THEORY OF ELASTICITY AND PLASTICITY

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MEASURES OF STRAIN

MEASURES OF STRAIN

We've already introduced following measures of deformation:

- Deformation gradient – information on **deformation of material fibres:**

$$\mathbf{F}(\mathbf{X}) = \left.\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right|_{\mathbf{X}}$$ (material description)

$$\mathbf{f}(\mathbf{x}) = \left.\frac{\partial \mathbf{X}}{\partial \mathbf{x}}\right|_{\mathbf{x}}$$ (spatial description)

- Displacement gradient – information on **deformation of material fibres:**

$$\mathbf{H}(\mathbf{X}) = \left.\frac{\partial \mathbf{u}}{\partial \mathbf{X}}\right|_{\mathbf{X}} - \mathbf{1}$$ (material description)

$$\mathbf{h}(\mathbf{x}) = \mathbf{1} - \left.\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right|_{\mathbf{x}}$$ (spatial description)

- Stretch tensors $\mathbf{U}, \mathbf{V}$– information on **elongation of material fibres:**
- Rotation tensor $\mathbf{R}$ – information on **rotation of material fibres:**

$$\mathbf{F} = \mathbf{R}\,\mathbf{U} = \mathbf{V}\,\mathbf{R}$$ (material description)

MEASURES OF STRAIN

REMARK:

Material deformation gradient is more proper to be used within **material description** since in this framework **spatial coordinates are functions of material coordinates** and components of the tensor are found by differentiation of the deformation relations with respect to chosen independent variables.:

$$F_{ij}(\mathbf{X}) = \frac{\partial}{\partial X_j}\left[x_i(X_1, X_2, X_3)\right]\Big|_{\mathbf{X}}$$

However, if we know the relation $\mathbf{X} = \mathbf{X}(\mathbf{x})$ as well as inverse relation $\mathbf{x} = \mathbf{x}(\mathbf{X})$, then it is possible to find the form of the **material deformation gradient** which is proper for **spatial description**:

$$F_{ij}(\mathbf{x}) = F_{ij}(\mathbf{X}(\mathbf{x}))$$

The above remar concerns all other tensors. If only the deformation relations are known **any tensor may be expressed as a function of either material coordinates** (material description) **or spatial coordinates** (spatial description). It is often very impractical approach.

DEFORMATION TENSOR

Let's calculate the **length of a material fibre after deformation in material description**, namely let's try to express this length in terms of quantities describing the fibre before deformation:

$$|\mathrm{d}\mathbf{x}| = \sqrt{\mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x}} = \sqrt{(\mathrm{d}\mathbf{x})^{\mathrm{T}}\mathrm{d}\mathbf{x}} = \sqrt{(\mathbf{F}\,\mathrm{d}\mathbf{X})^{\mathrm{T}}(\mathbf{F}\,\mathrm{d}\mathbf{X})} = \sqrt{(\mathrm{d}\mathbf{X})^{\mathrm{T}}\mathbf{F}^{\mathrm{T}}\mathbf{F}\,\mathrm{d}\mathbf{X}} = \sqrt{\underbrace{F_{ki}F_{kj}}_{=C_{ij}}\mathrm{d}X_i\,\mathrm{d}X_j} = \sqrt{C_{ij}\,\mathrm{d}X_i\,\mathrm{d}X_j}$$

We define the material deformation tensor (right Cauchy – Green deformation tensor) as follows:

$$C_{ij} = F_{ki}F_{kj} = \frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} \qquad \Leftrightarrow \qquad \mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$$

Summation written explicitly gives us:

$$C_{ij} = F_{1i}F_{1j} + F_{2i}F_{2j} + F_{3i}F_{3j} = \frac{\partial x_1}{\partial X_i}\frac{\partial x_1}{\partial X_j} + \frac{\partial x_2}{\partial X_i}\frac{\partial x_2}{\partial X_j} + \frac{\partial x_3}{\partial X_i}\frac{\partial x_3}{\partial X_j}$$

In matrix notation:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} F_{11} & F_{21} & F_{31} \\ F_{12} & F_{22} & F_{32} \\ F_{13} & F_{23} & F_{33} \end{bmatrix}\begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix}$$

DEFORMATION TENSOR

Multiplication is commutative, so the **deformation tensor is symmetric**:

$$C_{ij} = \frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} = \frac{\partial x_k}{\partial X_j}\frac{\partial x_k}{\partial X_i} = C_{ji}$$

Significance of the deformation tensor may be expressed as follows:

Square of length of a material fibre after deformation is a quadratic form* of components of a vector desribing that fibre before deformation, and the coefficients of that form are given by components of the deformation tensor:

$$|\mathrm{d}\,\mathbf{x}|^2 = C_{ij}\mathrm{d}X_i\mathrm{d}X_j =$$

$$= C_{11}\mathrm{d}X_1^2 + C_{22}\mathrm{d}X_2^2 + C_{33}\mathrm{d}X_3^2 + 2\left(C_{23}\mathrm{d}X_2\mathrm{d}X_3 + C_{31}\mathrm{d}X_3\mathrm{d}X_1 + C_{12}\mathrm{d}X_1\mathrm{d}X_2\right) =$$

$$= \begin{bmatrix}\mathrm{d}X_1\\ \mathrm{d}X_2\\ \mathrm{d}X_3\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}C_{11} & C_{12} & C_{13}\\ C_{21} & C_{22} & C_{23}\\ C_{31} & C_{32} & C_{33}\end{bmatrix}\begin{bmatrix}\mathrm{d}X_1\\ \mathrm{d}X_2\\ \mathrm{d}X_3\end{bmatrix}$$

***quadratic form** – a function which is assigning a scalar to a vector. This scalar is a sum of monomials of the 2nd degree (components of a vector occur as squares or as a product of two first powers of them)

DEFORMATION TENSOR

The use of the polar decomposition of material deformation gradient allow us to write:

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = (\mathbf{R}\,\mathbf{U})^{\mathrm{T}}(\mathbf{R}\,\mathbf{U}) = \mathbf{U}^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}\mathbf{R}\,\mathbf{U} = \mathbf{U}\,\mathbf{R}^{-1}\mathbf{R}\,\mathbf{U} = \mathbf{U}^{2}$$

Deformation tensor is a square of right stretch tensor (in the sense of product: $\mathbf{U}\mathbf{U}=\mathbf{U}^2$).

In the coordinate system the axes of which are parallel to its eigenvectors a tensor has a diagonal form, so:

$$\mathbf{U} = \begin{bmatrix} U_1 & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & U_3 \end{bmatrix} \quad \Rightarrow \quad \mathbf{C} = \mathbf{U}^2 = \begin{bmatrix} U_1^2 & 0 & 0 \\ 0 & U_2^2 & 0 \\ 0 & 0 & U_3^2 \end{bmatrix} = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & C_3 \end{bmatrix}$$

This means that:

- **tensors $\mathbf{C}$ and $\mathbf{U}$ have the same eigenaxes (eigenvectors)**
- **Eigenvalues of $\mathbf{C}$ are squares of eigenvalues of $\mathbf{U}$.**

These conclusions will help us in practical performance of polar decomposition.

