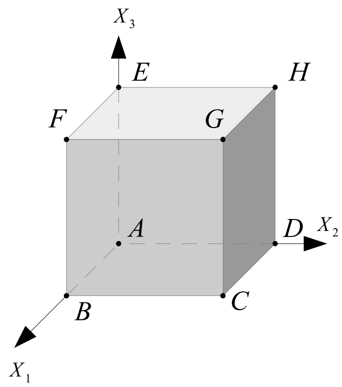
EXAMPLE 22



Deformation equations in material description are given for a cube of edge of unit length:

$$\begin{cases} x_1 = 2X_1 + 3X_2 - 4X_3 \\ x_2 = -3X_3 \\ x_3 = -X_1 - X_2 \end{cases}$$

Check if the above equations are invertible and find:

1. deformation equations in spatial description,
2. displacement vector in material and spatial description,
3. actual configuration – make a sketch,
4. material deformation gradient **F** and spatial deformation gradient **f** – perform polar decomposition of **F**,
5. material deformation tensor **C** and spatial deformation tensor **c**,
6. material strain tensor **E** and spatial strain tensor **e** as well as small strain tensor and small rotation tensor,
7. Piola-Kirchhoff stress tensor of the 1st and 2nd kind as well as the Cauchy stress tensor assuming linear constitutive relation of generalized Hooke's Law between material strain tensor and Piola-Kirchhoff stress tensor of the 2nd kind:

   $S_{ij} = 2G E_{ij} + \lambda E_{kk} \delta_{ij}$, Young modulus $E = 11 \text{ kPa}$, Poisson's ratio $\nu = 0{,}1$

8. actual surface load on BCGF face referred to actual configuration (true load),
9. actual surface load on BCGF face referred to reference configuration (nominal load),
10. surface area of BCGF face before and after deformation,
11. length of AG segment before and after deformation,
12. volume of the cube before and after deformation.

SOLUTION:

At first, let's determine **material deformation gradient**. The i, j component is equal derivative of i -th spatial coordinate $\mathbf{x}$ with respect to j -th material coordinate $\mathbf{X}$:

$$\mathbf{F} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} 2 & 3 & -4 \\ 0 & 0 & -3 \\ -1 & -1 & 0 \end{bmatrix} \tag{1}$$

We may check if the equations are invertible by finding the **determinant of jacobian matrix**:

$$J = \det \mathbf{F} = \begin{vmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{vmatrix} = \begin{vmatrix} 2 & 3 & -4 \\ 0 & 0 & -3 \\ -1 & -1 & 0 \end{vmatrix} = 3 \tag{2}$$

Determinant is greater than 0, so the equations are **locally invertible** in each point.

AD 1) DEFORMATION EQUATIONS IN SPATIAL DESCRIPTION

Deformation equations constitute a system of equations biding material and spatial coordinates – this system may be solved with respect to one or another set of coordinates. We want to express material coordinates $\mathbf{X}$ in terms of the spatial ones, so we look for $\mathbf{X}$ as a solution of this system. In our case this is a **linear system** – it can be solved easily with the use of the **Cramer formulae**:

$$\begin{cases} x_1 = 2X_1 + 3X_2 - 4X_3 \\ x_2 = -3X_3 \\ x_3 = -X_1 - X_2 \end{cases} \tag{3}$$

Determinants:

$$W = \det(\mathbf{F}) = 3 \tag{4}$$

$$W_1 = \begin{bmatrix} x_1 & 3 & -4 \\ x_2 & 0 & -3 \\ x_3 & -1 & 0 \end{bmatrix} = -3x_1 + 4x_2 - 9x_3 \tag{5}$$

$$W_2 = \begin{bmatrix} 2 & x_1 & -4 \\ 0 & x_2 & -3 \\ -1 & x_3 & 0 \end{bmatrix} = 3x_1 - 4x_2 + 6x_3 \tag{6}$$

$$W_3 = \begin{bmatrix} 2 & 3 & x_1 \\ 0 & 0 & x_2 \\ -1 & -1 & x_3 \end{bmatrix} = x_2 \tag{7}$$

Solution:

$$\begin{cases} X_1 = \frac{W_1}{W} = -x_1 + \frac{4}{3}x_2 - 3x_3 \\ X_2 = \frac{W_2}{W} = x_1 - \frac{4}{3}x_2 + 2x_3 \\ X_3 = \frac{W_3}{W} = -\frac{x_2}{3} \end{cases} \tag{8}$$

In case of non-linear relations such an approach is impossible and a non-linear system of equations must be solved.

AD 2) DISPLACEMENT VECTOR

Displacement vector is defined as: $\mathbf{u}=\mathbf{x}-\mathbf{X}$ (9)

Depending on that, in which description we want it to be express, we should express one of the coordinates in the above formula in terms of the other one, according to relations (3) or (8).

- **material description** – material coordinates $\mathbf{X}$ are independent variables now, so we need to express spatial coordinates $\mathbf{x}$ in terms of material ones according to (3):

$$\begin{cases} u_1(\mathbf{X}) = x_1((\mathbf{X}))-X_1 = X_1+3X_2-4X_3 \\ u_2(\mathbf{X}) = x_2((\mathbf{X}))-X_2 = -X_2-3X_3 \\ u_3(\mathbf{X}) = x_3((\mathbf{X}))-X_3 = -X_1-X_2-X_3 \end{cases} \tag{10}$$

- **spatial description**: – spatial coordinates $\mathbf{x}$ are independent variables now, so we need to express material coordinates $\mathbf{X}$ in terms of spatial ones according to (8):

$$\begin{cases} u_1(\mathbf{x}) = x_1-X_1(\mathbf{x}) = 2x_1-\frac{4}{3}x_2+3x_3 \\ u_2(\mathbf{x}) = x_2-X_2(\mathbf{x}) = -x_1+\frac{7}{3}x_2-2x_3 \\ u_3(\mathbf{x}) = x_3-X_3(\mathbf{x}) = x_3+\frac{x_2}{3} \end{cases} \tag{11}$$

