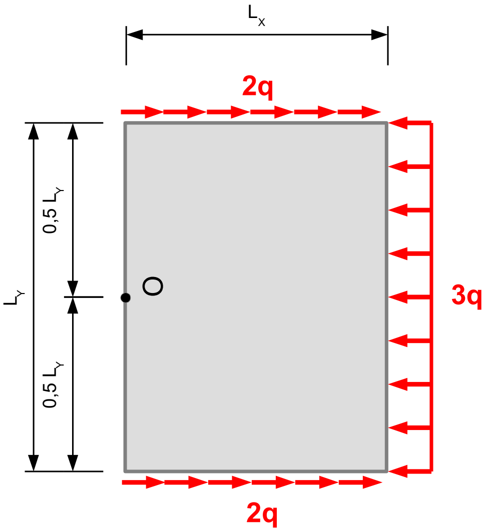
EXAMPLE

Find the values of the Airy stress function for a membrane loaded as depicted in the figure below – use the Finite Difference Method. Starting point should be chosen in point O. Tangent axis of local coordinate system should be oriented downwards. Making use of the symmetry of the system, account only for a half of this membrane. Find the components σ_{xx}, σ_{yy}, τ_{xy} of the stress tensor in two points which are closest to the middle of the membrane.



Dimensions: $L_X = 8\,\text{m}$

$L_Y = 6\,\text{m}$

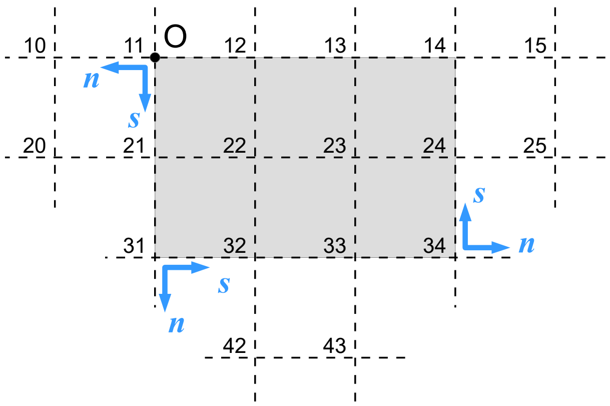
Thickness: $h = 20\,\text{cm}$

Load: $q = 100\,\text{kPa}$

FDM grid spacing: $s = \Delta x = \Delta y = 2\,\text{m}$

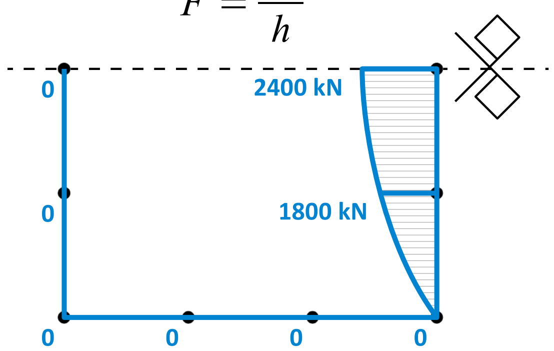
SOLUTION:

Assumed FDM mesh:

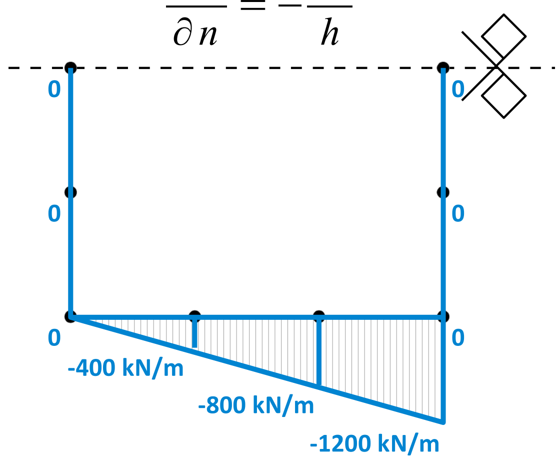


Vales of the Airy stress function and its directional derivative along external normal direction:

$$F = \frac{M}{h}$$



$$\frac{\partial F}{\partial n} = -\frac{Q_s}{h}$$



Equations for internal nodes: $\nabla^4 F = 0$

$$\begin{cases} 20\cdot F_{12}-8\cdot(F_{13}+F_{22}+F_{11}+F_{22})+2\cdot(F_{23}+F_{21}+F_{21}+F_{23})+1\cdot(F_{32}+F_{10}+F_{32}+F_{14})=0 \\ 20\cdot F_{13}-8\cdot(F_{23}+F_{12}+F_{23}+F_{14})+2\cdot(F_{24}+F_{22}+F_{22}+F_{24})+1\cdot(F_{33}+F_{11}+F_{33}+F_{15})=0 \\ 20\cdot F_{22}-8\cdot(F_{12}+F_{21}+F_{32}+F_{23})+2\cdot(F_{13}+F_{11}+F_{31}+F_{33})+1\cdot(F_{22}+F_{20}+F_{42}+F_{24})=0 \\ 20\cdot F_{23}-8\cdot(F_{13}+F_{22}+F_{33}+F_{24})+2\cdot(F_{14}+F_{12}+F_{32}+F_{34})+1\cdot(F_{23}+F_{21}+F_{43}+F_{25})=0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} F_{10}+20F_{12}-8F_{13}-16F_{22}+4F_{23}=-2400 \\ -8F_{12}+20F_{13}+F_{15}+4F_{22}-16F_{23}=12000 \\ -8F_{12}+2F_{13}+F_{20}+21F_{22}-8F_{23}+F_{42}=-1800 \\ 2F_{12}-8F_{13}-8F_{22}+21F_{23}+F_{25}+F_{43}=9600 \end{cases} \quad \text{[kN]}$$

Boundary conditions: $\dfrac{\partial F}{\partial n}=-\dfrac{Q_s}{h}$

$$\begin{cases} \left[\dfrac{(F_{10}-F_{12})}{2s}\right]=-\dfrac{Q_{S,11}}{h} \\ \left[\dfrac{(F_{20}-F_{22})}{2s}\right]=-\dfrac{Q_{S,21}}{h} \\ \left[\dfrac{(F_{42}-F_{22})}{2s}\right]=-\dfrac{Q_{S,32}}{h} \\ \left[\dfrac{(F_{43}-F_{23})}{2s}\right]=-\dfrac{Q_{S,33}}{h} \\ \left[\dfrac{(F_{25}-F_{23})}{2s}\right]=-\dfrac{Q_{S,24}}{h} \\ \left[\dfrac{(F_{15}-F_{13})}{2s}\right]=-\dfrac{Q_{S,14}}{h} \end{cases} \Rightarrow \begin{cases} F_{10}-F_{12}=0 \\ F_{20}-F_{22}=0 \\ F_{42}-F_{22}=-1600 \\ F_{43}-F_{23}=-3200 \\ F_{25}-F_{23}=0 \\ F_{15}-F_{13}=0 \end{cases} \quad \text{[kN]}$$

Linear system of equations of the FDM and its solution:

$$\begin{bmatrix} 1 & 20 & -8 & 0 & 0 & -16 & 4 & 0 & 0 & 0 \\ 0 & -8 & 20 & 1 & 0 & 4 & -16 & 0 & 0 & 0 \\ 0 & -8 & 2 & 0 & 1 & 21 & -8 & 0 & 1 & 0 \\ 0 & 2 & -8 & 0 & 0 & -8 & 21 & 1 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} F_{10} \\ F_{12} \\ F_{13} \\ F_{15} \\ F_{20} \\ F_{22} \\ F_{23} \\ F_{25} \\ F_{42} \\ F_{43} \end{bmatrix} = \begin{bmatrix} -2400 \\ 12000 \\ -1800 \\ 9600 \\ 0 \\ 0 \\ -1600 \\ -3200 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} F_{10}=694{,}707 \\ F_{12}=694{,}707 \\ F_{13}=1705{,}293 \\ F_{15}=1705{,}293 \\ F_{20}=527{,}325 \\ F_{22}=527{,}325 \\ F_{23}=1272{,}675 \\ F_{25}=1272{,}675 \\ F_{42}=-1072{,}675 \\ F_{43}=-1927{,}325 \end{cases} \quad \text{[kN]}$$

Stress state:

NODE 12:

$$\sigma_{xx}^{(12)} = \frac{\partial^2 F}{\partial y^2} = \frac{F_{22} - 2F_{12} + F_{22}}{s^2} = -83{,}691 \text{ kPa}$$

$$\sigma_{yy}^{(12)} = \frac{\partial^2 F}{\partial x^2} = \frac{F_{11} - 2F_{12} + F_{13}}{s^2} = 78{,}970 \text{ kPa}$$

$$\sigma_{xy}^{(12)} = -\frac{\partial^2 F}{\partial x \partial y} = \frac{F_{23} + F_{21} - F_{21} - F_{23}}{4s^2} = 0 \text{ kPa}$$