DEFORMATION TENSOR

Let's calculate the **length of a material fibre before deformation in spatial description**, namely let's try to express this length in terms of quantities describing the fibre after deformation:

$$|\mathrm{d}\mathbf{X}| = \sqrt{\mathrm{d}\mathbf{X}\cdot\mathrm{d}\mathbf{X}} = \sqrt{(\mathrm{d}\mathbf{X})^{\mathrm{T}}\mathrm{d}\mathbf{X}} = \sqrt{(\mathbf{f}\,\mathrm{d}\mathbf{x})^{\mathrm{T}}(\mathbf{f}\,\mathrm{d}\mathbf{x})} = \sqrt{(\mathrm{d}\mathbf{x})^{\mathrm{T}}\mathbf{f}^{\mathbf{T}}\mathbf{f}\,\mathrm{d}\mathbf{x}} = \sqrt{\underbrace{f_{ki}f_{kj}}_{=c_{ij}}\mathrm{d}x_i\mathrm{d}x_j} = \sqrt{c_{ij}\mathrm{d}x_i\mathrm{d}x_j}$$

We define spatial deformation tensor as follows:

$$c_{ij} = f_{ki}f_{kj} = \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j} \qquad \Leftrightarrow \qquad \mathbf{c} = \mathbf{f}^{\mathbf{T}}\mathbf{f}$$

DEFORMATION TENSOR

Following tensorial measures of deformation emerge to be useful in solving certain problems:

- Material deformation tensor (right Cauchy – Green deformation tensor)

$$C_{ij} = F_{ki} F_{kj} = \frac{\partial x_k}{\partial X_i} \frac{\partial x_k}{\partial X_j} \quad \Leftrightarrow \quad \mathbf{C} = \mathbf{F}^{\mathrm{T}} \mathbf{F} = \mathbf{U}^2$$

material description

- Left Cauchy – Green deformation tensor

$$B_{ij} = F_{ik} F_{jk} = \frac{\partial x_i}{\partial X_k} \frac{\partial x_j}{\partial X_k} \quad \Leftrightarrow \quad \mathbf{B} = \mathbf{F} \mathbf{F}^{\mathrm{T}} = \mathbf{V}^2$$

material description

- Spatial deformation tensor (Cauchy deformation tensor)

$$c_{ij} = f_{ki} f_{kj} = \frac{\partial X_k}{\partial x_i} \frac{\partial X_k}{\partial x_j} \quad \Leftrightarrow \quad \mathbf{c} = \mathbf{f}^{\mathrm{T}} \mathbf{f}$$

spatial description

- Finger's tensor

$$b_{ij} = f_{ik} f_{jk} = \frac{\partial X_i}{\partial x_k} \frac{\partial X_j}{\partial x_k} \quad \Leftrightarrow \quad \mathbf{b} = \mathbf{f} \mathbf{f}^{\mathrm{T}}$$

spatial description

TENSOR DEFORMACJI

REMARKS:

- When **no deformation** occurs, then:

$$\mathbf{F} = \mathbf{1} \qquad \Rightarrow \qquad \mathbf{C} = \mathbf{1}$$

$$\mathbf{f} = \mathbf{1} \qquad \Rightarrow \qquad \mathbf{c} = \mathbf{1}$$

- The relation between material and spatial deformation gradients allow us to write:

$$\mathbf{C}\,\mathbf{b} = \left(\mathbf{F}^{\mathrm{T}}\mathbf{F}\right)\left(\mathbf{f}\,\mathbf{f}^{\mathrm{T}}\right) = \mathbf{F}^{\mathrm{T}}\underbrace{\left(\mathbf{F}\,\mathbf{f}\right)}_{=\mathbf{1}}\mathbf{f}^{\mathrm{T}} = \mathbf{F}^{\mathrm{T}}\mathbf{f}^{\mathrm{T}} = \underbrace{\left(\mathbf{f}\,\mathbf{F}\right)}_{=\mathbf{1}}^{\mathrm{T}} = \mathbf{1} \qquad \Rightarrow \qquad \boxed{\mathbf{b} = \mathbf{C}^{-1}\,, \quad \mathbf{C} = \mathbf{b}^{-1}}$$

$$\mathbf{B}\,\mathbf{c} = \left(\mathbf{F}\,\mathbf{F}^{\mathrm{T}}\right)\left(\mathbf{f}^{\mathrm{T}}\mathbf{f}\right) = \mathbf{F}\left(\mathbf{F}^{\mathrm{T}}\mathbf{f}^{\mathrm{T}}\right)\mathbf{f} = \mathbf{F}\underbrace{\left(\mathbf{f}\,\mathbf{F}\right)}_{=\mathbf{1}}^{\mathrm{T}}\mathbf{f} = \underbrace{\mathbf{f}\,\mathbf{F}}_{=\mathbf{1}} = \mathbf{1} \qquad \Rightarrow \qquad \boxed{\mathbf{c} = \mathbf{B}^{-1}\,, \quad \mathbf{B} = \mathbf{c}^{-1}}$$

STRAIN TENSOR

Let's calculate the **difference of squares of lengths of a material fibre after and before deformation**:

$$|\mathrm{d}\,\mathbf{x}|^2 - |\mathrm{d}\,\mathbf{X}|^2$$

This provides us with a **new measure of deformation**:

- If the fibre become **elongated** → this measure is **positive**
- If the fibre become **shortened** → this measure is **negative**
- If the fibre does not change its length → this measure is **zero**

The above measure will be termed strain.

STRAIN TENSOR

In **material descritpion:**

$$|\mathrm{d}\mathbf{x}|^2 - |\mathrm{d}\mathbf{X}|^2 = \mathrm{d}x_i \mathrm{d}x_i - \mathrm{d}X_i \mathrm{d}X_i = C_{ij}\mathrm{d}X_i \mathrm{d}X_j - \delta_{ij}\mathrm{d}X_i \mathrm{d}X_j =$$

$$= (C_{ij} - \delta_{ij})\mathrm{d}X_i \mathrm{d}X_j = 2 \cdot \underbrace{\frac{C_{ij} - \delta_{ij}}{2}}_{= E_{ij}} \mathrm{d}X_i \mathrm{d}X_j = 2E_{ij}\mathrm{d}X_i \mathrm{d}X_j$$

Material strain tensor (Green – de Saint-Venant strain tensor, Green – Lagrange strain tensor) is defined as follows:

$$E_{ij} = \frac{1}{2}(C_{ij} - \delta_{ij}) \quad \Leftrightarrow \quad \mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{1})$$

REMARKS:

- Strain tensor is **symmetric:** $E_{ij} = E_{ji}$
- If **no deformation** occurs, strain tensor is a **zero tensor**: $\mathbf{F} = \mathbf{1} \Rightarrow \mathbf{C} = \mathbf{1} \Rightarrow \mathbf{E} = \mathbf{0}$

TENSOR ODKSZTAŁCENIA

In **spatial description:**

$$|\mathrm{d}\mathbf{x}|^2 - |\mathrm{d}\mathbf{X}|^2 = \mathrm{d}x_i \mathrm{d}x_i - \mathrm{d}X_i \mathrm{d}X_i = \delta_{ij} \mathrm{d}x_i \mathrm{d}x_j - c_{ij} \mathrm{d}x_i \mathrm{d}x_j =$$

$$= (\delta_{ij} - c_{ij}) \mathrm{d}x_i \mathrm{d}x_j = 2 \cdot \underbrace{\frac{\delta_{ij} - c_{ij}}{2}}_{= e_{ij}} \mathrm{d}x_i \mathrm{d}x_j = 2 e_{ij} \mathrm{d}x_i \mathrm{d}x_j$$

Spatial strain tensor (Almansi – Hamel strain tensor) is defined as follows:

$$e_{ij} = \frac{1}{2}(\delta_{ij} - c_{ij}) \quad \Leftrightarrow \quad \mathbf{e} = \frac{1}{2}(\mathbf{1} - \mathbf{c})$$

REMARKS:

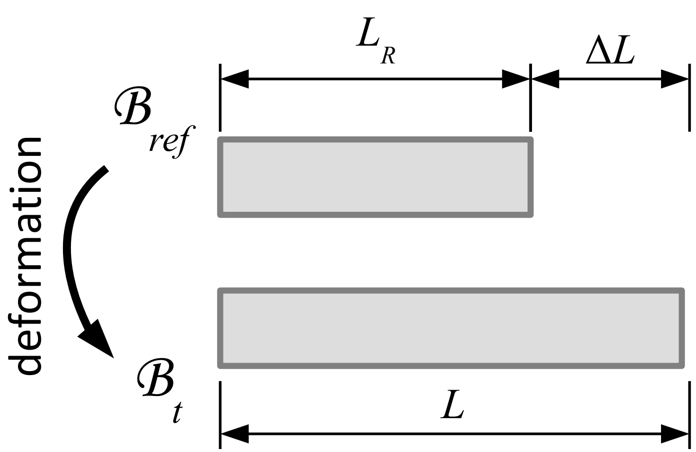
- Strain tensor is **symmetric:** $e_{ij} = e_{ji}$
- If **no deformation** occurs, strain tensor is a **zero tensor**: $\mathbf{f} = \mathbf{1} \;\Rightarrow\; \mathbf{c} = \mathbf{1} \;\Rightarrow\; \mathbf{e} = \mathbf{0}$

PHYSICAL INTERPRETATION OF STRAIN TENSOR COMPONENTS

DIAGONAL COMPONENTS – RELATIVE ELONGATIONS

Stretch is the **ratio of the length after defromation and the original length.**

Relative elongation is the **ratio of the increment of length and the original length.**



$$\lambda = \frac{L}{L_R}$$ – stretch

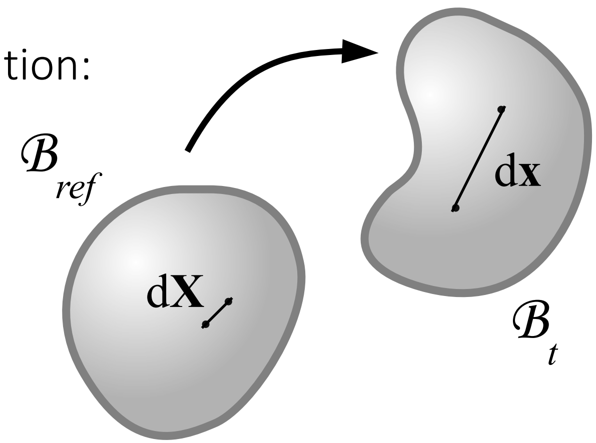
$$\varepsilon = \frac{\Delta L}{L_R}$$ – relative elongation

$$\lambda = 1 + \varepsilon$$

PHYSICAL INTERPRETATION OF STRAIN TENSOR COMPONENTS

DIAGONAL COMPONENTS – RELATIVE ELONGATIONS

Let's consider a material fibre which was parallel to X_1 axis before deformation:



- Material fibre: $\mathrm{d}\mathbf{X} = [\mathrm{d}X_1 \,;\, 0 \,;\, 0]$
- Length before deformation: $|\mathrm{d}\mathbf{X}| = \mathrm{d}X_1$
- Length after deformation: $|\mathrm{d}\mathbf{x}| = \sqrt{C_{ij}\mathrm{d}X_i\mathrm{d}X_j} = \sqrt{C_{11}}\mathrm{d}X_1$
- Relative elongation: $\varepsilon_1 = \dfrac{|\mathrm{d}\mathbf{x}|-|\mathrm{d}\mathbf{X}|}{|\mathrm{d}\mathbf{X}|} = \sqrt{C_{11}}-1 = \sqrt{2E_{11}+1}-1$

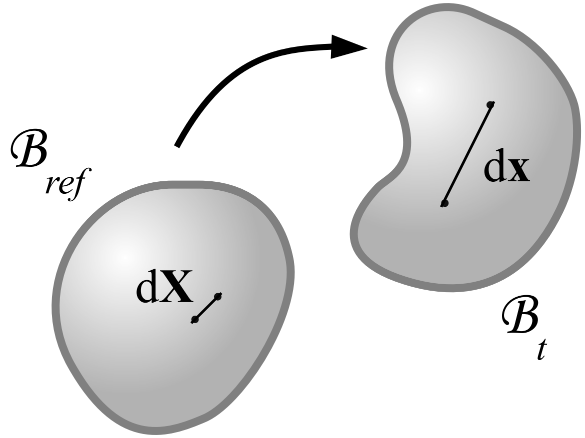
$$\varepsilon_1 = \sqrt{2E_{11}+1}-1$$

Relative elongations of material fibres which are parallel to *i*-th axis of the considered coordinate system depend only on diagonal component E_{ii} (no summation wrt. i) of the **strain tensor**.

PHYSICAL INTERPRETATION OF STRAIN TENSOR COMPONENTS

DIAGONAL COMPONENTS – RELATIVE ELONGATIONS

If the strain is small, then the obtained relation may be expanded Into a Taylor power series in the neighbourhood of $E_{11}=0$



$$\varepsilon_1=\sqrt{2E_{11}+1}-1=E_{11}-\frac{E_{11}^2}{2}+\frac{E_{11}^3}{2}+\ldots\approx E_{11}$$

Finally:

$$\varepsilon_1\approx E_{11}$$

If the deformation is small, then the diagonal components of the strain tensor are approximately equal to the relative elongations of material fibres which are parallel to the respective axis of the coordinate system.

PHYSICAL INTERPRETATION OF STRAIN TENSOR COMPONENTS

OFF-DIAGONAL COMPONENTS – CHANGE OF AN ANGLE BETWEEN FIBRES

An angle between two vectors of known lengths may be calculated with the use of the dot product:

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\sphericalangle(\mathbf{a};\mathbf{b})$$

Cosine of the angle between two deformed material fibres may be calculated as follows:

$$\cos\phi_{12} = \frac{\mathrm{d}\,\mathbf{x}^{(1)}\cdot\mathrm{d}\,\mathbf{x}^{(2)}}{|\mathrm{d}\,\mathbf{x}^{(1)}|\,|\mathrm{d}\,\mathbf{x}^{(2)}|}$$

PHYSICAL INTERPRETATION OF STRAIN TENSOR COMPONENTS

OFF-DIAGONAL COMPONENTS – CHANGE OF AN ANGLE BETWEEN FIBRES

Let's consider two material fibres, one of which was parallel to the axis X_1 before deformation, while the second one was parallel to the axis X_2:

- Material fibres **before deformation**:

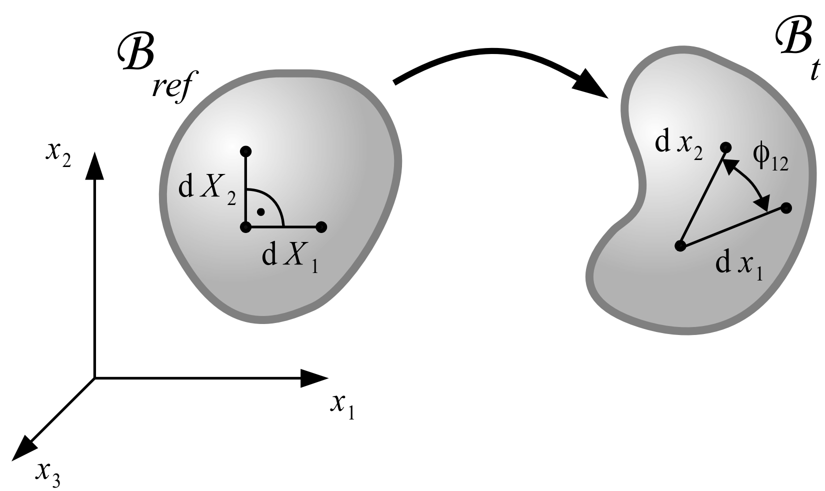
$$\mathrm{d}\,\mathbf{X}^{(1)} = [\mathrm{d}\,X_1\ ;\ 0\ ;\ 0]$$

$$\mathrm{d}\,\mathbf{X}^{(2)} = [0\ ;\ \mathrm{d}\,X_2\ ;\ 0]$$

- Material fibres **after deformation**:

$$\mathrm{d}\,\mathbf{x}^{(1)} = \mathbf{F}\,\mathrm{d}\,\mathbf{X}^{(1)} = \mathrm{d}\,X_1[F_{11}\ ;\ F_{21}\ ;\ F_{31}]$$

$$\mathrm{d}\,\mathbf{x}^{(2)} = \mathbf{F}\,\mathrm{d}\,\mathbf{X}^{(2)} = \mathrm{d}\,X_2[F_{12}\ ;\ F_{22}\ ;\ F_{32}]$$