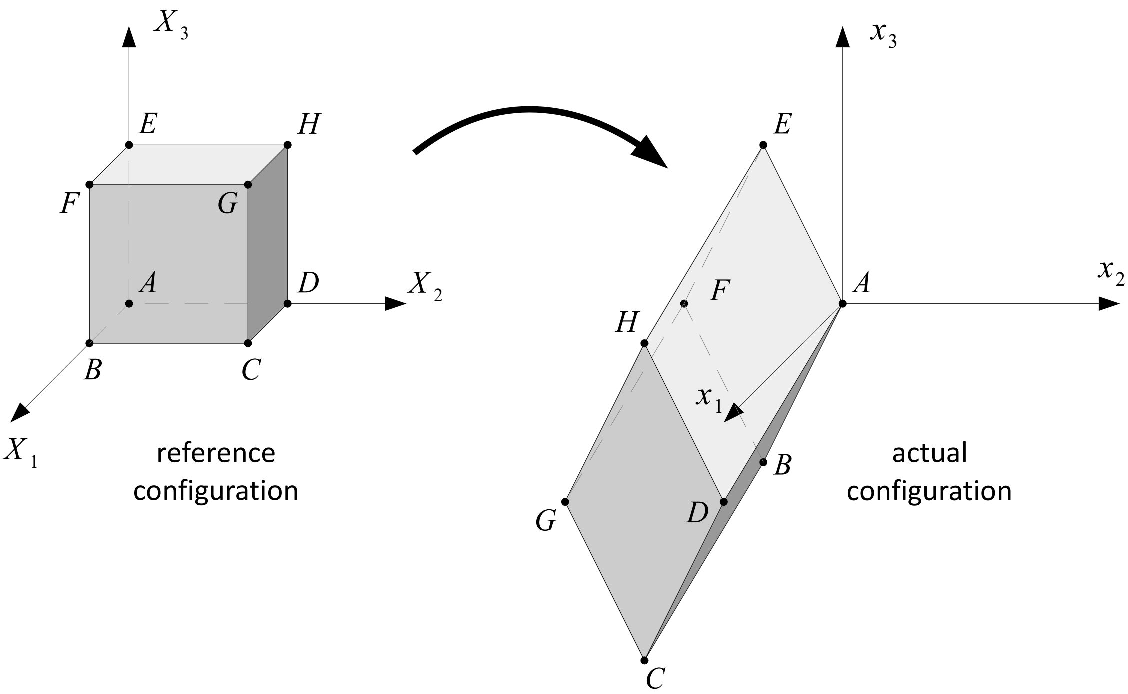
It may be easily checked that substituting (8) in (10) results in (11) and substituting (3) in (11) gives us (10).

AD 3) ACTUAL CONFIGURATION OF THE CUBE

Deformation equations are **linear functions**, which means that **each straight line segment is transformed into a straight segment**, however, in general its position, orientation and length is changed. Since the reference configuration is a cube, then **actual configuration will be a parallelepiped** – so **it is enough to find location of its corners** and connect them with straight line segments.

Position in actual configuration is given by spatial coordinates $\mathbf{x}$. Let's make use of relations (3) and for each corner in reference configuration we read its initial position (material coordinates $\mathbf{X}$) and substitute appropriate coordinates in relations (3). We obtain:

$$\begin{aligned} &A:\ \mathbf{x}(0;0;0)=[0;0;0]^T && B:\ \mathbf{x}(1;0;0)=[2;0;-1]^T \\ &C:\ \mathbf{x}(1;1;0)=[5;0;-2]^T && D:\ \mathbf{x}(0;1;0)=[3;0;-1]^T \\ &E:\ \mathbf{x}(0;0;1)=[-4;-3;0]^T && F:\ \mathbf{x}(1;0;1)=[-2;-3;-1]^T \\ &G:\ \mathbf{x}(1;1;1)=[1;-3;-2]^T && H:\ \mathbf{x}(0;1;1)=[-1;-3;-1]^T \end{aligned}$$



AD 4) POLAR DECOMPOSITION OF MATERIAL DEFORMATION GRADIENT

Polar decomposition of material deformation gradient is a possibility to express the gradient as a product of a symmetric tensor and of an orthogonal tensor in one of two forms:

$$\mathbf{F} = \mathbf{R}\,\mathbf{U} = \mathbf{V}\,\mathbf{R} \tag{12}$$

where:

- $\mathbf{R}$ rotation tensor – **orthogonal**: $\det(\mathbf{R}) = 1\,,\quad \mathbf{R}^{\mathrm{T}} = \mathbf{R}^{-1}$
- $\mathbf{U}$ right stretch tensor – **symmetric:** $\mathbf{U}^{\mathrm{T}} = \mathbf{U}$
- $\mathbf{V}$ left stretch tensor – **symmetric:** $\mathbf{V}^{\mathrm{T}} = \mathbf{V}$

An algorithm for finding the components of this decomposition is as follows:

1. find **material deformation gradient:** $\mathbf{F}$
2. find **deformation tensor:** $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$
3. find **eigenvalues** C_1, C_2, C_3 and **eigenvectors** $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3$ of **deformation tensor**. Deformation tensor in its eigenvector coordinate system has a diagonal form:

$$\mathbf{C}_{[\omega]} = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & C_3 \end{bmatrix}$$

4. find **transformation matrix**:

$$\mathbf{A} = \begin{bmatrix} \boldsymbol{\omega}^{(1)} \\ \boldsymbol{\omega}^{(2)} \\ \boldsymbol{\omega}^{(3)} \end{bmatrix} = \begin{bmatrix} \omega_1^{(1)} & \omega_2^{(1)} & \omega_3^{(1)} \\ \omega_1^{(2)} & \omega_2^{(2)} & \omega_3^{(2)} \\ \omega_1^{(3)} & \omega_2^{(3)} & \omega_3^{(3)} \end{bmatrix}$$

5. find **right stretch tensor in its eigenvector coordinate system**:

$$\mathbf{U}_{[\omega]} = \begin{bmatrix} \sqrt{C_1} & 0 & 0 \\ 0 & \sqrt{C_2} & 0 \\ 0 & 0 & \sqrt{C_3} \end{bmatrix}$$

6. find **inverse of the right stretch tensor in its eigenvector coordinate system**:

$$\mathbf{U}^{-1}{}_{[\omega]} = \begin{bmatrix} \frac{1}{\sqrt{C_1}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{C_2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{C_3}} \end{bmatrix}$$

7. find **right stretch tensor in the original coordinate system**:

$$\mathbf{U} = \mathbf{A}^{\mathrm{T}} \mathbf{U}_{[\omega]} \mathbf{A}$$

8. find **inverse of the right stretch tensor in the original coordinate system**:

$$\mathbf{U}^{-1} = \mathbf{A}^{\mathrm{T}} \mathbf{U}^{-1}{}_{[\omega]} \mathbf{A}$$

9. find **rotation tensor**: $\mathbf{R} = \mathbf{F}\,\mathbf{U}^{-1}$
10. find **left stretch tensor**: $\mathbf{V} = \mathbf{R}\,\mathbf{U}\,\mathbf{R}^{\mathrm{T}} = \mathbf{F}\,\mathbf{R}^{\mathrm{T}}$

Deformation tensor:

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}} \cdot \mathbf{F} = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 0 & -1 \\ -4 & -3 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 & -4 \\ 0 & 0 & -3 \\ -1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 7 & -8 \\ 7 & 10 & -12 \\ -8 & -12 & 25 \end{bmatrix} \tag{13}$$

In order to find eigenvalues of deformation tensor we need to solve the **secular equation**:

$$C^3 - I_1 C^2 + I_2 C - I_3 = 0 \tag{14}$$

the coefficients of which are given by **invariants of deformation tensor**:

- **the first invariant – trace of the tensor**:

$$I_1 = \mathrm{tr}(\mathbf{C}) = C_{11} + C_{22} + C_{33} = 40 \tag{15}$$

- **the second invariant**:

$$I_2 = \begin{vmatrix} C_{22} & C_{23} \\ C_{32} & C_{33} \end{vmatrix} + \begin{vmatrix} C_{11} & C_{13} \\ C_{31} & C_{33} \end{vmatrix} + \begin{vmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{vmatrix} = 168 \tag{16}$$

- **the third invariant – determinant of the tensor**:

$$I_3 = \det(\mathbf{C}) = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix} = 9 \tag{17}$$

