THEORY OF ELASTICITY AND PLASTICITY

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NOTATION IN TENSOR CALCULUS

TYPES OF NOTATION AND SUMMATION CONVENTION

	index notation (operations on components of tensor in the given coord. sys.)	**matrix notation** (operations on matrices representing tensors in the given coordinate system)	**absolute notation** (operations on tensors without any reference to components or any representation of a tensor)
scalar	α	α	α
vector	v_i	$[v]=\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$	$\mathbf{v}$
tensor	T_{ij}	$[T]=\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$	$\mathbf{T}$
Dot product of a tensor and a vector	$\sum_{j=1}^{3} \sigma_{ij} n_j = p_i$	$[\sigma][n]=[p] \quad \Leftrightarrow \quad \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}=\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$	$\boldsymbol{\sigma}\,\mathbf{n}=\mathbf{p}$
dot product of vectors	$\alpha=\sum_{j=1}^{3} v_i w_i$	$\alpha=[v]^{\mathrm{T}}[w] \quad \Leftrightarrow \quad \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$	$\alpha=\mathbf{v}\cdot\mathbf{w}$
gradient of a vector	$F_{ij}=X_{i,j}=\dfrac{\partial X_i}{\partial x_j}$	$[F]=\begin{bmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} & \frac{\partial X_1}{\partial x_3} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} & \frac{\partial X_2}{\partial x_3} \\ \frac{\partial X_3}{\partial x_1} & \frac{\partial X_3}{\partial x_2} & \frac{\partial X_3}{\partial x_3} \end{bmatrix}$	$\mathbf{F}=\mathbf{X}\otimes\nabla_{\mathbf{x}}$

TYPES OF NOTATION AND SUMMATION CONVENTION

We will make use of the following symbols

- Kronecker's delta

$$\delta_{ij} = \begin{cases} 1 & \Leftrightarrow \quad i = j \\ 0 & \Leftrightarrow \quad i \neq j \end{cases}$$

- Levi-Civita permutation symbol

$$\epsilon_{ijk} = \begin{cases} 1 & \Leftrightarrow \quad (i,j,k) \in \{(1,2,3);(2,3,1);(3,1,2)\} \quad \leftarrow \text{even permutations of } (1,2,3) \\ -1 & \Leftrightarrow \quad (i,j,k) \in \{(2,1,3);(1,3,2);(3,2,1)\} \quad \leftarrow \text{odd permutations of } (1,2,3) \\ 0 & \Leftrightarrow \quad \text{values of any two indices are the same} \end{cases}$$

Kronecker's delta will be used as an index notation of a **unit matrix** which is a representation of a **unit tensor** (**identity operator**).

$$\mathbf{1} = [\delta_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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TYPES OF NOTATION AND SUMMATION CONVENTION

In order to simplify complex expressions so called Einstein's summation convention is used. It is originally formulated as follows:

„If in a single expression certain index is repeated exactly twice, once in a superscript and once in a subscript, this means summation with respect to that index for all values the index may take"

In case when only Cartesian coordinates and orthogonal transformations are considered there is no necessity of distinguishing between superscripts and subscripts – we will write all indices in subscript, and repeated index will mean that summation should be performed.

Examples:

$$|\mathbf{v}| = \sqrt{v_i v_i} \quad \Leftrightarrow \quad |\mathbf{v}| = \sqrt{\sum_{i=1}^{3} v_i v_i} = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

$$\sigma_{ij} n_j \quad \Leftrightarrow \quad \sum_{j=1}^{3} \sigma_{ij} n_j$$

$$\theta = \delta_{ik} \varepsilon_{ik} = \varepsilon_{kk} \quad \Leftrightarrow \quad \theta = \sum_{k=1}^{3} \delta_{ik} \varepsilon_{ik} = \sum_{k=1}^{3} \varepsilon_{kk} \text{¿} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$

$$\sigma_{ij,j} + b_i = \ddot{u}_i \quad \Leftrightarrow \quad \sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} + b_i = u_i$$

DEFORMABLE SOLIDS

DEFORMABLE SOLIDS

Deformable solid – physical object which may be attributed with mass and which **may change its volume and shape**. **Distance between any two particles** belonging to the deformable solid **may change in time**.

Its **contrary** is the model of **rigid body** – an object in which the distance between any two points belonging to that body is fixed in time.

We may distinguish **4 basic types of materials** the deformable solids may be made of:

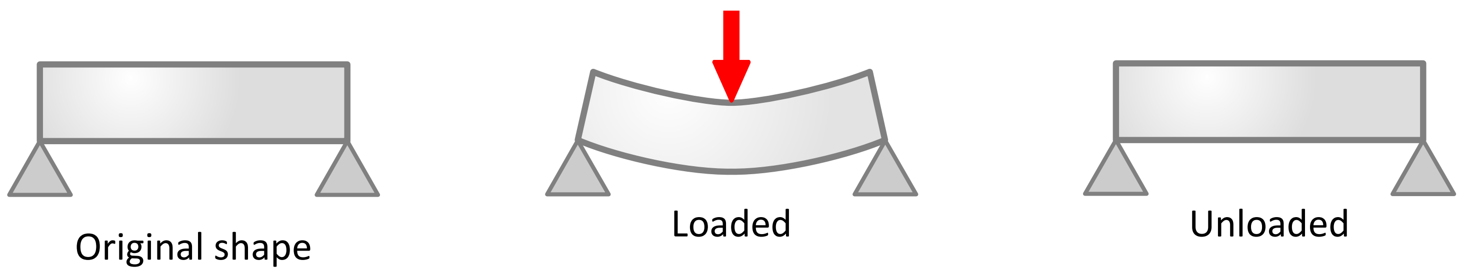
- Elastic materials
- Plastic materials
- Viscous materials
- Brittle materials

Any **real material actually has the properties characteristic for each of those types**, but e.g. the way of using of that material results in that the properties of a single type surpass the properties of other types:

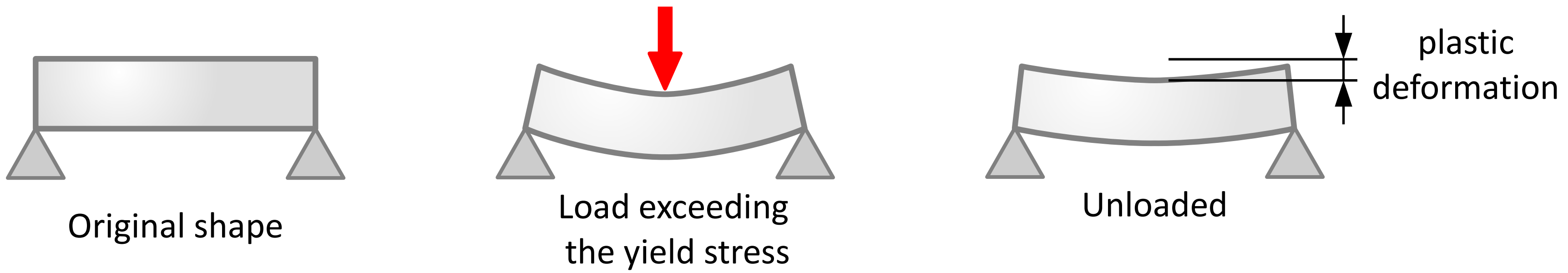
> If the **load and deformation is sufficiently small**, they are **applied slowly for a short time**, then most of **constructional materials** (metals, concrete, timber etc.) exhibit primarily **elastic properties**.

DEFORMABLE SOLIDS

Elastic materials – they deform when loaded (with a force or excited displacement), but **when the load is removed they recover its original shape**.

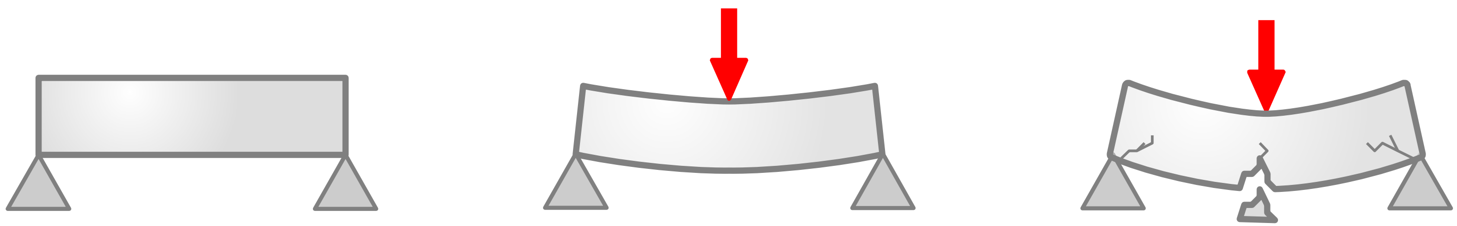


Plastic materials – they deform when loaded and when the load is removed they preserve the deformed shape. Usually we deal with **elastic-plastic materials**. They deform elastically and after a specific magnitude of load is exceeded they deform also plastically. **After the load in removed the elastic deformation vanish while the plastic one is preserved**.