PHYSICAL INTERPRETATION OF STRAIN TENSOR COMPONENTS

OFF-DIAGONAL COMPONENTS – CHANGE OF AN ANGLE BETWEEN FIBRES

Cosine of the angle between deformed fibres

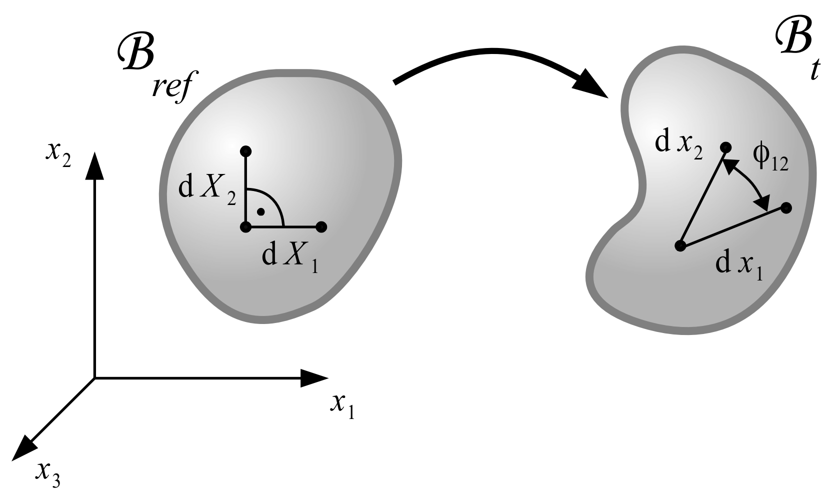
$$\cos\phi_{12} = \frac{\mathrm{d}\,\mathbf{x}^{(1)}\cdot\mathrm{d}\,\mathbf{x}^{(2)}}{|\mathrm{d}\,\mathbf{x}^{(1)}|\,|\mathrm{d}\,\mathbf{x}^{(2)}|} = \frac{\mathrm{d}\,X_1\,\mathrm{d}\,X_2\left(F_{11}F_{12}+F_{21}F_{22}+F_{31}F_{32}\right)}{\mathrm{d}\,X_1\sqrt{F_{11}^2+F_{21}^2+F_{31}^2}\cdot\mathrm{d}\,X_2\sqrt{F_{12}^2+F_{22}^2+F_{32}^2}}$$

Let's make use of the definition of the deformation tensor:

$$C_{ij} = F_{ki}F_{kj} = F_{1i}F_{1j} + F_{2i}F_{2j} + F_{3i}F_{3j}$$

We may write then:

$$\cos\phi_{12} = \frac{C_{12}}{\sqrt{C_{11}}\cdot\sqrt{C_{22}}} = \frac{2E_{12}}{\sqrt{(2E_{11}+1)(2E_{22}+1)}}$$

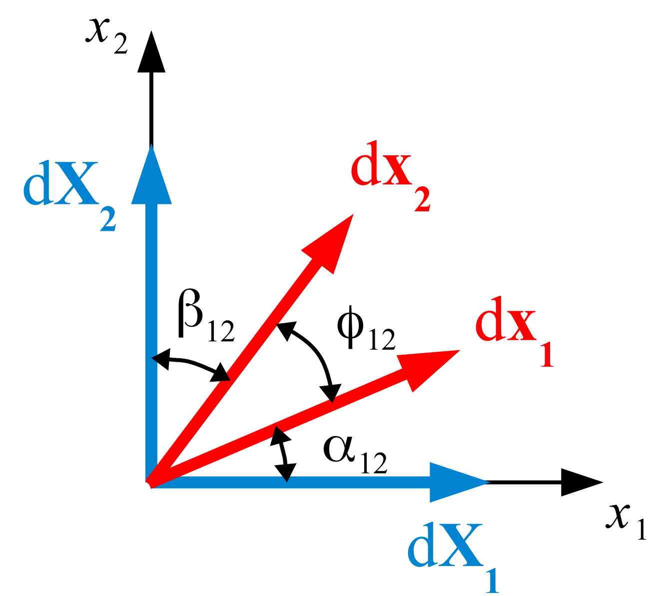


PHYSICAL INTERPRETATION OF STRAIN TENSOR COMPONENTS

OFF-DIAGONAL COMPONENTS – CHANGE OF AN ANGLE BETWEEN FIBRES

Let's define the **change of initially right angle between two fibres**:

$$\gamma_{12} = \alpha_{12}+\beta_{12} = 90^\circ - \phi_{12}$$



Trigonometric identities give us:

$$\cos\phi_{12} = \cos(90^\circ - \gamma_{12}) = \sin\gamma_{12}$$

so:

$$\sin\gamma_{12} = \frac{2E_{12}}{\sqrt{(2E_{11}+1)(2E_{22}+1)}}$$

Sine of change of initially right angle between material fibres which before deformation were parallel to axes X_i and X_j of considered coordinate system depend on the off-diagonal component E_{ij} of the strain tensor.

PHYSICAL INTERPRETATION OF STRAIN TENSOR COMPONENTS

OFF-DIAGONAL COMPONENTS – CHANGE OF AN ANGLE BETWEEN FIBRES

Relation

$$\sin\gamma_{12} = \frac{2E_{12}}{\sqrt{(2E_{11}+1)(2E_{22}+1)}}$$

may be interpreted as depending on 4 variables. If the deformation is small, then each of that variables is close to 0 and in the neighbourhood of such a point both sides of that relation may be expanded into a power series. Accounting only for the 1st degree terms gives us:

$$\left.\begin{aligned} \sin\gamma_{12} &= \gamma_{12} + \ldots \\ \frac{2E_{12}}{\sqrt{(2E_{11}+1)(2E_{22}+1)}} &= 2E_{12} + \ldots \end{aligned}\right\} \Rightarrow \quad \gamma_{12} \approx 2E_{12}$$

Therefore for small deformation:

$$E_{12} \approx \frac{\gamma_{12}}{2}$$

In case of small deformation the off-diagonal components of the strain tensor are approximately equal to the half the change of an angle between two initially perpendicular material fibres.

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KINEMATIC RELATIONS

KINEMATIC RELATIONS

Kinematic relations (geometric r.) are the relations between components of the **strain tensor** and components of the **displacement vector**.