Secular equation: $$C^3 - 40C^2 + 168C - 9 = 0 \quad (18)$$

It may be shown that due to symmetry and positive definiteness of this tensor the above equation has **exactly three real and positive roots**. We may find them with the use of **Cardano formulae** (as below) or **numerically** with the use of calculator or computer:

Analytical computation require finding following parameters:

$$p = \frac{1}{3}(C_{11} + C_{22} + C_{33}) = 13{,}333 \quad (19)$$

$$J_2 = \frac{1}{6}\left[(C_{22} - C_{33})^2 + (C_{33} - C_{11})^2 + (C_{11} - C_{22})^2\right] + (C_{23}^2 + C_{31}^2 + C_{12}^2) = 365{,}333 \quad (20)$$

$$\begin{aligned} J_3 &= (C_{11} - p)(C_{22} - p)(C_{33} - p) + 2C_{23}C_{31}C_{12} - \\ &- (C_{11} - p)C_{23}^2 - (C_{22} - p)C_{31}^2 - (C_{33} - p)C_{12}^2 = 2509{,}741 \end{aligned} \quad (21)$$

$$q = \sqrt{2J_2} = 27{,}031 \quad (22)$$

$$\theta = \frac{1}{3}\arccos\left(\frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}}\right) = 0{,}122\,\text{rad} \quad (23)$$

Roots of the secular equation:

$$C_1 = p + \sqrt{\frac{2}{3}}q\cos(\theta) = 35{,}240 \quad (24)$$

$$C_2 = p + \sqrt{\frac{2}{3}}q\cos\left(\theta + \frac{2\pi}{3}\right) = 0{,}0543 \quad (25)$$

$$C_3 = p + \sqrt{\frac{2}{3}}q\cos\left(\theta + \frac{4\pi}{3}\right) = 4{,}706 \quad (26)$$

Numbering of the above roots may be chosen arbitrary. It is commonly done in such a way that the smallest one is the first one, the intermediate is the second one and the largest one is the third one (or exactly the other way round):

Eigenvalues of deformation tensor: $$C_1 = 0{,}0543 \qquad C_2 = 4{,}706 \qquad C_3 = 35{,}240 \quad (27)$$

For each single eigenvalue there is one principal direction and any vector parallel to that direction is an eigenvector corresponding with that eigenvalue. Among an infinite number of possible vectors we will choose **normalized vectors** (of **unit length**), the **sense** (**orientation**) **of two eigenvectors will be chosen arbitrary** and the **sense of the third eigenvector will be chosen in such a way that those vectors constituted a right-handed set**. This set will be used to construct a transformation matrix allowing us to change the original coordinate system into an eigenvector coordinate system. Due to symmetry of the tensor it is known that eigenvectors are mutually orthogonal. Normalization and preserving proper orientation will provide us with a right-handed **orthonormal** (**Crtesian**) **coordinate system**.

In order to find first eigenvector $\boldsymbol{\omega}^{(1)}$ corresponding with the first eigenvalue C_1 let's write down the following expression:

$$\mathbf{C}\,\boldsymbol{\omega}^{(1)} - C_1\boldsymbol{\omega}^{(1)} = \mathbf{0} \quad \Rightarrow \quad (\mathbf{C}-C_1\mathbf{1})\boldsymbol{\omega}^{(1)} = \mathbf{0}$$
$$\begin{bmatrix} 5-0{,}0543 & 7 & -8 \\ 7 & 10-0{,}0543 & -12 \\ -8 & -12 & 25-0{,}0543 \end{bmatrix} \begin{bmatrix} \omega_1^{(1)} \\ \omega_2^{(1)} \\ \omega_3^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{28}$$

The above vector equation is satisfied by definition of the eigenvector. It corresponds with a system of homogeneous (with right hand side equal 0) linear equations. Such a system has a non-zero solution if the determinant of matrix of coefficients is equal 0. Zero determinant may be interpreted as a zero value of a triple product of vectors, the components of which are described by three rows of the matrix of coefficients of that system. This in turn means that these vectors lie in a single plane. Simultaneously each equation in the above system may be interpreted as a dot product of one of those vectors and eigenvector $\boldsymbol{\omega}^{(1)}$ - since the right hand side of that equation is 0, this means that those vectors are perpendicular. It means that eigenvector $\boldsymbol{\omega}^{(1)}$ is perpendicular to the plane determined by vectors corresponding with rows of the coefficient matrix. Such a perpendicular vector may be determined as a cross product of any two vectors lying in that plane, e.g. First two vectors:

$$\begin{bmatrix} 4{,}946 & 7 & -8 \\ 7 & 9{,}946 & -12 \\ -8 & -12 & 24{,}946 \end{bmatrix} \Rightarrow \begin{array}{rc} & [4{,}946\ ;\ 7\ ;\ -8] \\ \times & [7\ ;\ 9{,}946\ ;\ -12] \\ \hline \mathbf{v}^{(1)} = & [-4{,}434\,;3{,}349\,;0{,}189] \end{array} \tag{29}$$

Eigenvector is obtained by normalization of the above result:

$$\boldsymbol{\omega}^{(1)} = \frac{\mathbf{v}^{(1)}}{|\mathbf{v}^{(1)}|} = \frac{[-4{,}434\,;3{,}349\,;0{,}189]}{\sqrt{(-4{,}434)^2 + (3{,}349)^2 + (0{,}189)^2}} = [-0{,}798\,;0{,}603\,;0{,}034] \tag{30}$$

In an analogous way we may find the second eigenvector $\boldsymbol{\omega}^{(2)}$ corresponding with the second eigenvalue C_2 .

$$(\mathbf{C}-C_2\mathbf{1})\boldsymbol{\omega}^{(2)} = \mathbf{0}$$
$$\begin{bmatrix} 5-4{,}706 & 7 & -8 \\ 7 & 10-4{,}706 & -12 \\ -8 & -12 & 25-4{,}706 \end{bmatrix} \begin{bmatrix} \omega_1^{(2)} \\ \omega_2^{(2)} \\ \omega_3^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{31}$$

$$\begin{bmatrix} 0{,}294 & 7 & -8 \\ 7 & 5{,}294 & -12 \\ -8 & -12 & 20{,}294 \end{bmatrix} \Rightarrow \begin{array}{rc} & [0{,}294\ ;\ 7\ ;\ -8] \\ \times & [7\ ;\ 5{,}294\ ;\ -12] \\ \hline \mathbf{v}^{(2)} = & [-41{,}646\,;-52{,}470\,;-47{,}442] \end{array} \tag{32}$$

$$\boldsymbol{\omega}^{(2)} = \frac{\mathbf{v}^{(2)}}{|\mathbf{v}^{(2)}|} = \frac{[-41{,}646\,;-52{,}470\,;-47{,}442]}{\sqrt{(-41{,}646)^2 + (-52{,}470)^2 + (-47{,}442)^2}} = [-0{,}507\,;-0{,}639\,;-0{,}578] \quad (33)$$

