THEORY OF ELASTICITY AND PLASTICITY

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COUNTINUITY OF MASS FLOW
AND
CONSERVATION OF MASS

CONSERVATION OF MASS

Mass should be assigned to each particle – since the particales are infinitely small (we're dealing with continuum), **mass density is assigned to each particle** (point).

Density may be a function of
- **Point** (partilce) (inhomogeneous medium)
- **Time** (density changes due to deformation process)

In general the **density in reference configuration** $\rho_R(\mathbf{X})$ (which is independent of time) may be different from **density in current configuration** $\rho(\mathbf{x},t)$, however, according to the **principle of conservation of mass**, **total mass of the body cannot change due to deformation**:

$$m_{ref} = m_t \quad \Rightarrow \quad \iiint_{V_R} \rho_R \,\mathrm{d}V_R = \iiint_V \rho \,\mathrm{d}V$$

CONSERVATION OF MASS

$$m_{ref} = m_t \quad \Rightarrow \quad \iiint_{V_R} \rho_R \, \mathrm{d}V_R = \iiint_{V} \rho \, \mathrm{d}V$$

Let's make use of the known relation between $\mathrm{d}V$ and $\mathrm{d}V_R$:

$$\iiint_{V_R} \rho_R \, \mathrm{d}V_R = \iiint_{V_R} \rho J \, \mathrm{d}V_R$$

The above relation must hold true for any V_R , so it must be:

$$\boxed{\rho_R = \rho \cdot J}$$

CONTINUITY EQUATION

An amount of mass which flows through a closed surface S (in a point o **external unit normal vector** $\mathbf{n}$) with velocity $\mathbf{v}$, may be expressed be integration of the mass flux vector $\boldsymbol{\Phi}_m$ (mass flows from inside the considered surface):

$$\frac{\partial m}{\partial t} = \iint\limits_{\Delta S} \underbrace{\rho(\mathbf{x};t)\mathbf{v}(\mathbf{x};t)\cdot\mathbf{n}}_{\boldsymbol{\Phi}_m} \mathrm{d}S$$

Increment of mass contained in region V bounded by surface S is equal:

$$\frac{\partial m}{\partial t} = \iiint\limits_{\Delta V} \frac{\partial \rho}{\partial t} \mathrm{d}V$$

Amount of mass flowing through the boundary of region must be equal to the increment of mass within this boundary:

$$\iiint\limits_{\Delta V} \frac{\partial \rho}{\partial t} \mathrm{d}V = -\iint\limits_{\Delta S} \rho\,\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}S$$

RÓWNANIE CIĄGŁOŚCI

Volume integral may be transformed into a surface integral according to the Green – Gauss – Ostrogradski theorem:

$$\iiint_V \nabla\cdot\mathbf{f}\,\mathrm{d}V = \oiint_S \mathbf{f}\cdot\mathbf{n}\,\mathrm{d}S$$

We have then:

$$\iiint_{\Delta V} \frac{\partial\rho}{\partial t}\mathrm{d}V = -\iiint_{\Delta V} \nabla\cdot(\rho\,\mathbf{v})\mathrm{d}V$$

Integrals are additive:

$$\iiint_{\Delta V} \left[\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\,\mathbf{v})\right]\mathrm{d}V = 0$$

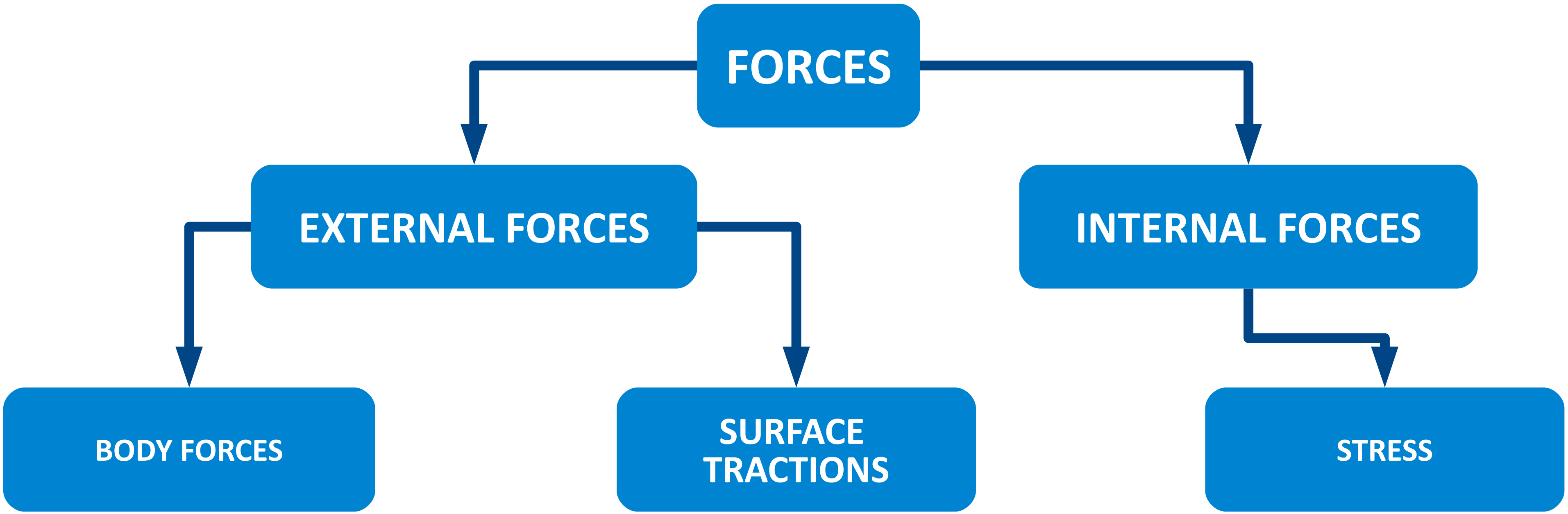
This relation must hold true for any region V, so

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\,\mathbf{v}) = 0 \qquad \Leftrightarrow \qquad \frac{\partial\rho}{\partial t}+(\rho\,v_k)_{,k} = 0$$

The above equation is termed continuity equation – it is fundamental equation of fluid dynamics.

EXTERNAL AND INTERNAL FORCES

FORCE POSTULATES



BODY FORCES

- Action of an **external environment** on the body.
- They **act directly on particles inside the body** (they penetrate the boundary of the body)
- Examples: gravity, EM field

SURFACE TRACTIONS

- Action of an **external environment** on the body.
- They **act directly only on particles at the surface of the body**
- Examples: contact stress, fluid pressure on a body which is immersed in that fluid

STRESS

- **Interaction between particles**.