- Strain tensor may be expressed in terms of deformation tensor: $E_{ij} = \frac{1}{2}(C_{ij} - \delta_{ij})$
- Deformation tensor may be expressed in terms of deformation gradient: $C_{ij} = F_{ki} F_{kj}$
- Deformation gradient may be expressed in terms of displacement gradient: $F_{ij} = \frac{\partial u_i}{\partial X_j} + \delta_{ij}$

$$E_{ij} = \frac{1}{2}(C_{ij} - \delta_{ij}) = \frac{1}{2}(F_{ki} F_{kj} - \delta_{ij}) = \frac{1}{2}\left[(u_{k,i} + \delta_{ki})(u_{k,j} + \delta_{kj}) - \delta_{ij}\right] =$$

$$= \frac{1}{2}\left[u_{k,i}\delta_{kj} + u_{k,j}\delta_{ki} + u_{k,i}u_{k,j} + \delta_{ki}\delta_{kj} - \delta_{ij}\right] = = \frac{1}{2}\left[u_{j,i} + u_{i,j} + u_{k,i}u_{k,j} + \delta_{ij} - \delta_{ij}\right] =$$

$$= \frac{1}{2}\left[u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}\right]$$

$$E_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}\right) \quad \Leftrightarrow \quad \mathbf{E} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^{\mathrm{T}} + \mathbf{H}^{\mathrm{T}}\mathbf{H}\right)$$

KINEMATIC RELATIONS

Kinematic relations constitute a system of non-linear partial differential equations accounting for components of strain tensor and displacement vector:

$$E_{11} = \frac{1}{2}\left[2\frac{\partial u_1}{\partial X_1} + \left(\frac{\partial u_1}{\partial X_1}\right)^2 + \left(\frac{\partial u_2}{\partial X_1}\right)^2 + \left(\frac{\partial u_3}{\partial X_1}\right)^2\right]$$

$$E_{22} = \frac{1}{2}\left[2\frac{\partial u_2}{\partial X_2} + \left(\frac{\partial u_1}{\partial X_2}\right)^2 + \left(\frac{\partial u_2}{\partial X_2}\right)^2 + \left(\frac{\partial u_3}{\partial X_2}\right)^2\right]$$

$$\ldots$$

$$E_{12} = \frac{1}{2}\left[\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} + \frac{\partial u_1}{\partial X_1}\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\frac{\partial u_2}{\partial X_2} + \frac{\partial u_3}{\partial X_1}\frac{\partial u_3}{\partial X_2}\right]$$

Systems of non-linear partial differential equations are difficult to solve...

Very difficult...

We'll need to do something about it.

KINEMATIC RELATIONS

The same considerations may be done in **spatial description**:

- Strain tensor may be expressed in terms of deformation tensor: $e_{ij}=\frac{1}{2}(\delta_{ij}-c_{ij})$
- Deformation tensor may be expressed in terms of deformation gradient: $c_{ij}=f_{ki}f_{kj}$
- Deformation gradient may be expressed in terms of displacement gradient: $f_{ij}=\delta_{ij}-\frac{\partial u_i}{\partial x_j}$

$$e_{ij}=\frac{1}{2}(\delta_{ij}-c_{ij})=\frac{1}{2}(\delta_{ij}-f_{ki}f_{kj})=\frac{1}{2}\left[\delta_{ij}-(\delta_{ki}-u_{k,i})(\delta_{kj}-u_{k,j})\right]=$$

$$=\frac{1}{2}\left[u_{k,i}\delta_{kj}+u_{k,j}\delta_{ki}-u_{k,i}u_{k,j}-\delta_{ki}\delta_{kj}+\delta_{ij}\right]=\frac{1}{2}\left[u_{j,i}+u_{i,j}-u_{k,i}u_{k,j}-\delta_{ij}+\delta_{ij}\right]$$

$$=\frac{1}{2}\left[u_{i,j}+u_{j,i}-u_{k,i}u_{k,j}\right]$$

$$e_{ij}=\frac{1}{2}\left(u_{i,j}+u_{j,i}-u_{k,i}u_{k,j}\right)\quad\Leftrightarrow\quad \mathbf{e}=\frac{1}{2}\left(\mathbf{h}+\mathbf{h}^{\mathrm{T}}-\mathbf{h}^{\mathrm{T}}\mathbf{h}\right)$$

CAUTION: in **material description**: $u_{i,j}=\frac{\partial u_i}{\partial X_j}$ in **spatial description**: $u_{i,j}=\frac{\partial u_i}{\partial x_j}$

DEFORMATION OF INFINITESIMAL AND FINITE ELEMENTS

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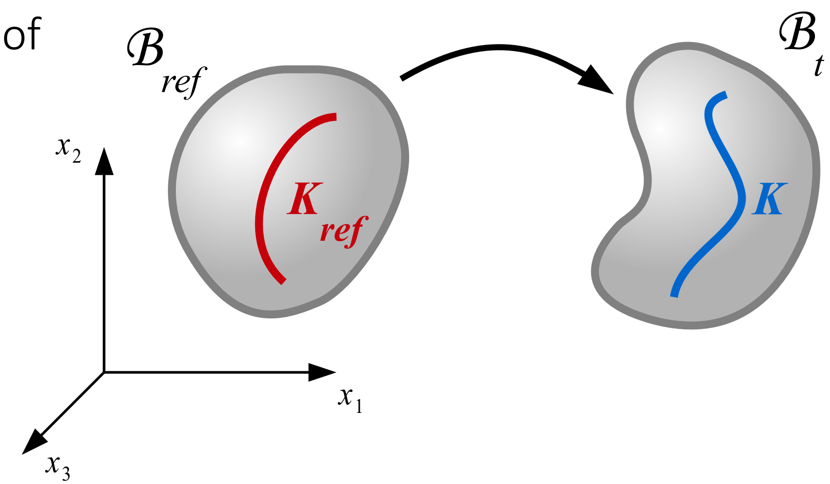
DEFORMATION OF INFINITESIMAL AND FINITE ELEMENTS

The length of an infinitesimal arc element:

- before deformation: $\mathrm{d}L_R = |\mathrm{d}\mathbf{X}| = \sqrt{\mathrm{d}X_i \mathrm{d}X_i}$
- after deformation: $\mathrm{d}L = |\mathrm{d}\mathbf{x}| = \sqrt{\mathrm{d}x_i \mathrm{d}x_i} = \sqrt{C_{ij}\mathrm{d}X_i \mathrm{d}X_j} = \dfrac{\sqrt{C_{ij}\mathrm{d}X_i \mathrm{d}X_j}}{\sqrt{\mathrm{d}X_k \mathrm{d}X_k}}\mathrm{d}L_R$

DEFORMATION OF INFINITESIMAL AND FINITE ELEMENTS

The length of a material curve (finite arc element) which in the reference configuration is given by a system of parametric equations:



$$K_{ref}: \begin{cases} X_1 = X_1(\lambda) \\ X_2 = X_2(\lambda) \\ X_3 = X_3(\lambda) \end{cases}$$

- **before deformation:**

$$\underbrace{L_R = \int_{K_{ref}} \mathrm{d}L_R = \int_{K_{ref}} \sqrt{\mathrm{d}X_i \mathrm{d}X_i} = \int_{\lambda_0}^{\lambda_1} \sqrt{\frac{\mathrm{d}X_i}{\mathrm{d}\lambda}\frac{\mathrm{d}X_i}{\mathrm{d}\lambda}}\,\mathrm{d}\lambda}_{\text{integration over } \mathcal{B}_{ref}}$$

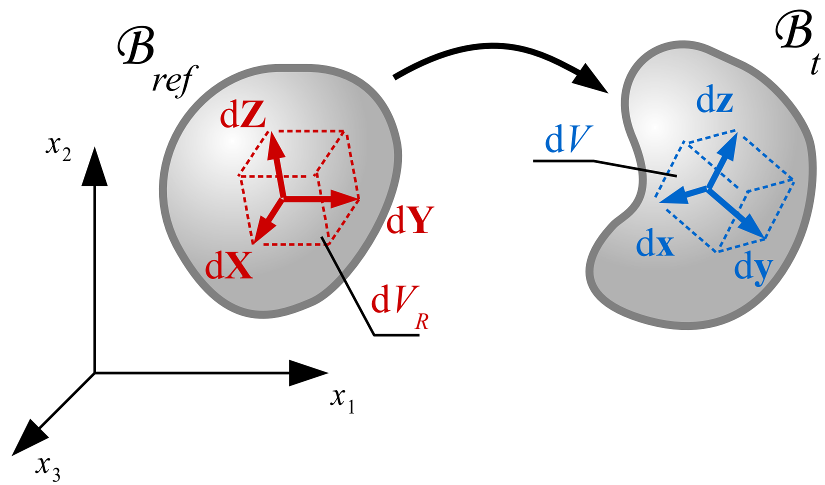
- **after deformation:**

$$L = \underbrace{\int_{K} \mathrm{d}L = \int_{K} \sqrt{\mathrm{d}x_i \mathrm{d}x_i}}_{\text{integration over } \mathcal{B}_t} = \underbrace{\int_{K_{ref}} \sqrt{C_{ij}\mathrm{d}X_i \mathrm{d}X_j} = \int_{\lambda_0}^{\lambda_1} \sqrt{C_{ij}\frac{\mathrm{d}X_i}{\mathrm{d}\lambda}\frac{\mathrm{d}X_i}{\mathrm{d}\lambda}}\,\mathrm{d}\lambda}_{\text{integration over } \mathcal{B}_{ref}}$$