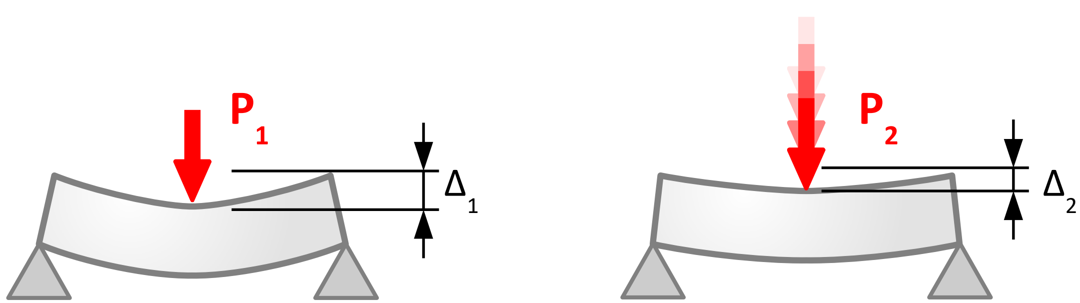
DEFORMABLE SOLIDS

Brittle materials – the deform when loaded but **when the load magnitude exceeds certain limit value, they loose cohesion and continuity**.



Original shape | Loaded | Load exceeding the strength of material

Viscous materials (**rheological materials**) – character of the deformation depends not only on the magnitude of the load but also on its **history**, namely: **load rate**, **load duration**, **variation of load in time**. It is necessary to account for the factor of **time**.



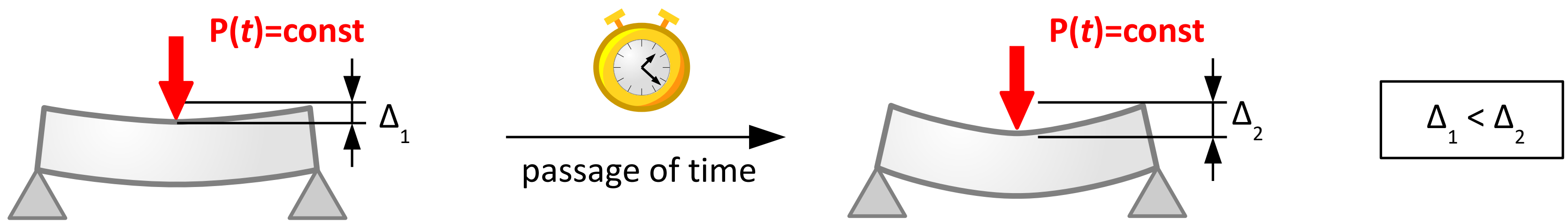
small load rate | large load rate

$$P_1 = P_2$$
$$\frac{\mathrm{d}P_1}{\mathrm{d}t} < \frac{\mathrm{d}P_2}{\mathrm{d}t}$$
$$\Delta_1 > \Delta_2$$

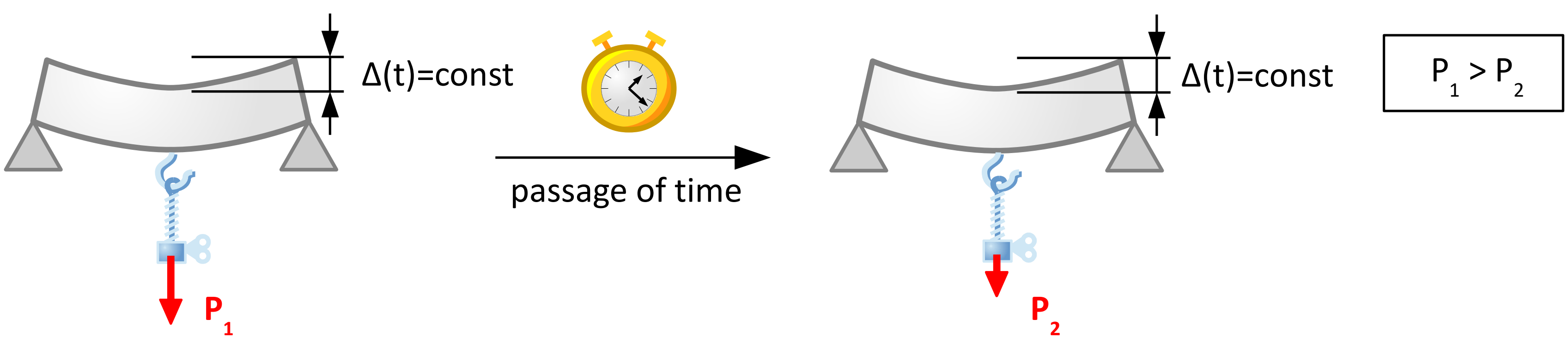
DEFORMABLE SOLIDS

Viscous materials (**rheological materials**) – cont.

- creep – increase of the strain when the load is fixed (standard rules for calculating "final deflection")



- relaxation – decrease in stress when forced displacement is fixed (loss of pre-stressing force)



- Change of mechanical properties of material in time, e.g. of the strength of materials, Young's modulus (concrete curing)

DEFORMABLE SOLIDS

Mathematical models accounting for the properties of many of the above types are also considered:

- Elastic-plastic materials
- Viscoelastic materials
- Visco-elastic-plastic materials
- Elastic-plastic-brittle materials
- …

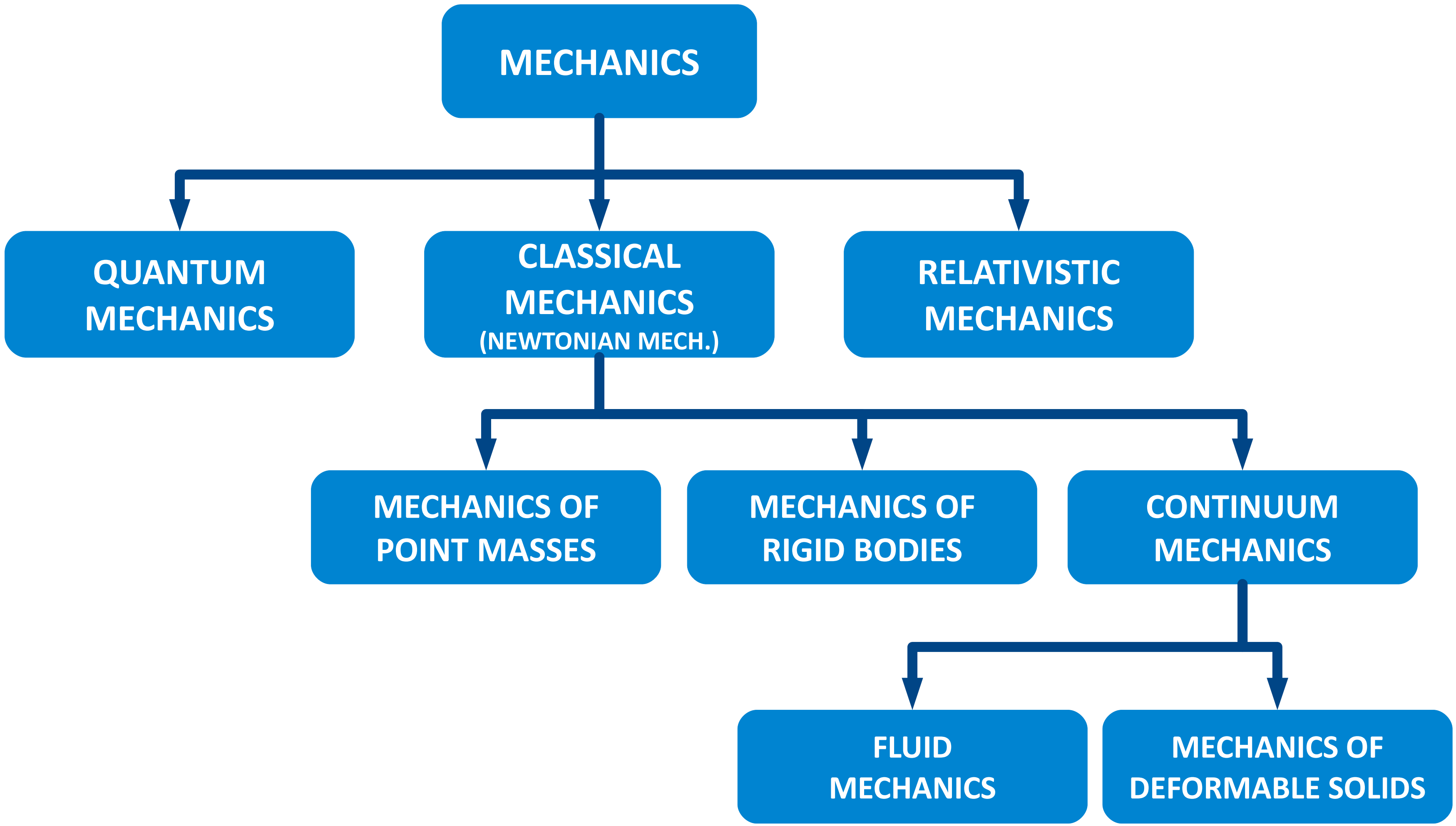
These models are usually so complicated that the only practical way of using them is the use of approximate numerical methods (e.g. Finite Element Method).

