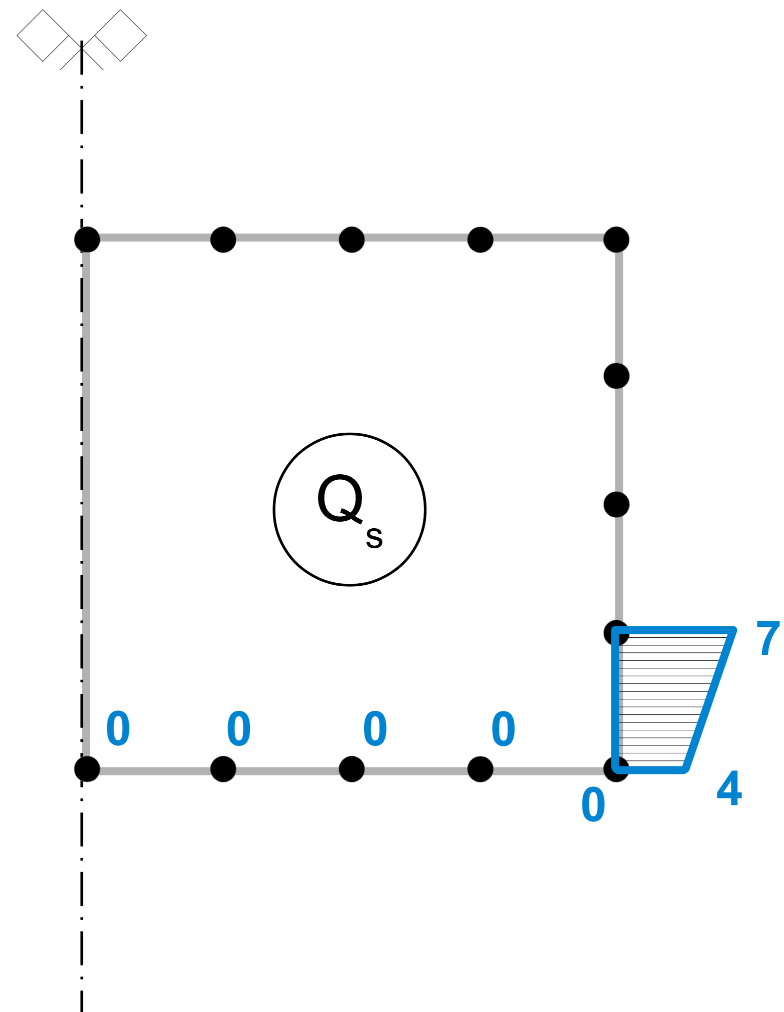
## EXAMPLE

**We take point P in node 25.**

Sum of all forces parallel to axis s applied to the segment of the boundary between point $P_0$ (węzeł 11) and point P (węzeł 25):

$$Q_{s,25} = 2\cdot 2 + 3\cdot 1 = 7\,\mathrm{kN}$$



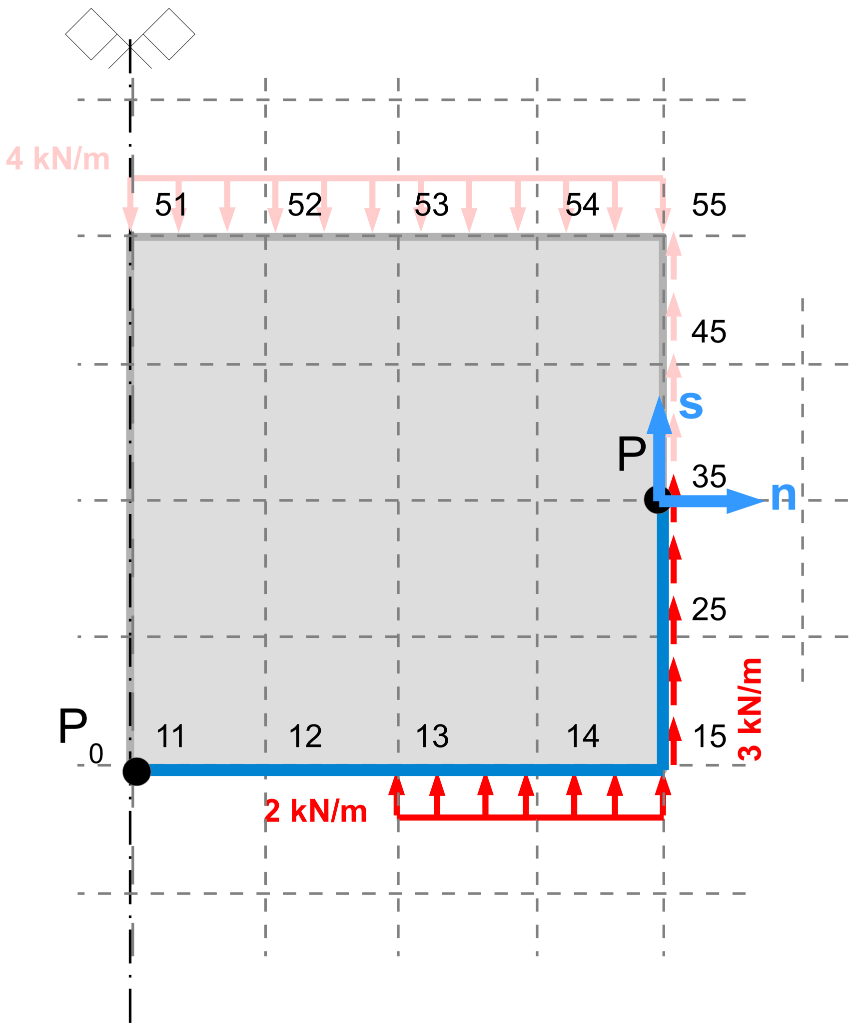



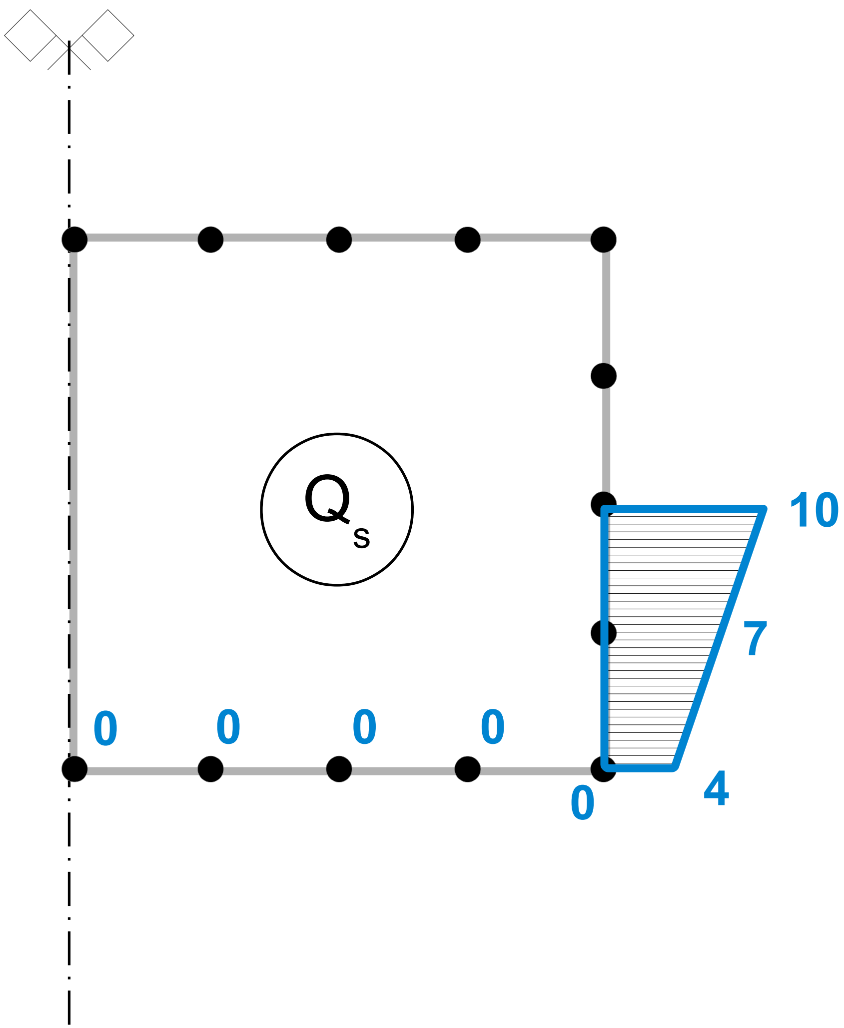
## EXAMPLE

**We take point P in node 35.**

Sum of all forces parallel to axis s applied to the segment of the boundary between point $P_0$ (węzeł 11) and point P (węzeł 35):

$$Q_{s,35} = 2 \cdot 2 + 3 \cdot 2 = 10 \text{ kN}$$



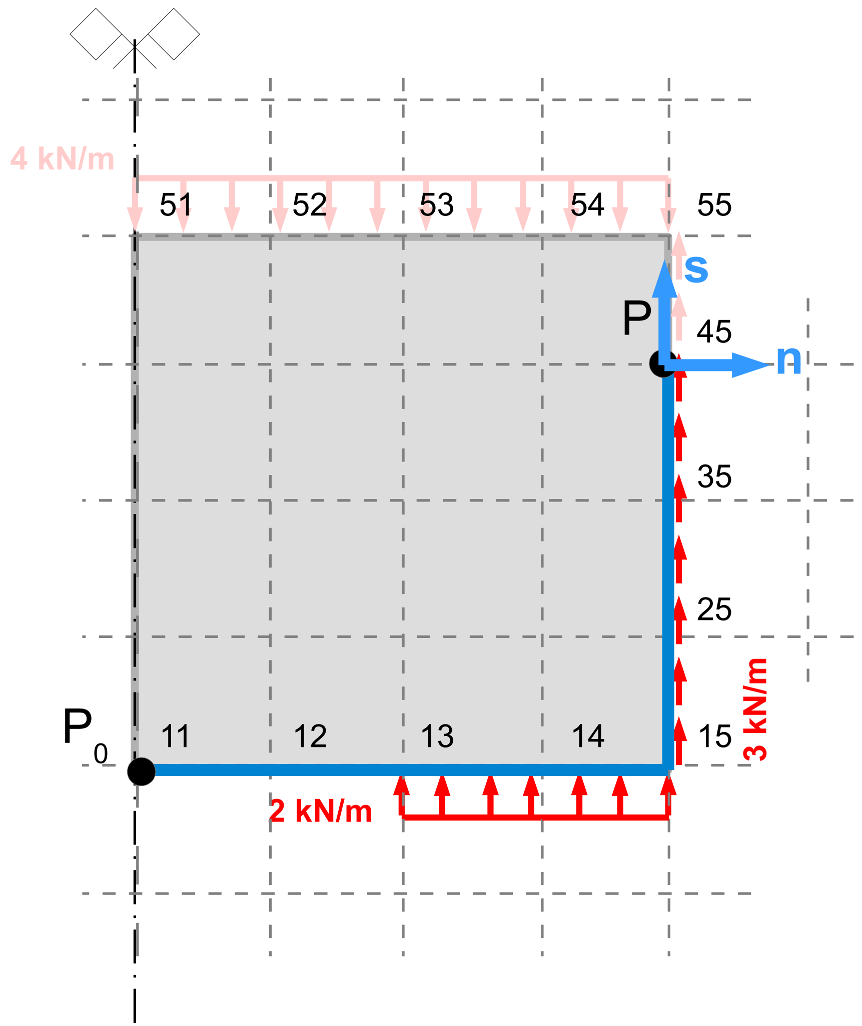



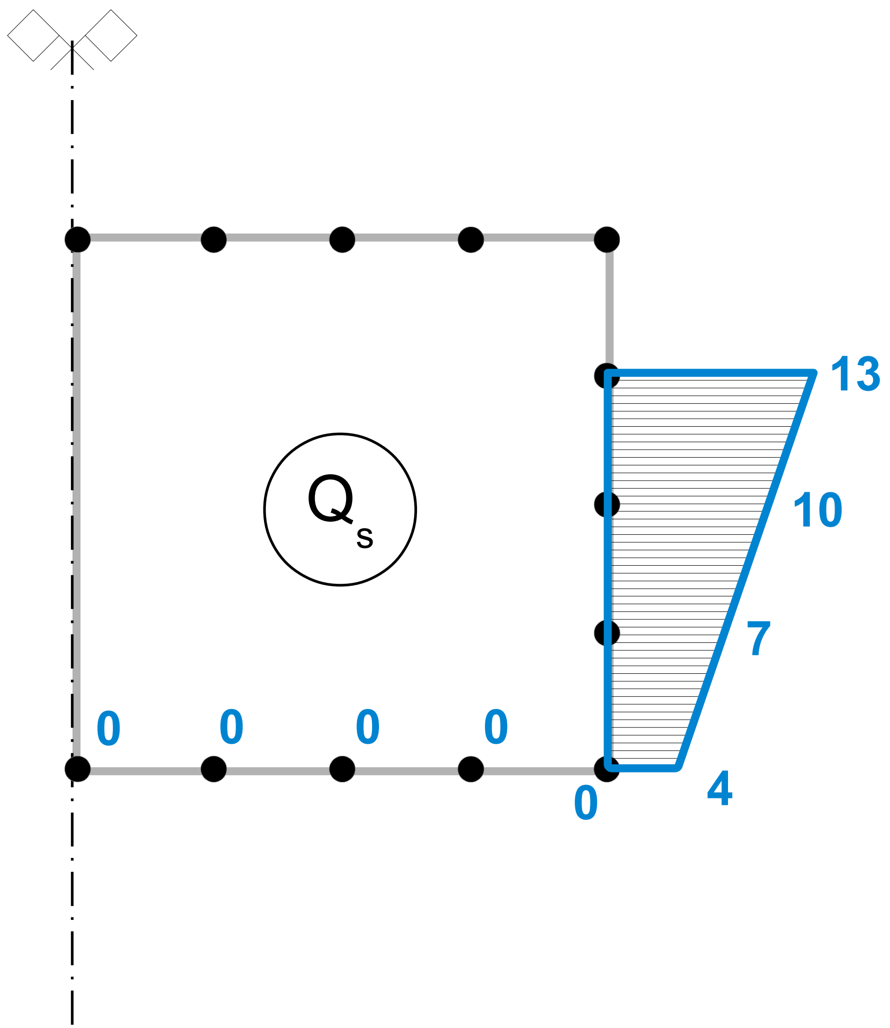
## EXAMPLE

**We take point P in node 45.**

Sum of all forces parallel to axis s applied to the segment of the boundary between point $P_0$ (węzeł 11) and point P (węzeł 45):