Reference configuration is often "easier" to be integrated over, and it is constant in time

DEFORMATION OF INFINITESIMAL AND FINITE ELEMENTS

The **volume of an infinitesimal volume element ("brick" element)** is approximated by the **volume of an infinitesimal parallelepiped**, the edges of which are given by 3 non-coplanar material fibres. Such a volume is equal the **triple product** of the respective vectors:



- **before deformation**:

$$\mathrm{d}V_R = [\mathrm{d}\mathbf{X}, \mathrm{d}\mathbf{Y}, \mathrm{d}\mathbf{Z}] = (\mathrm{d}\mathbf{X} \times \mathrm{d}\mathbf{Y}) \cdot \mathrm{d}\mathbf{Z}$$

- **after deformations**

$$\mathrm{d}V = [\mathrm{d}\mathbf{x}, \mathrm{d}\mathbf{y}, \mathrm{d}\mathbf{z}] = (\mathrm{d}\mathbf{x} \times \mathrm{d}\mathbf{y}) \cdot \mathrm{d}\mathbf{z}$$

$$= ((\mathbf{F}\cdot\mathrm{d}\mathbf{X})\times(\mathbf{F}\cdot\mathrm{d}\mathbf{Y}))\cdot(\mathbf{F}\cdot\mathrm{d}\mathbf{Z}) =$$

$$= \det\mathbf{F}(\mathrm{d}\mathbf{X} \times \mathrm{d}\mathbf{Y}) \cdot \mathrm{d}\mathbf{Z} = J\,\mathrm{d}V_R \qquad \Rightarrow \qquad \boxed{\mathrm{d}V = J\,\mathrm{d}V_R}$$

DEFORMATION OF INFINITESIMAL AND FINITE ELEMENTS

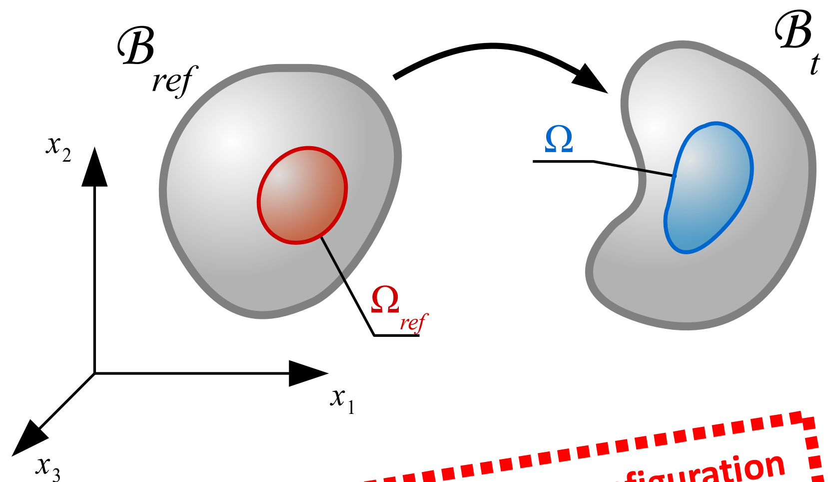
Volume of a finite subregion Ω of body's configuration:

- before deformation:

$$V_R = \underbrace{\iiint\limits_{\Omega_{ref}} \mathrm{d}V_R}_{\text{integration over } \mathcal{B}_{ref}}$$

- after deformation:

$$V = \underbrace{\iiint\limits_{\Omega} \mathrm{d}V}_{\text{integration over } \mathcal{B}_t} = \underbrace{\iiint\limits_{\Omega_{ref}} J \,\mathrm{d}V_R}_{\text{integration over } \mathcal{B}_{ref}}$$



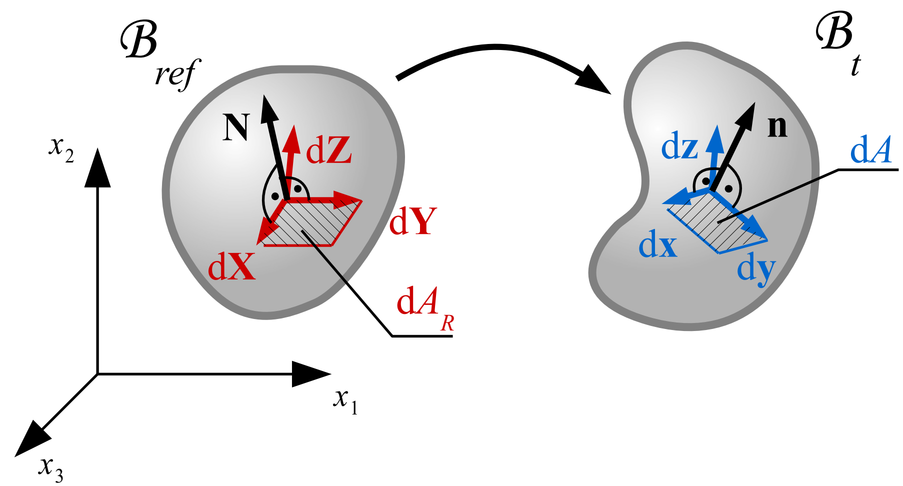
Reference configuration is often "easier" to be integrated over, and it is constant in time

REMARKS:

- **Jacobian** is a function of $\mathbf{X}$ and is a **measure of** local volume change (in point $\mathbf{X}$) due to deformation.
- $J = 0$ means „vanishing of particles"
- $J < 0$ means that the region is turned "inside out" (negative volume)

DEFORMATION OF INFINITESIMAL AND FINITE ELEMENTS

Surface area of an infinitesimal surface element is approximated by an **infinitesimal parallelogram,** the edges of which are given by two non-collinear material fibres. The surface area of such a parallelogram in equal the length of a vector being a result of a **cross product** of vectors representing those material fibres.



Let's use the just found relation between volume elements before and after deformation. **Triple product** may be expressed as a **composition** of **cross product** and of a **dot product**:

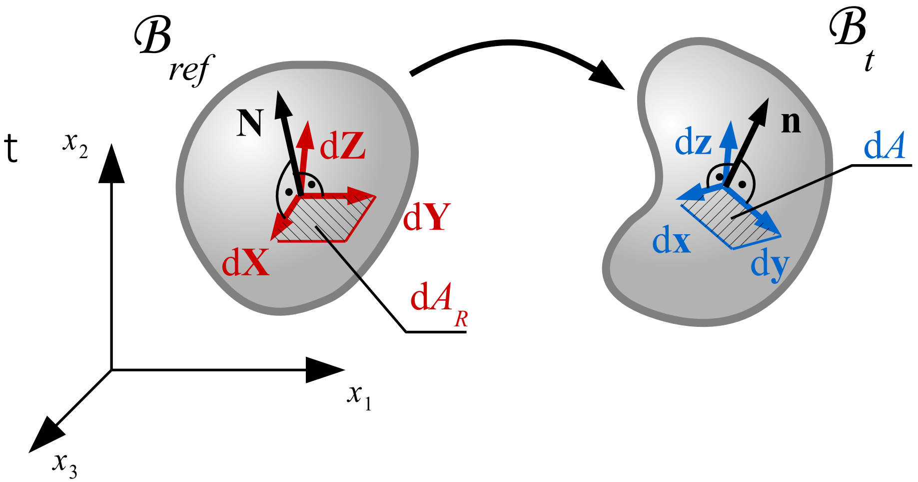
- **before deformation**: $\mathrm{d}V_R = [\mathrm{d}\mathbf{X}, \mathrm{d}\mathbf{Y}, \mathrm{d}\mathbf{Z}] = (\mathrm{d}\mathbf{X} \times \mathrm{d}\mathbf{Y}) \cdot \mathrm{d}\mathbf{Z} = \mathrm{d}A_R \mathbf{N} \cdot \mathrm{d}\mathbf{Z}$ $\qquad |\mathbf{N}| = 1$