In the majority of basic problems of civil engineering, mechanical and material engineering as well as construction of machines the use of the model of an **elastic material** gives sufficiently precise results. The **theory of elasticity** deal with mathematical modelling of elastic materials:

- It provides the basic methods of decription and analysis of deformation of solids
- It provides first approximations of solutions of real engineering problems
- Most of the basic structural code recommendations are derived from the theory of elasticity

CONTINUUM MECHANICS

CONTINUUM MECHANICS



ASSUMPTIONS:

- Newton's principles of motion are in force
- True solids are modelled with the model of continuum

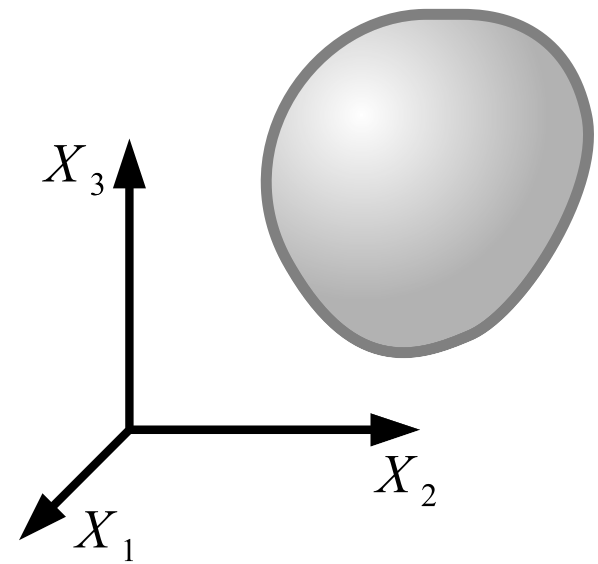
CONTINUUM MECHANICS

Continuum – mathematical model of the deformable solid. The body is represented by a **subregion of Euclidean space** such that it is possible to **define on that region functions** which will be **continuous and differentiable** in each point of that region (**differential manifold**).

- **The point belonging to the continuum** will be termed particle – it has nothing to do with atoms, particles or molecules.
- There is an **uncountably infinite number** of particles. In any arbitrary small neighbourhood of any particle there is also infinite number of particles. That's why we call it continuum.

The functions that will be defined in that region will describe the state of the solid, e.g.:

- Scalar field of temperatura $\theta(\mathbf{X})$
- Vector field of displacement $\mathbf{u}(\mathbf{X})$
- Tensor field of strain $\boldsymbol{\varepsilon}(\mathbf{X})$
- Tensor field of stress $\boldsymbol{\sigma}(\mathbf{X})$
- ...



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CONTINUUM MECHANICS

CONTINUUM MECHANICS

KINEMATICS

- Description of motion without looking for causes of that motion or trying to predict it
- Definition of the strain
- Determining the relation between strain and displacement (geometric relations)

STATICS

- Definition of types of forces
- Definition of equilibrium conditions for internal and external forces
- Definition of the stress
- Determining the statical boundary conditions

DYNAMICS

- Determining the relation between factors enforcing the motion and the resulting motion (equations of motion)

CONSTITUTIVE MODELLING

- Determining the relation between the stress and the strain (constitutive relations)

KINEMATICS

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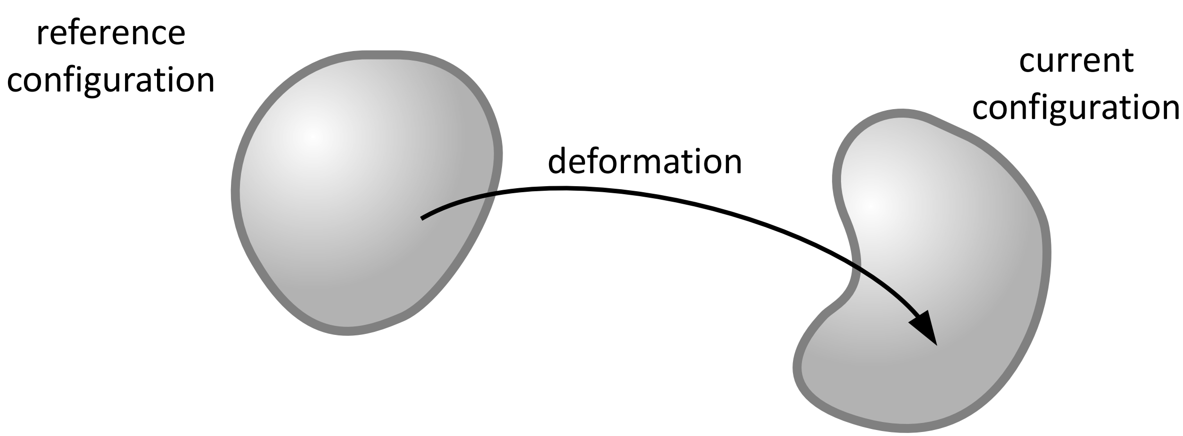
KINEMATICS OF CONTINUUM

Configuration – subregion in the space, in which point belonging to the continuum are placed.

Reference configuration (initial conf.) – configuration in the **initial instant of time** t_0 ⟶ $\mathcal{B}_0$, $\mathcal{B}_{ref}$

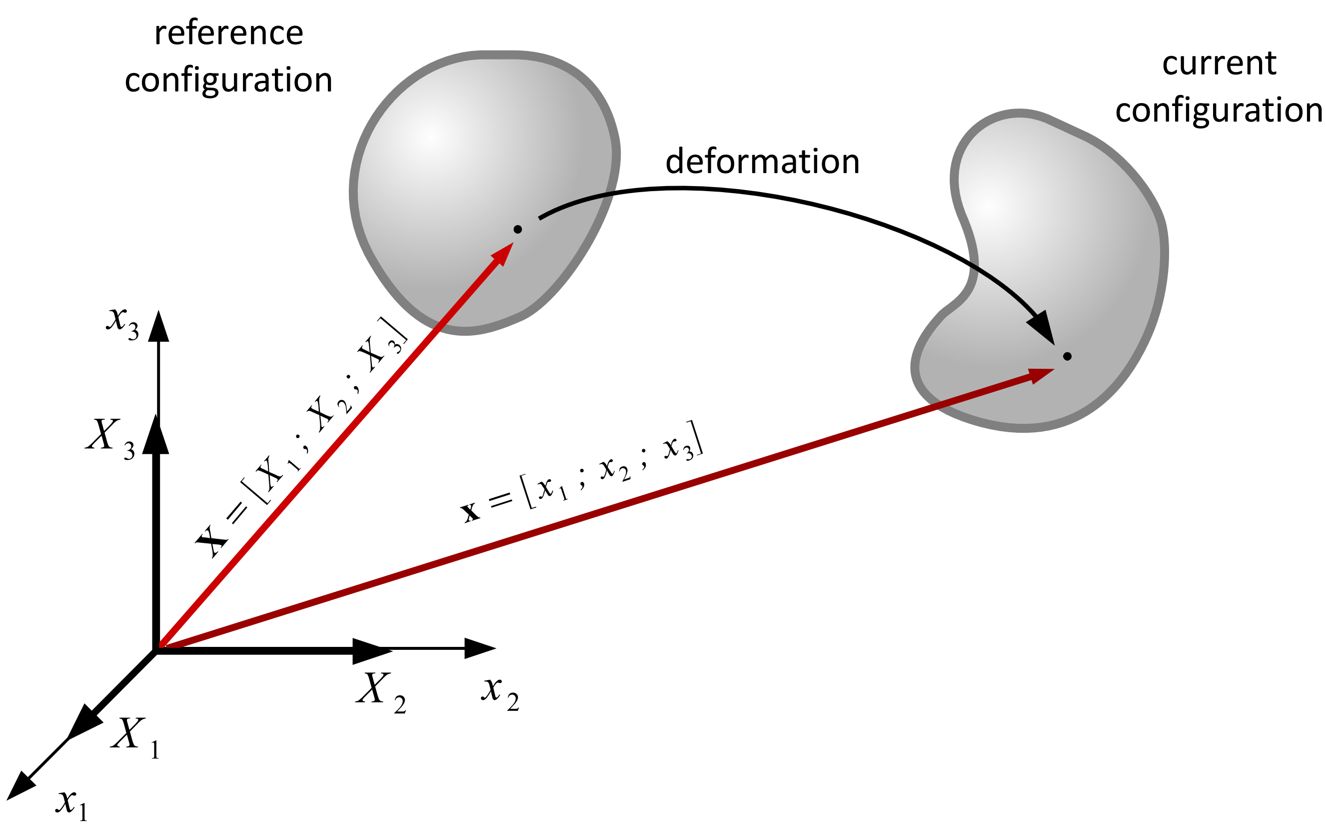
Current configuration– configuration in the **current instant of time** t ⟶ $\mathcal{B}_t$, $\mathcal{B}(t)$