$$Q_{s,45} = 2 \cdot 2 + 3 \cdot 3 = 13 \text{ kN}$$



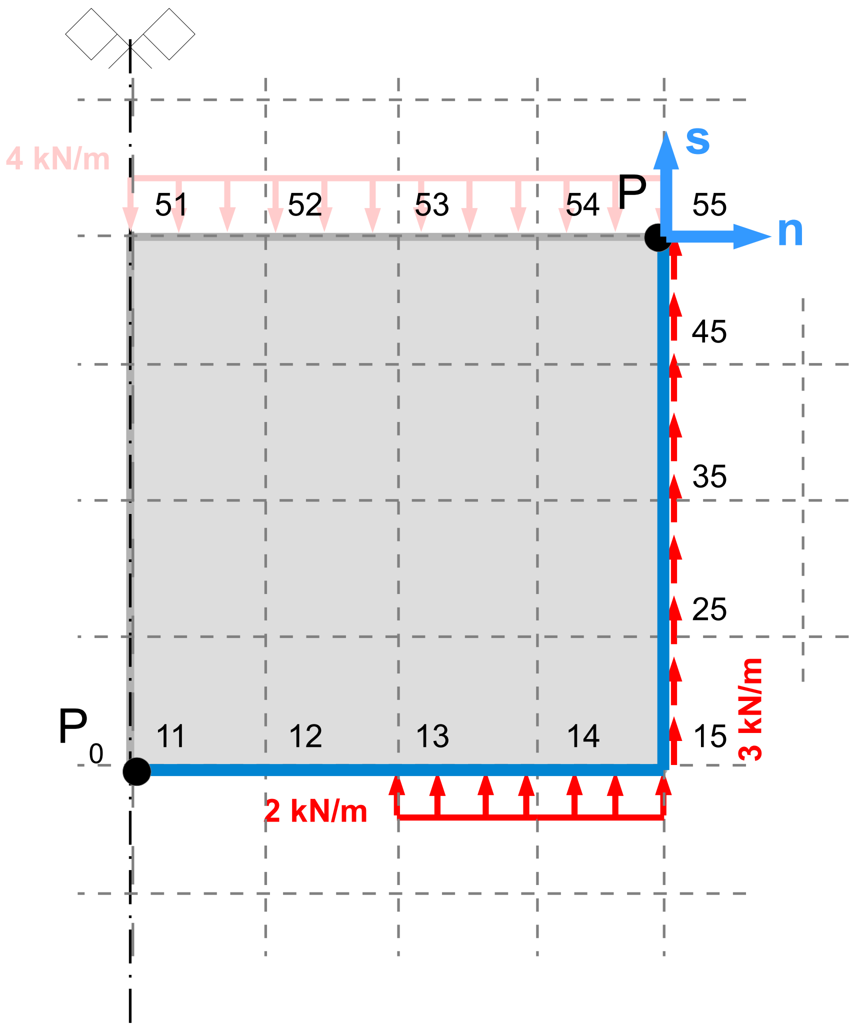
## EXAMPLE

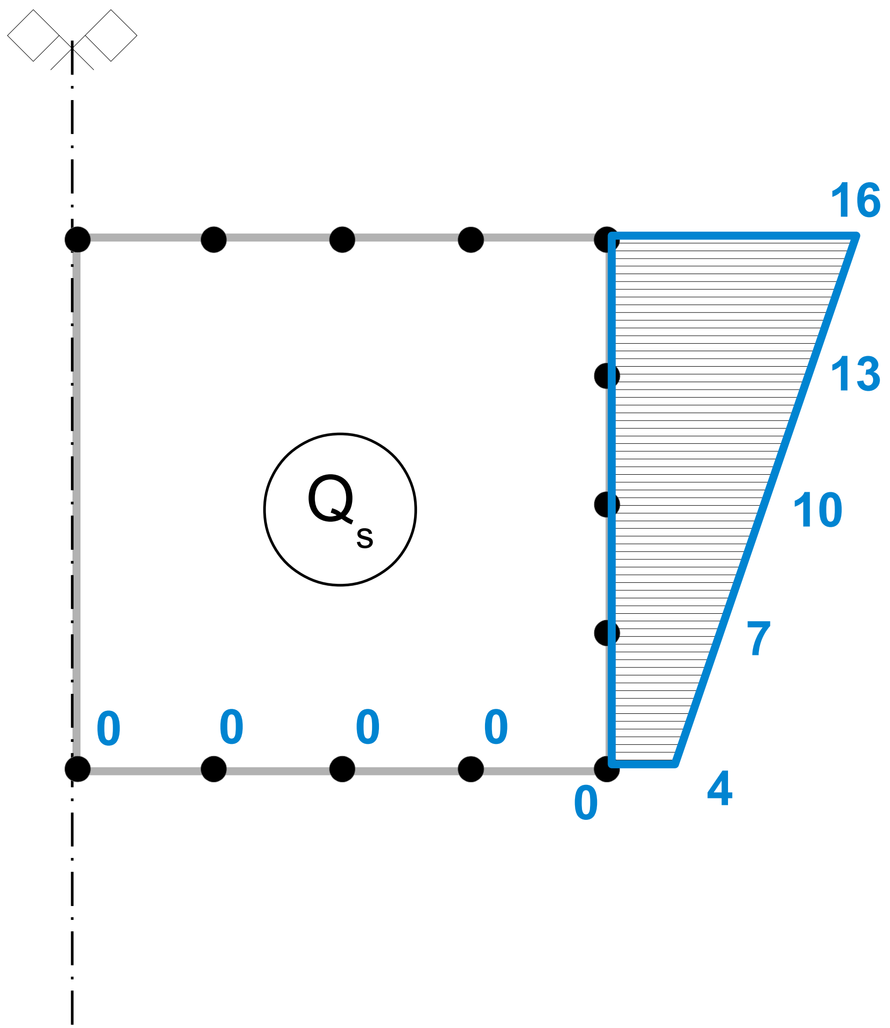
**We take point P in node 55.**

Sum of all forces parallel to axis s applied to the segment of the boundary between point $P_0$ (węzeł 11) and point P (węzeł 55):

$$Q^{-}_{s,55} = 2 \cdot 2 + 3 \cdot 4 = 16 \text{ kN}$$

(**one-sided left value**)



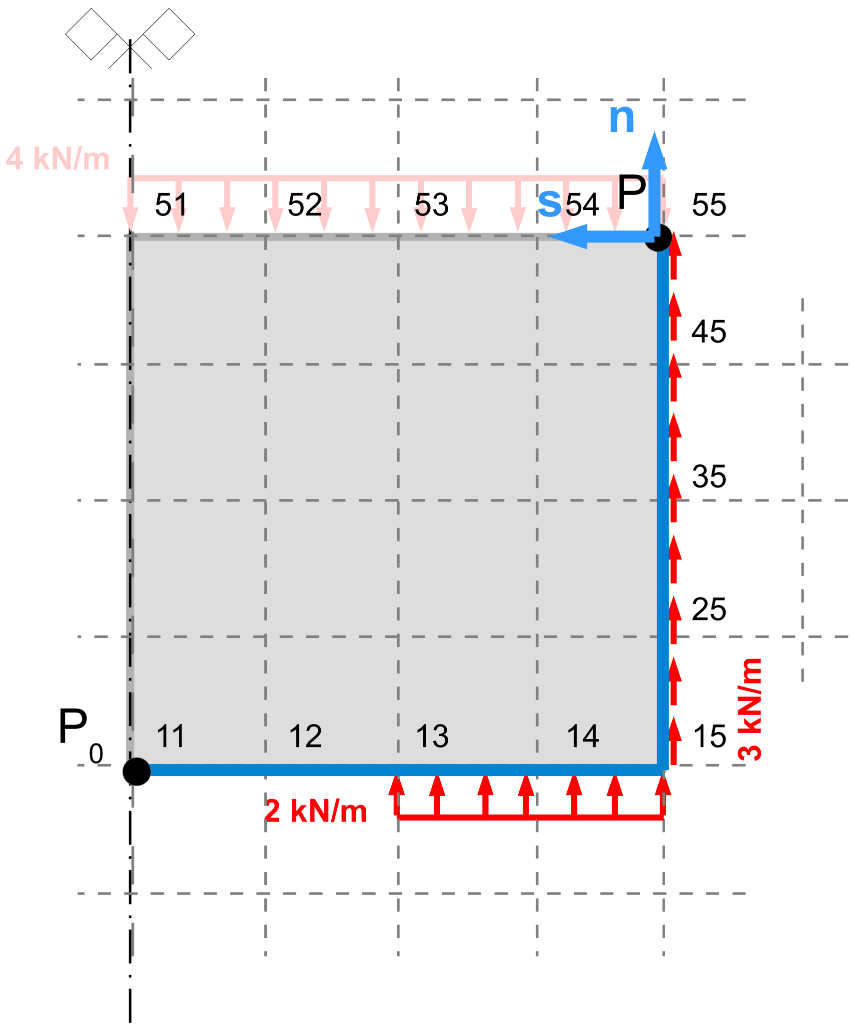
## EXAMPLE

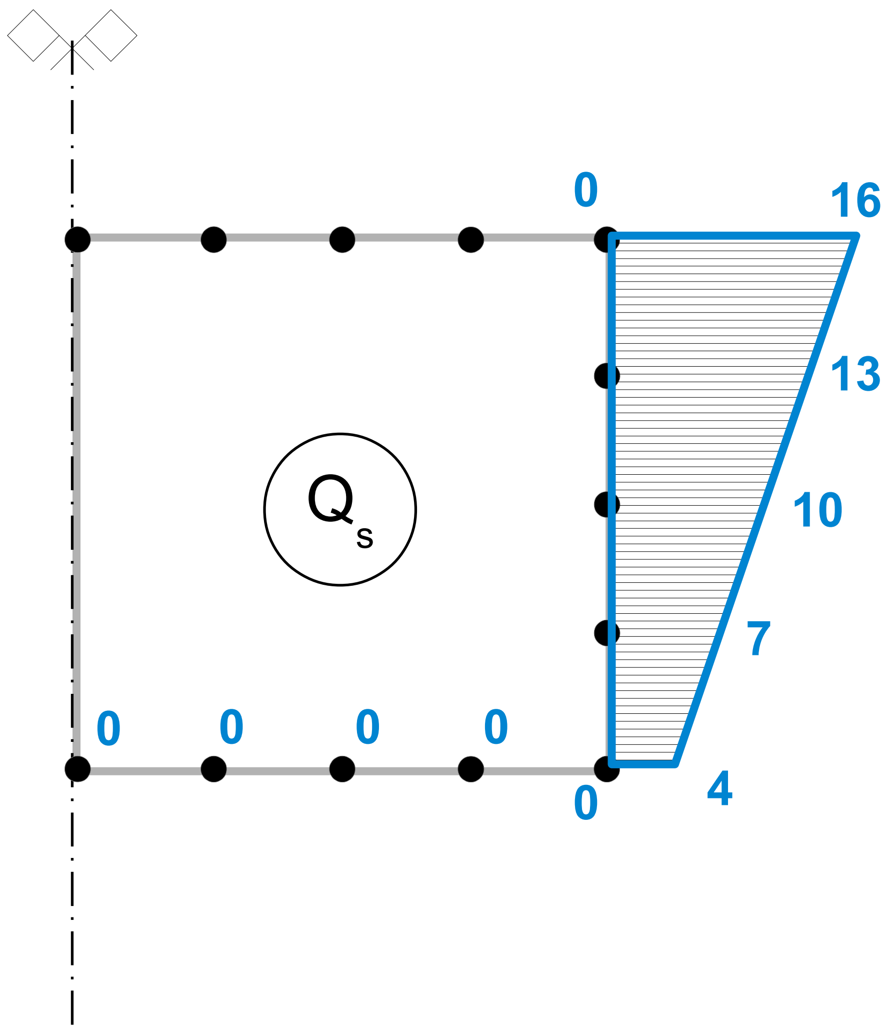
**We take point P in node 55.**

Sum of all forces parallel to axis s applied to the segment of the boundary between point $P_0$ (węzeł 11) and point P (węzeł 55):

$$Q^{+}_{s,55} = 0\ \text{kN}$$

(**one-sided right value**)



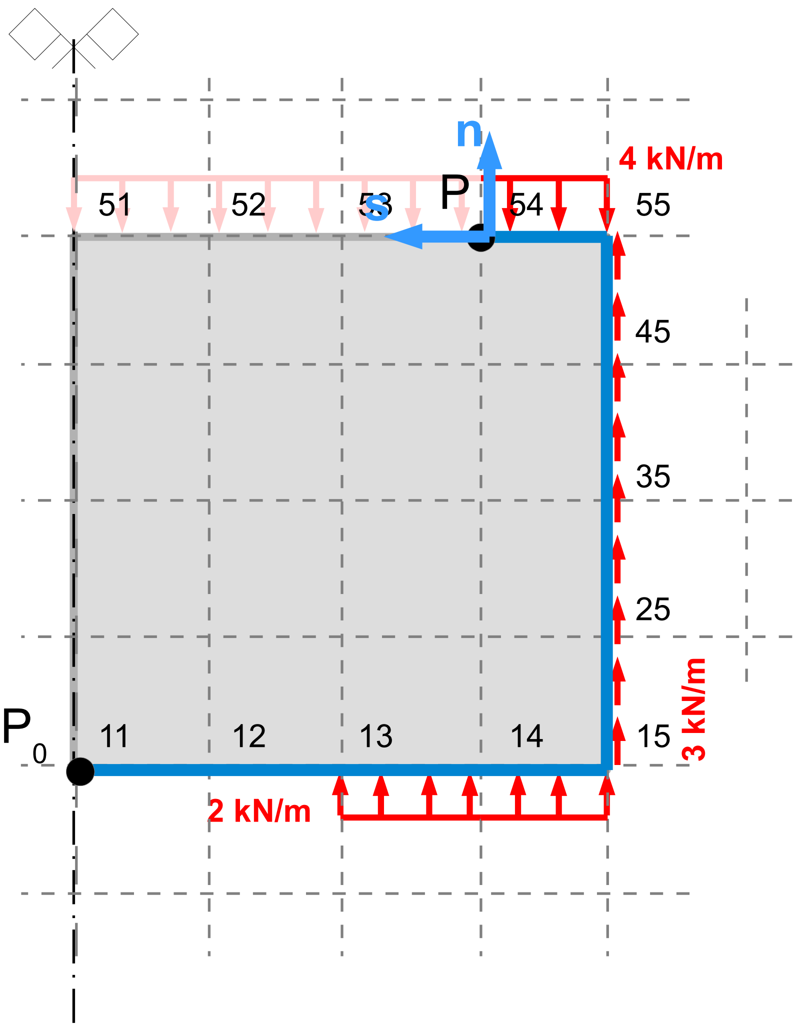



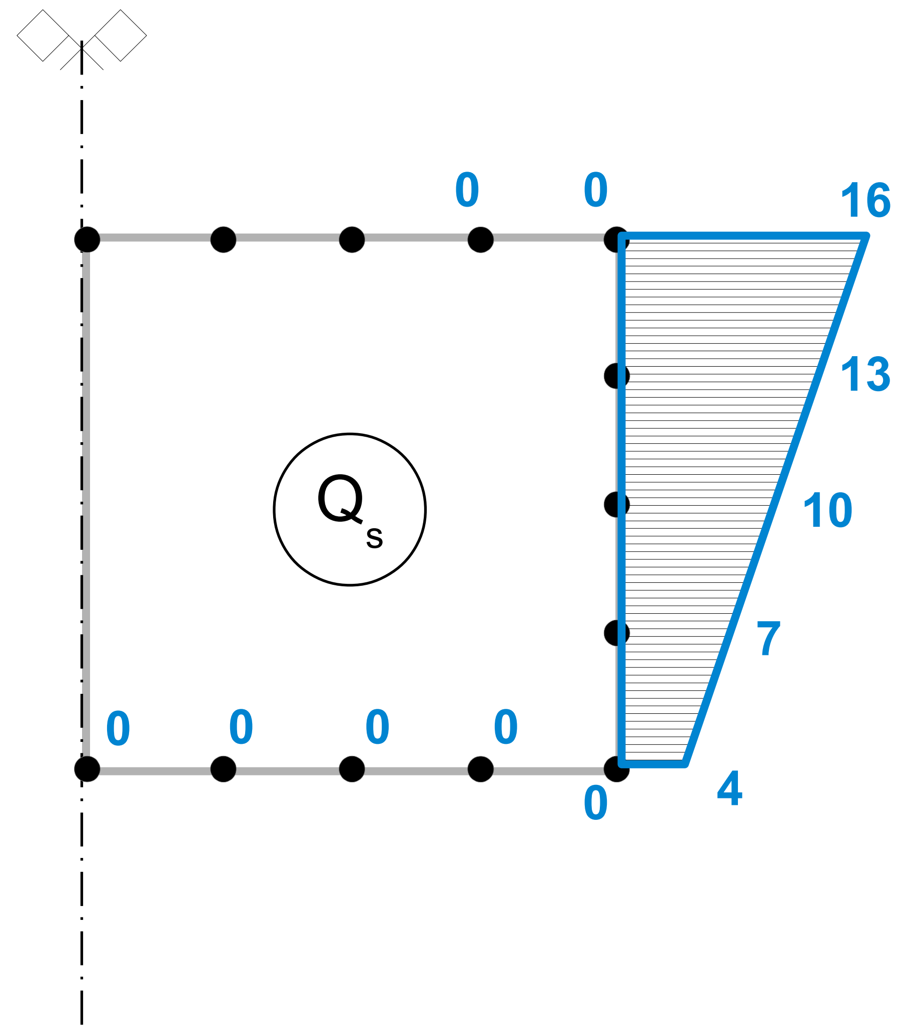
## EXAMPLE

**We take point P in node 54.**

Sum of all forces parallel to axis s applied to the segment of the boundary between point $P_0$ (węzeł 11) and point P (węzeł 54):