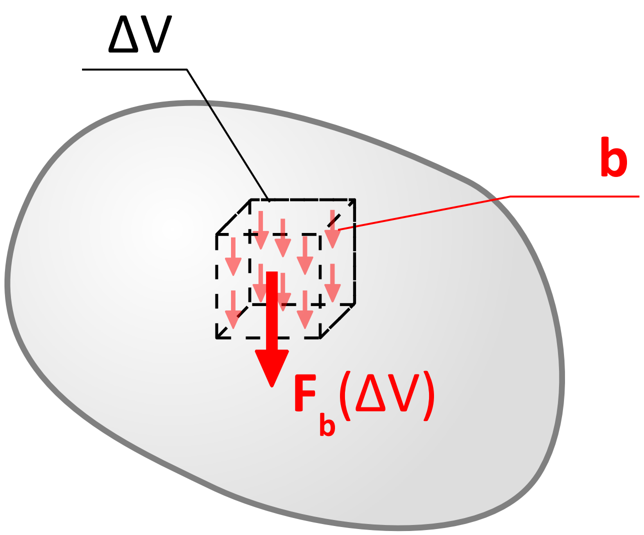
BODY FORCES

Body forces are described by body force density vector $\mathbf{b}(\mathbf{x},t)$ (or simply body forces vector), such that sum of such a continuous system of forces is equal

$$\mathbf{F}_b = \iiint\limits_{\Delta V} \mathbf{b}(\mathbf{x},t)\,\mathrm{d}V$$

Body force density vector may be interpreted as a **limit case** when the **sum of a continuous system of body forces** integrated over a certain region ΔV is then **divided by the size of that region** ΔV, and then this size tends in limit to 0: $\Delta V \to 0$

$$\mathbf{b}(\mathbf{x}) = \lim_{\Delta V \to 0} \frac{\mathbf{F}_b(\Delta V)}{\Delta V}$$



Physical dimension: $[\mathbf{b}] = \mathrm{N/m^3}$

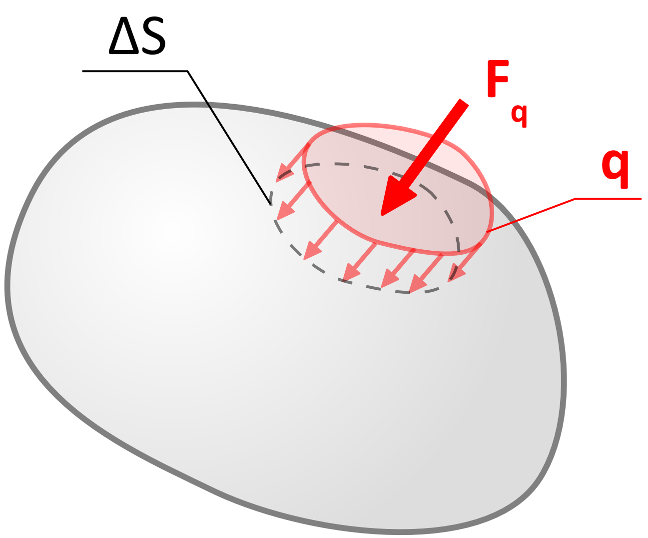
SURFACE TRACTIONS

Surface tractions are described by surface traction vector $\mathbf{q}(\mathbf{x},t)$, such that sum of such a continuous system of forces is equal

$$\mathbf{F}_q = \iint_{\Delta S} \mathbf{q}(\mathbf{x},t)\,\mathrm{d}S$$

Surface traction vector may be interpreted as a **limit case** when the **sum of a continuous system of surface tractions** integrated over a certain surface ΔS is then **divided by the area of that surface** ΔS, and then this area tends in limit to 0: $\Delta S \to 0$

$$\mathbf{q}(\mathbf{x}) = \lim_{\Delta S \to 0} \frac{\mathbf{F}_q(\Delta S)}{\Delta S}$$



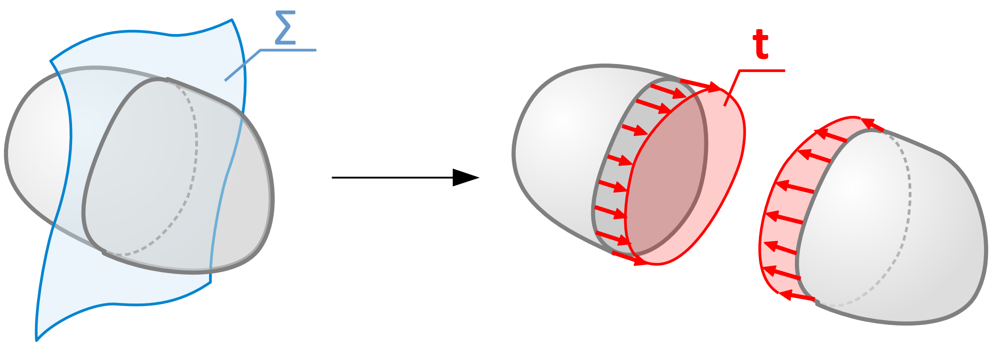
Physical dimension: $[\mathbf{q}] = \mathrm{N/m^2} = \mathrm{Pa}$

SURFACE DENSITY OF INTERNAL FORCES

The concept of internal forces emerge naturally if we assume that **the cause of any motion is presence of forces** and if we notice that **elastic bodies may perform motion** (e.g. elastic vibration) **even without presence of any external forces**. There must exist forces of another kind.

Our knowledge concerning atomic structure of matter allow us to interpret these forces in terms of interatomic interactions, however the model of continuum reject the concept of countable number of discrete particles of finite dimensions for which such forces could be determined. For these reasons we will rather consider the **surface density of internal forces**.

As we cannot distinguish discrete particles, let **distinguish a part of a body** in a different way. We could perform an **imaginary cut, which divides the body into to interacting parts** with the use of certain surface Σ.



The system of forces providing cohesion of the body will be termed internal surface forces or stresses – they depend both on the **chosen point** in a body as well as on the **choice of the surface** Σ of the imaginary cut.

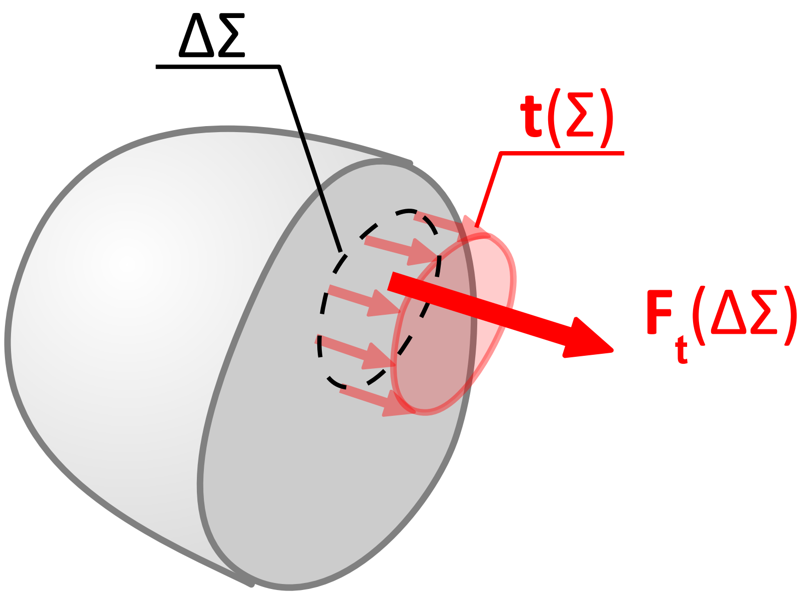
STRESS

Internal surface forces are described by internal surface force density vector $\mathbf{t}(\mathbf{x}, t)$ (stress vector), such that sum of such a continuous system of forces is equal

$$\mathbf{F}_t = \iint_{\Delta\Sigma} \mathbf{t}(\mathbf{x}, t)\, \mathrm{d}\Sigma$$

Stress vector may be interpreted as a **limit case** when the **sum of a continuous system of internal surface forces** integrated over a certain surface $\Delta\Sigma$ is then **divided by the area of that surface** $\Delta\Sigma$, and then this area tends in limit to 0: $\Delta\Sigma \to 0$

$$\mathbf{t}(\mathbf{x}, \Sigma) = \lim_{\Delta\Sigma \to 0} \frac{\mathbf{F}_t(\Delta\Sigma)}{\Delta\Sigma}$$



Physical dimension: $[\mathbf{t}] = \mathrm{N/m^2} = \mathrm{Pa}$

LAWS OF MOTION

LAWS OF MOTION

Theory of elasticity is a part of **classical mechanics**, what means, that **Newton's Laws of Motion** are in force.

1st LAW OF MOTION

There exist so called **inertial frames of reference**, in which, if no force act on a body or the acting forces are in equilibrium, than the body remains still or it performs uniform motion (along a straight line and with constant speed)

LAWS OF MOTION

Theory of elasticity is a part of **classical mechanics**, what means, that **Newton's Laws of Motion** are in force.