NODE 12:

$$\sigma_{xx}^{(13)} = \frac{\partial^2 F}{\partial y^2} = \frac{F_{23} - 2F_{13} + F_{2}}{s^2} = -216{,}309 \text{ kPa}$$

$$\sigma_{yy}^{(13)} = \frac{\partial^2 F}{\partial x^2} = \frac{F_{12} - 2F_{13} + F_{14}}{s^2} = -78{,}970 \text{ kPa}$$

$$\sigma_{xy}^{(13)} = -\frac{\partial^2 F}{\partial x \partial y} = \frac{F_{24} + F_{22} - F_{22} - F_{24}}{4s^2} = 0 \text{ kPa}$$

wxMaxima script:

(%i1)	/* CONSTANTS */ Lx:6; Ly:8; h:20e-2; q:100e3; s:2;
(%i2)	/* MOMENTS OF EXTERNAL LOAD ABOUT BOUNDARY NODES */ M11:0; M21:0; M31:0; M32:0; M33:0; M34:0; M24:2*q*h*3*s*s-3*q*h*s*s/2; M14:2*q*h*3*s*2*s-3*q*h*2*s*s;
(%i3)	/* TANGENT FORCE IN BOUNDARY NODES */ QS11:0; QS21:0; QS32:2*q*h*s; QS33:2*q*h*2*s; QS24:0; QS14:0;
(%i4)	/* BOUNDARY VALUES OF AIRY STRESS FUNCTION */ F11:M11/h; F21:M21/h; F31:M31/h; F32:M32/h; F33:M33/h; F34:M34/h; F24:M24/h; F14:M14/h;

(%i5)	/* EQUATIONS FOR INTERNAL NODES: D^4 F = 0 */ eq1:20*F12-8*(F13+F22+F11+F22)+2*(F23+F21+F21+F23)+1*(F32+F10+F32+F14); eq2:20*F13-8*(F23+F12+F23+F14)+2*(F24+F22+F22+F24)+1*(F33+F11+F33+F15); eq3:20*F22-8*(F12+F21+F32+F23)+2*(F13+F11+F31+F33)+1*(F22+F20+F42+F24); eq4:20*F23-8*(F13+F22+F33+F24)+2*(F14+F12+F32+F34)+1*(F23+F21+F43+F25);
(%i6)	/* BOUNDARY CONDITIONS: -dF//dn = QS/h */ eq5:-((F10-F12)/2/s)-QS11/h; eq6:-((F20-F22)/2/s)-QS21/h; eq7:-((F42-F22)/2/s)-QS32/h; eq8:-((F43-F23)/2/s)-QS33/h; eq9:-((F25-F23)/2/s)-QS24/h; eq10:-((F15-F13)/2/s)-QS14/h;
(%i7)	/* SOLUTION OF THE SYSTEM OF EQUATIONS */ W:solve([eq1,eq2,eq3,eq4,eq5,eq6,eq7,eq8,eq9,eq10],[F10,F20,F12,F22,F42,F13,F23,F43,F15,F25]);
(%i8)	F10:float(W[1][1]); F20:float(W[1][2]); F12:float(W[1][3]); F22:float(W[1][4]); F42:float(W[1][5]); F13:float(W[1][6]); F23:float(W[1][7]); F43:float(W[1][8]); F15:float(W[1][9]); F25:float(W[1][10]);
(%i9)	/* STRESS STATE IN NODE 12 */ SXX12:(F22-2*F12+F22)/s^2; SYY12:(F11-2*F12+F13)/s^2; SXY12:-(F23+F21-F21-F23)/4/s^2;
(%i10)	/* STRESS STATE IN NODE 13 */ SXX31:(F23-2*F13+F323)/s^2; SYY31:(F12-2*F13+F14)/s^2; SXY31:-(F24+F22-F22-F24)/4/s^2;

REMARKS:

- function *float* changes a vulgar fraction into a decimal one;
- function *solve([e1,e2,...][x1,x2,...])* solves a system of equations of the form e1=0, e2=0, … with respect to unknowns x1, x2, …;

OBTAINED RESULTS:

F10=694706.7238912733
F20=527324.7496423462
F12=694706.7238912733
F22=527324.7496423462
F42=-1072675.250357654
F13=1705293.276108727
F23=1272675.250357654
F43=-1927324.749642346
F15=1705293.276108727
F25=1272675.250357654

SXX12=-83690.98712446354
SYY12=78969.95708154509
SXY12=0.0

SXX31=-216309.0128755365
SYY31=-78969.95708154514
SXY31=0.0