$$Q_{s,54} = 0 \text{ kN}$$



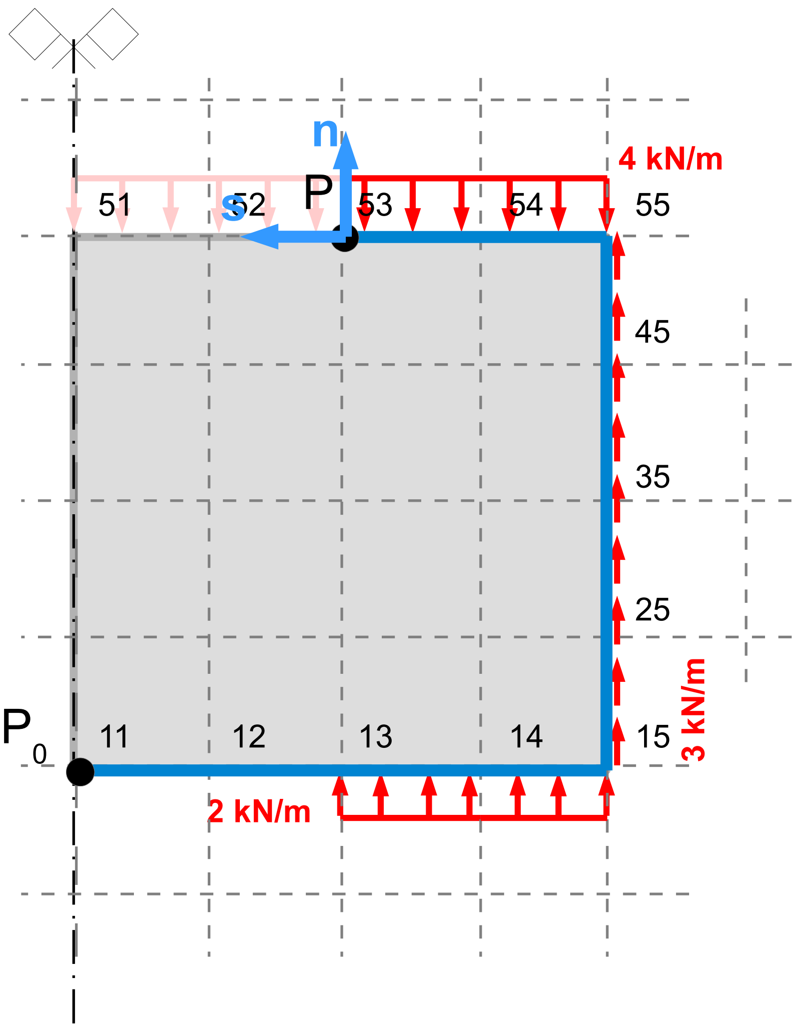



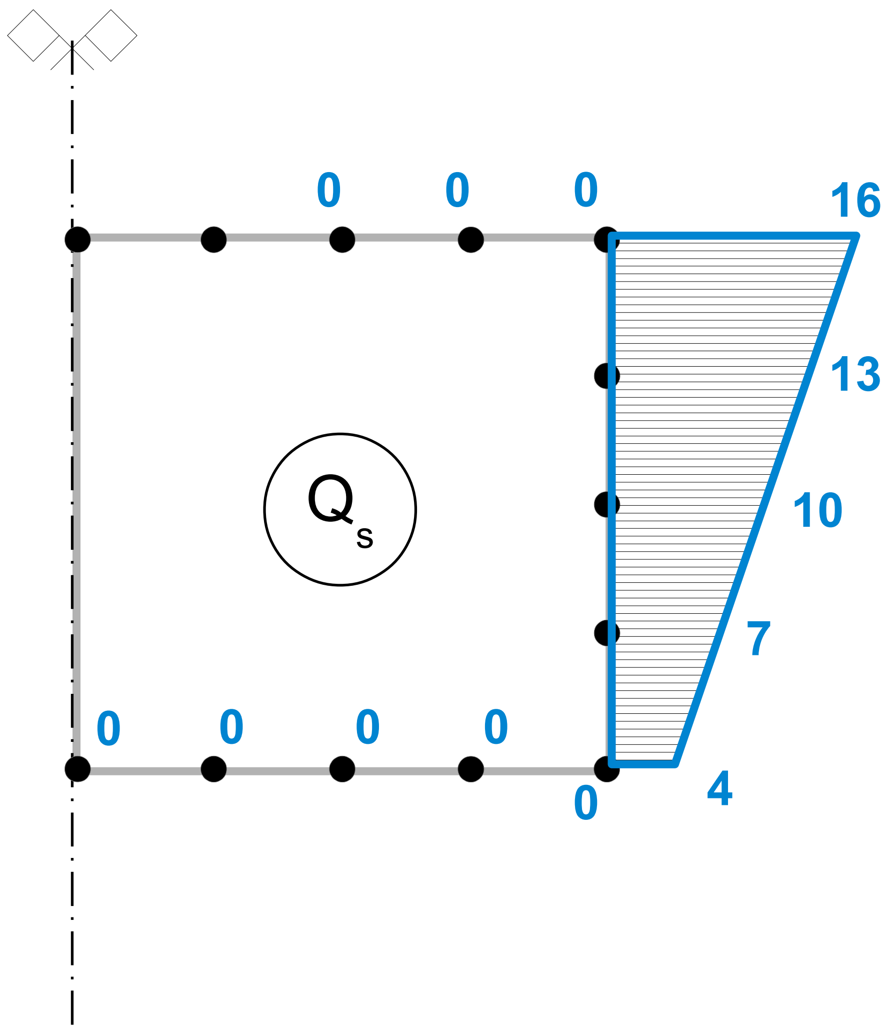
**EXAMPLE**

**We take point P in node 53.**

Sum of all forces parallel to axis s applied to the segment of the boundary between point $P_0$ (węzeł 11) and point P (węzeł 53):

$$Q_{s,53} = 0 \text{ kN}$$



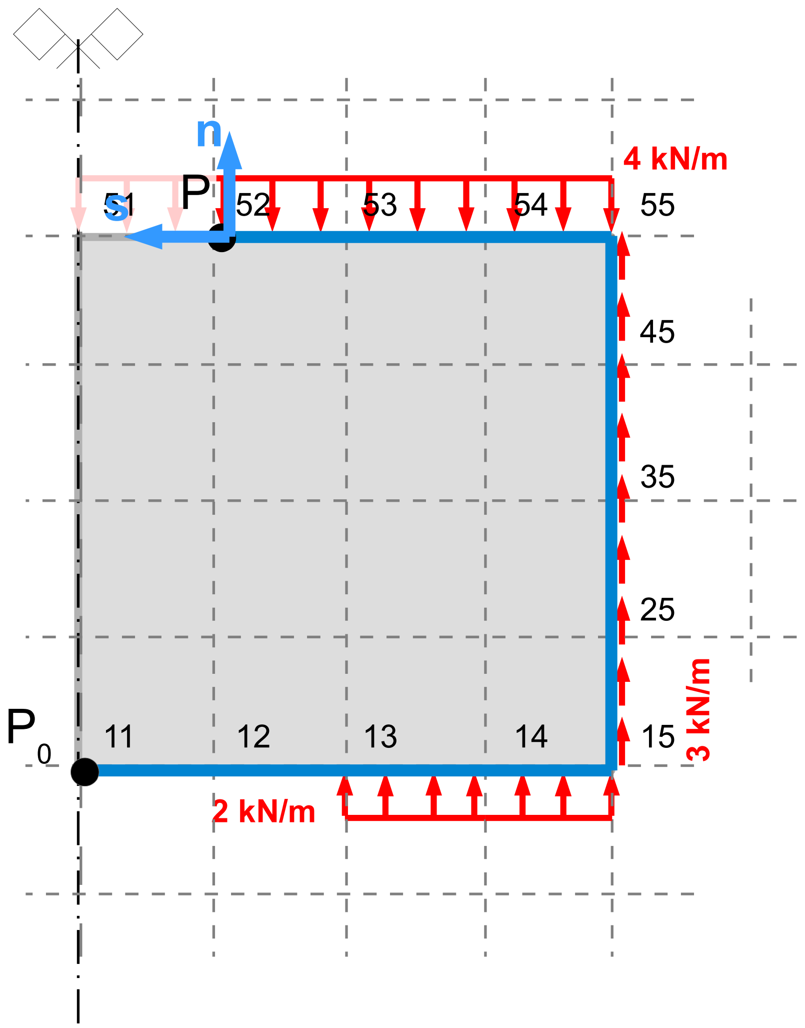



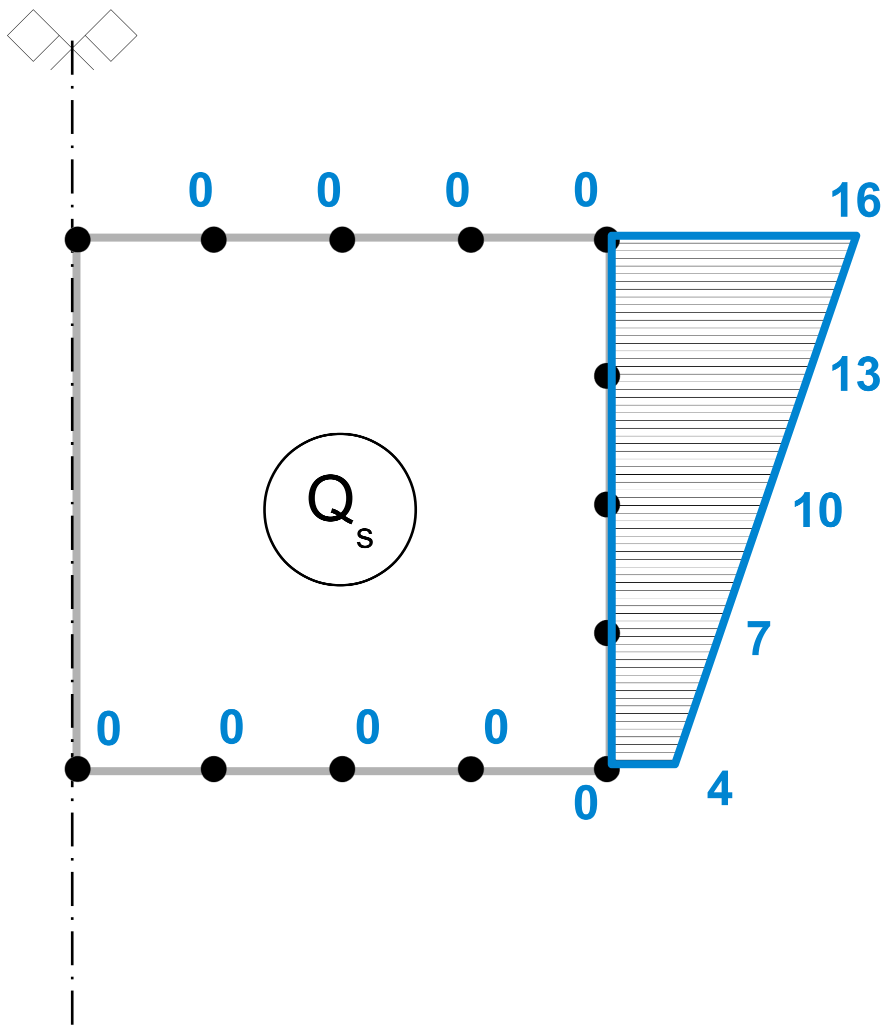
## EXAMPLE

**We take point P in node 52.**

Sum of all forces parallel to axis s applied to the segment of the boundary between point $P_0$ (węzeł 11) and point P (węzeł 52):

$$Q_{s,52} = 0 \text{ Nm}$$



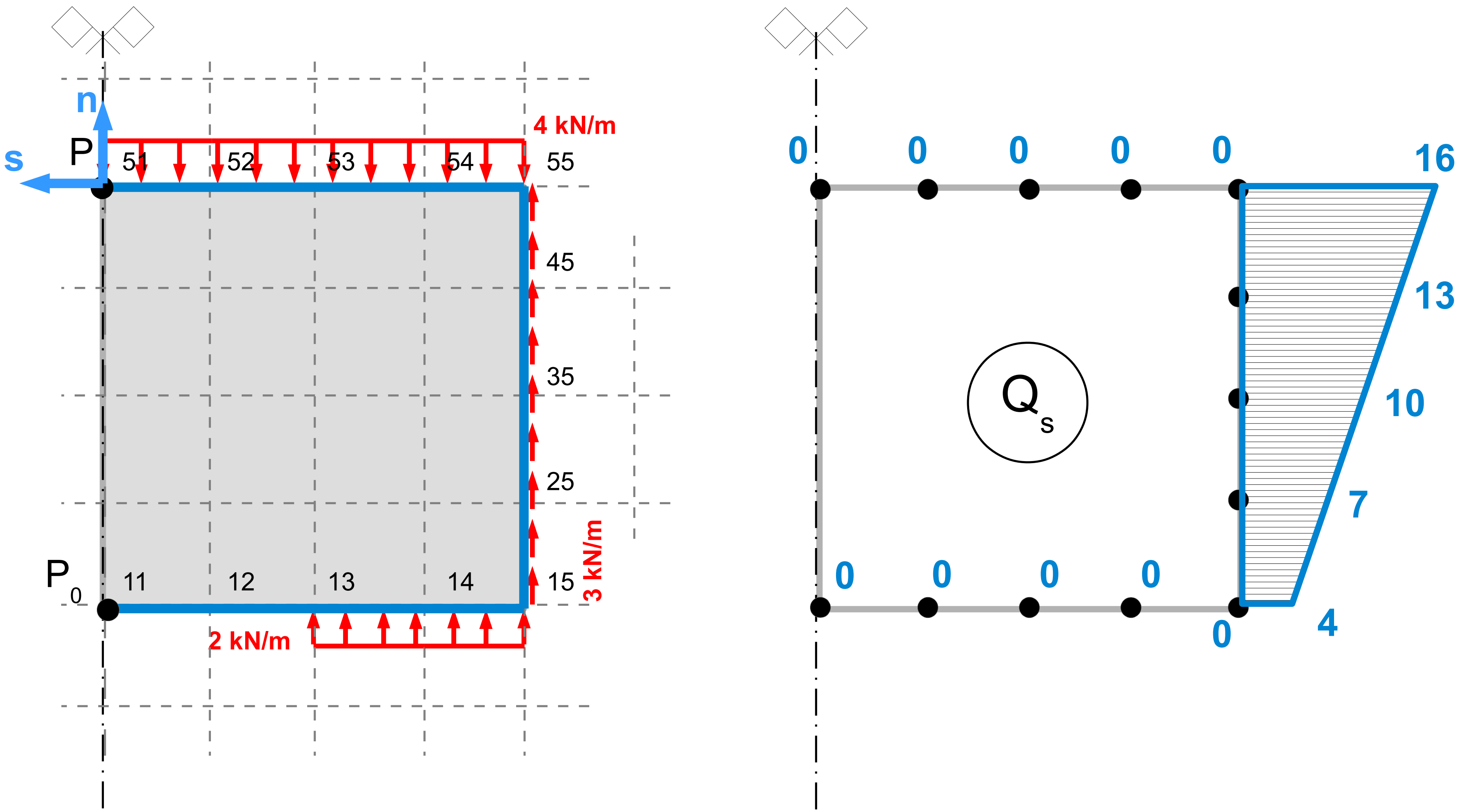



**EXAMPLE**

**We take point P in node 51.**

Sum of all forces parallel to axis s applied to the segment of the boundary between point $P_0$ (węzeł 11) and point P (węzeł 51):