2nd LAW OF MOTION

In inertial frames of reference:

- concerning **translation**, the rate of change of momentum is equal to the sum of forces acting on the body (principle of momentum):

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint\limits_V \rho\,\mathbf{v}\,\mathrm{d}V = \mathbf{S} \qquad \Leftrightarrow \qquad \dot{\mathbf{P}} = \mathbf{S}$$

- concerning **rotation**, the rate of change of angular momentum about fixed point O is equal to the moment of force forces acting on the body about point O (principle of angular momentum):

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint\limits_V \rho\,\mathbf{v}\times(\mathbf{x}_O-\mathbf{x})\,\mathrm{d}V = \mathbf{M}_O \qquad \Leftrightarrow \qquad \dot{\mathbf{K}}_O = \mathbf{M}_O$$

LAWS OF MOTION

Theory of elasticity is a part of **classical mechanics**, what means, that **Newton's Laws of Motion** are in force.

3rd LAW OF MOTION

If a body A acts on body B with a certain force, then body B acts on body A with the force of the same magnitude, the same direction but opposite orientation.

EQUILIBRIUM OF INTERNAL FORCES

TETRAHEDRON CONDITIONS

Let's consider a small part of an elastic solid in the form of tetrahedron, such that 3 of its faces are perpendicular to the axes of considered coordinate system, while the last one is perpendicular to the unit normal vector $\mathbf{v}$ $(|\mathbf{v}| = 1)$.

$$\mathbf{v} = [\nu_1 \; ; \; \nu_2 \; ; \; \nu_3]$$

$$\mathbf{t} = [t_1 \; ; \; t_2 \; ; \; t_3]$$

$$\mathbf{t}_i = [t_{i1} \; ; \; t_{i2} \; ; \; t_{i3}]$$

$$\mathbf{b} = [b_1 \; ; \; b_2 \; ; \; b_3]$$

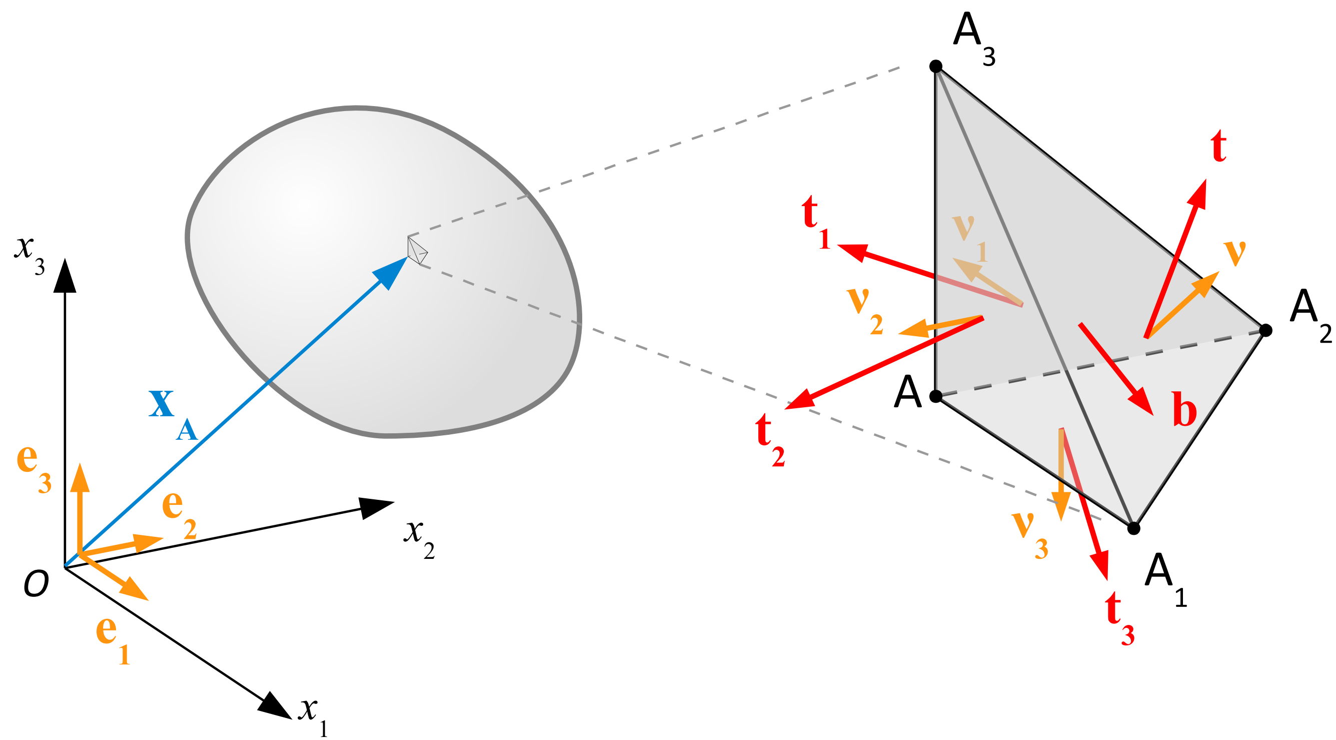
S – area of triangle $\Delta A_1 A_2 A_3$

S_{ij} – area of triangle $\Delta A A_i A_j$

h – height of the tetrahedron perpendicular to $\Delta A_1 A_2 A_3$

Volume of the tetrahedron:

$$V = \frac{1}{3} h S$$

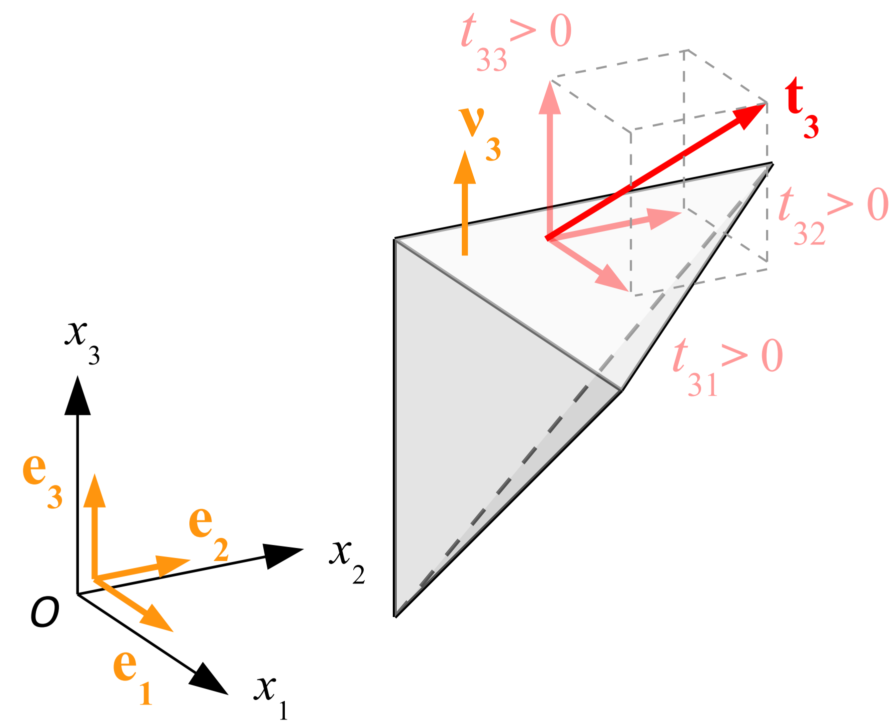


TETRAHEDRON CONDITIONS

Symbol t_{ij} denotes j-th component of the stress vector which is applied to the face which is perpendicular to the i-th axis of considered coordinate system.