The third eigenvector is determined in a different way. Since we know that it must be perpendicular to two others and we want it to be normalized and oriented in such a way so that it constituted with the rest of eigenvectors a right-hand set, then all those features are provided by a **vector which is a cross product of two other eigenvectors**:

$$\begin{array}{l} \boldsymbol{\omega}^{(1)} = [-0{,}798\ ;\ 0{,}602\ ;\ 0{,}034] \\ \boldsymbol{\omega}^{(2)} = [-0{,}507\ ;\ -0{,}639\ ;\ -0{,}578] \\ \hline \boldsymbol{\omega}^{(3)} = \boldsymbol{\omega}^{(1)} \times \boldsymbol{\omega}^{(2)} \quad [-0{,}326\,;-0{,}478\,;0{,}815] \end{array} \quad (34)$$

Eigenvectors of deformation tensor:

$$\boldsymbol{\omega}^{(1)} = [-0{,}798\ ;\ 0{,}602\ ;\ 0{,}034]$$
$$\boldsymbol{\omega}^{(2)} = [-0{,}507\ ;\ -0{,}639\ ;\ -0{,}578]$$
$$\boldsymbol{\omega}^{(3)} = [-0{,}326\ ;\ -0{,}478\ ;\ 0{,}815]$$

The i,j **-th component** of transformation matrix is equal j **-th component of** i **-th eigenvector**:

Transformation matrix:

$$\mathbf{A} = \begin{bmatrix} -0{,}798 & 0{,}602 & 0{,}034 \\ -0{,}507 & -0{,}639 & -0{,}578 \\ -0{,}326 & -0{,}478 & 0{,}815 \end{bmatrix} \quad (35)$$

Right stretch tensor in eigenvector coordinate system:

$$\mathbf{U}_{[\omega]} = \begin{bmatrix} \sqrt{C_1} & 0 & 0 \\ 0 & \sqrt{C_2} & 0 \\ 0 & 0 & \sqrt{C_3} \end{bmatrix} = \begin{bmatrix} 0{,}233 & 0 & 0 \\ 0 & 2{,}169 & 0 \\ 0 & 0 & 5{,}936 \end{bmatrix} \quad (36)$$

Inverse of $\mathbf{U}$ in eigenvector coordinate system:

$$\mathbf{U}_{[\omega]}^{-1} = \begin{bmatrix} \frac{1}{\sqrt{C_1}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{C_2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{C_3}} \end{bmatrix} = \begin{bmatrix} 4{,}293 & 0 & 0 \\ 0 & 0{,}461 & 0 \\ 0 & 0 & 0{,}169 \end{bmatrix} \quad (37)$$

Right stretch tensor $\mathbf{U}$ in the original coordinate system:

$$\mathbf{U}=\mathbf{A}^{T}\mathbf{U}_{[\omega]}\mathbf{A}=$$

$$=\begin{bmatrix}-0{,}798 & -0{,}507 & -0{,}326\\ 0{,}602 & -0{,}639 & -0{,}478\\ 0{,}034 & -0{,}578 & 0{,}815\end{bmatrix}\begin{bmatrix}0{,}233 & 0 & 0\\ 0 & 2{,}169 & 0\\ 0 & 0 & 5{,}936\end{bmatrix}\begin{bmatrix}-0{,}798 & 0{,}602 & 0{,}034\\ -0{,}507 & -0{,}639 & -0{,}578\\ -0{,}326 & -0{,}478 & 0{,}815\end{bmatrix}=$$

$$=\begin{bmatrix}-0{,}186 & -1.100 & -1.935\\ 0{,}140 & -1.386 & -2.837\\ 0{,}00792 & -1.254 & 4.838\end{bmatrix}\begin{bmatrix}-0{,}798 & 0{,}602 & 0{,}034\\ -0{,}507 & -0{,}639 & -0{,}578\\ -0{,}326 & -0{,}478 & 0{,}815\end{bmatrix}=\begin{bmatrix}1{,}339 & 1{,}518 & -0{,}950\\ 1{,}518 & 2{,}328 & -1{,}508\\ -0{,}950 & -1{,}508 & 4{,}671\end{bmatrix} \qquad (38)$$

Inverse of the right stretch tensor $\mathbf{U}^{-1}$ in the original coordinate system:

$$\mathbf{U}^{-1}=\mathbf{A}^{T}\mathbf{U}^{-1}{}_{[\omega]}\mathbf{A}=$$

$$=\begin{bmatrix}-0{,}798 & -0{,}507 & -0{,}326\\ 0{,}602 & -0{,}639 & -0{,}478\\ 0{,}034 & -0{,}578 & 0{,}815\end{bmatrix}\begin{bmatrix}4{,}293 & 0 & 0\\ 0 & 0{,}461 & 0\\ 0 & 0 & 0{,}169\end{bmatrix}\begin{bmatrix}-0{,}798 & 0{,}602 & 0{,}034\\ -0{,}507 & -0{,}639 & -0{,}578\\ -0{,}326 & -0{,}478 & 0{,}815\end{bmatrix}=$$

$$=\begin{bmatrix}-3{,}423 & -0{,}234 & -0{,}055\\ 2{,}585 & -0{,}295 & -0{,}081\\ 0{,}146 & -0{,}266 & 0{,}137\end{bmatrix}\begin{bmatrix}-0{,}798 & 0{,}602 & 0{,}034\\ -0{,}507 & -0{,}639 & -0{,}578\\ -0{,}326 & -0{,}478 & 0{,}815\end{bmatrix}=\begin{bmatrix}2{,}867 & -1{,}886 & -0{,}026\\ -1{,}886 & 1{,}784 & 0{,}192\\ -0{,}026 & 0{,}192 & 0{,}271\end{bmatrix} \qquad (39)$$

Rotation tensor:

$$\mathbf{R}=\mathbf{F}\,\mathbf{U}^{-1}=$$

$$=\begin{bmatrix}2 & 3 & -4\\ 0 & 0 & -3\\ -1 & -1 & 0\end{bmatrix}\begin{bmatrix}2{,}867 & -1{,}886 & -0{,}026\\ -1{,}886 & 1{,}784 & 0{,}192\\ -0{,}026 & 0{,}192 & 0{,}271\end{bmatrix}=\begin{bmatrix}0{,}179 & 0{,}810 & -0{,}558\\ 0{,}078 & -0{,}577 & -0{,}813\\ -0{,}981 & 0{,}102 & -0{,}167\end{bmatrix} \qquad (40)$$

Left stretch tensor:

$$\mathbf{V}=\mathbf{R}\,\mathbf{U}\,\mathbf{R}^{T}=\mathbf{F}\,\mathbf{R}^{T}=$$

$$=\begin{bmatrix}2 & 3 & -4\\ 0 & 0 & -3\\ -1 & -1 & 0\end{bmatrix}\begin{bmatrix}0{,}179 & 0{,}078 & -0{,}981\\ 0{,}810 & -0{,}577 & 0{,}102\\ -0{,}558 & -0{,}813 & -0{,}167\end{bmatrix}=\begin{bmatrix}5{,}018 & 1{,}676 & -0{,}988\\ 1{,}676 & 2{,}440 & 0{,}500\\ -0{,}988 & 0{,}500 & 0{,}879\end{bmatrix} \qquad (41)$$

Check:

$\mathbf{U}^{T}=\mathbf{U}$ - tensor $\mathbf{U}$ is symmetric

$\mathbf{V}^{T}=\mathbf{V}$ - tensor $\mathbf{V}$ is symmetric

$\det(\mathbf{R})=1$

$$\mathbf{R}\,\mathbf{R}^{T}=\begin{bmatrix}0{,}179 & 0{,}810 & -0{,}558\\ 0{,}078 & -0{,}577 & -0{,}813\\ -0{,}981 & 0{,}102 & -0{,}167\end{bmatrix}\begin{bmatrix}0{,}179 & 0{,}078 & -0{,}981\\ 0{,}810 & -0{,}577 & 0{,}102\\ -0{,}558 & -0{,}813 & -0{,}167\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\mathbf{R}\,\mathbf{U}=\begin{bmatrix}0{,}179 & 0{,}810 & -0{,}558\\ 0{,}078 & -0{,}577 & -0{,}813\\ -0{,}981 & 0{,}102 & -0{,}167\end{bmatrix}\begin{bmatrix}1{,}339 & 1{,}518 & -0{,}950\\ 1{,}518 & 2{,}328 & -1{,}508\\ -0{,}950 & -1{,}508 & 4{,}671\end{bmatrix}=\begin{bmatrix}2 & 3 & -4\\ 0 & 0 & -3\\ -1 & -1 & 0\end{bmatrix}=\mathbf{F}$$

DEFORMATION OF MATERIAL FIBRE $\mathrm{d}\mathbf{X} = [0\,;1\,;0]$

Stretching before rotation:

$$\mathbf{U}\cdot\mathrm{d}\mathbf{X} = \begin{bmatrix} 1{,}339 & 1{,}518 & -0{,}950 \\ 1{,}518 & 2{,}328 & -1{,}508 \\ -0{,}950 & -1{,}508 & 4{,}671 \end{bmatrix}\cdot\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1{,}518 \\ 2{,}328 \\ -1{,}508 \end{bmatrix}$$

Rotation after stretching:

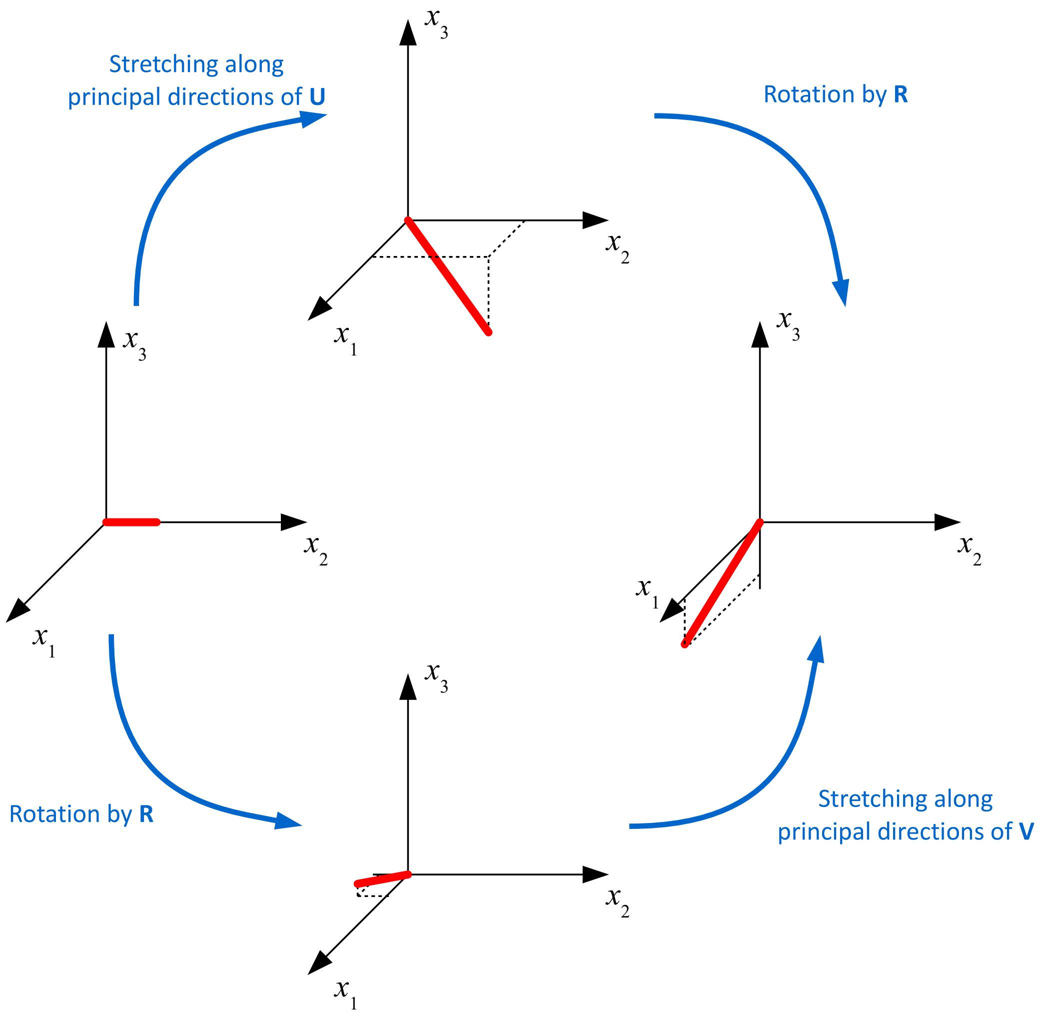
$$\mathbf{F}\ \mathrm{d}\mathbf{X} = \mathbf{R}\left(\mathbf{U}\ \mathrm{d}\mathbf{X}\right) = \begin{bmatrix} 0{,}179 & 0{,}810 & -0{,}558 \\ 0{,}078 & -0{,}577 & -0{,}813 \\ -0{,}981 & 0{,}102 & -0{,}167 \end{bmatrix}\cdot\begin{bmatrix} 1{,}518 \\ 2{,}328 \\ -1{,}508 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$$

Rotation before stretching:

$$\mathbf{R}\cdot\mathrm{d}\mathbf{X} = \begin{bmatrix} 0{,}179 & 0{,}810 & -0{,}558 \\ 0{,}078 & -0{,}577 & -0{,}813 \\ -0{,}981 & 0{,}102 & -0{,}167 \end{bmatrix}\cdot\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0{,}810 \\ -0{,}577 \\ 0{,}102 \end{bmatrix}$$

Stretching after rotation:

$$\mathbf{F}\ \mathrm{d}\mathbf{X} = \mathbf{V}\left(\mathbf{R}\ \mathrm{d}\mathbf{X}\right) = \begin{bmatrix} 5{,}018 & 1{,}676 & -0{,}988 \\ 1{,}676 & 2{,}440 & 0{,}500 \\ -0{,}988 & 0{,}500 & 0{,}879 \end{bmatrix}\cdot\begin{bmatrix} 0{,}810 \\ -0{,}577 \\ 0{,}102 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$$



AD 5) DEFORMATION TENSOR

In order to find spatial deformation tensor we need to determine spatial deformation gradient:

Spatial deformation gradient:

$$\mathbf{f} = \mathbf{F}^{-1} = \frac{(\operatorname{cof}(\mathbf{F}))^{\mathrm{T}}}{\det(\mathbf{F})} =$$

$$= \frac{1}{3}\begin{bmatrix} (-1)^{(1+1)}\begin{vmatrix} 0 & -3 \\ -1 & 0 \end{vmatrix} & (-1)^{(1+2)}\begin{vmatrix} 0 & -3 \\ -1 & 0 \end{vmatrix} & (-1)^{(1+3)}\begin{vmatrix} 0 & 0 \\ -1 & -1 \end{vmatrix} \\ (-1)^{(2+1)}\begin{vmatrix} 3 & -4 \\ -1 & 0 \end{vmatrix} & (-1)^{(2+2)}\begin{vmatrix} 2 & -4 \\ -1 & 0 \end{vmatrix} & (-1)^{(2+3)}\begin{vmatrix} 2 & 3 \\ -1 & -1 \end{vmatrix} \\ (-1)^{(3+1)}\begin{vmatrix} 3 & -4 \\ 0 & -3 \end{vmatrix} & (-1)^{(3+2)}\begin{vmatrix} 2 & -4 \\ 0 & -3 \end{vmatrix} & (-1)^{(3+3)}\begin{vmatrix} 2 & 3 \\ 0 & 0 \end{vmatrix} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} -1 & \frac{4}{3} & -3 \\ 1 & -\frac{4}{3} & 2 \\ 0 & -\frac{1}{3} & 0 \end{bmatrix} \tag{42}$$

It may be as well found by definition with the use of relations (8):

$$\mathbf{f} = \begin{bmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} & \frac{\partial X_1}{\partial x_3} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} & \frac{\partial X_2}{\partial x_3} \\ \frac{\partial X_3}{\partial x_1} & \frac{\partial X_3}{\partial x_2} & \frac{\partial X_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} -1 & \frac{4}{3} & -3 \\ 1 & -\frac{4}{3} & 2 \\ 0 & -\frac{1}{3} & 0 \end{bmatrix} \tag{43}$$

Right Cauchy-Green deformation tensor (**material deformation tensor**):

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}} \cdot \mathbf{F} = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 0 & -1 \\ -4 & -3 & 0 \end{bmatrix}\begin{bmatrix} 2 & 3 & -4 \\ 0 & 0 & -3 \\ -1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 7 & -8 \\ 7 & 10 & -12 \\ -8 & -12 & 25 \end{bmatrix} \tag{44}$$

Cauchy deformation tensor (**spatial deformation tensor**):

$$\mathbf{c} = \mathbf{f}^{\mathrm{T}} \cdot \mathbf{f} =$$

$$= \begin{bmatrix} -1 & 1 & 0 \\ 1{,}333 & -1{,}333 & -0{,}333 \\ -3 & 2 & 0 \end{bmatrix}\begin{bmatrix} -1 & 1{,}333 & -3 \\ 1 & -1{,}333 & 2 \\ 0 & -0{,}333 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2{,}667 & 5 \\ -2{,}667 & 3{,}667 & -6{,}667 \\ 5 & -6{,}667 & 13 \end{bmatrix} \tag{45}$$

AD 6) STRAIN TENSOR

Green – de Saint-Venant strain tensor (material strain tensor):

$$\mathbf{E} = \frac{1}{2}(\mathbf{C}-\mathbf{1}) = \frac{1}{2}\left(\begin{bmatrix} 5 & 7 & -8 \\ 7 & 10 & -12 \\ -8 & -12 & 25 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 2 & 3{,}5 & -4 \\ 3{,}5 & 4{,}5 & -6 \\ -4 & -6 & 12 \end{bmatrix} \tag{46}$$

Almansi-Hamel strain tensor (spatial strain tensor)

$$\mathbf{e} = \frac{1}{2}(\mathbf{1}-\mathbf{c}) = \frac{1}{2}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -2{,}667 & 5 \\ -2{,}667 & 3{,}667 & -6{,}667 \\ 5 & -6{,}667 & 13 \end{bmatrix}\right) = \begin{bmatrix} -0{,}5 & 1{,}333 & -2{,}5 \\ 1{,}333 & -1{,}333 & 3{,}333 \\ -2{,}5 & 3{,}333 & -6 \end{bmatrix} \tag{47}$$

In order to find small strain tensor and small rotation tensor we need to find **material displacement gradient**:

$$\mathbf{H} = \mathbf{F}-\mathbf{1} = \begin{bmatrix} 2 & 3 & -4 \\ 0 & 0 & -3 \\ -1 & -1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -4 \\ 0 & -1 & -3 \\ -1 & -1 & -1 \end{bmatrix} \tag{48}$$

Small strain tensor:

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\mathbf{H}+\mathbf{H}^{\mathrm{T}}) = \frac{1}{2}\left(\begin{bmatrix} 1 & 3 & -4 \\ 0 & -1 & -3 \\ -1 & -1 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -1 \\ 3 & -1 & -1 \\ -4 & -3 & -1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1{,}5 & -2{,}5 \\ 1{,}5 & -1 & -2 \\ -2{,}5 & -2 & -1 \end{bmatrix} \tag{49}$$

Small rotation tensor:

$$\boldsymbol{\omega} = \frac{1}{2}(\mathbf{H}-\mathbf{H}^{\mathrm{T}}) = \frac{1}{2}\left(\begin{bmatrix} 1 & 3 & -4 \\ 0 & -1 & -3 \\ -1 & -1 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & -1 \\ 3 & -1 & -1 \\ -4 & -3 & -1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1{,}5 & -1{,}5 \\ -1{,}5 & 0 & -1 \\ 1{,}5 & 1 & 0 \end{bmatrix} \tag{50}$$

AD 7) STRESS TENSOR

Piola-Kirchhoff stress tensor of the 2nd kind is determined with the use of given constitutive relations, parameters of which are found according to known values of the Young modulus and Poisson's ratio:

- **Young modulus**: $E = 11\ \text{kPa}$ (51)
- **Poisson's ratio**: $\nu = 0{,}1$ (52)
- **Kirchhoff modulus**: $G = \dfrac{E}{2(1+\nu)} = 5\ \text{kPa}$ (53)
- **Lame parameter**: $\lambda = \dfrac{E\nu}{(1+\nu)(1-2\nu)} = 1{,}25\ \text{kPa}$ (54)

Piola-Kirchhoff stress tensor of the 2nd kind (material stress tensor)

$$\mathbf{T}_S = 2G\,\mathbf{E} + \lambda\,\mathrm{tr}\,\mathbf{E}\,\mathbf{1} = 2\cdot 5\cdot\begin{bmatrix} 2 & 3{,}5 & -4 \\ 3{,}5 & 4{,}5 & -6 \\ -4 & -6 & 12 \end{bmatrix} + 1{,}25\cdot(2+4{,}5+12)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 43{,}125 & 35 & -40 \\ 35 & 68{,}125 & -60 \\ -40 & -60 & 143{,}125 \end{bmatrix}\ [\text{kPa}] \quad (55)$$