$$Q_{s,51} = 0 \text{ Nm}$$



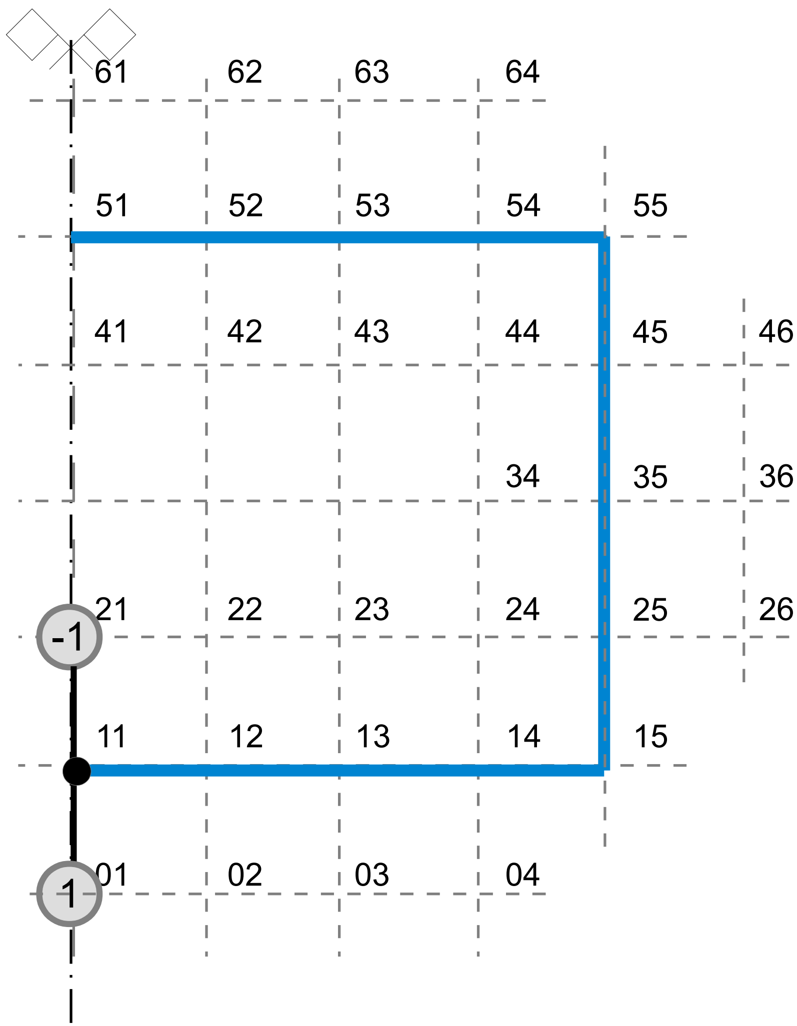
# EXAMPLE

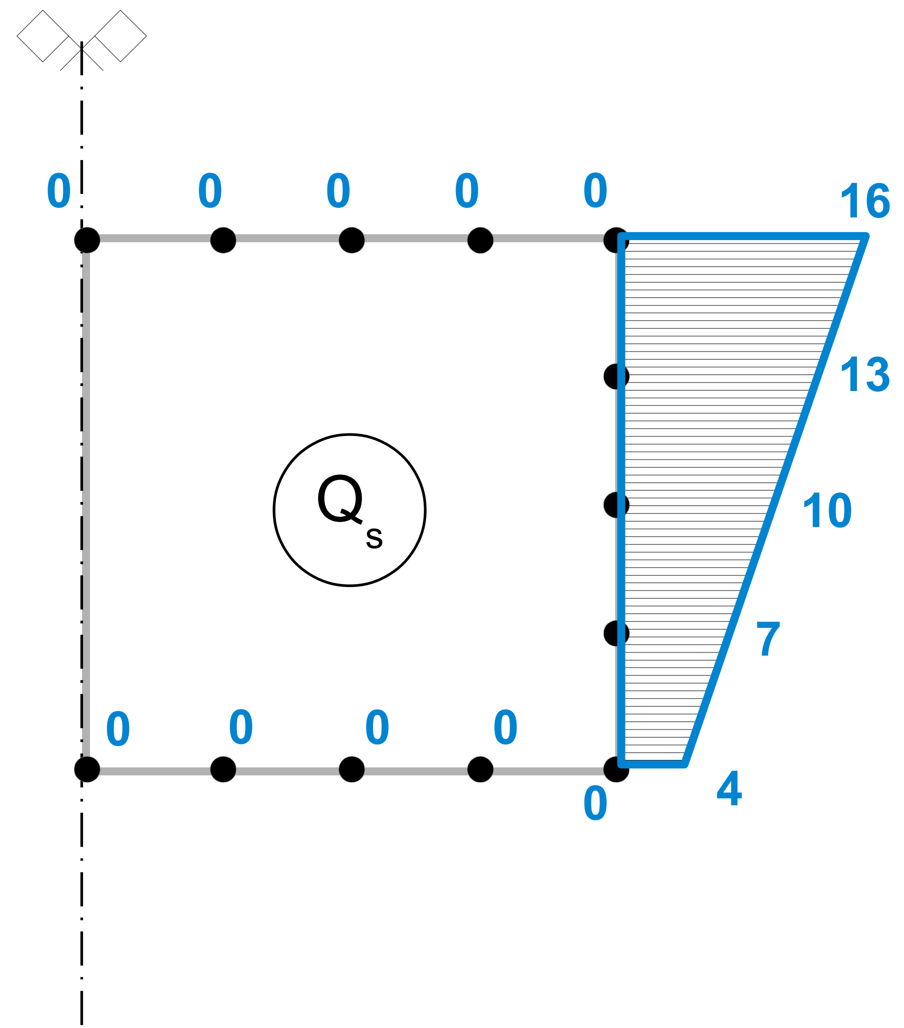
(in nodes corresponding with **zero tangent force**, boundary condition for derivative allow us to **reduce the number of unknown nodal values**)

## Boundary conditions in node 11.

$$\left.\frac{\partial F}{\partial n}\right|_P = -\frac{\left.Q_s\right|_P}{h}$$

$$\frac{1}{2s}\left(F_{01}-F_{21}\right) = -\frac{0\,\mathrm{kN}}{0{,}2\,\mathrm{m}} = 0 \quad \Rightarrow \quad F_{01} = F_{21}$$

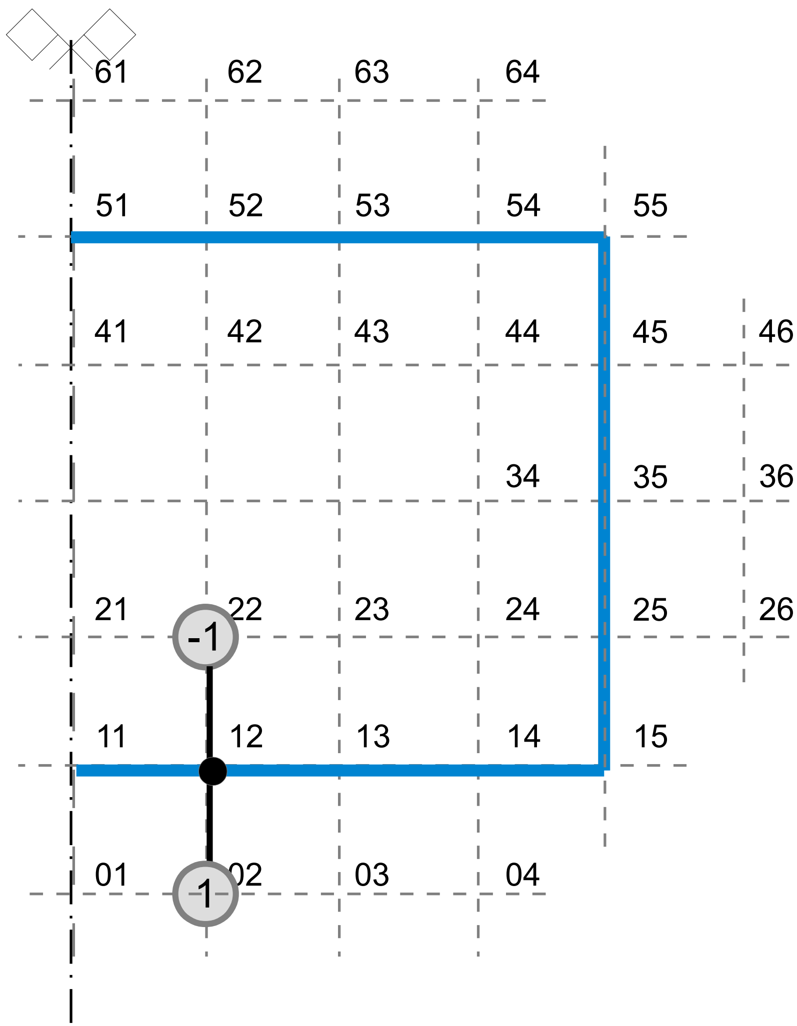


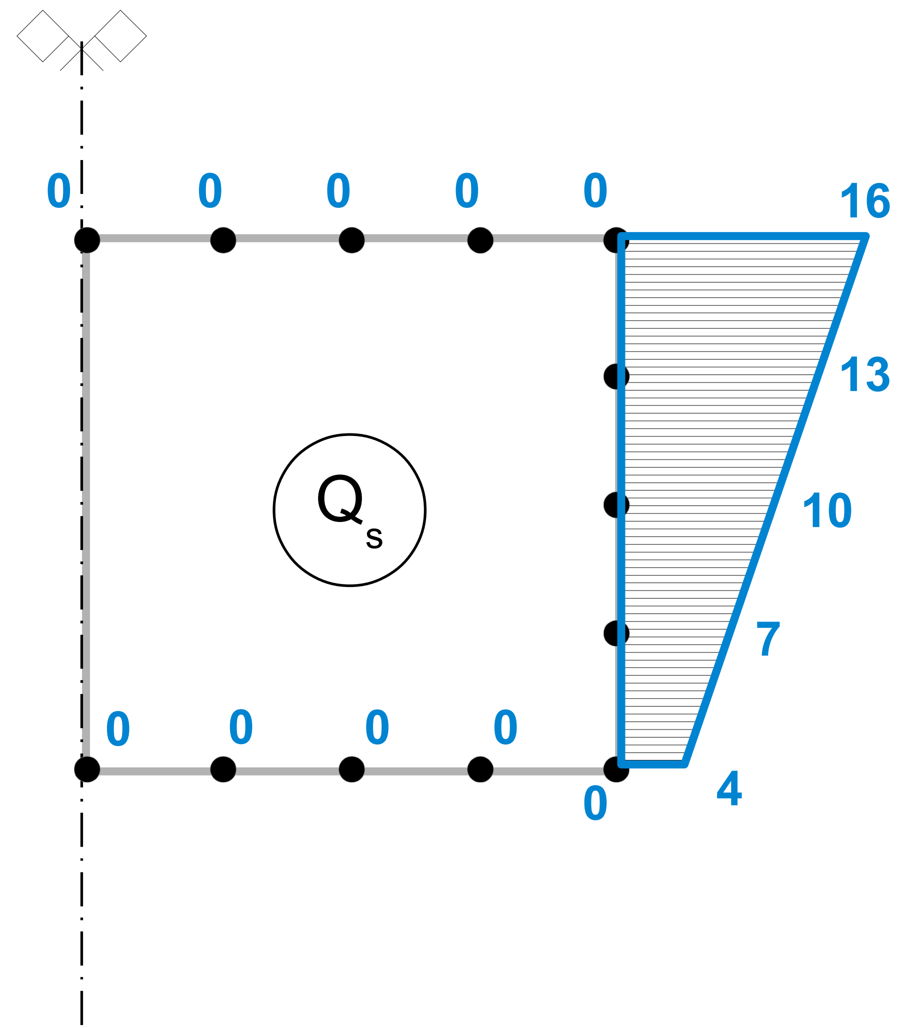



## Boundary conditions in node 12.

$$\left.\frac{\partial F}{\partial n}\right|_P = -\frac{Q_s|_P}{h}$$

$$\frac{1}{2s}\left(F_{02}-F_{22}\right) = -\frac{0\,\text{kN}}{0{,}2\,\text{m}} = 0 \quad \Rightarrow \quad F_{02} = F_{22}$$



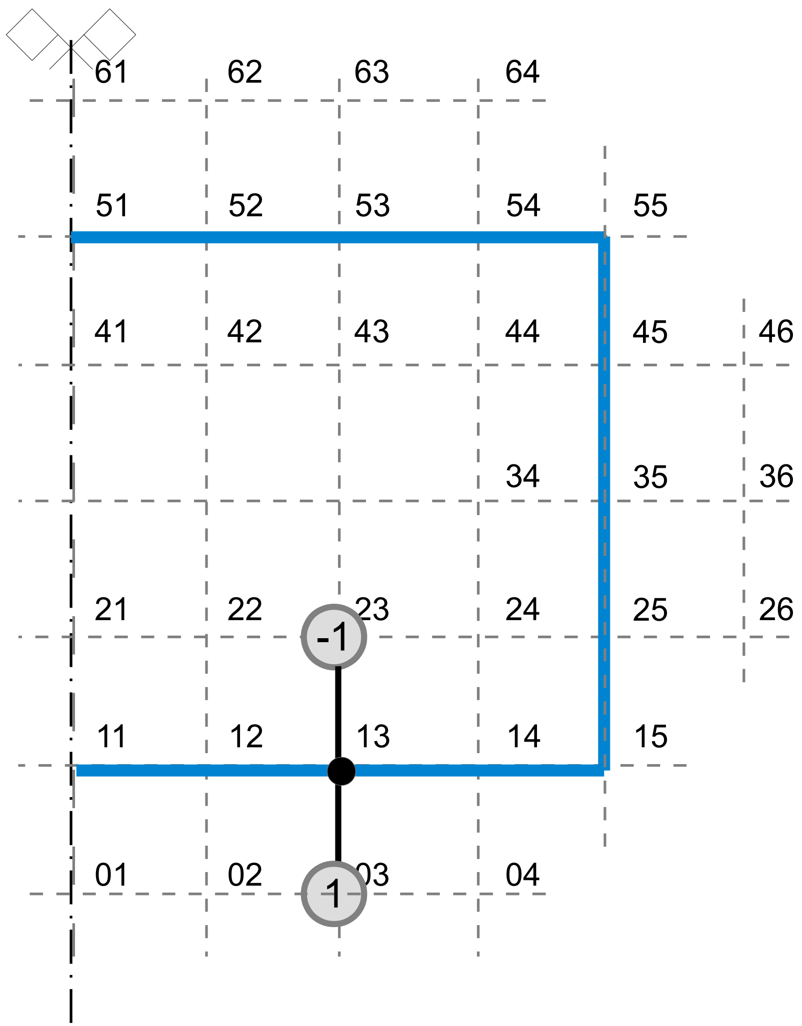



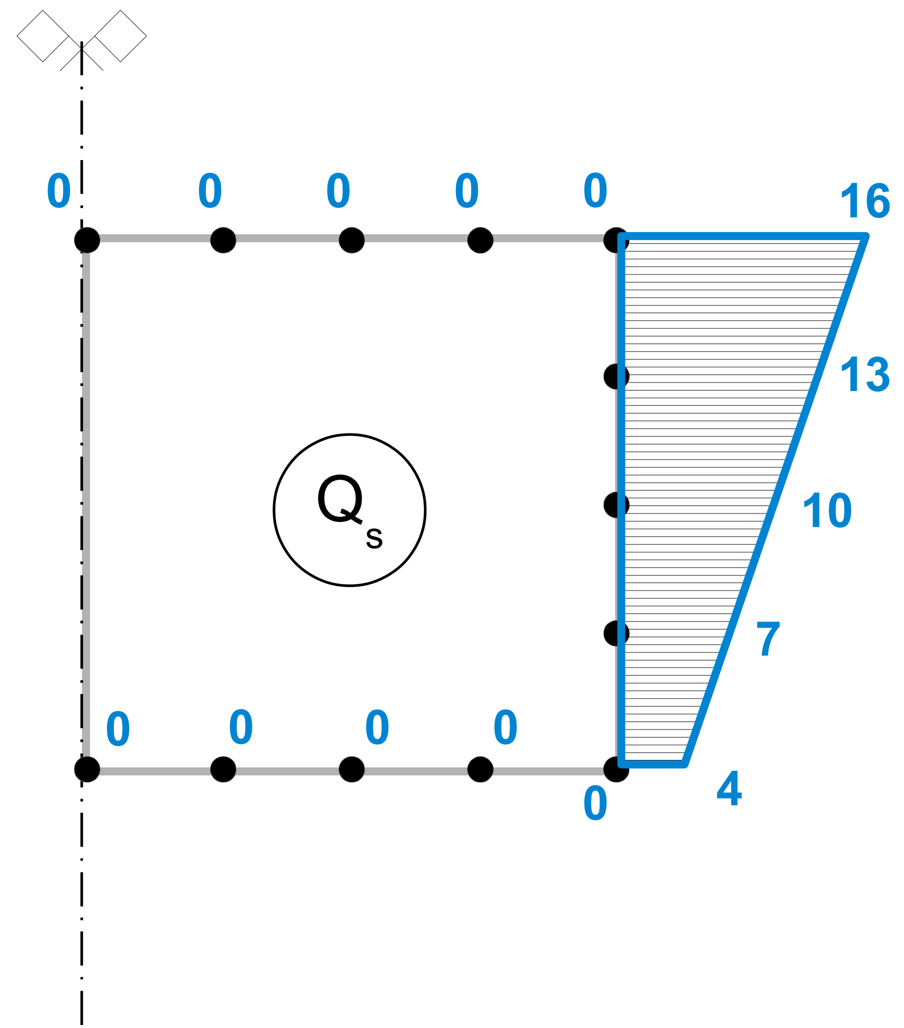
EXAMPLE

## Boundary conditions in node 13.

$$\left.\frac{\partial F}{\partial n}\right|_P = -\frac{\left.Q_s\right|_P}{h}$$

$$\frac{1}{2s}\left(F_{03}-F_{23}\right) = -\frac{0\,\mathrm{kN}}{0{,}2\,\mathrm{m}} = 0 \quad \Rightarrow \quad F_{03} = F_{23}$$