We will introduce the following **convention concerning positive values of stress**:

If the **external normal** $\mathbf{v}_i$ of the face is oriented in the same way as **versor** $\mathbf{e}_i$ **of respective coordinate axis,** then the positive stress component is the one which is oriented in the same way **as versor of corresponding coordinate axis**.



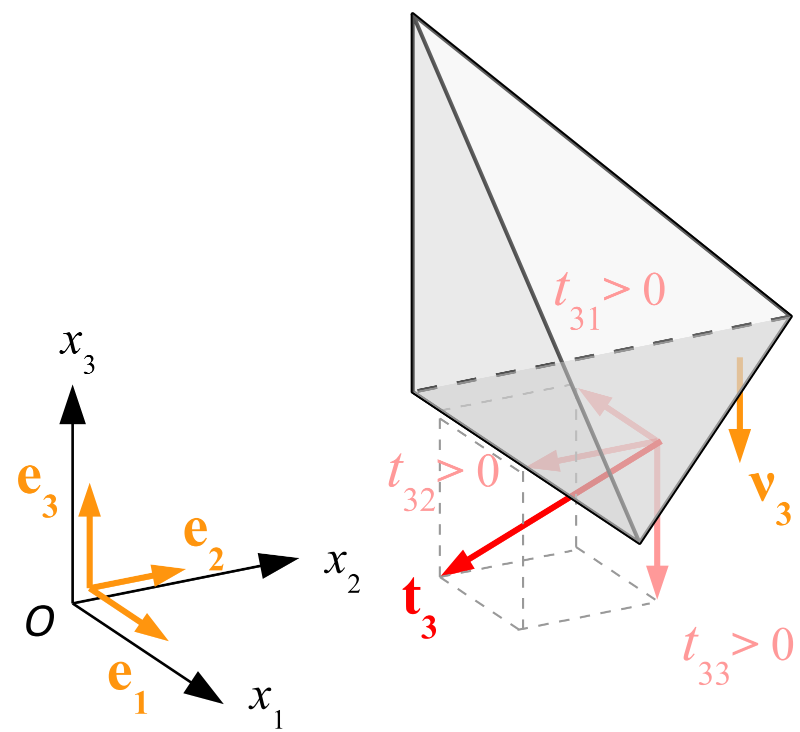
TETRAHEDRON CONDITIONS

Symbol t_{ij} denotes j-th component of the stress vector which is applied to the face which is perpendicular to the i-th axis of considered coordinate system.

We will introduce the following **convention concerning positive values of stress**:

If the **external normal** $\mathbf{v}_\mathbf{i}$ of the face is oriented an opposite way as **versor** $\mathbf{e}_\mathbf{i}$ **of respective coordinate axis,** then the positive stress component is the one which is oriented in an opposite way **as versor of corresponding coordinate axis**.

A stress which is normal to the face is <u>positive</u> when it is a <u>tensile stress</u>



TETRAHEDRON CONDITIONS

Let's write down the principle of momentum **for the tetrahedron**. Sum of all forces acting on it is equal:

$$\mathbf{S} = \iiint_V \mathbf{b}\,\mathrm{d}V + \iint_\Sigma \mathbf{t}\,\mathrm{d}S + \iint_{\Sigma_1} \mathbf{t_1}\,\mathrm{d}S + \iint_{\Sigma_2} \mathbf{t_2}\,\mathrm{d}S + \iint_{\Sigma_3} \mathbf{t_3}\,\mathrm{d}S$$

Time derivative of momentum: $\dot{\mathbf{P}} = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_V \rho\,\mathbf{v}\,\mathrm{d}V$

Time derivative concerns a function which is an integral over a configuration which also changes in time – this change should be accounted for. We can change the domain of integration:

$$\dot{\mathbf{P}} = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_V \rho\,\mathbf{v}\,\mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_{V_R} \rho\,\mathbf{v}\,J\,\mathrm{d}V_R$$

Reference configuration is constant in time – we may put the derivative sign inside the integral:

$$\dot{\mathbf{P}} = \iiint_{V_R} \frac{\mathrm{d}}{\mathrm{d}t}\left[\rho\,\mathbf{v}\,J\right]\mathrm{d}V_R$$

Let's use the **product rule**:

$$\dot{\mathbf{P}} = \iiint_{V_R} \left[\frac{\mathrm{d}}{\mathrm{d}t}(\rho\,J)\cdot\mathbf{v} + (\rho\,J)\cdot\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v} \right]\mathrm{d}V_R$$

TETRAHEDRON CONDITIONS

Time derivative of momentum: $\dot{\mathbf{P}} = \iiint\limits_{V_R} \frac{\mathrm{d}}{\mathrm{d}t}(\rho J)\cdot \mathbf{v}\,\mathrm{d}V_R + \iiint\limits_{V_R} (\rho J)\cdot \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}\,\mathrm{d}V_R$

Due to conservation of mass, we have $\rho J = \rho_R$, and also $\rho_R(t) = const$, so:

$$\dot{\mathbf{P}} = \iiint\limits_{V_R} \underbrace{\frac{\mathrm{d}}{\mathrm{d}t}(\rho_R)}_{=0}\cdot \mathbf{v}\,\mathrm{d}V_R + \iiint\limits_{V_R} \rho\cdot \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\mathbf{v}\,\underbrace{J\,\mathrm{d}V_R}_{\mathrm{d}V} = \iiint\limits_{V} \rho\,\mathbf{a}\,\mathrm{d}V$$

Momentum principle:

$$\dot{\mathbf{P}} = \mathbf{S} \quad \Leftrightarrow \quad \iiint\limits_{V} \rho\,\mathbf{a}\,\mathrm{d}V = \iiint\limits_{V} \mathbf{b}\,\mathrm{d}V + \iint\limits_{\Sigma} \mathbf{t}\,\mathrm{d}S + \iint\limits_{\Sigma_1} \mathbf{t}_1\,\mathrm{d}S + \iint\limits_{\Sigma_2} \mathbf{t}_2\,\mathrm{d}S + \iint\limits_{\Sigma_3} \mathbf{t}_3\,\mathrm{d}S$$

For each of the above integrals we can apply the **mean value theorem for integrals**, according to which, for any definite integral of continuous function $f(\mathbf{x})$ over region Ω of size $|\Omega|$ there exists a point $\mathbf{x}_0 \in \Omega$, such that

$$\int\limits_{\Omega} f(\mathbf{x})\,\mathrm{d}\Omega = f(\mathbf{x}_0)|\Omega|$$

TETRAHEDRON CONDITIONS

Principle of momentum:

$$\rho\,\mathbf{a}(\hat{\mathbf{x}})\cdot V = \mathbf{b}(\tilde{\mathbf{x}})\cdot V + \mathbf{t}(\breve{\mathbf{x}})\cdot S + \mathbf{t}_1(\mathbf{x}')\cdot S_{23} + \mathbf{t}_2(\mathbf{x}'')\cdot S_{31} + \mathbf{t}_3(\mathbf{x}''')\cdot S_{12}$$

Let's use the formula for the volume of the tetrahedron:

$$\rho\,\mathbf{a}(\hat{\mathbf{x}})\cdot \frac{1}{3}hS = \mathbf{b}(\tilde{\mathbf{x}})\cdot \frac{1}{3}hS + \mathbf{t}(\breve{\mathbf{x}})\cdot S + \mathbf{t}_1(\mathbf{x}')\cdot S_{23} + \mathbf{t}_2(\mathbf{x}'')\cdot S_{31} + \mathbf{t}_3(\mathbf{x}''')\cdot S_{12}$$