Piola-Kirchhoff stress tensor of the 1st kind (nominal stress tensor)

$$\mathbf{T}_R = \mathbf{F}\cdot\mathbf{T}_S = \begin{bmatrix} 2 & 3 & -4 \\ 0 & 0 & -3 \\ -1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 43{,}125 & 35 & -40 \\ 35 & 68{,}125 & -60 \\ -40 & -60 & 143{,}125 \end{bmatrix} = \begin{bmatrix} 351{,}25 & 514{,}375 & -832{,}5 \\ 120 & 180 & -429{,}375 \\ -78{,}125 & -103{,}125 & 100 \end{bmatrix}\ [\text{kPa}] \quad (56)$$

Cauchy stress tensor (true stress tensor)

$$\mathbf{T}_\sigma = \frac{1}{J}\mathbf{T}_R\cdot\mathbf{F}^{\mathrm{T}} = \frac{1}{3}\begin{bmatrix} 2 & 0 & -1 \\ 3 & 0 & -1 \\ -4 & -3 & 0 \end{bmatrix}\begin{bmatrix} 351{,}25 & 514{,}375 & -832{,}5 \\ 120 & 180 & -429{,}375 \\ -78{,}125 & -103{,}125 & 100 \end{bmatrix} = \begin{bmatrix} 1858{,}542 & 832{,}5 & -288{,}542 \\ 832{,}5 & 429{,}375 & -100 \\ -288{,}542 & -100 & 60{,}417 \end{bmatrix}\ [\text{kPa}] \quad (57)$$

AD 8) ACTUAL LOAD ON BCGF FACE REFERRED TO ACTUAL CONFIGURATION

Face BCGF before deformation lied in a plane given by equation:

$$A_R: \quad X_1 - 1 = 0 \tag{58}$$

Equation of the face after deformation may be obtained by substitution of (8) into (58):

$$A: \quad X_1(x_1, x_2, x_3) - 1 = -x_1 + \frac{4}{3}x_2 - 3x_3 - 1 = 0 \tag{59}$$

Exterior unit normal for face BCGF is found as a **normalized gradient of the function describing the form of deformed face**:

$$\mathbf{n} = \frac{\nabla_{\mathbf{x}} A}{|\nabla_{\mathbf{x}} A|} = \frac{\left[-1\ ;\ \frac{4}{3}\ ;\ -3\right]}{\sqrt{(-1)^2 + \left(\frac{4}{3}\right)^2 + (-3)^2}} = [-0{,}291\ ;\ 0{,}389\ ;\ -0{,}874] \tag{60}$$

True load vector on BCGF face:

$$\mathbf{q} = \mathbf{T}_\sigma \cdot \mathbf{n} = \begin{bmatrix} 1858{,}542 & 832{,}5 & -288{,}542 \\ 832{,}5 & 429{,}375 & -100 \\ -288{,}542 & -100 & 60{,}417 \end{bmatrix} \begin{bmatrix} -0{,}291 \\ 0{,}398 \\ -0{,}874 \end{bmatrix} = \begin{bmatrix} 34{,}116 \\ 11{,}655 \\ -7{,}588 \end{bmatrix} [\mathrm{kPa}] \tag{61}$$

AD 9) ACTUAL LOAD ON BCGF FACE REFERRED TO REFERENCE CONFIGURATION

Face BCGF before deformation lied in a plane given by equation:

$$A_R: \quad X_1 - 1 = 0 \tag{62}$$

Exterior unit normal for face BCGF is found as a **normalized gradient of the function describing the undeformed face**:

$$\mathbf{N} = \frac{\nabla_{\mathbf{X}} A_R}{|\nabla_{\mathbf{X}} A_R|} = \frac{[1\ ;\ 0\ ;\ 0]}{\sqrt{1^2 + 0^2 + 0^2}} = [1\ ;\ 0\ ;\ 0] \tag{63}$$

Nominal load vector on BCGF face:

$$\mathbf{Q} = \mathbf{T}_R \cdot \mathbf{N} = \begin{bmatrix} 351{,}25 & 514{,}375 & -832{,}5 \\ 120 & 180 & -429{,}375 \\ -78{,}125 & -103{,}125 & 100 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 351{,}25 \\ 120 \\ -78{,}125 \end{bmatrix} [\mathrm{kPa}] \tag{64}$$

AD 10) SURFACE AREA OF FACE BCGF

Surface area of a part of any curvilinear surface parametrized by two parameters may be calculated as a **double integral in which integrand is a constant unit function**:

$$A=\iint_A \mathrm{d}A=\iint_A \sqrt{\left(\begin{vmatrix}\frac{\partial X_2}{\partial\alpha} & \frac{\partial X_2}{\partial\beta}\\ \frac{\partial X_3}{\partial\alpha} & \frac{\partial X_3}{\partial\beta}\end{vmatrix}\right)^2+\left(\begin{vmatrix}\frac{\partial X_3}{\partial\alpha} & \frac{\partial X_3}{\partial\beta}\\ \frac{\partial X_1}{\partial\alpha} & \frac{\partial X_1}{\partial\beta}\end{vmatrix}\right)^2+\left(\begin{vmatrix}\frac{\partial X_1}{\partial\alpha} & \frac{\partial X_1}{\partial\beta}\\ \frac{\partial X_2}{\partial\alpha} & \frac{\partial X_2}{\partial\beta}\end{vmatrix}\right)^2}\,\mathrm{d}\alpha\,\mathrm{d}\beta \tag{65}$$

In our case the considered surface is a plane perpendicular to X_1 axis, so it is easy to parametrize it with coordinates X_2 , X_3 :

$$A_R=\{\mathbf{X}: X_1=1 \wedge X_2\in\langle 0;1\rangle \wedge X_3\in\langle 0;1\rangle\} \tag{66}$$

Surface ares of face BCGF before deformation:

$$A_R=\iint_{BCGF}\mathrm{d}A_R=\int_{X_2=0}^{1}\int_{X_3=0}^{1}\mathrm{d}X_2\,\mathrm{d}X_3=1 \tag{67}$$

Surface area of deformed face may be calculated in an analogous way – the difference is that we need to integrate deformed surface elements. We will use the relation between undeformed and deformed infinitesimal surface elements:

$$\mathrm{d}A=J\sqrt{(\mathbf{N}^{\mathbf{T}}\mathbf{F}^{-1})\cdot(\mathbf{N}^{\mathbf{T}}\mathbf{F}^{-1})^{\mathrm{T}}}\,\mathrm{d}A_R \tag{68}$$

Then:

$$A=\iint_{BCGF}\mathrm{d}A=\iint_{BCGF}J\sqrt{(\mathbf{N}^{\mathbf{T}}\mathbf{F}^{-1})\cdot(\mathbf{N}^{\mathbf{T}}\mathbf{F}^{-1})^{\mathrm{T}}}\,\mathrm{d}A_R \tag{69}$$

We calculate:

$$\mathbf{N}^{\mathbf{T}}\mathbf{F}^{-1}=[1;0;0]\begin{bmatrix}-1 & \frac{4}{3} & -3\\ 1 & -\frac{4}{3} & 2\\ 0 & -\frac{1}{3} & 0\end{bmatrix}=\left[-1\,;\,\frac{4}{3}\,;\,-3\right] \tag{70}$$

$$(\mathbf{N}^{\mathbf{T}}\mathbf{F}^{-1})\cdot(\mathbf{N}^{\mathbf{T}}\mathbf{F}^{-1})^{\mathrm{T}}=\left[-1\,;\,\frac{4}{3}\,;\,-3\right]\cdot\begin{bmatrix}-1\\ \frac{4}{3}\\ -3\end{bmatrix}=(-1)^2+\left(\frac{4}{3}\right)^2+(-3)^2=\frac{106}{9}\approx 11{,}778 \tag{71}$$