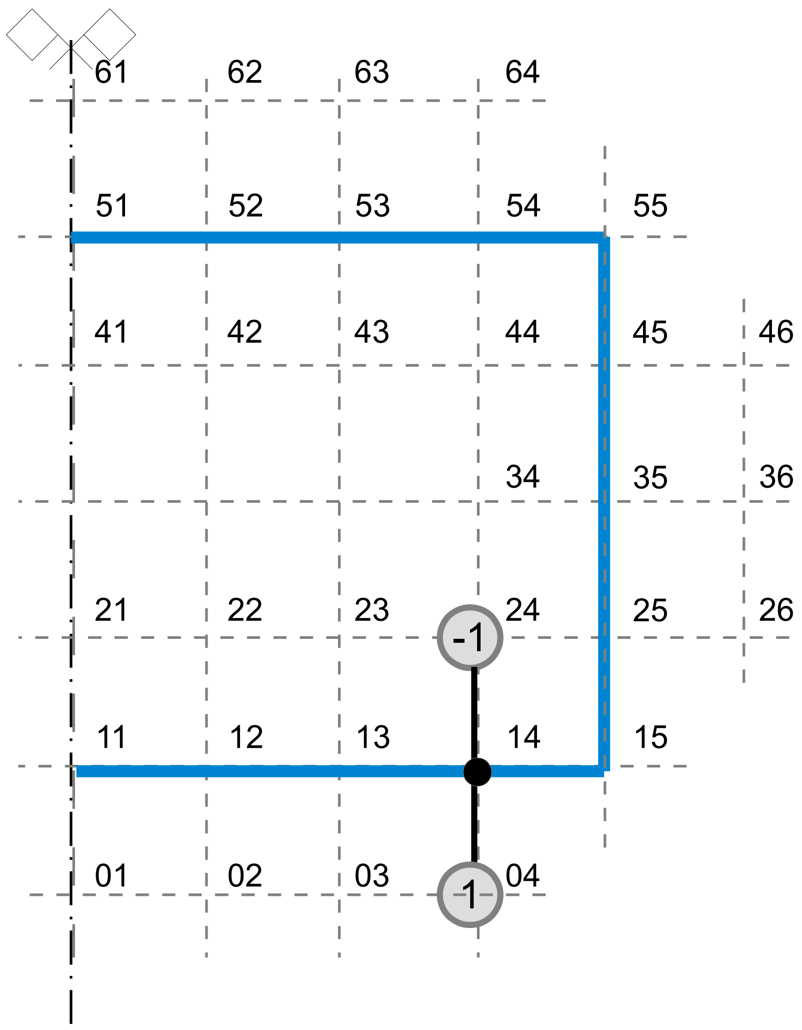



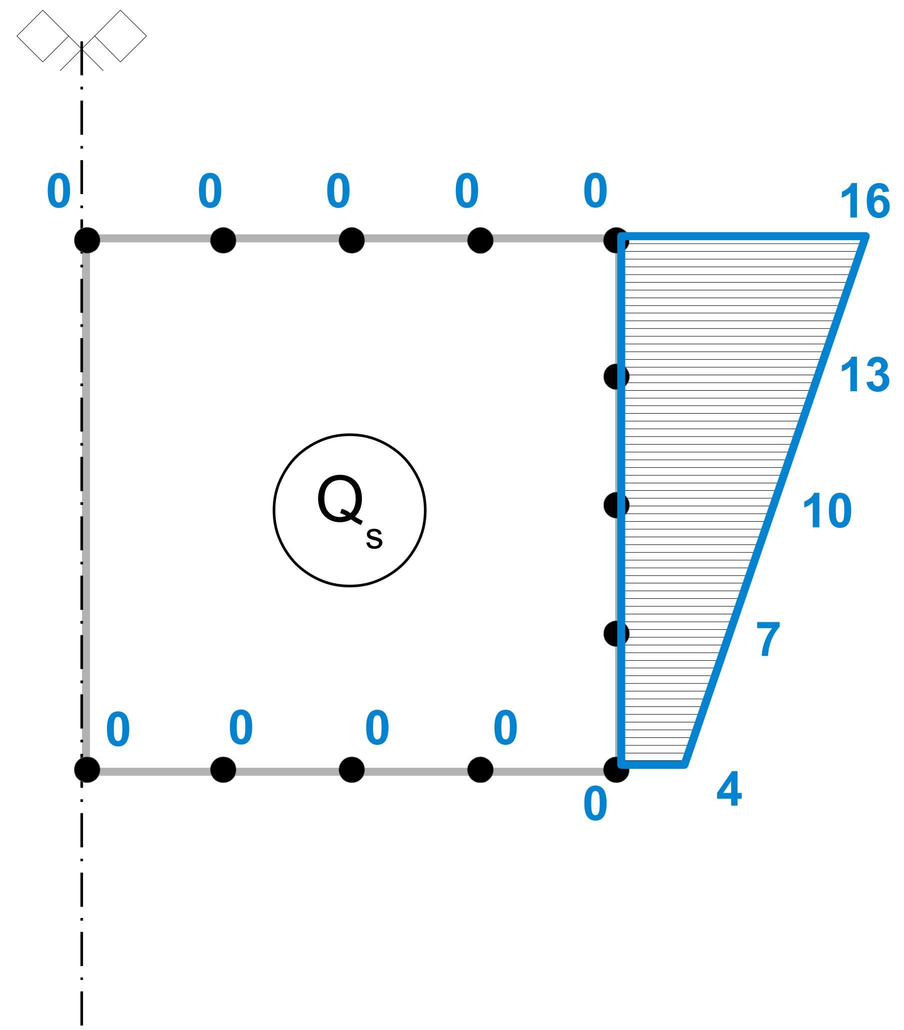
# EXAMPLE

## Boundary conditions in node 14.

$$\left.\frac{\partial F}{\partial n}\right|_P = -\frac{\left.Q_s\right|_P}{h}$$

$$\frac{1}{2\,s}\left(F_{04} - F_{24}\right) = -\frac{0\,\text{kN}}{0{,}2\,\text{m}} = 0 \quad \Rightarrow \quad F_{04} = F_{24}$$



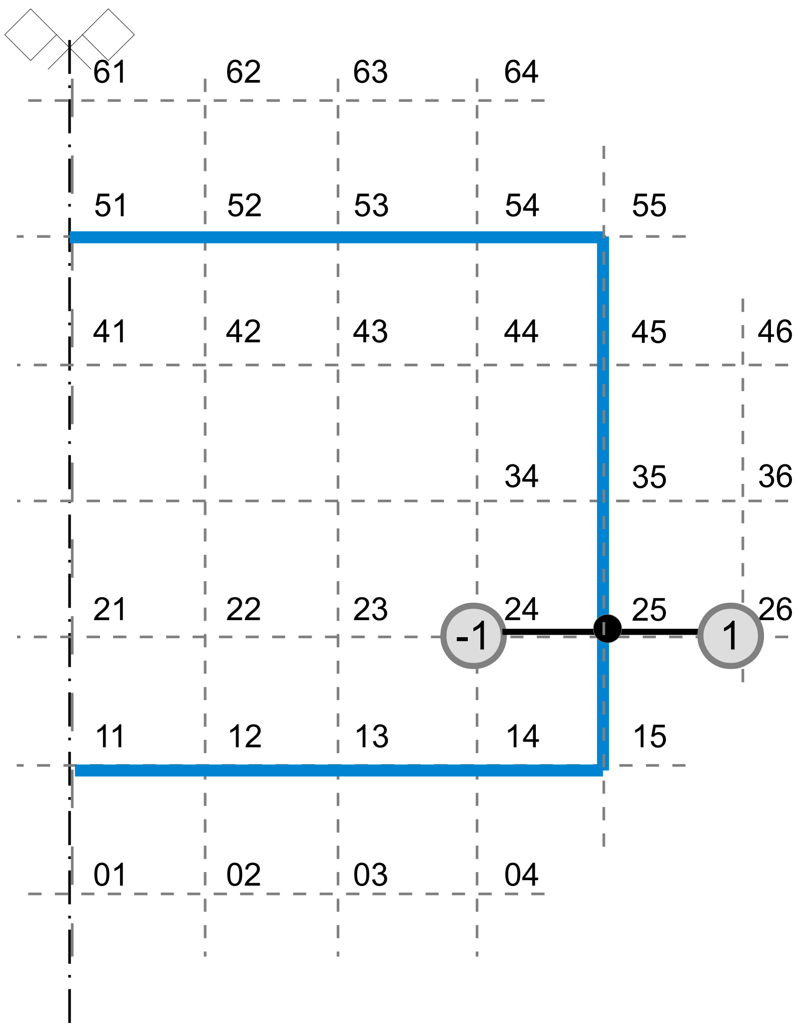
**EXAMPLE**

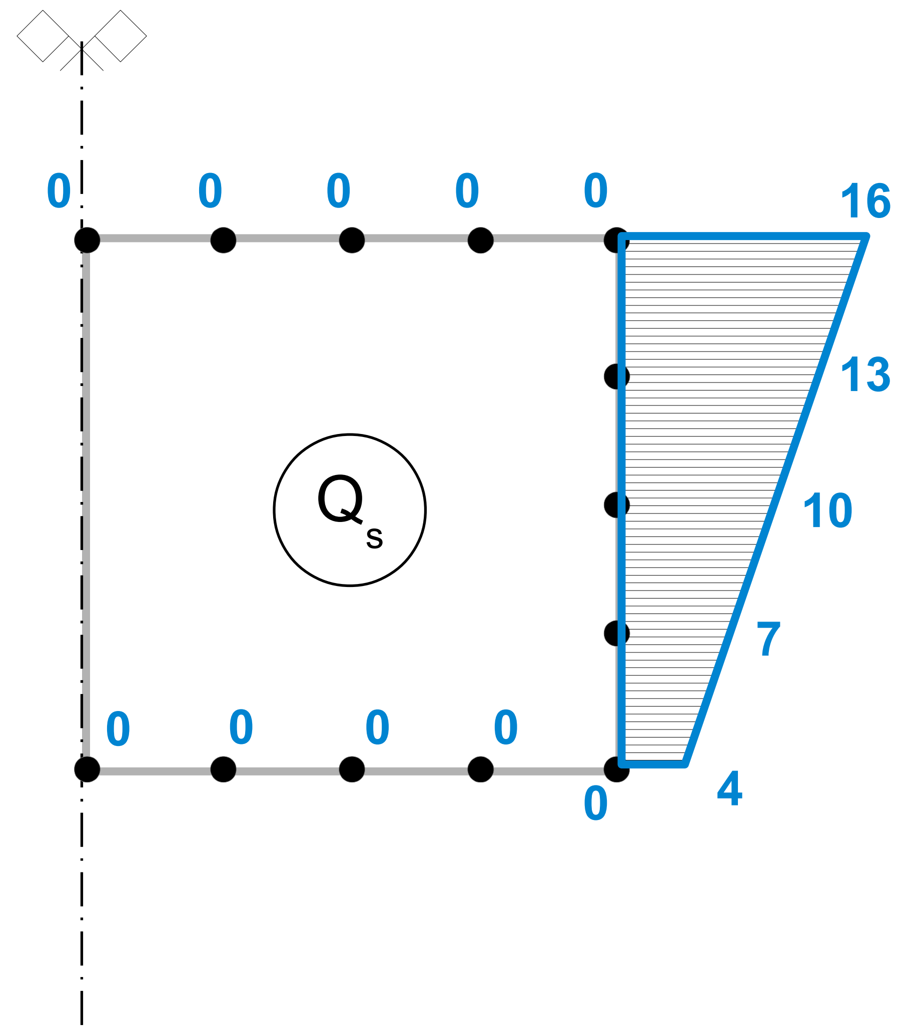
**Boundary condition for derivatives of F is not written down in corner points.**

**Boundary conditions in node 25.**

$$\left.\frac{\partial F}{\partial n}\right|_P = -\frac{Q_s|_P}{h}$$

$$\frac{1}{2s}\left(F_{26} - F_{24}\right) = -\frac{7\,\mathrm{kN}}{0{,}2\,\mathrm{m}} = -35\frac{\mathrm{kN}}{\mathrm{m}} \Rightarrow \boxed{F_{26} - F_{24} = -70\,\mathrm{kN}}$$



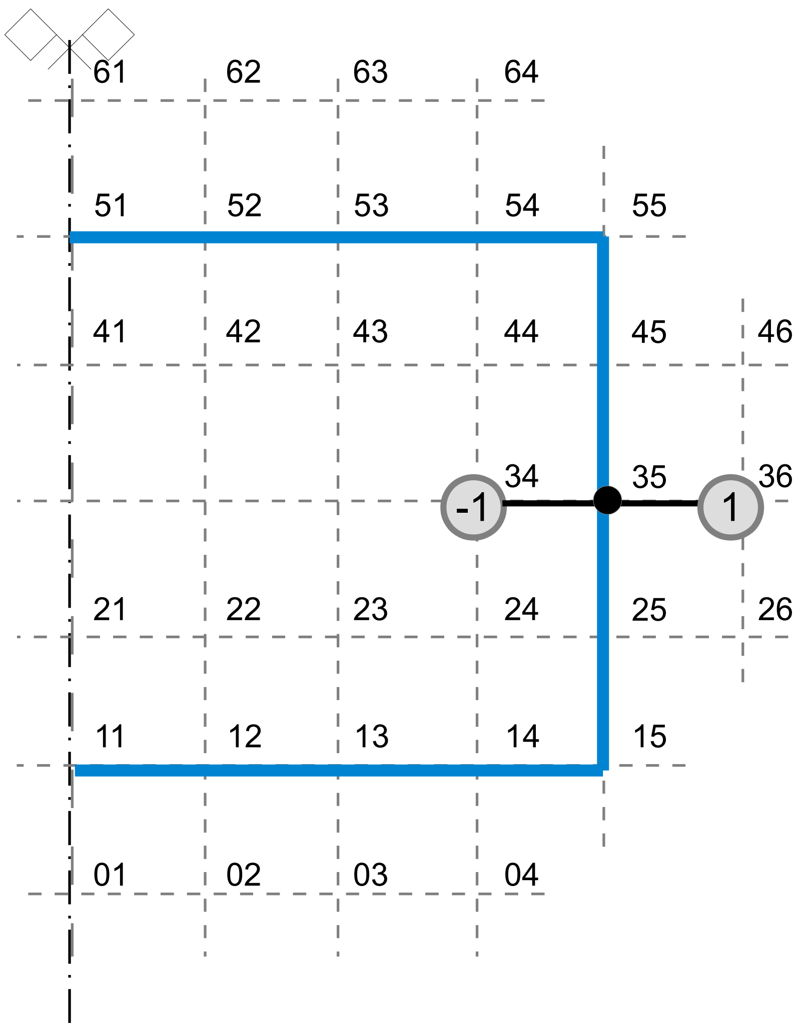



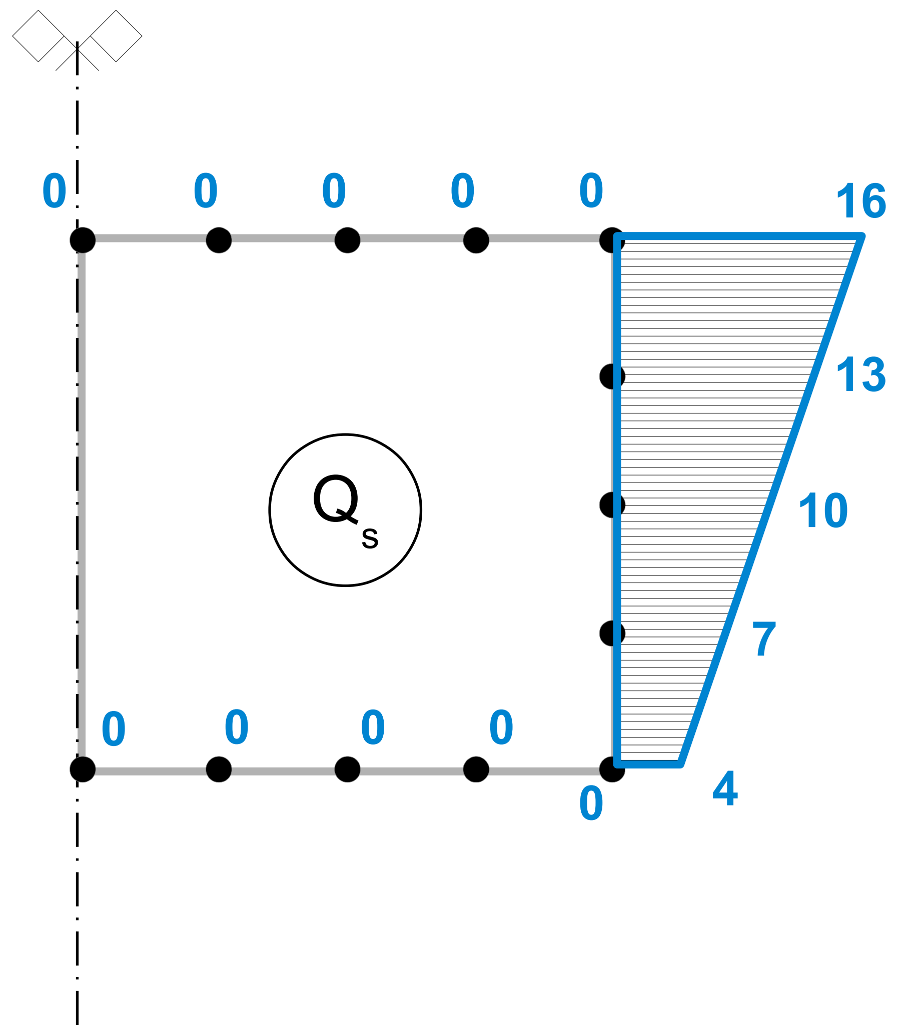
## EXAMPLE

**Boundary conditions in node 35.**

$$\left.\frac{\partial F}{\partial n}\right|_P = -\frac{Q_s|_P}{h}$$

$$\frac{1}{2s}\left(F_{36}-F_{34}\right) = -\frac{10\,\text{kN}}{0{,}2\,\text{m}} = -50\,\frac{\text{kN}}{\text{m}} \Rightarrow F_{36}-F_{34} = -100\,\text{kN}$$