KINEMATICS OF CONTINUUM

We introduce two sets of coordinates:

- material (Lagrange) coordinates – position of particles in **reference configuration** $\mathbf{X}$
- spatial (Euler) coordinates – position of particles in **current configuration** $\mathbf{x}(t)$



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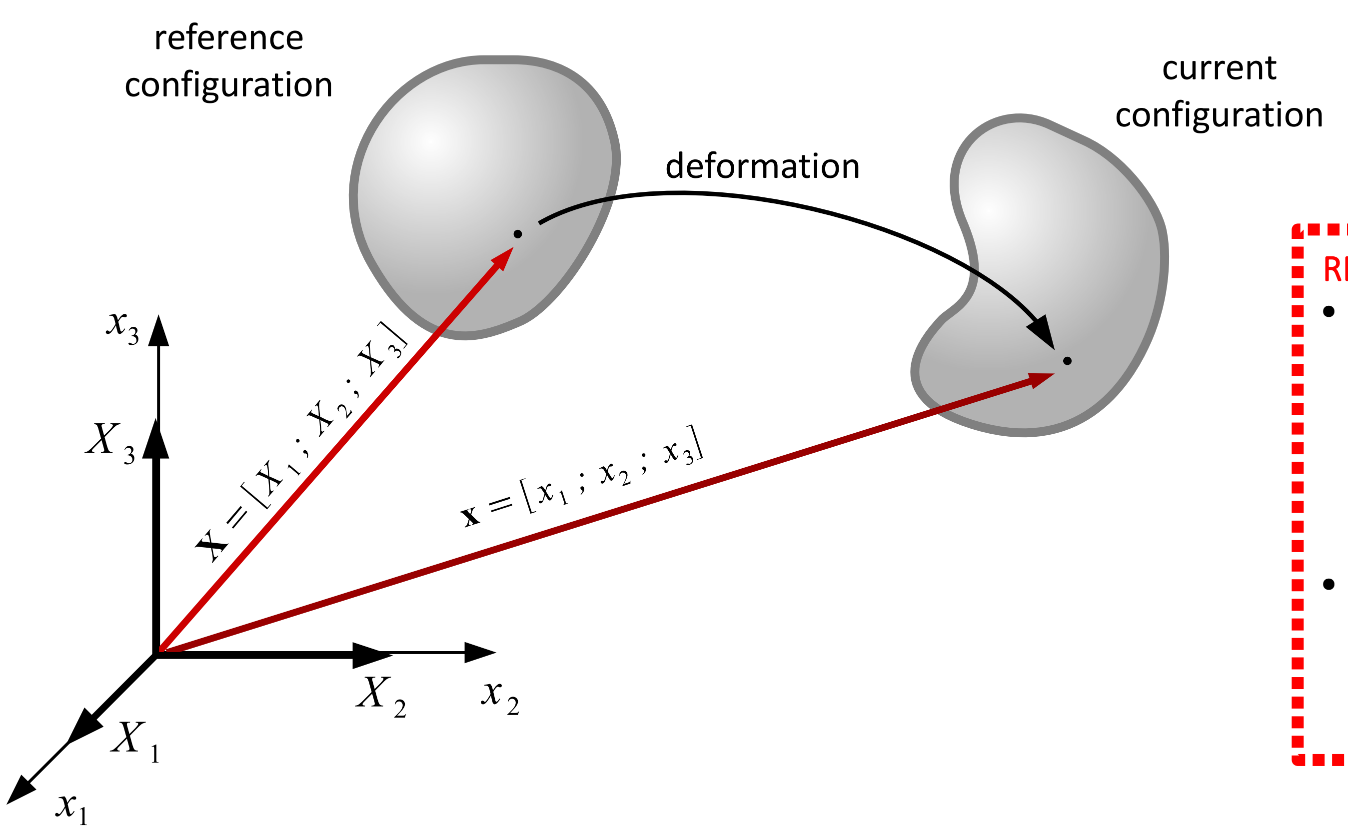

REMARKS:

- Material and spatial coordinates may be different sets of coordinates (different origins, one set may be orthonormal coordinates while the others maybe curvilinear coordinates)
- For simplicity we will assume that both coordinate ststems are Cartesian ones and both have the same origin. We will draw only one set of axes.

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KINEMATICS OF CONTINUUM

Material coordinates – they **identify a particle in a body**
Spatial coordinates – they **identify a point in space**

Solving the problem of continuum mechanics means finding the relation between material coordinates and spatial coordinates (deformation relations) for any instant of time:

$$\mathbf{X} \xleftrightarrow{\ ?\ } \mathbf{x}$$

This relation must be BIJECTIVE (one-to-one) so it also must be INVERTIBLE

It could be done in two ways:

- find $\mathbf{x}$ (dependent variable) as a function of $\mathbf{X}$ and t (independent variables) $\rightarrow$ material description

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$$

- find $\mathbf{X}$ (dependent variable) as a function of $\mathbf{x}$ and t (independent variables) $\rightarrow$ spatial description

$$\mathbf{X} = \mathbf{X}(\mathbf{x}, t)$$

MATERIAL DESCRIPTION

Material description (Lagrange description, particle tracking)

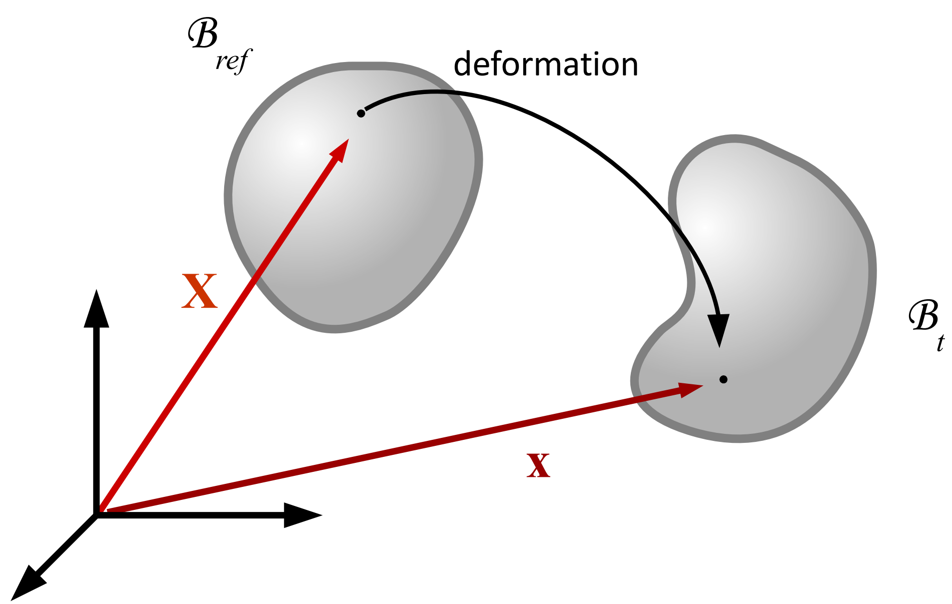
- Independent variables - material coordinates $\mathbf{X}$
  - time t
- dependent variable - spatial coordinates $\mathbf{x}$

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$$

We are answering the following question:

Let's consider a certain particle which at the beginning of the motion was placed in point $\mathbf{X}$. In what point of space $\mathbf{x}$ is it located now (in the considered instant of time t)?