Let's divide both sides with S:

$$\rho\,\mathbf{a}(\hat{\mathbf{x}})\cdot \frac{h}{3} = \mathbf{b}(\tilde{\mathbf{x}})\cdot \frac{h}{3} + \mathbf{t}(\breve{\mathbf{x}}) + \mathbf{t}_1(\mathbf{x}')\cdot\frac{S_{23}}{S} + \mathbf{t}_2(\mathbf{x}'')\cdot\frac{S_{31}}{S} + \mathbf{t}_3(\mathbf{x}''')\cdot\frac{S_{12}}{S}$$

It can be shown that

$$\nu_1 = \frac{S_{23}}{S}, \qquad \nu_2 = \frac{S_{31}}{S}, \qquad \nu_3 = \frac{S_{12}}{S}$$

TETRAHEDRON CONDITIONS

Principle of momentum:

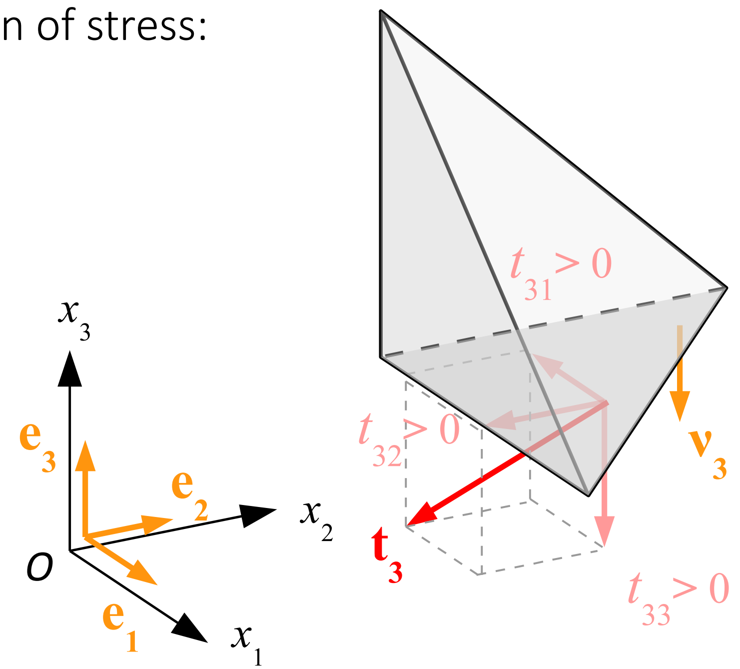
$$\rho\,\mathbf{a}(\hat{\mathbf{x}})\cdot\frac{h}{3} = \mathbf{b}(\tilde{\mathbf{x}})\cdot\frac{h}{3} + \mathbf{t}(\check{\mathbf{x}}) + \mathbf{t}_1(\mathbf{x}')\cdot\nu_1 + \mathbf{t}_2(\mathbf{x}'')\cdot\nu_2 + \mathbf{t}_3(\mathbf{x}''')\cdot\nu_3$$

Relation between components (index notation):

$$\rho\,a_i(\hat{\mathbf{x}})\cdot\frac{h}{3} = b_i(\tilde{\mathbf{x}})\cdot\frac{h}{3} + t_i(\check{\mathbf{x}}) - t_{1i}(\mathbf{x}')\cdot\nu_1 - t_{2i}(\mathbf{x}'')\cdot\nu_2 - t_{3i}(\mathbf{x}''')\cdot\nu_3 \qquad i=1,2,3$$

Minus signs are due to assumed convention concerning the sign of stress:

- Unit normals of the perpendicular faces are oriented in an opposite way as respective coordinate axes
- Positive stress components are also oriented in an opposite way.
- Integration of positive stresses result in forces which are oriented in an opposite was as the respective coordinate axes, to force components are **negative**.



TETRAHEDRON CONDITIONS

Let's consider a **limit case** $h \to 0$.

$$\lim_{h\to 0} \rho\, a_i(\hat{\mathbf{x}})\cdot\frac{h}{3} = \lim_{h\to 0}\left[b_i(\tilde{\mathbf{x}})\cdot\frac{h}{3} + t_i(\check{\mathbf{x}}) - t_{1i}(\mathbf{x}')\cdot\nu_1 - t_{2i}(\mathbf{x}'')\cdot\nu_2 - t_{3i}(\mathbf{x}''')\cdot\nu_3 \right] \qquad i=1,2,3$$

In such a case the intertial term as well as the body forces term vanish and points of the mean values all converge to a single point – the apex of the tetrahedron.

$$\hat{\mathbf{x}}, \tilde{\mathbf{x}}, \check{\mathbf{x}}, \mathbf{x}', \mathbf{x}'', \mathbf{x}''' \quad \overset{h\to 0}{\longrightarrow} \quad \mathbf{x}_A$$

We obtain:

$$t_i(\mathbf{x}_A) - t_{1i}(\mathbf{x}_A)\nu_1 - t_{2i}(\mathbf{x}_A)\nu_2 - t_{3i}(\mathbf{x}_A)\nu_3 = 0$$

This can be written in the following way:

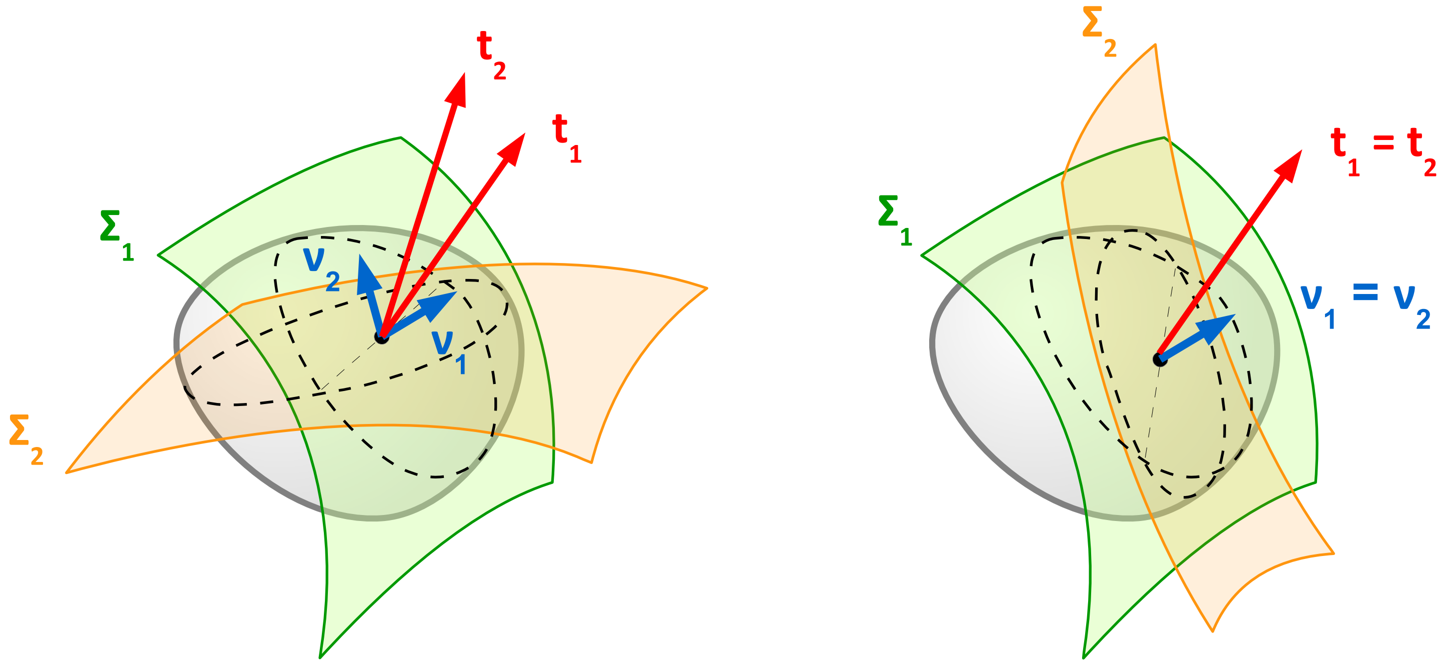
$$t_i = t_{ji}\nu_j \quad \Leftrightarrow \quad \mathbf{t} = (\mathbf{T}_\sigma)^{\mathrm{T}}\cdot\boldsymbol{\nu} \quad \Leftrightarrow \quad \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} t_{11} & t_{21} & t_{31} \\ t_{12} & t_{22} & t_{32} \\ t_{13} & t_{23} & t_{33} \end{bmatrix}\cdot\begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix}$$

It can be proved that object $\mathbf{T}_\sigma$ is a **tensor**. It will be termed the Cauchy stress tensor.