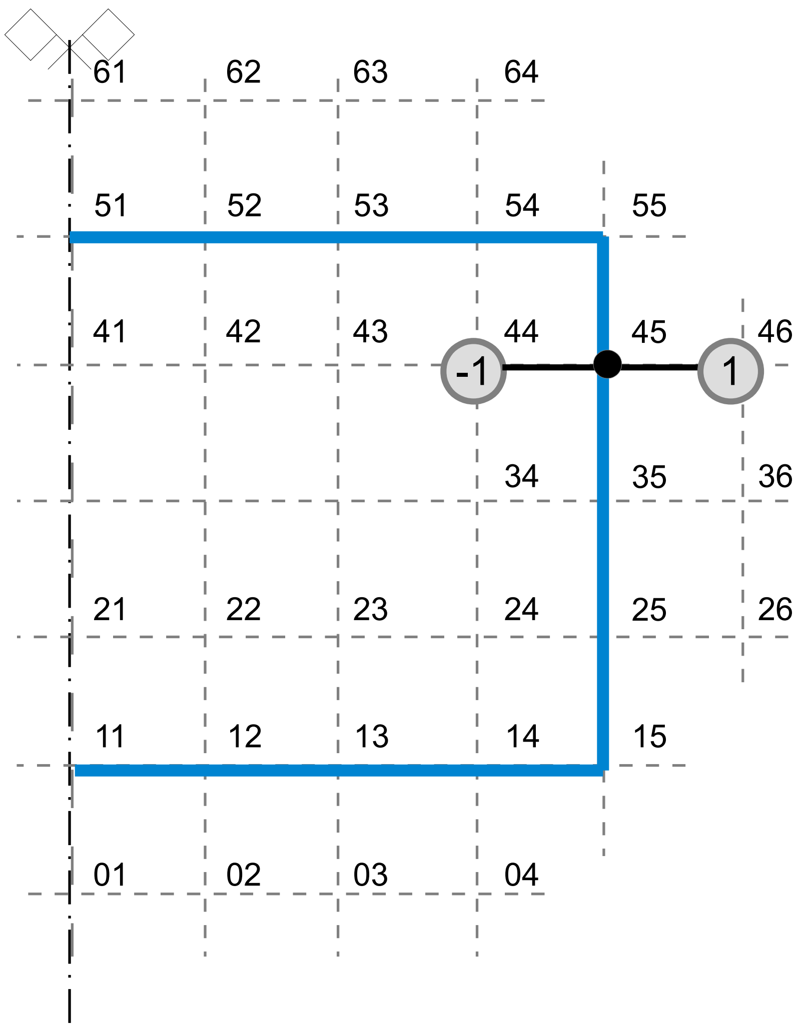



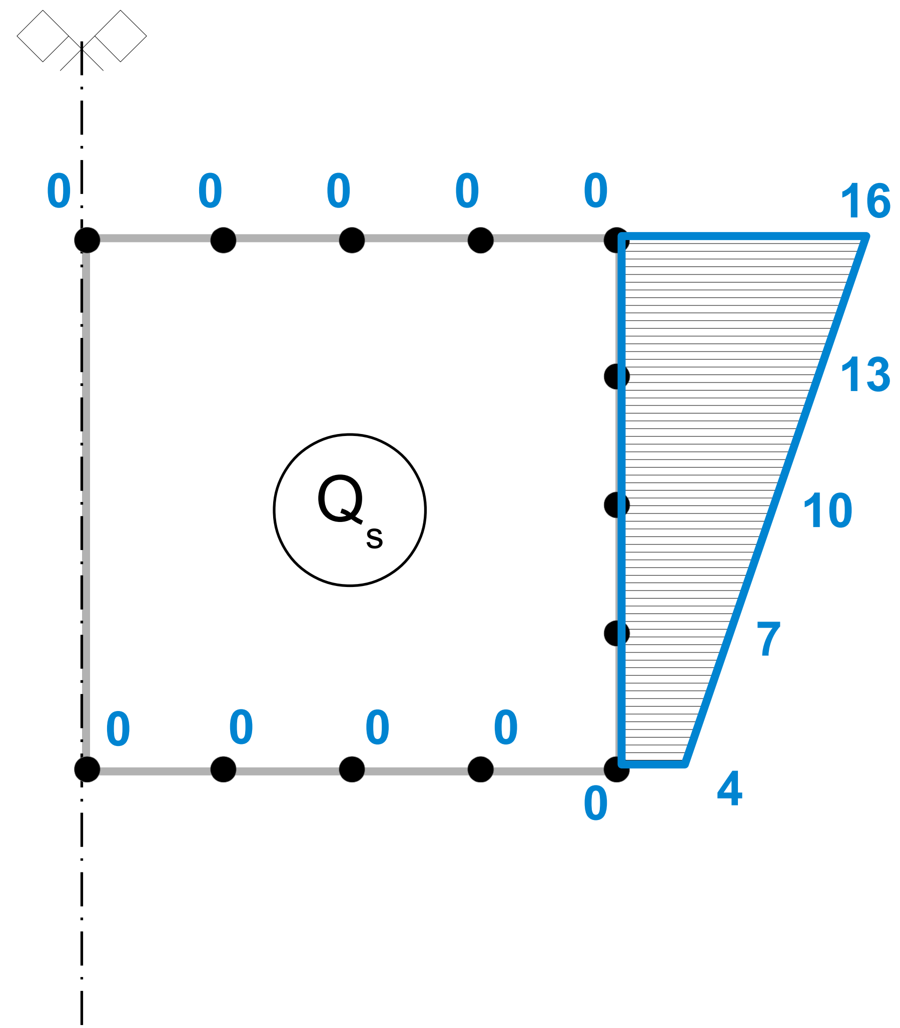
## EXAMPLE

**Boundary conditions in node 45.**

$$\left.\frac{\partial F}{\partial n}\right|_P = -\frac{Q_s|_P}{h}$$

$$\frac{1}{2s}\left(F_{46}-F_{44}\right) = -\frac{13\,\mathrm{kN}}{0{,}2\,\mathrm{m}} = -65\frac{\mathrm{kN}}{\mathrm{m}} \Rightarrow \boxed{F_{46}-F_{44}=-130\,\mathrm{kN}}$$



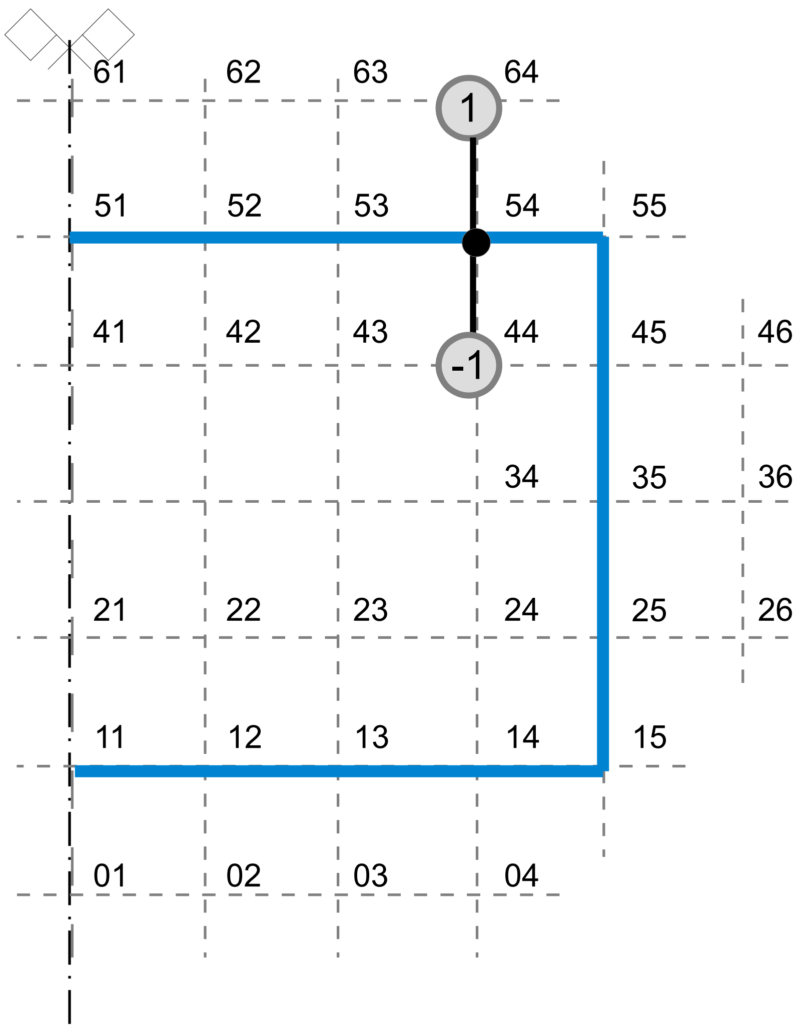
# EXAMPLE

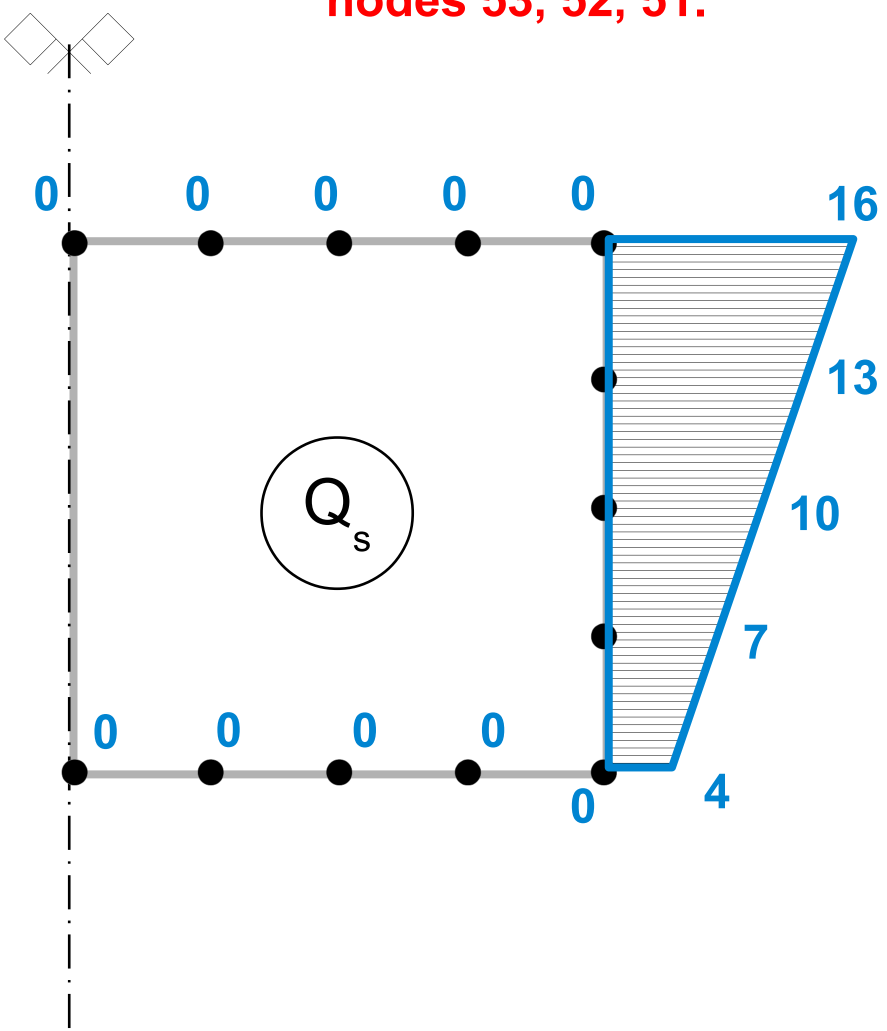
**Boundary conditions in node 54.**

$$\left.\frac{\partial F}{\partial n}\right|_P = -\frac{Q_s|_P}{h}$$

$$\frac{1}{2s}\left(F_{64} - F_{44}\right) = -\frac{0\,\mathrm{kN}}{0{,}2\,\mathrm{m}} = 0\,\frac{\mathrm{kN}}{\mathrm{m}} \quad \Rightarrow \quad F_{64} = F_{44}$$