We assume that **we know the initial configuration** of the body (**domain of $\mathbf{X}$ variable**). Material description Is useful in the description of **solids**.



SPATIAL DESCRIPTION

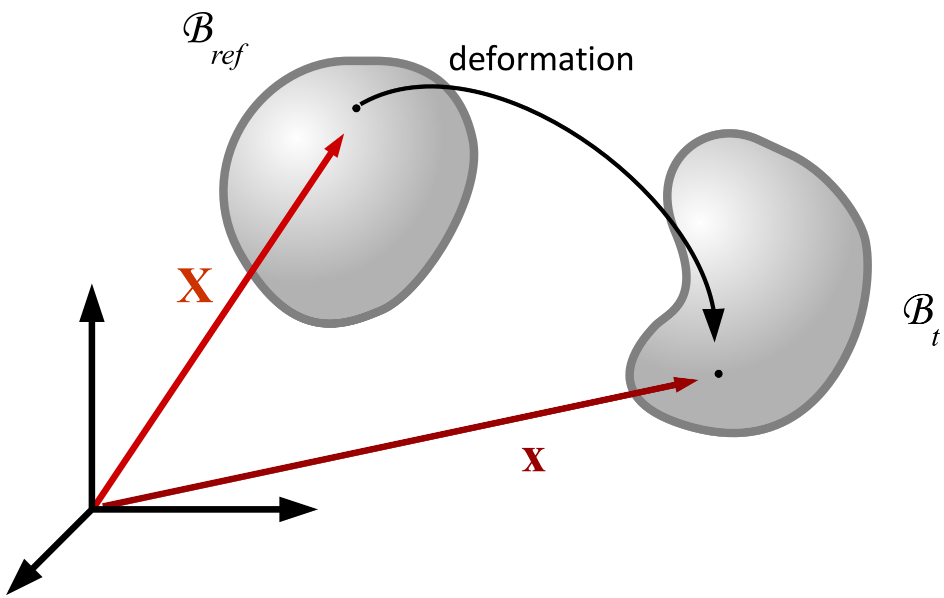
Spatial description (Euler description, local analysis)

- independent variables - spatial coordinates $\mathbf{x}$
  - time t
- dependent variable - material coordinates $\mathbf{X}$

$$\mathbf{X} = \mathbf{X}(\mathbf{x}, t)$$

We are answering the following question:

Let's consider point $\mathbf{x}$ in space. What particle occupies this place in space now (in the considered instant of time t) – what was its position $\mathbf{X}$ at the beginning of the motion?



We assume that we know the region in space, a set of Points in space which may be occupied by particles e.g. **shape of container** (**domain od $\mathbf{x}$ variable**). Spatial description s useful in **fluid** mechanics.

MEASURES OF DISPLACEMENT

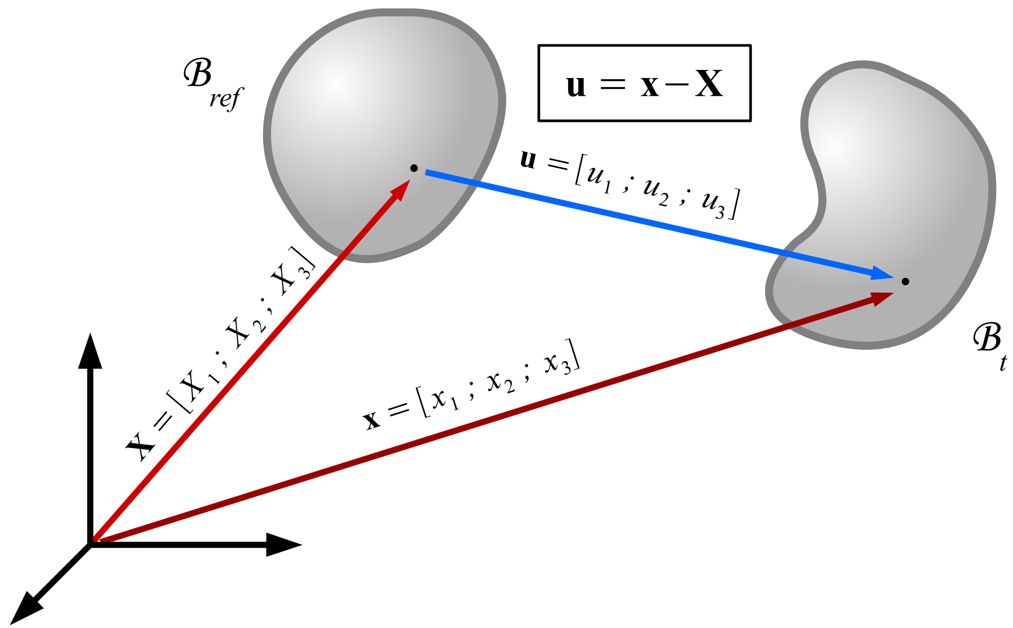
Independently of the chosen description we may define

- Displacement vector

$$\mathbf{u} \overset{\text{df.}}{=} \mathbf{x} - \mathbf{X}$$

In material description it is a function of $\mathbf{X}$: $\mathbf{u}(\mathbf{X}) = \mathbf{x}(\mathbf{X}) - \mathbf{X}$

In spatial description it is a function of $\mathbf{x}$: $\mathbf{u}(\mathbf{x}) = \mathbf{x} - \mathbf{X}(\mathbf{x})$



MEASURES OF DISPLACEMENT

Independently of the chosen description we may define

- Velocity vector $\mathbf{v} = \frac{\mathrm{d}\,\mathbf{u}}{\mathrm{d}\,t}$
- Acceleration vector $\mathbf{a} = \frac{\mathrm{d}\,\mathbf{v}}{\mathrm{d}\,t}$

Operator $\frac{\mathrm{d}}{\mathrm{d}\,t}$ is the **total derivative operator**, which must account also for **implicit time-dependence**, namely for **indirect dependence on time** via the **time-dependence of the rest of independent variables**.

Spatial description – spatial coordinates are time-dependent $\mathbf{x} = \mathbf{x}(t)$

$$\frac{\mathrm{d}}{\mathrm{d}\,t} = \frac{\partial}{\partial\,t} + \frac{\partial\,x_1}{\partial\,t}\frac{\partial}{\partial\,x_1} + \frac{\partial\,x_2}{\partial\,t}\frac{\partial}{\partial\,x_2} + \frac{\partial\,x_2}{\partial\,t}\frac{\partial}{\partial\,x_2} = \frac{\partial}{\partial\,t} + \sum_{i=1}^{3}\frac{\partial\,x_i}{\partial\,t}\frac{\partial}{\partial\,x_i}$$

Material description – material coordinates are time-independent $\mathbf{X}(t) = const.$

$$\frac{\mathrm{d}}{\mathrm{d}\,t} = \frac{\partial}{\partial\,t}$$

DEFORMATION GRADIENT

A linear material element or a material fibre – infinitely small (infinitesimal) segment connecting two particles.