TETRAHEDRON CONDITIONS

CONCLUSIONS:

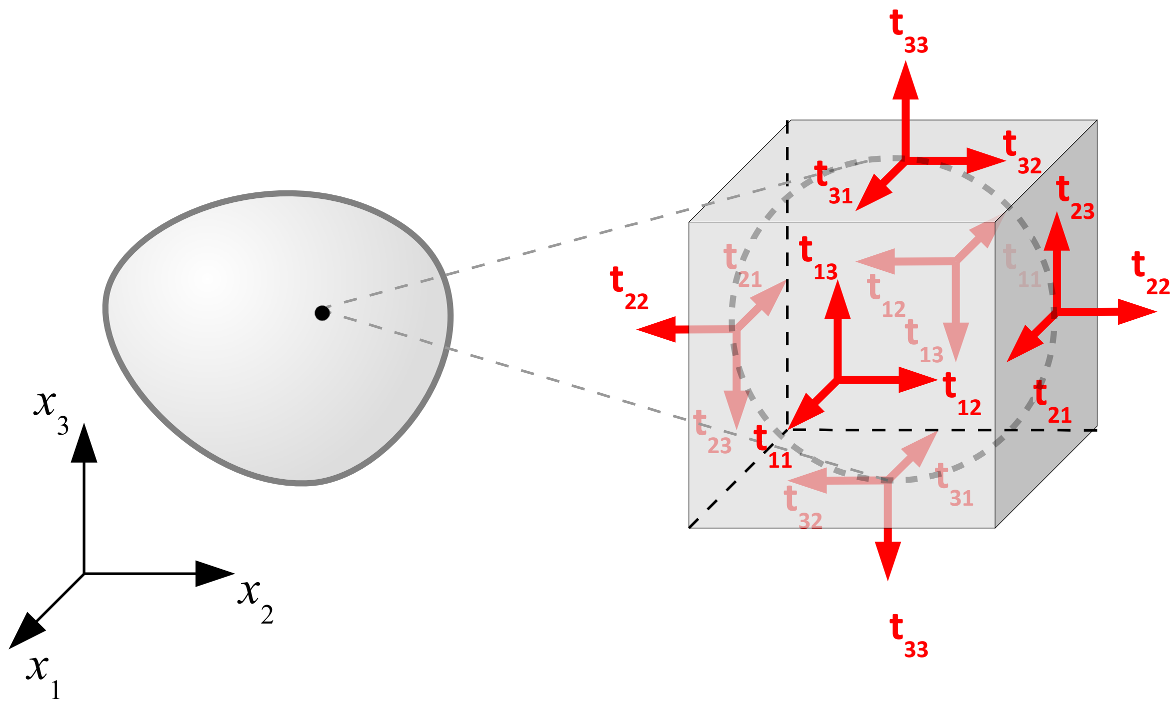
- **Stress vector** in a given point and for given surface of imaginary cut **does not on the surface itself but only on its normal vector** in the considered point.



- **In order to determine a stress vector** in given point and for given unit normal, it is enough to know the distribution od stress corresponding to the surfaces perpendicular to the considered coordinate axes, namely the **components of the stress tensor**. We say that it determines the stress state in the body.

CAUCHY STRESS TENSOR

Graphical illustration of physical interpretation of components of the stress tensor id as follows:



- **Diagonal components** t_{ij} $(i = j)$ will be termed normal stresses – these are tensile **(positive)** or compressive **(negative)** stresses.

- **Off-diagonal components** t_{ij} $(i \neq j)$ will be termed shear stresses.

EQUATIONS OF MOTION

EQUATIONS OF MOTION

Let's consider a small subregion of a continuum and let's write down the principle of momentum:

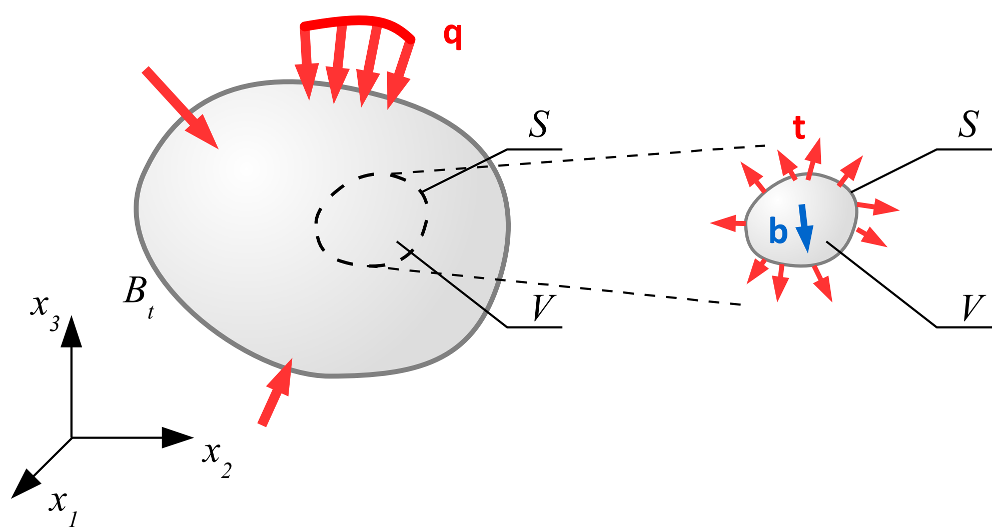
$$\dot{\mathbf{P}} = \mathbf{S} \quad \Leftrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} \iiint_V \rho \, \mathbf{v} \, \mathrm{d}V = \iiint_V \mathbf{b} \, \mathrm{d}V + \iint_S \mathbf{t} \, \mathrm{d}S$$

Time derivative of the momentum will be calculated as in the case of **tetrahedron conditions. Stress vector** will be expressed in terms of the **stress tensor** and a **unit normal** of surface S. bounding the considered region. In index notation:

$$\iiint_V \rho a_i \, \mathrm{d}V = \iiint_V b_i \, \mathrm{d}V + \iint_S t_{ji} \nu_j \, \mathrm{d}S$$

We shall now apply the **Green – Gauss – Ostrogradski theorem** to the third integral:

$$\iiint_V \rho a_i \, \mathrm{d}V = \iiint_V b_i \, \mathrm{d}V + \iiint_V t_{ji,j} \, \mathrm{d}V$$



EQUATIONS OF MOTION

Since integral is additive:

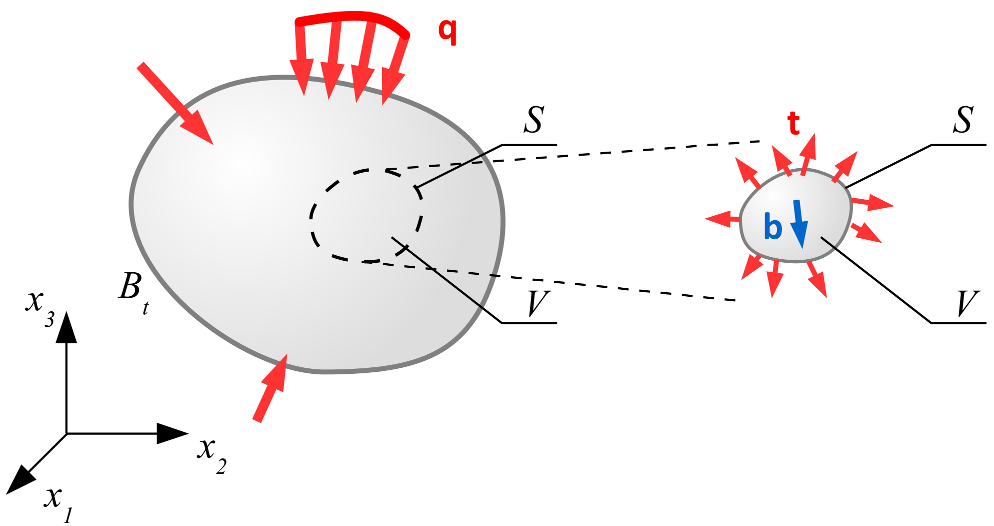
$$\iiint_V \left(\rho \ddot{u}_i - b_i - t_{ji,j}\right) \mathrm{d}V = 0$$

The above relation must hold true for any region V, therefore:

$$t_{ji,j} + b_i = \rho \ddot{u}_i \qquad i = 1,2,3$$

This is the **system of** equations of motion **of continuum** in spatial description.

CONCLUSION: **Cauchy stress tensor** is a proper measure of stress for the **spatial description** since in the equation of motion **it is differentiated with respect to spatial coordinates**.



EQUATIONS OF MOTION

Equations of motion:

$$t_{ji,j} + b_i = \rho \ddot{u}_i \quad \Leftrightarrow \quad \begin{cases} \dfrac{\partial t_{11}}{\partial x_1} + \dfrac{\partial t_{21}}{\partial x_2} + \dfrac{\partial t_{31}}{\partial x_3} + b_1 = \rho \dfrac{\partial^2 u_1}{\partial t^2} \\ \dfrac{\partial t_{12}}{\partial x_1} + \dfrac{\partial t_{22}}{\partial x_2} + \dfrac{\partial t_{32}}{\partial x_3} + b_2 = \rho \dfrac{\partial^2 u_2}{\partial t^2} \\ \dfrac{\partial t_{13}}{\partial x_1} + \dfrac{\partial t_{23}}{\partial x_2} + \dfrac{\partial t_{33}}{\partial x_3} + b_3 = \rho \dfrac{\partial^2 u_3}{\partial t^2} \end{cases}$$

This is a **system of 3 partial differential equations of the 2nd order** for **9 components of the stress tensor** and **3 components of the displacement vector**.

Together with kinematic relations (or strain compatibility conditions) binding 3 components of displacement vector and 6 components of the strain tensor, we have 9 equations for 18 unknowns.

If the right-hand side is assumed to be 0 (**no inertial forces** – **quasistatic problems**), the above equations are often termed the **Navier equilibrium equations**.

EQUATIONS OF MOTION

Let's write down the angular momentum principle for the considered subregion of continuum, assuming that the point about which the angular momentum and moment of forces is calculated is the **origin of the considered coordinate system** ($\mathbf{x}_o = \mathbf{0}$).

$$\dot{\mathbf{K}}_O = \mathbf{M}_O \quad \Leftrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\iiint_V \rho\, \mathbf{v}\times(-\mathbf{x})\,\mathrm{d}V = \iiint_V \mathbf{b}\times(-\mathbf{x})\,\mathrm{d}V + \iint_S \mathbf{t}\times(-\mathbf{x})\,\mathrm{d}S$$

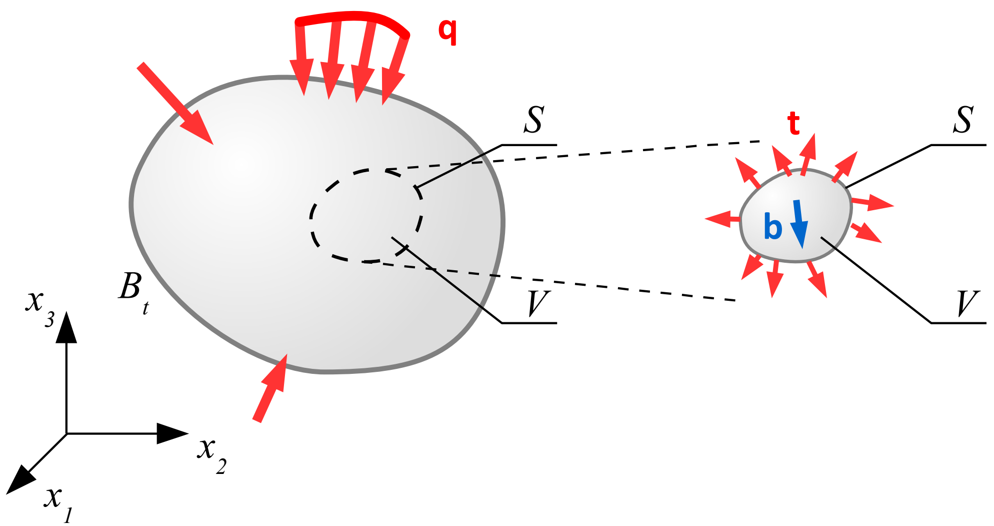
Time derivative from a function defined on a current configuration is calculated as previously:

$$\dot{\mathbf{K}}_O = -\frac{\mathrm{d}}{\mathrm{d}t}\iiint_V \rho\, \mathbf{v}\times\mathbf{x}\,\mathrm{d}V = -\iiint_V \rho\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}\times\mathbf{x})\,\mathrm{d}V$$

Product rule gives us:

$$\dot{\mathbf{K}}_O = -\iiint_V \rho(\dot{\mathbf{v}}\times\mathbf{x} + \underbrace{\mathbf{v}\times\dot{\mathbf{x}}}_{=\mathbf{v}\times\mathbf{v}=\mathbf{0}})\,\mathrm{d}V$$

$$\dot{\mathbf{K}}_O = -\iiint_V \rho\,\mathbf{a}\times\mathbf{x}\,\mathrm{d}V$$



EQUATIONS OF MOTION

Principle of angular momentum:

$$\iiint_V \rho \mathbf{a} \times \mathbf{x} \, \mathrm{d}V = \iiint_V \mathbf{b} \times \mathbf{x} \, \mathrm{d}V + \iint_S \mathbf{t} \times \mathbf{x} \, \mathrm{d}S$$

In index notation:

$$\iiint_V \rho \epsilon_{ijk} a_j x_k \, \mathrm{d}V = \iiint_V \epsilon_{ijk} b_j x_k \, \mathrm{d}V + \iint_S \epsilon_{ijk} t_j x_k \, \mathrm{d}S \qquad i = 1,2,3$$

Let's account for the tetrahedron conditions:

$$\iiint_V \rho \epsilon_{ijk} a_j x_k \, \mathrm{d}V = \iiint_V \epsilon_{ijk} b_j x_k \, \mathrm{d}V + \iint_S \epsilon_{ijk} t_{lj} \nu_l x_k \, \mathrm{d}S$$

Let's use the Green – Gauss – Ostrogradski theorem:

$$\iiint_V \rho \epsilon_{ijk} a_j x_k \, \mathrm{d}V = \iiint_V \epsilon_{ijk} b_j x_k \, \mathrm{d}V + \iiint_V \left(\epsilon_{ijk} t_{lj} x_k \right)_{,l} \mathrm{d}V$$

Let's deal with the last integral.