Surface ares of face BCGF after deformation:

$$A = \iint_{BCGF} \mathrm{d}A = \iint_{BCGF} 3\cdot\sqrt{11{,}778}\,\mathrm{d}A_R = 10{,}296 \int_{X_2=0}^{1} \int_{X_3=0}^{1} \mathrm{d}X_2 \mathrm{d}X_3 = 10{,}296\cdot 1 = 10{,}296 \quad [\mathrm{m}^2] \tag{72}$$

AD 11) LENGTH OF SEGMENT AG

Arc length of a curve may be calculated as a **line integral in which the integrand is constant unit function**:

$$L_R = \int_{L_R} \mathrm{d}S = \int_{L_R} \sqrt{\mathrm{d}X_1^2 + \mathrm{d}X_2^2 + \mathrm{d}X_3^2} = \int_{L_R} \sqrt{\left(\frac{\mathrm{d}X_1}{\mathrm{d}\lambda}\right)^2 + \left(\frac{\mathrm{d}X_2}{\mathrm{d}\lambda}\right)^2 + \left(\frac{\mathrm{d}X_3}{\mathrm{d}\lambda}\right)^2}\,\mathrm{d}\lambda \tag{73}$$

Line containing segment AG before deformation is given by parametric equations:

$$\mathbf{X}_{AG}(\lambda) = \mathbf{X}_A + \lambda(\mathbf{X}_G - \mathbf{X}_A) = \begin{cases} X_1(\lambda) = X_1^A + \lambda(X_1^G - X_1^A) = 0 + \lambda(1-0) = \lambda \\ X_2(\lambda) = X_2^A + \lambda(X_2^G - X_1^A) = 0 + \lambda(1-0) = \lambda \\ X_3(\lambda) = X_3^A + \lambda(X_3^G - X_1^A) = 0 + \lambda(1-0) = \lambda \end{cases} \tag{74}$$

Segment AG corresponds with the following values of parameter: $\lambda \in \langle 0\,;\,1 \rangle$

Derivatives of coordinates of point of the curve with respect to parameter:

$$\frac{\mathrm{d}X_1}{\mathrm{d}\lambda} = 1\,,\quad \frac{\mathrm{d}X_2}{\mathrm{d}\lambda} = 1\,,\quad \frac{\mathrm{d}X_3}{\mathrm{d}\lambda} = 1 \tag{75}$$

Length of segment AG before deformation:

$$L_R = \int_{AG} \mathrm{d}S = \int_{AG} \sqrt{\mathrm{d}X_1^2 + \mathrm{d}X_2^2 + \mathrm{d}X_3^2} = \int_0^1 \sqrt{\left(\frac{\mathrm{d}X_1}{\mathrm{d}\lambda}\right)^2 + \left(\frac{\mathrm{d}X_2}{\mathrm{d}\lambda}\right)^2 + \left(\frac{\mathrm{d}X_3}{\mathrm{d}\lambda}\right)^2}\,\mathrm{d}\lambda =$$
$$= \int_0^1 \sqrt{(1)^2 + (1)^2 + (1)^2}\,\mathrm{d}\lambda = \int_0^1 \sqrt{3}\,\mathrm{d}\lambda = \sqrt{3}\int_0^1 \mathrm{d}\lambda = \sqrt{3}\cdot 1 = \sqrt{3} \approx 1{,}732 \quad [\mathrm{m}] \tag{76}$$

Length of deformed curve may be calculated in an analogous way – the difference is that we need to integrate deformed line elements. We will use the relation between undeformed and deformed infinitesimal line elements:

$$\mathrm{d}s = \sqrt{C_{ij}\,\mathrm{d}X_i\,\mathrm{d}X_j} \tag{77}$$

Length of segment AG after deformation:

$$L=\int_L \mathrm{d}s=\int_{AG}\sqrt{C_{ij}\mathrm{d}X_i\mathrm{d}X_j}=\int_{AG}\sqrt{C_{ij}\frac{\mathrm{d}X_i}{\mathrm{d}\lambda}\frac{\mathrm{d}X_j}{\mathrm{d}\lambda}}\mathrm{d}\lambda=$$

$$=\int_0^1\sqrt{C_{11}\left(\frac{\mathrm{d}X_1}{\mathrm{d}\lambda}\right)^2+C_{22}\left(\frac{\mathrm{d}X_2}{\mathrm{d}\lambda}\right)^2+C_{33}\left(\frac{\mathrm{d}X_3}{\mathrm{d}\lambda}\right)^2+2C_{23}\frac{\mathrm{d}X_2}{\mathrm{d}\lambda}\frac{\mathrm{d}X_3}{\mathrm{d}\lambda}+2C_{31}\frac{\mathrm{d}X_3}{\mathrm{d}\lambda}\frac{\mathrm{d}X_1}{\mathrm{d}\lambda}+2C_{12}\frac{\mathrm{d}X_1}{\mathrm{d}\lambda}\frac{\mathrm{d}X_2}{\mathrm{d}\lambda}}\mathrm{d}\lambda=$$

$$=\int_0^1\sqrt{5\cdot(1)^2+10\cdot(1)^2+25\cdot(1)^2+2\cdot(-12)\cdot1\cdot1+2\cdot(-8)\cdot1\cdot1+2\cdot7\cdot1\cdot1}\,\mathrm{d}\lambda=\int_0^1\sqrt{14}\mathrm{d}\lambda=$$

$$=\sqrt{14}\int_0^1\mathrm{d}\lambda=\sqrt{14}\cdot1=\sqrt{14}\approx3{,}742 \qquad (78)$$

AD 12) VOLUME

Volume of a block may be calculated as a triple integral. Reference configuration is defined as:

$$V_R=\{\mathbf{X}:X_1\in\langle0;1\rangle\wedge X_2\in\langle0;1\rangle\wedge X_3\in\langle0;1\rangle\} \qquad (79)$$

Volume before deformation:

$$V_R=\iiint_V \mathrm{d}V_R=\int_{X_1=0}^1\int_{X_2=0}^1\int_{X_3=0}^1\mathrm{d}X_1\mathrm{d}X_2\mathrm{d}X_3=1$$

Volume of deformed block may be calculated in an analogous way – the difference is that we need to integrate deformed volume elements. We will use the relation between undeformed and deformed infinitesimal volume elements:

$$\mathrm{d}V=J\,\mathrm{d}V_R \qquad (80)$$

Volume after deformation:

$$V=\iiint_V\mathrm{d}V=\iiint_V J\,\mathrm{d}V_R=\int_{X_1=0}^1\int_{X_2=0}^1\int_{X_3=0}^1(3)\mathrm{d}X_1\mathrm{d}X_2\mathrm{d}X_3 \qquad (81)$$

$$=3\int_{X_1=0}^1\int_{X_2=0}^1\int_{X_3=0}^1\mathrm{d}X_1\mathrm{d}X_2\mathrm{d}X_3=3\cdot1=3$$