**In a similar way for nodes 53, 52, 51.**

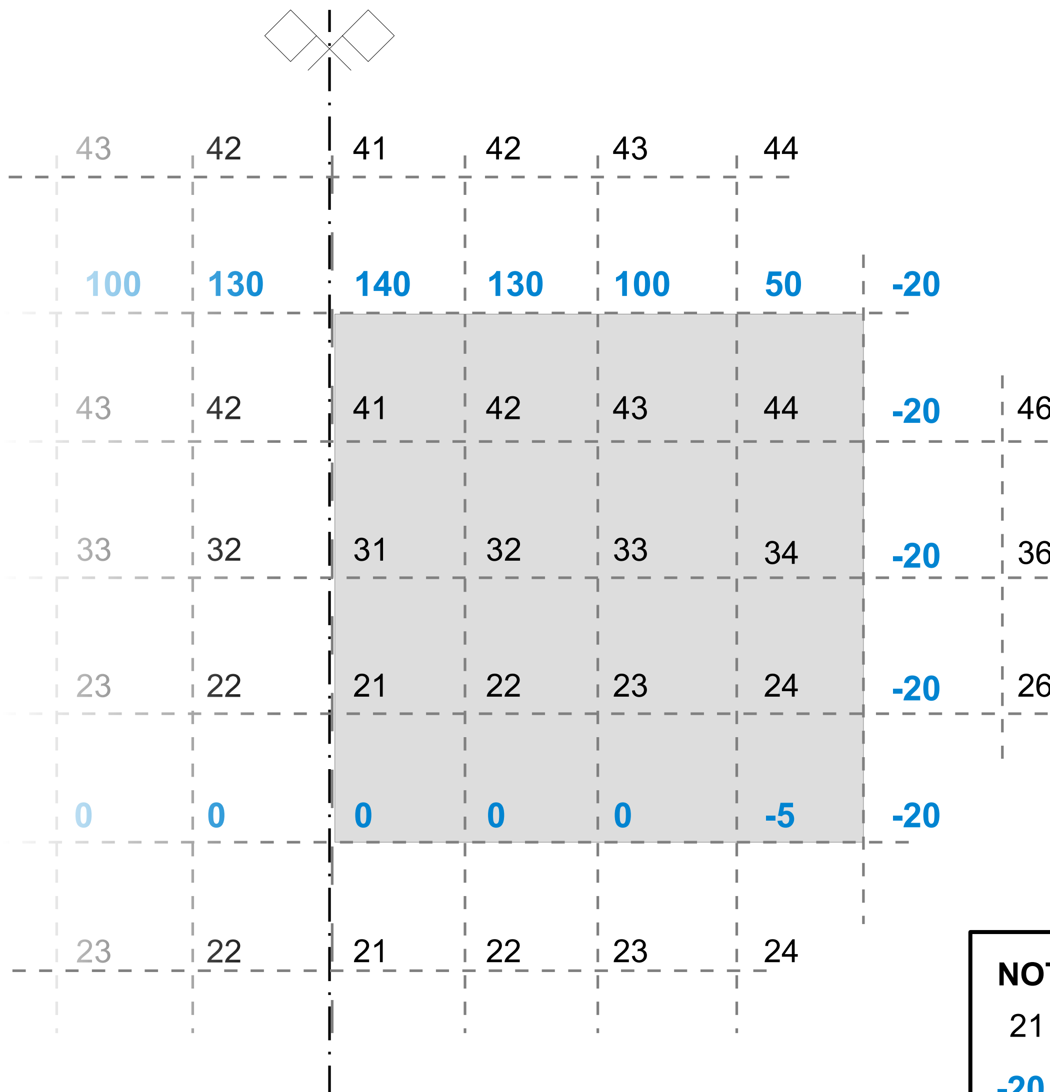


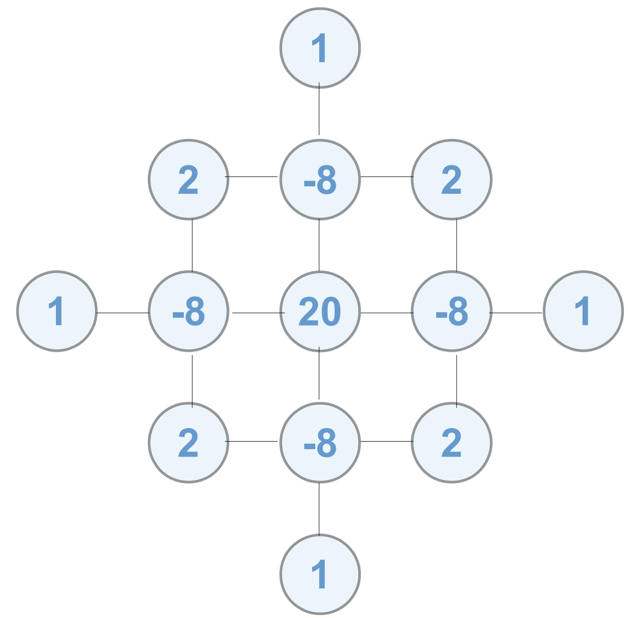
## EXAMPLE

Making use of the boundary conditions the FDM mesh may be simplified as shown below.

Let's write down the governing equation $\nabla^4 F = 0$ in **each internal node** with the use of finite difference operator.




$$\nabla^4 \approx \frac{1}{s^4}\cdot$$




**NOTATION:**

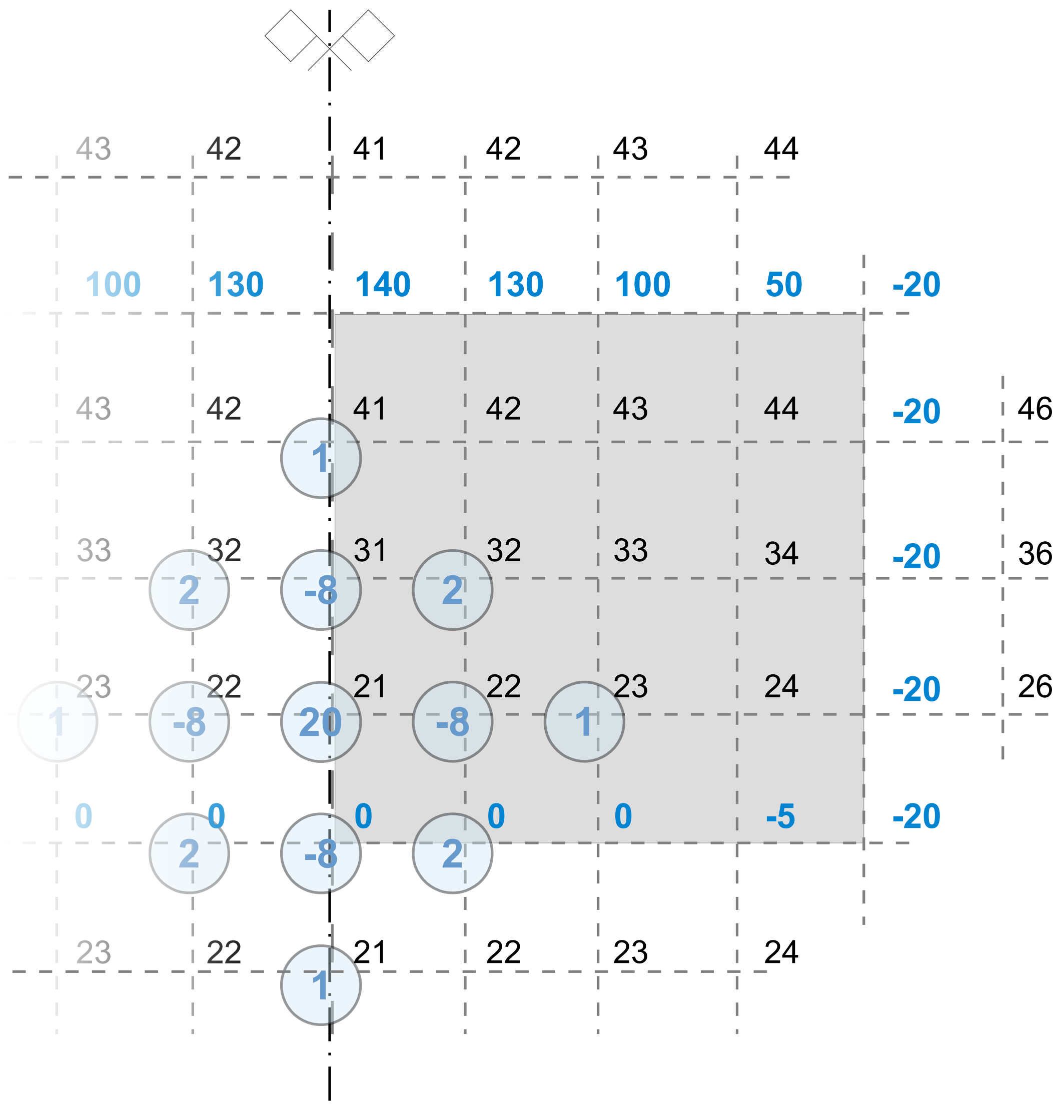
21 - node number

-20 - value of Airy stress function

## EXAMPLE

### Equation in node 21:

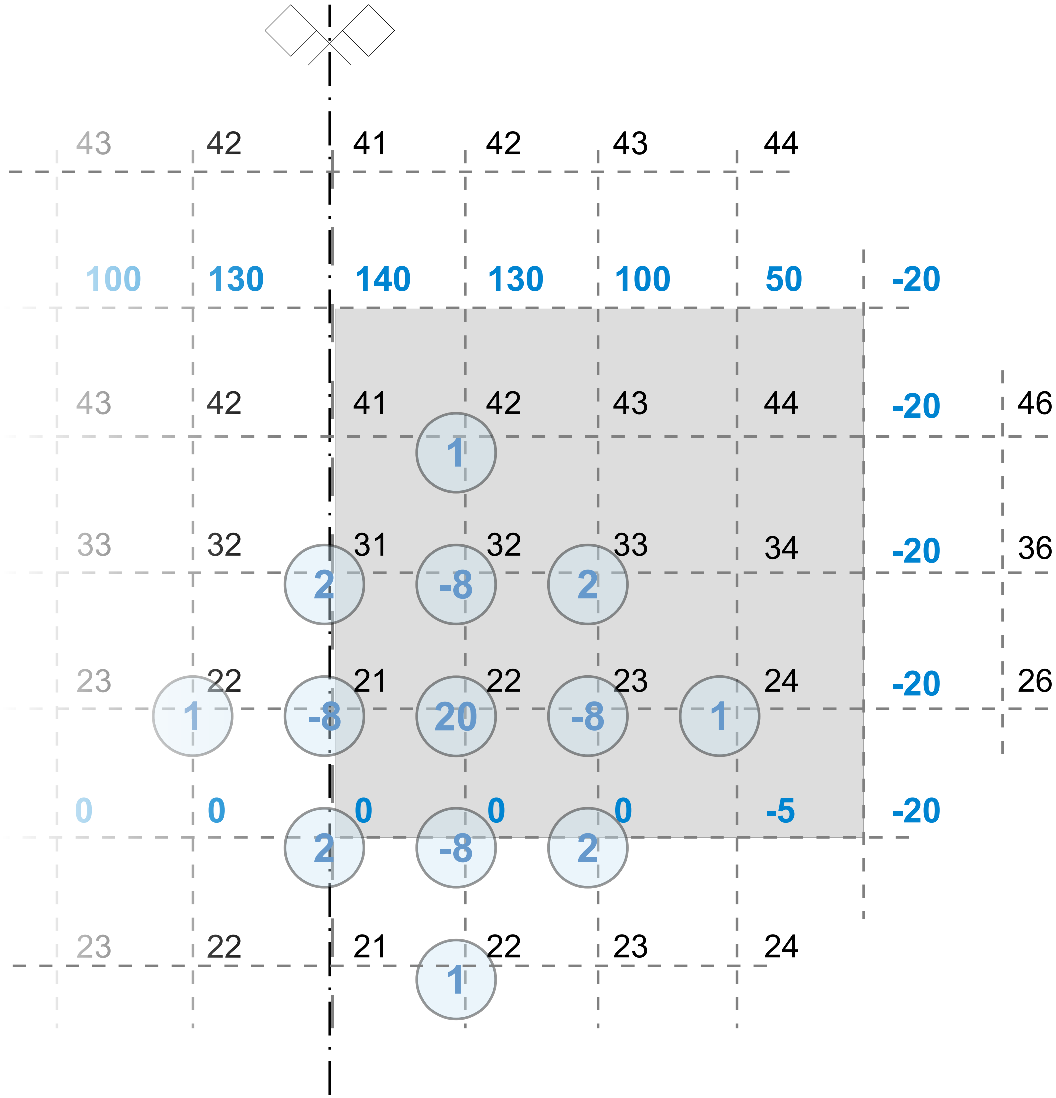
$$\frac{1}{s^4}\left[20\cdot(F_{21}) - 8\cdot(F_{22}+F_{22}+F_{31}+0) + 2\cdot(F_{32}+F_{32}+0+0) + 1\cdot(F_{21}+F_{23}+F_{23}+F_{41})\right]=0$$



## EXAMPLE

### Equation in node 22:

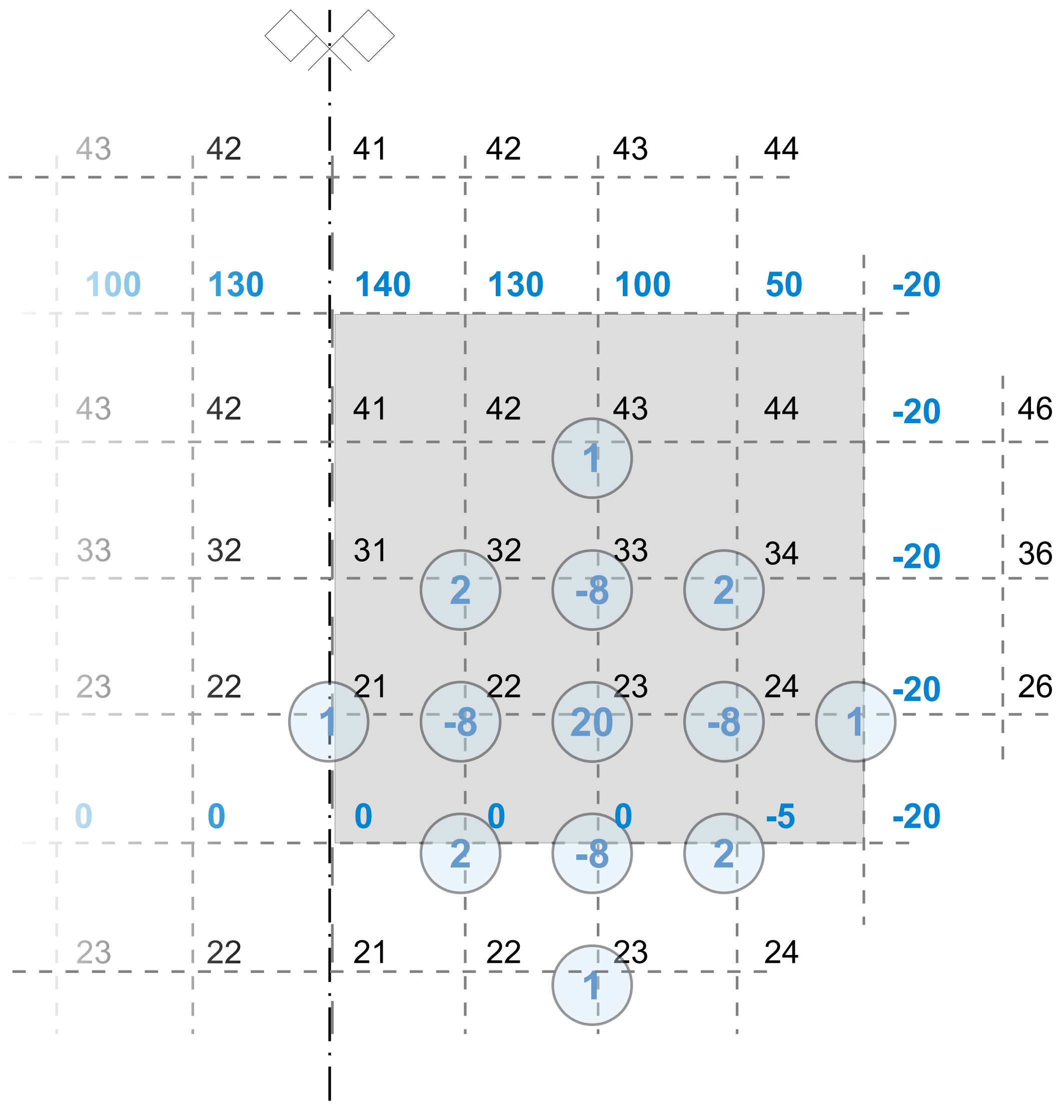
$$\frac{1}{s^4}\left[20\cdot(F_{22}) - 8\cdot(F_{21}+F_{23}+F_{32}+0) + 2\cdot(F_{31}+F_{33}+0+0) + 1\cdot(F_{24}+F_{22}+F_{22} + F_{42})\right] = 0$$



**EXAMPLE**

**Equation in node 23:**

$$\frac{1}{s^4}\left[20\cdot(F_{23}) - 8\cdot(F_{22}+F_{24}+F_{33}+0) + 2\cdot(F_{32}+F_{34}+0+(-5)) + 1\cdot(F_{21}+F_{23}+F_{43} + (-20))\right] = 0$$