Position of the 1st end-point of fibre before deformation: $\mathbf{X}$
Position of the 2nd end-point of fibre before deformation: $\mathbf{X}+\mathrm{d}\,\mathbf{X}$

Position of the 1st end-point of fibre after deformation: $\mathbf{x}$
Position of the 2nd end-point of fibre after deformation:: $\mathbf{x}+\mathrm{d}\,\mathbf{x}$

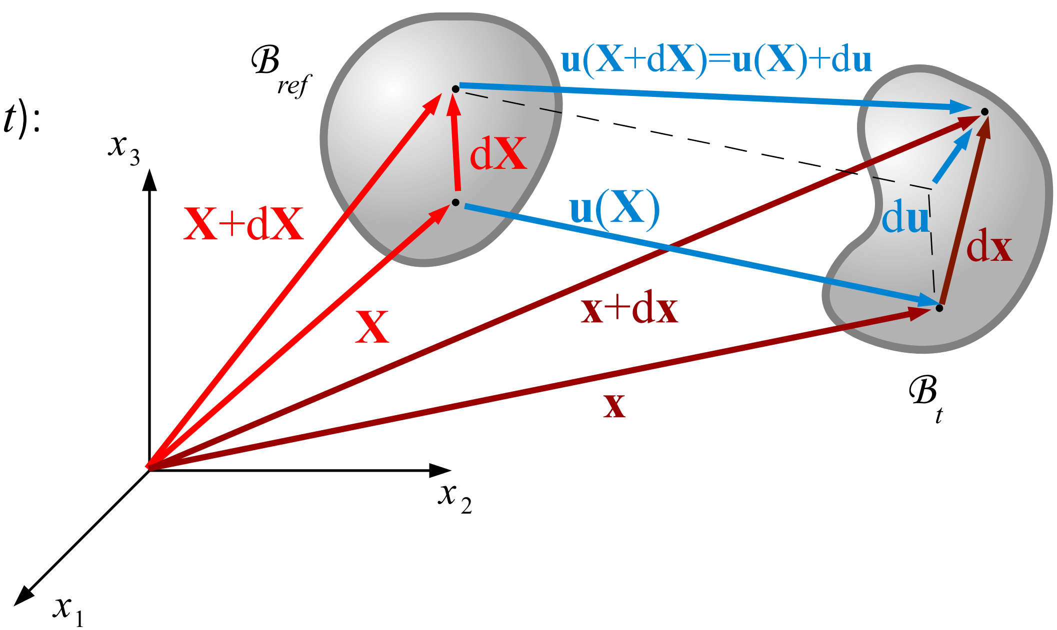
In the material description we may write
(for simplicity we're not denoting dependence on t):

$$\mathbf{x} = \mathbf{x}(\mathbf{X})$$

$$\mathbf{x} + \mathrm{d}\,\mathbf{x} = \mathbf{x}(\mathbf{X}+\mathrm{d}\,\mathbf{X})$$

As a result:

$$\mathrm{d}\,\mathbf{x} = \mathbf{x}(\mathbf{X}+\mathrm{d}\,\mathbf{X}) - \mathbf{x}(\mathbf{X})$$



DEFORMATION GRADIENT

Let's consider the expression for **material fibre after deformation** (in index notation):

$$\mathrm{d}x_i = x_i(\mathbf{X}+\mathrm{d}\mathbf{X}) - x_i(\mathbf{X}) \qquad i=1,2,3$$

If $\mathrm{d}\mathbf{X}$ is small, the the first term on the right-hand side may be expanded into a **Taylor power series**:

$$x_i(\mathbf{X}+\mathrm{d}\mathbf{X}) = x_i(\mathbf{X}) + \left.\frac{\partial x_i}{\partial X_1}\right|_{\mathbf{X}} \mathrm{d}X_1 + \left.\frac{\partial x_i}{\partial X_2}\right|_{\mathbf{X}} \mathrm{d}X_2 + \left.\frac{\partial x_i}{\partial X_3}\right|_{\mathbf{X}} \mathrm{d}X_3 + \ldots$$

Further terms of that expansion depend on products of components of $\mathrm{d}\mathbf{X}$. If those components are much smaller than 1, then their products are close to 0. If we neglect those small further terms, we will obtain:

$$\mathrm{d}x_i \approx \sum_{j=1}^{3} \frac{\partial x_i}{\partial X_j} \mathrm{d}X_j$$

We shall write:

$$\mathrm{d}x_i \approx F_{ij}\,\mathrm{d}X_j \qquad i=1,2,3 \qquad \text{where} \qquad F_{ij} = \frac{\partial x_i}{\partial X_j} \qquad i,j=1,2,3$$

DEFORMATION GRADIENT

In **matrix notation** this relation may be written as follows:

$$\begin{bmatrix} \mathrm{d}x_1 \\ \mathrm{d}x_2 \\ \mathrm{d}x_3 \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} \mathrm{d}X_1 \\ \mathrm{d}X_2 \\ \mathrm{d}X_3 \end{bmatrix} \qquad \text{where} \qquad F_{ij} = \frac{\partial x_i}{\partial X_j}$$

In **absolute notation**:

$$\mathrm{d}\mathbf{x} = \mathbf{F}\,\mathrm{d}\mathbf{X}$$

The object defined as follows:

$$F_{ij} = \frac{\partial x_i}{\partial X_j} \quad \Leftrightarrow \quad \mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}$$

Is termed material deformation gradient. It may be shown that it is a **tensor** (when coordinates are changed, its components transform according to the rule specific for tensors)– in general it is **not a symmetric tensor**.

DEFORMATION GRADIENT

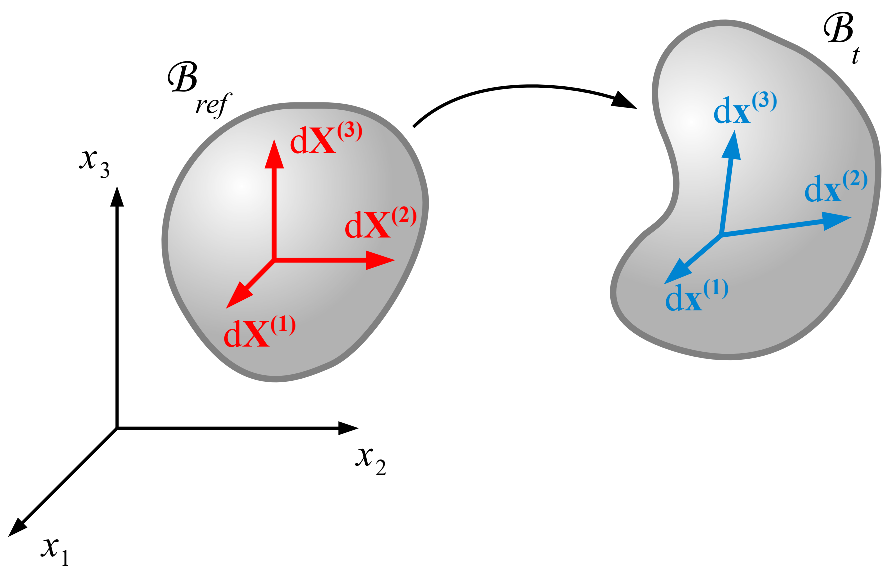
Components of deformation gradient are **functions of particle**:

$$F_{ij}(\mathbf{X}) = \left.\frac{\partial x_i}{\partial X_j}\right|_{\mathbf{X}}$$

The do not contain any information of displacements which are independent of material coordinates, namely on constant displacement (common for all particles) – they vanish in the process of differentiation. **We loose information on parallel translation** (without rotation and strain).

Components of i-th column of deformation gradient are the components of a vector describing the deformed material fibre which was parallel to the i-th axis of the assumed coordinate system before deformation.

$$\mathrm{d}\mathbf{X}^{(1)}=[1;0;0] \qquad \mathrm{d}\mathbf{x}^{(1)}=[F_{11};F_{21};F_{31}]$$
$$\mathrm{d}\mathbf{X}^{(2)}=[0;1;0] \qquad \mathrm{d}\mathbf{x}^{(2)}=[F_{12};F_{22};F_{32}]$$
$$\mathrm{d}\mathbf{X}^{(3)}=[0;0;1] \qquad \mathrm{d}\mathbf{x}^{(3)}=[F_{13};F_{23};F_{33}]$$



DEFORMATION GRADIENT

In analogous way one may define spatial deformation gradient:

$$f_{ij} = \frac{\partial X_i}{\partial x_j} \qquad \Leftrightarrow \qquad \mathbf{f} = \frac{\partial \mathbf{X}}{\partial \mathbf{x}}$$

We will have then:

$$\mathrm{d}\mathbf{X} = \mathbf{f}\;\mathrm{d}\mathbf{x}$$

Accounting for both derived relations allow us to write:

$$\mathrm{d}\mathbf{x} = \mathbf{F}\;\mathrm{d}\mathbf{X} = \mathbf{F}\left(\mathbf{f}\;\mathrm{d}\mathbf{x}\right) = \left(\mathbf{F}\mathbf{f}\right)\mathrm{d}\mathbf{x}$$

$$\mathrm{d}\mathbf{X} = \mathbf{f}\;\mathrm{d}\mathbf{x} = \mathbf{f}\left(\mathbf{F}\;\mathrm{d}\mathbf{X}\right) = \left(\mathbf{f}\mathbf{F}\right)\mathrm{d}\mathbf{X}$$