- **after deformation**: $\mathrm{d}V = [\mathrm{d}\mathbf{x}, \mathrm{d}\mathbf{y}, \mathrm{d}\mathbf{z}] = (\mathrm{d}\mathbf{x} \times \mathrm{d}\mathbf{y}) \cdot \mathrm{d}\mathbf{z} = \mathrm{d}A \mathbf{n} \cdot \mathrm{d}\mathbf{z}$ $\qquad |\mathbf{n}| = 1$

$$\mathrm{d}V = J\,\mathrm{d}V_R \qquad \Rightarrow \qquad \mathrm{d}A\,\mathbf{n} \cdot \mathrm{d}\mathbf{z} = J\,\mathrm{d}A_R \mathbf{N} \cdot \mathrm{d}\mathbf{Z}$$

DEFORMATION OF INFINITESIMAL AND FINITE ELEMENTS

$$\mathrm{d}A\,\mathbf{n}\cdot\mathrm{d}\mathbf{z} = J\,\mathrm{d}A_R\mathbf{N}\cdot\mathrm{d}\mathbf{Z}$$



Fibre $\mathbf{dz}$ may be expressed by $\mathbf{dZ}$ and deformation gradient

$$\mathrm{d}A\,\mathbf{n}\cdot\mathbf{F}\,\mathrm{d}\mathbf{Z} = J\,\mathrm{d}A_R\mathbf{N}\cdot\mathrm{d}\mathbf{Z}$$

In index notation it is easier to notice that both sides of the above equation may be interpreted as certain maps acting on vector $\mathbf{dZ}$:

$$\left(\mathrm{d}A\,n_j F_{ji}\right)\,\mathrm{dZ}_i = \left(J\,\mathrm{d}A_R N_i\right)\,\mathrm{dZ}_i$$

This relation must hold true **for any $\mathrm{d}\mathbf{Z}$**, so both **linear operators must be equal**:

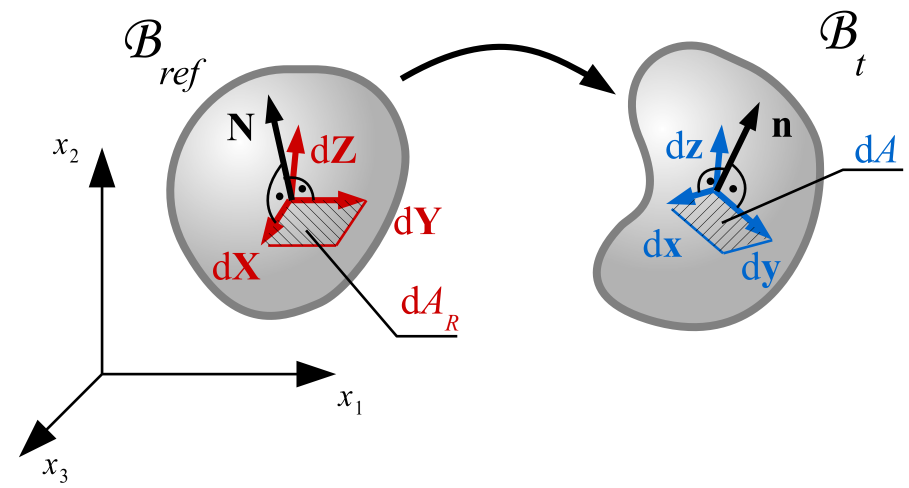
$$\mathrm{d}A\,\mathbf{n}\mathbf{F} = J\,\mathrm{d}A_R\mathbf{N}$$

After multiplying by $\mathbf{F}^{-1}$ we obtain so called Nanson's formula:

$$\boxed{\mathrm{d}A\,\mathbf{n} = J\,\mathrm{d}A_R\mathbf{N}\,\mathbf{F}^{-1}}$$

DEFORMATION OF INFINITESIMAL AND FINITE ELEMENTS

Nanson's formula is an identity of two vectors:



$$\mathrm{d}A\,\mathbf{n} = J\,\mathrm{d}A_R\,\mathbf{N}\,\mathbf{F}^{-1}$$

In particular those vectors must have equal lengths:

$$|\mathrm{d}A\,\mathbf{n}| = |J\,\mathrm{d}A_R\,\mathbf{N}\,\mathbf{F}^{-1}|$$

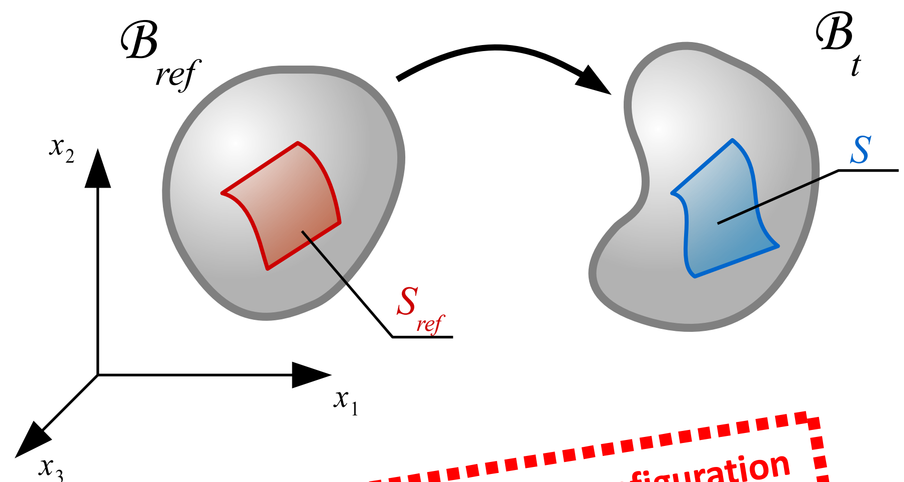
$$\mathrm{d}A|\mathbf{n}| = J\,\mathrm{d}A_R|\mathbf{N}\,\mathbf{F}^{-1}|$$

Normal vector is assumed to be a unit vector, so $|\mathbf{n}| = 1$, as a result we obtain the **relation between the surface areas of an infitesimal surface element before and after deformation**:

$$\mathrm{d}A = J\,|\mathbf{N}\,\mathbf{F}^{-1}|\,\mathrm{d}A_R$$

DEFORMATION OF INFINITESIMAL AND FINITE ELEMENTS

Surface area of a finite surface S contained in the configuration of a body::



- **before deformation**:

$$A_R = \underbrace{\iint\limits_{S_{ref}} \mathrm{d}\,A_R}_{\text{integration over } \mathcal{B}_{ref}}$$

- **after deformation**:

$$A = \underbrace{\iint\limits_{S} \mathrm{d}\,A}_{\text{integration over } \mathcal{B}_t} = \underbrace{\iint\limits_{S_{ref}} J\left|\mathbf{N}\,\mathbf{F}^{-1}\right|\mathrm{d}\,A_R}_{\text{integration over } \mathcal{B}_{ref}}$$

Reference configuration is often "easier" to be integrated over, and it is constant in time

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SMALL STRAIN THEORY

SMALL STRAIN THEORY

We have already shown that in case of **small deformation**:

- diagonal **components of the strain tensor** are the **measures of** relative elongations **of fibres**
- off-diagonal **components of the strain tensor** are the **measures of** change of angle **between fibres**

If the derivatives of displacements (displacement increments, strains) are small, namely, $u_{i,j} \ll 1$ then geometric relation may be simplified:

$$E_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}\right) \approx \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) = \varepsilon_{ij}$$

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^{\mathrm{T}} + \mathbf{H}^{\mathrm{T}}\mathbf{H}\right) \approx \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^{\mathrm{T}}\right) = \boldsymbol{\varepsilon}$$

Tensor $\boldsymbol{\varepsilon}$, which is **symmetric part of the displacement gradient**, is termed the Cauchy small strain tensor. Its relation to the derivatives of displacements are termed Cauchy kinematic relations.