**In a similar way for nodes 24, 31, 32, 33, 34, 31, 42, 43, 44**

## EXAMPLE

### FDM system of equation:

$$20\cdot(F_{21}) - 8\cdot(F_{22}+F_{22}+F_{31}+0) + 2\cdot(F_{32}+F_{32}+0+0) + 1\cdot(F_{21}+F_{23}+F_{23} + F_{41}) = 0$$

$$20\cdot(F_{22}) - 8\cdot(F_{21}+F_{23}+F_{32}+0) + 2\cdot(F_{31}+F_{33}+0+0) + 1\cdot(F_{24}+F_{22}+F_{22} + F_{42}) = 0$$

$$20\cdot(F_{23}) - 8\cdot(F_{22}+F_{24}+F_{33}+0) + 2\cdot(F_{32}+F_{34}+0+(-5)) + 1\cdot(F_{21}+F_{23}+F_{43} + (-20)) = 0$$

$$20\cdot(F_{24}) - 8\cdot(F_{23}+F_{34}+(-5)+(-20)) + 2\cdot(F_{33}+0+(-20)+(-20)) + 1\cdot(F_{24}+F_{22}+F_{26} + F_{44}) = 0$$

$$20\cdot(F_{31}) - 8\cdot(F_{32}+F_{32}+F_{21}+F_{41}) + 2\cdot(F_{22}+F_{22}+F_{42}+F_{42}) + 1\cdot(F_{33}+F_{33}+0 + 140) = 0$$

$$20\cdot(F_{32}) - 8\cdot(F_{31}+F_{33}+F_{22}+F_{42}) + 2\cdot(F_{41}+F_{21}+F_{43}+F_{23}) + 1\cdot(F_{32}+F_{34}+0 + 130) = 0$$

$$20\cdot(F_{33}) - 8\cdot(F_{32}+F_{34}+F_{23}+F_{43}) + 2\cdot(F_{22}+F_{24}+F_{42}+F_{44}) + 1\cdot(F_{31}+(-20)+0 + 100) = 0$$

$$20\cdot(F_{34}) - 8\cdot(F_{33}+F_{24}+F_{44}+(-20)) + 2\cdot(F_{23}+F_{43}+(-20)+(-20)) + 1\cdot(F_{32}+F_{36}+(-5) + 50) = 0$$

$$20\cdot(F_{41}) - 8\cdot(F_{42}+F_{42}+F_{31}+140) + 2\cdot(F_{32}+F_{32}+130+130) + 1\cdot(F_{21}+F_{43}+F_{43} + F_{41}) = 0$$

$$20\cdot(F_{42}) - 8\cdot(F_{41}+F_{43}+F_{32}+130) + 2\cdot(F_{31}+F_{33}+140+100) + 1\cdot(F_{42}+F_{44}+F_{22} + F_{42}) = 0$$

$$20\cdot(F_{43}) - 8\cdot(F_{42}+F_{44}+F_{33}+100) + 2\cdot(F_{32}+F_{34}+130+50) + 1\cdot(F_{41}+F_{23}+F_{43} + (-20)) = 0$$

$$20\cdot(F_{44}) - 8\cdot(F_{43}+F_{34}+50+(-20)) + 2\cdot(F_{33}+(-20)+(-20)+100) + 1\cdot(F_{42}+F_{24}+F_{46} + F_{44}) = 0$$

$$F_{26} - F_{24} = -70$$

$$F_{36} - F_{34} = -100$$

$$F_{46} - F_{44} = -130$$

**Governing equation** written down in **internal points**

**Boundary conditions for derivative** written down in **boundary nodes** in which **tangent force is not equal 0**.

# EXAMPLE

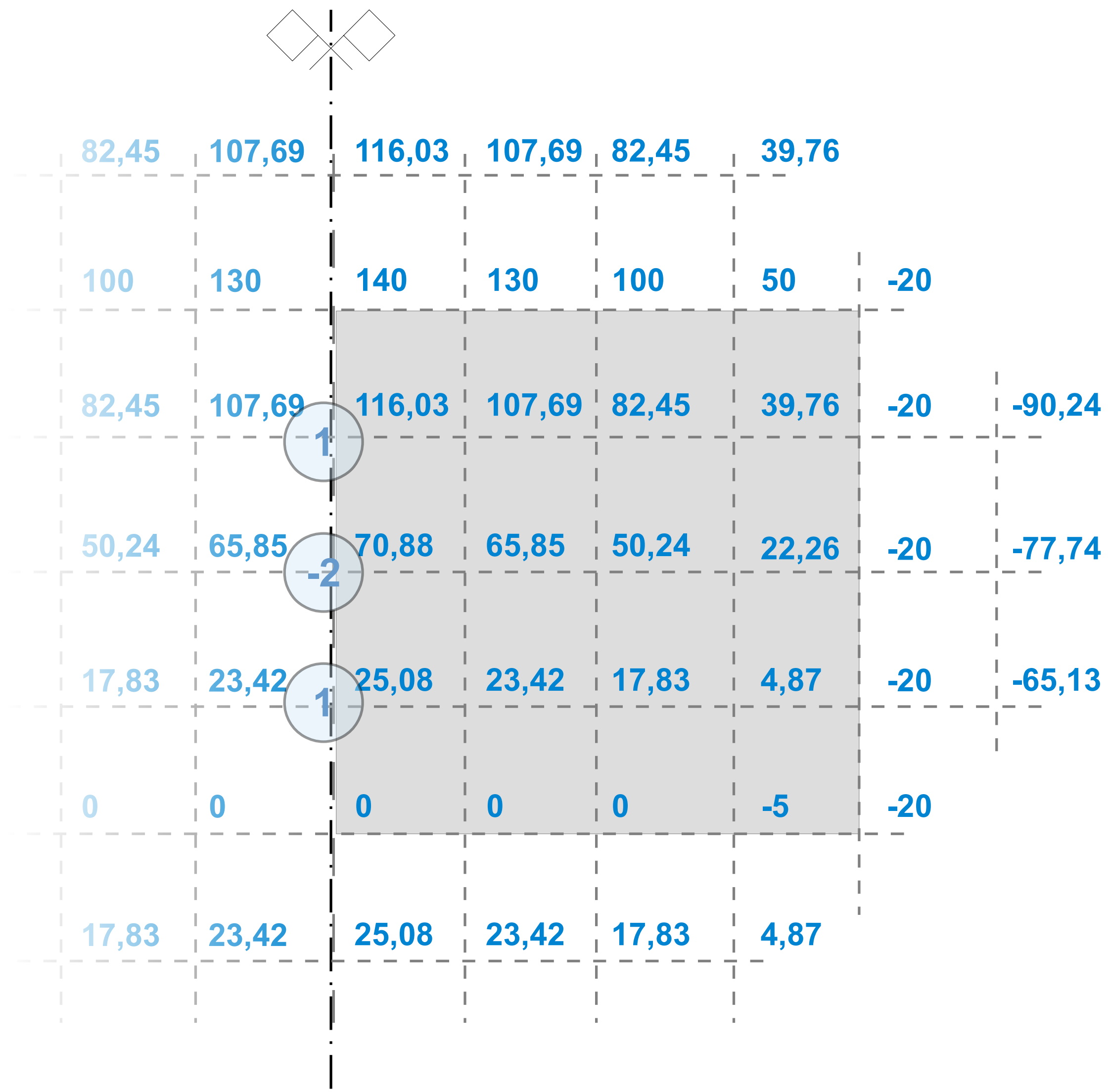
## SOLUTION:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 82,45 | 107,69 | 116,03 | 107,69 | 82,45 | 39,76 | | |
| 100 | 130 | 140 | 130 | 100 | 50 | -20 | |
| 82,45 | 107,69 | 116,03 | 107,69 | 82,45 | 39,76 | -20 | -90,24 |
| 50,24 | 65,85 | 70,88 | 65,85 | 50,24 | 22,26 | -20 | -77,74 |
| 17,83 | 23,42 | 25,08 | 23,42 | 17,83 | 4,87 | -20 | -65,13 |
| 0 | 0 | 0 | 0 | 0 | -5 | -20 | |
| 17,83 | 23,42 | 25,08 | 23,42 | 17,83 | 4,87 | | |

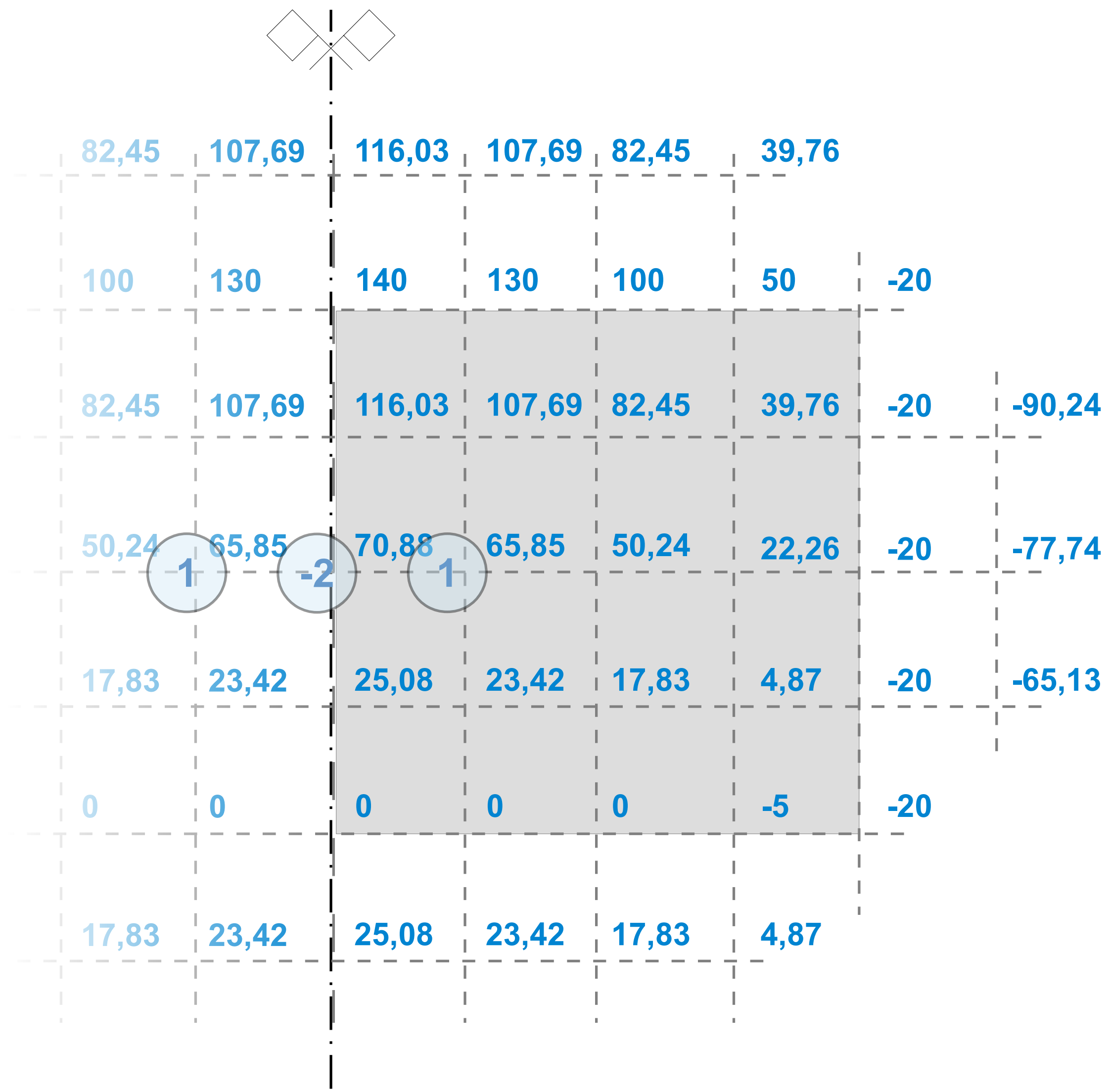
## Stress state in the middle of the membrane - node 31:

$$\sigma_{xx} = \frac{\partial^2 F}{\partial y^2} \approx \frac{1}{s^2}\left(F_{21} - 2F_{31} + F_{41}\right) =$$

$$= \frac{1}{1^2}\left(25{,}08 - 2\cdot 70{,}88 + 116{,}03\right) = -0{,}65 \text{ kPa}$$



## Stress state in the middle of the membrane - node 31:



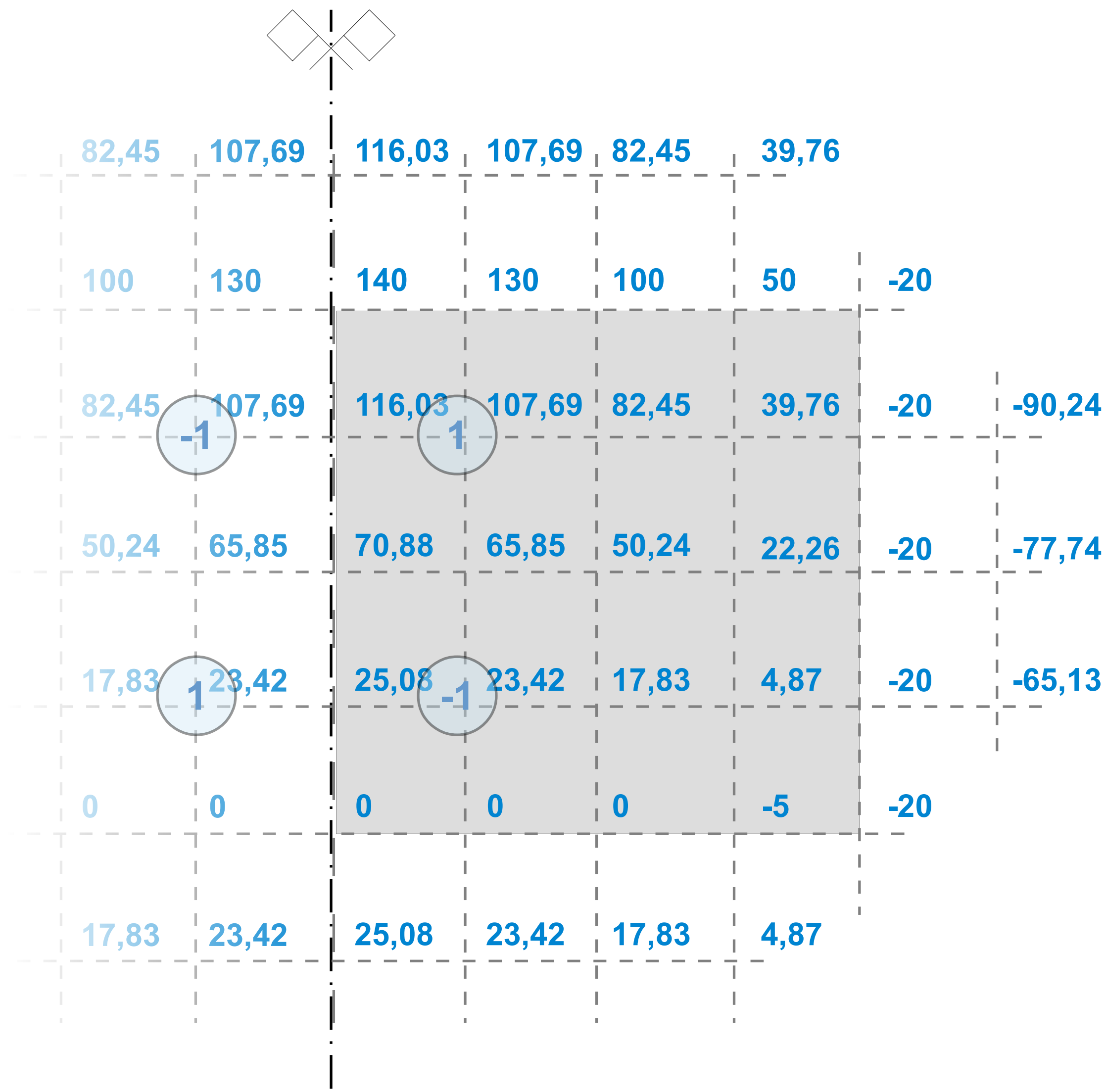

$$\sigma_{xx} = \frac{\partial^2 F}{\partial y^2} \approx \frac{1}{s^2}\left(F_{21} - 2F_{31} + F_{41}\right) =$$

$$= \frac{1}{1^2}\left(25{,}08 - 2\cdot 70{,}88 + 116{,}03\right) = -0{,}65 \text{ kPa}$$

$$\sigma_{yy} = \frac{\partial^2 F}{\partial x^2} \approx \frac{1}{s^2}\left(F_{32} - 2F_{31} + F_{32}\right) =$$

$$= \frac{1}{1^2}\left(65{,}85 - 2\cdot 70{,}88 + 65{,}85\right) = -10{,}06 \text{ kPa}$$

## EXAMPLE




**Stress state in the middle of the membrane - node 31:**

$$\sigma_{xx} = \frac{\partial^2 F}{\partial y^2} \approx \frac{1}{s^2}\left(F_{21} - 2F_{31} + F_{41}\right) =$$

$$= \frac{1}{1^2}\left(25{,}08 - 2\cdot 70{,}88 + 116{,}03\right) = -0{,}65 \text{ kPa}$$

$$\sigma_{yy} = \frac{\partial^2 F}{\partial x^2} \approx \frac{1}{s^2}\left(F_{32} - 2F_{31} + F_{32}\right) =$$

$$= \frac{1}{1^2}\left(65{,}85 - 2\cdot 70{,}88 + 65{,}85\right) = -10{,}06 \text{ kPa}$$

$$\sigma_{xy} = -\frac{\partial^2 F}{\partial x \partial y} \approx -\frac{1}{4s^2}\left(F_{22} + F_{42} - F_{22} - F_{42}\right) =$$

$$= 0 \text{ kPa}$$

## EXAMPLE

**Stress state in the middle of the membrane - node 31:**

$$\boldsymbol{\sigma} = \begin{bmatrix} -0{,}65 & 0 \\ 0 & -10{,}06 \end{bmatrix} \text{kPa}$$