Since this must be true for any material fibres, then:

$$\mathbf{F}\mathbf{f} = \mathbf{f}\mathbf{F} = \mathbf{1} \qquad \Leftrightarrow \qquad \mathbf{f} = \mathbf{F}^{-1} \;\wedge\; \mathbf{F} = \mathbf{f}^{-1}$$

DEFORMATION GRADIENT

If a tensor (or a matrix) is to be **invertible**, then **its determinant must not be equal 0**. Determinant of material deformation gradient will be termed simply the jacobian:

$$J = \det(\mathbf{F}) \neq 0$$

Determinant of inverse matrix is the inverse of determinant of original matrix

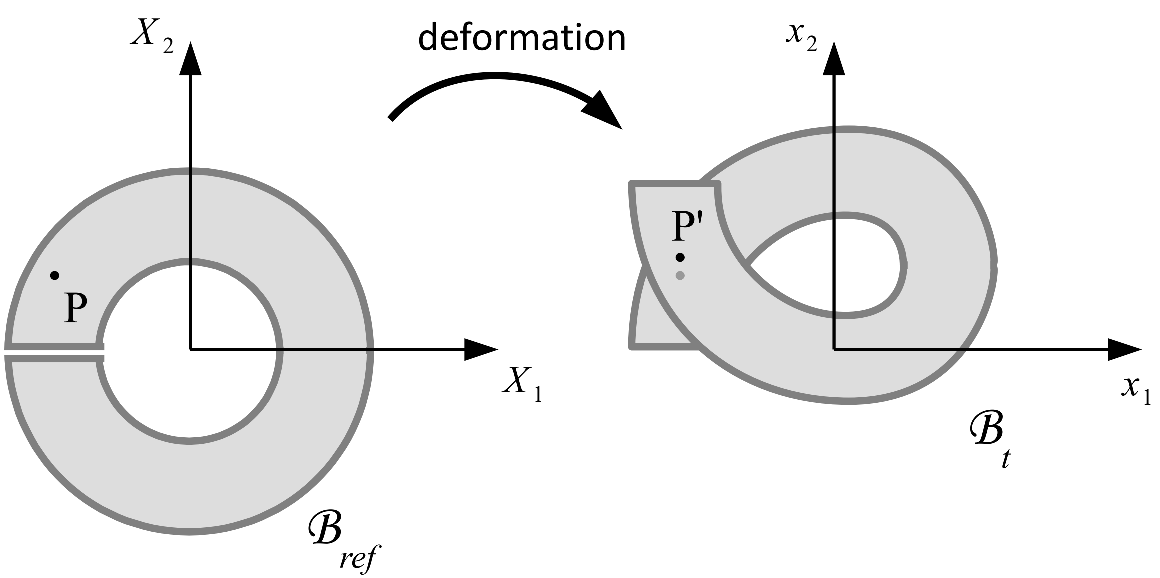
$$\det(\mathbf{f}) = \det(\mathbf{F}^{-1}) = \frac{1}{J} \neq 0$$

The requirement that the deformation relations must be bijective (one-to-one) and thus invertible needs that jacobian is not equal 0. It emerges that negative values of jacobian occur when as a result of deformation the body is turned "inside out" (the measure of volume is negative) – we reject such situations. For this reason we will demand that **jacobian** to be **positive:**

$$J > 0$$

DEFORMATION GRADIENT

Jacobian not being equal 0 guarantees only local **invertibility** of deformation relations. Those relations may be still globally uninvertible, e.g. when distant parts of a body after deformation will "intersect", namely they will occupy the same region in space.



$$J > 0$$

Such situations occurs in the case of large displacements or large strains. **In this basic formulation the equations of theory of elasticity do not guarantee global invertibility**. It requires tracking the deformation concerning possible contact between distant parts of a body. Such an analysis is done in advanced numerical methods of solving the problems of finite deformations.

POLAR DECOMPOSITION OF THE DEFORMATION GRADIENT

Following theorem may be proved:

THEOREM ON POLAR DECOMPOSITION OF MATERIAL DEFORMATION GRADIENT

For any positive-determinate ($J > 0$) material deformation gradient $\mathbf{F}$ following tensor are determined **uniquely**:

- $\mathbf{U}$ – right stretch tensor - **symmetric**: $\mathbf{U}^{\mathrm{T}} = \mathbf{U} \quad \Leftrightarrow \quad U_{ij} = U_{ji}$
- $\mathbf{V}$ – left stretch tensor - **symmetric**: $\mathbf{V}^{\mathrm{T}} = \mathbf{V} \quad \Leftrightarrow \quad V_{ij} = V_{ji}$
- $\mathbf{R}$ – rotation tensor - **orthogonal**: $\mathbf{R}^{\mathrm{T}} = \mathbf{R}^{-1} \quad \Leftrightarrow \quad \det \mathbf{R} = 1$

and:

$$\mathbf{F} = \mathbf{R}\,\mathbf{U} = \mathbf{V}\,\mathbf{R}$$

We have then:

$$\det \mathbf{U} = \det \mathbf{F} = J > 0$$

POLAR DECOMPOSITION OF THE DEFORMATION GRADIENT

PHYSICAL INTERPRETATION OF THE ROTATION TENSOR

Let's consider a situation when **stretch tensor is a unit tensor. Deformation gradient is equivalent to the rotation tensor**.

$$\mathbf{U} = \mathbf{1} \quad \Rightarrow \quad \mathbf{F} = \mathbf{R}$$

The length of any material fibre before deformation may be calculated as follows:

$$|\mathrm{d}\mathbf{X}| = \sqrt{\mathrm{d}\mathbf{X}\cdot\mathrm{d}\mathbf{X}} = \sqrt{(\mathrm{d}\mathbf{X})^{\mathrm{T}}\mathrm{d}\mathbf{X}}$$

The same **fibre after deformation** is described by a vector: $\mathrm{d}\mathbf{x} = \mathbf{F}\ \mathrm{d}\mathbf{X} = \mathbf{R}\ \mathrm{d}\mathbf{X}$

Its length is equal:

$$|\mathrm{d}\mathbf{x}| = |\mathbf{R}\,\mathrm{d}\mathbf{X}| = \sqrt{(\mathbf{R}\,\mathrm{d}\mathbf{X})^{\mathrm{T}}\mathbf{R}\,\mathrm{d}\mathbf{X}} = \sqrt{(\mathrm{d}\mathbf{X})^{\mathrm{T}}\underbrace{\mathbf{R}^{\mathrm{T}}\mathbf{R}}_{=\mathbf{1}}\mathrm{d}\mathbf{X}} = \sqrt{(\mathrm{d}\mathbf{X})^{\mathrm{T}}\mathrm{d}\mathbf{X}} = |\mathrm{d}\mathbf{X}|$$

Deformed fibre **did not change its original length** – it underwent only **rotation** (and/or translation).

POLAR DECOMPOSITION OF THE DEFORMATION GRADIENT

PHYSICAL INTERPRETATION OF THE STRETCH TENSOR

Let's consider a situation when **rotation tensor is a unit tensor. Deformation gradient is equivalent to the stretch tensor**.

$$\mathbf{R} = \mathbf{1} \quad \Rightarrow \quad \mathbf{F} = \mathbf{U} = \mathbf{V}$$

Any material fibre after deformation may be calculated as follows:

$$\mathrm{d}\,\mathbf{x} = \mathbf{F}\ \mathrm{d}\,\mathbf{X} = \mathbf{U}\ \mathrm{d}\,\mathbf{X}$$

Since the tensor $\mathbf{U}$ is **symmetric**, we know that the solutions of its eigenproblem is such that:

- there exist 3 **real** numbers λ_k (eigenvalues of $\mathbf{U}$)
- And there exist 3 mutually perpendicular vectors $\mathbf{u}^{(k)}$ (eigenvectors of $\mathbf{U}$) (k=1,2,3), such that

$$\mathbf{U}\,\mathbf{u}^{(k)} = \lambda_k\,\mathbf{u}^{(k)}\ , \quad k=1,2,3$$

Eigenvectors of $\mathbf{U}$ do not change their direction but only their lengths. They are λ times elongated.