SMALL STRAIN THEORY

Small strain measures:

- **Small strain tensor** (symmetric part of the displacement gradient):

$$\underbrace{\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)}_{\text{Cauchy kinematic relations}} \quad \Leftrightarrow \quad \boldsymbol{\varepsilon} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^{\mathrm{T}}\right) \qquad \boldsymbol{\varepsilon}^{\mathrm{T}} = \boldsymbol{\varepsilon}$$

- **Small rotation tensor** (**skew-symmetric** (anti-symmetric) **part of the displacement gradient**):

$$\omega_{ij} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right) \quad \Leftrightarrow \quad \boldsymbol{\omega} = \frac{1}{2}\left(\mathbf{H} - \mathbf{H}^{\mathrm{T}}\right) \qquad \boldsymbol{\omega}^{\mathrm{T}} = -\boldsymbol{\omega}$$

Displacement gradient may be decomposed into its symmetric and skew-symmetric part: $\mathbf{H} = \boldsymbol{\varepsilon} + \boldsymbol{\omega}$

Strain tensor (finite strains) may be expressed in the following way:

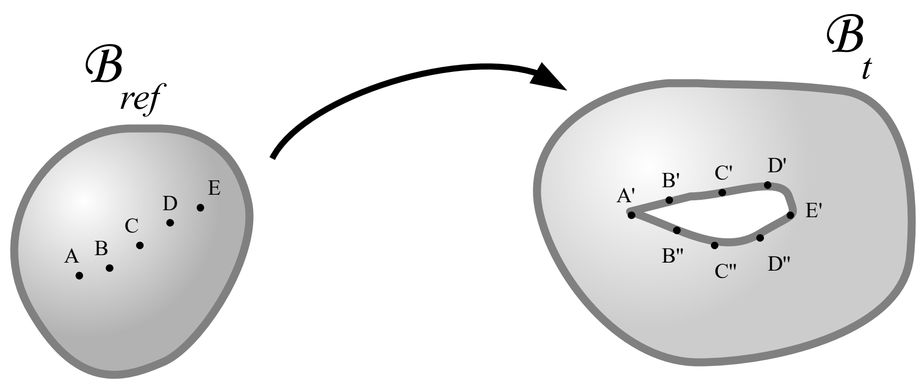
$$\mathbf{E} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^{\mathrm{T}} + \mathbf{H}^{\mathrm{T}} \cdot \mathbf{H}\right) = \boldsymbol{\varepsilon} + \frac{1}{2}\left(\boldsymbol{\varepsilon} + \boldsymbol{\omega}\right)^{\mathrm{T}}\left(\boldsymbol{\varepsilon} + \boldsymbol{\omega}\right)$$

STRAIN COMPATIBILITY CONDITIONS

STRAIN COMPATIBILITY CONDITIONS

Non-linear kinematic relations constitute a system of **6 non-linear partial differential equations**, in which for known strains there are **3 unknown components of the displacement vector**.

In general such a system is “overdetermined” (we have more equations than unknowns) and it may happen that it has no solution at all or the solution is not unique (there are few of them). In the latter case it could happen that multiple displacements could correspond to given strain.



The conditions which guarantee **integrability of kinematic relations** are termed the strain compatibility conditions.

STRAIN COMPATIBILITY CONDITIONS

The proof is quite a complicated one, but it can be shown that the strain compatibility conditions are:

- Finite strain theory:

$$\nabla_{\mathbf{X}} \times \mathbf{F} = \mathbf{0} \quad \Leftrightarrow \quad \epsilon_{pqi} F_{jq,p} = 0$$

$$\frac{\partial F_{13}}{\partial X_2} - \frac{\partial F_{12}}{\partial X_3} = 0 \qquad \frac{\partial F_{23}}{\partial X_2} - \frac{\partial F_{22}}{\partial X_3} = 0 \qquad \frac{\partial F_{33}}{\partial X_2} - \frac{\partial F_{32}}{\partial X_3} = 0$$

$$\frac{\partial F_{11}}{\partial X_3} - \frac{\partial F_{13}}{\partial X_1} = 0 \qquad \frac{\partial F_{21}}{\partial X_3} - \frac{\partial F_{23}}{\partial X_1} = 0 \qquad \frac{\partial F_{31}}{\partial X_3} - \frac{\partial F_{33}}{\partial X_1} = 0$$

$$\frac{\partial F_{12}}{\partial X_1} - \frac{\partial F_{11}}{\partial X_2} = 0 \qquad \frac{\partial F_{22}}{\partial X_1} - \frac{\partial F_{21}}{\partial X_2} = 0 \qquad \frac{\partial F_{32}}{\partial X_1} - \frac{\partial F_{31}}{\partial X_2} = 0$$

- Small strain theory:

$$\nabla_{\mathbf{X}} \times (\nabla_{\mathbf{X}} \times \boldsymbol{\varepsilon}) = \mathbf{0} \quad \Leftrightarrow \quad \epsilon_{pri} \epsilon_{qsj} \varepsilon_{pq,rs} = 0$$

$$\frac{\partial^2 \varepsilon_{22}}{\partial X_3^2} - 2\frac{\partial^2 \varepsilon_{23}}{\partial X_2 \partial X_3} + \frac{\partial^2 \varepsilon_{33}}{\partial X_2^2} = 0 \qquad \frac{\partial^2 \varepsilon_{11}}{\partial X_2 \partial X_3} - \frac{\partial}{\partial X_1}\left[-\frac{\partial \varepsilon_{23}}{\partial X_1} + \frac{\partial \varepsilon_{31}}{\partial X_2} + \frac{\partial \varepsilon_{12}}{\partial X_3}\right] = 0$$

$$\frac{\partial^2 \varepsilon_{33}}{\partial X_1^2} - 2\frac{\partial^2 \varepsilon_{31}}{\partial X_3 \partial X_1} + \frac{\partial^2 \varepsilon_{11}}{\partial X_3^2} = 0 \qquad \frac{\partial^2 \varepsilon_{22}}{\partial X_3 \partial X_1} - \frac{\partial}{\partial X_2}\left[\frac{\partial \varepsilon_{23}}{\partial X_1} - \frac{\partial \varepsilon_{31}}{\partial X_2} + \frac{\partial \varepsilon_{12}}{\partial X_3}\right] = 0$$

$$\frac{\partial^2 \varepsilon_{11}}{\partial X_2^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial X_1 \partial X_2} + \frac{\partial^2 \varepsilon_{22}}{\partial X_1^2} = 0 \qquad \frac{\partial^2 \varepsilon_{33}}{\partial X_1 \partial X_2} - \frac{\partial}{\partial X_3}\left[\frac{\partial \varepsilon_{23}}{\partial X_1} + \frac{\partial \varepsilon_{31}}{\partial X_2} - \frac{\partial \varepsilon_{12}}{\partial X_3}\right] = 0$$

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THANK YOU FOR YOUR ATTENTION