POLAR DECOMPOSITION OF THE DEFORMATION GRADIENT

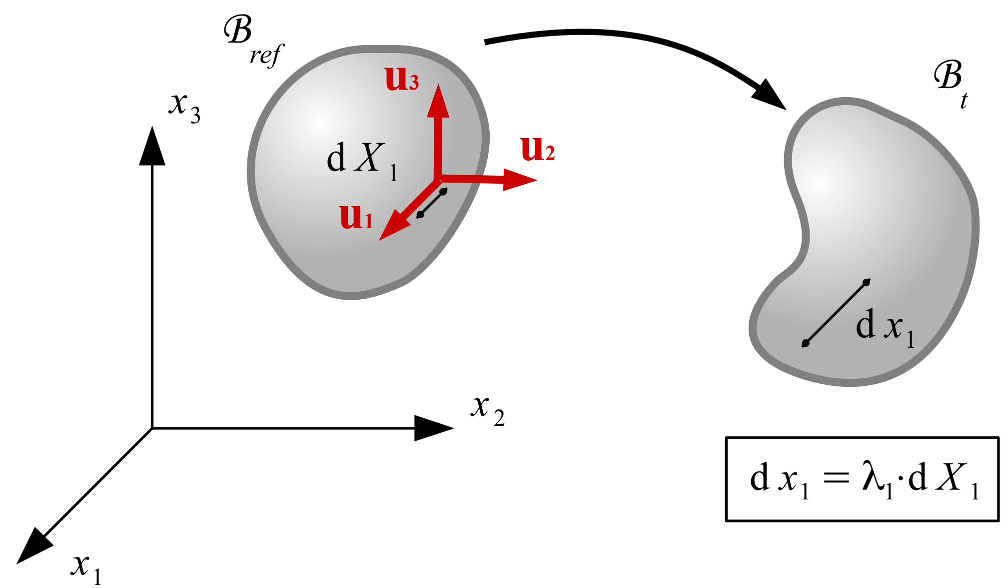
PHYSICAL INTERPRETATION OF THE STRETCH TENSOR

Still considering a situation when **rotation tensor is a unit tensor and deformation gradient is equivalent to the stretch tensor**, if we chose a coordinate system in such a way that its axes are parallel to the eigenaxes of the stretch tensor, then:

$$\mathrm{d}\mathbf{x}_1 = \mathbf{F}\,\mathrm{d}\mathbf{X}_1 = \mathbf{U}\cdot\mathrm{d}\mathbf{X}_1 = \lambda_1\cdot\mathrm{d}\mathbf{X}_1$$

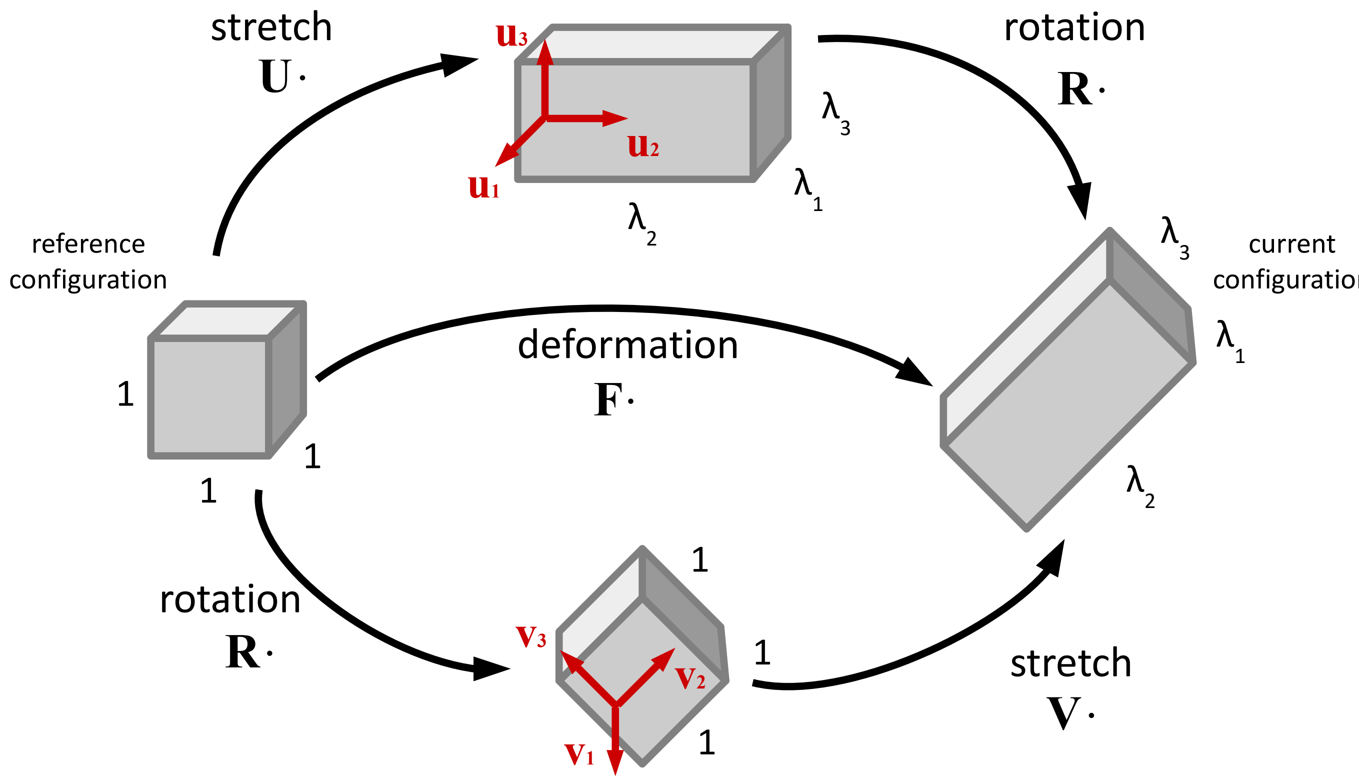
$$\mathrm{d}\mathbf{x}_2 = \mathbf{F}\,\mathrm{d}\mathbf{X}_2 = \mathbf{U}\cdot\mathrm{d}\mathbf{X}_2 = \lambda_3\cdot\mathrm{d}\mathbf{X}_2$$

$$\mathrm{d}\mathbf{x}_3 = \mathbf{F}\,\mathrm{d}\mathbf{X}_3 = \mathbf{U}\cdot\mathrm{d}\mathbf{X}_3 = \lambda_2\cdot\mathrm{d}\mathbf{X}_3$$



Material fibres which are parallel to the eigenaxes of the stretch tensor do not change their orientation but only their lengths. Eigenvalues of the stretch tensor are termed principal elongations.

POLAR DECOMPOSITION OF THE DEFORMATION GRADIENT



REMARKS:

- Eigenvalues of tensors $\mathbf{U}$ and $\mathbf{V}$ are the same.
- Tensors $\mathbf{U}$ are $\mathbf{V}$ distinguished by the orientation of eigenvectors which is different for each of them.
- Tensor $\mathbf{U}$ may be related with $\mathbf{V}$ with the operation of rotation given by rotation tensor $\mathbf{R}$:

$$\mathbf{U} = \mathbf{R}^T \mathbf{V} \mathbf{R} \qquad \mathbf{V} = \mathbf{R} \mathbf{U} \mathbf{R}^T$$

Paweł Szeptyński PhD, Eng.

POLAR DECOMPOSITION OF THE DEFORMATION GRADIENT

Summarizing the above considerations we may formulate the following

FUNDAMENTAL THEOREM OF KINEMATICS OF CONTINUA

Deformation of any material fibre in the neighbourhood of any point of the body is a composition of ***parallel translation****,* ***rotation*** *and* ***stretching*** */* ***schrinkage*** *along directions of eigenvectors of the stretch tensor.*

DISPLACEMENT GRADIENT

We define:

- Material displacement gradient

$$H_{ij} = \frac{\partial u_i}{\partial X_j} \quad \Leftrightarrow \quad \mathbf{H} = \frac{\partial \mathbf{u}}{\partial \mathbf{X}}$$

- Spatial displacement gradient

$$h_{ij} = \frac{\partial u_i}{\partial x_j} \quad \Leftrightarrow \quad \mathbf{h} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}}$$

The use of the definition of the displacement vector allow us to write:

$$H_{ij} = \frac{\partial}{\partial X_j}(x_i - X_i) = \frac{\partial x_i}{\partial X_j} - \frac{\partial X_i}{\partial X_j} = F_{ij} - \delta_{ij} \quad \Rightarrow \quad \mathbf{H} = \mathbf{F} - \mathbf{1}$$

$$h_{ij} = \frac{\partial}{\partial x_j}(x_i - X_i) = \frac{\partial x_i}{\partial x_j} - \frac{\partial X_i}{\partial x_j} = \delta_{ij} - f_{ij} \quad \Rightarrow \quad \mathbf{h} = \mathbf{1} - \mathbf{f}$$

THANK YOU FOR YOUR ATTENTION