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Permutation symbol in the integrand **is not affected by differentiation**. The rest of the integrand again the product rule may be applied:

$$\left(\epsilon_{ijk} t_{lj} x_k\right)_{,l} = \epsilon_{ijk}\left(t_{lj} x_k\right)_{,l} = \epsilon_{ijk}\left(t_{lj,l} x_k + t_{lj} x_{k,l}\right) = \epsilon_{ijk}\left(t_{lj,l} x_k + t_{lj}\delta_{kl}\right) = \epsilon_{ijk}\left(t_{lj,l} x_k + t_{kj}\right)$$

Obtained result is now substituted into the angular momentum principle:

$$\iiint\limits_V \rho\,\epsilon_{ijk} a_j x_k \,\mathrm{d}V = \iiint\limits_V \epsilon_{ijk} b_j x_k \,\mathrm{d}V + \iiint\limits_V \left(\epsilon_{ijk} t_{lj,l} x_k + \epsilon_{ijk} t_{kj}\right)\mathrm{d}V$$

The first integral on the right-hand side is put on the other side and additivity of integrals is accounted for:

$$\iiint\limits_V \epsilon_{ijk} x_k \underbrace{\left(\rho a_j - b_j - t_{lj,l}\right)}_{0}\mathrm{d}V = \iiint\limits_V \epsilon_{ijk} t_{kj}\,\mathrm{d}V$$

Integrand on the left-hand side must be equal to 0, since the equations of motion (momentum principle) must be satisfied. We're left with:

$$\iiint\limits_V \epsilon_{ijk} t_{kj}\,\mathrm{d}V = 0$$

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This relation must be satisfied for any region V, therefore:

$$\epsilon_{ijk} t_{kj} = 0$$

Summation may be explicitly expanded:

$$i=1: \quad \epsilon_{1jk} t_{kj} = t_{32} - t_{23} = 0 \quad \Rightarrow \quad t_{23} = t_{32}$$
$$i=2: \quad \epsilon_{2jk} t_{kj} = t_{13} - t_{31} = 0 \quad \Rightarrow \quad t_{31} = t_{13}$$
$$i=3: \quad \epsilon_{3jk} t_{kj} = t_{21} - t_{12} = 0 \quad \Rightarrow \quad t_{12} = t_{21}$$

CONCLUSION: The consequence of the fact that the **angular momentum principle must be saisfied** is that the Cauchy stress tensor **must be a** symmetric tensor:

$$t_{ij} = t_{ji}$$

BOUNDARY AND INITIAL CONDITIONS

BOUNDARY AND INITIAL CONDITIONS

- System of equations of motion is a system of **differential equations**. In order to obtain a **unique solution** it is necessary to formulate additional **boundary** and **initial conditions**.

- It is a system depending on temporal variable (2nd order derivatives) and spatial variables (1st order derivatives). It is necessary to formulate:

  - 2 initial conditions (corresponding with with differentiation wrt **temporal variable**)
    - Initial position – condition on **displacement** field (**unknown functions**) in initial instant of time
    - Initial velocity – condition on the **1st derivatives of unknown functions** in initial instant of time

  - Boundary conditions at the boundary – surface bounding the configuration (corresponding with with differentiation wrt **spatial variables**). We distinguish two types of such conditions:
    - Kinematic boundary conditions – prescribed displacements of points of the boundary (supports)
    - Static boundary conditions – prescribed surface traction at points of the boundary. These are conditions on spatial derivatives of displacements – these derivatives are strains, strain can be expressed in terms of stress (as we will show soon), and stress is related to the surface traction.

REMARK: If we prescribe conditions only on derivatives (velocity, stress) then the solution will be non-unique.

BOUNDARY AND INITIAL CONDITIONS

INITIAL CONDITIONS

- Initial position of each particle in instant t_0: $$\mathbf{u}(\mathbf{x}, t_0) = \mathbf{u_0}(\mathbf{x})$$
- Initial velocity of each particle in instant t_0: $$\dot{\mathbf{u}}(\mathbf{x}, t_0) = \mathbf{v_0}(\mathbf{x})$$

KINEMATIC BOUNDARY CONDITIONS

- Prescribed displacement of particles at each instant of time at part of the boundary S_u:

$$\mathbf{u}(\mathbf{x}, t) = \hat{\mathbf{u}}_0(\mathbf{x}, t) \qquad \mathbf{x} \in S_u$$

BOUNDARY AND INITIAL CONDITIONS

In order to formulate static boundary conditions, let's write down the principle of momentum **for the whole body**:

$$\dot{\mathbf{P}} = \mathbf{S} \quad \Leftrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\iiint_{B_t} \rho \mathbf{v}\,\mathrm{d}V = \iiint_{B_t} \mathbf{b}\,\mathrm{d}V + \iint_{S_t} \mathbf{q}\,\mathrm{d}S$$

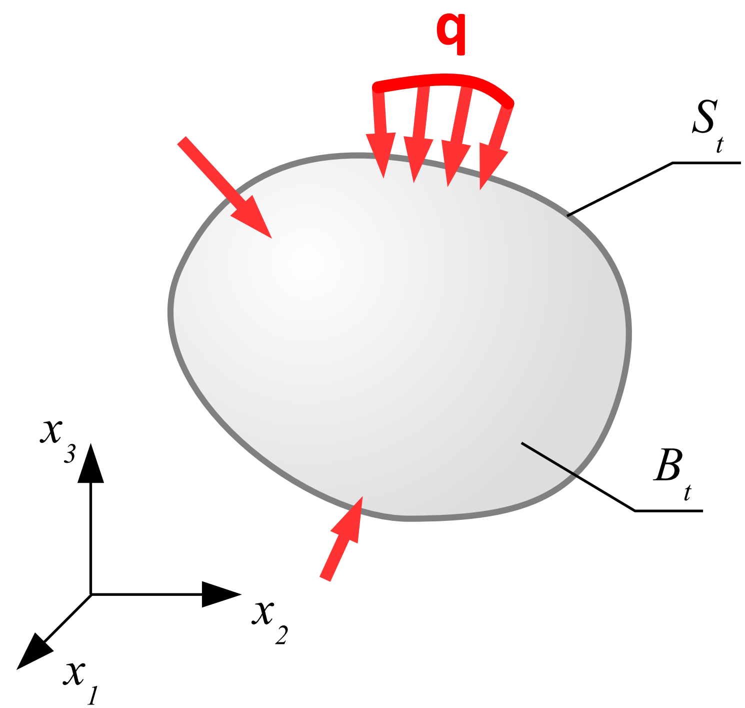
Time derivative of the function depending on current configuration is calculated as previously. In index notation:

$$\iiint_{B_t} \rho \ddot{u}_i\,\mathrm{d}V = \iiint_{B_t} b_i\,\mathrm{d}V + \iint_{S_t} q_i\,\mathrm{d}S$$

Body forces vector may be expressed via the substitution of equations of motion:

$$t_{ji,j} + b_i = \rho \ddot{u}_i \quad \Rightarrow \quad b_i = \rho \ddot{u}_i - t_{ji,j}$$

$$\iiint_{B_t} \rho \ddot{u}_i\,\mathrm{d}V = \iiint_{B_t} (\rho \ddot{u}_i - t_{ji,j})\,\mathrm{d}V + \iint_{S_t} q_i\,\mathrm{d}S$$



BOUNDARY AND INITIAL CONDITIONS

The first integral on the right-hand side is put on the other side of equation and an account for **additivity of integrals** is made:

$$\iiint_{B_t} t_{ji,j}\,\mathrm{d}V = \iint_{S_t} q_i\,\mathrm{d}S$$

We will now apply the **Green – Gauss – Ostrogradski theorem** to the integral on the left-hand side:

$$\iint_{S_t} t_{ji}\nu_j\,\mathrm{d}S = \iint_{S_t} q_i\,\mathrm{d}S \quad \Rightarrow \quad \iint_{S_t} (t_{ji}\nu_j - q)\,\mathrm{d}S = 0$$

This relation **must hold true for any surface** S_t, so:

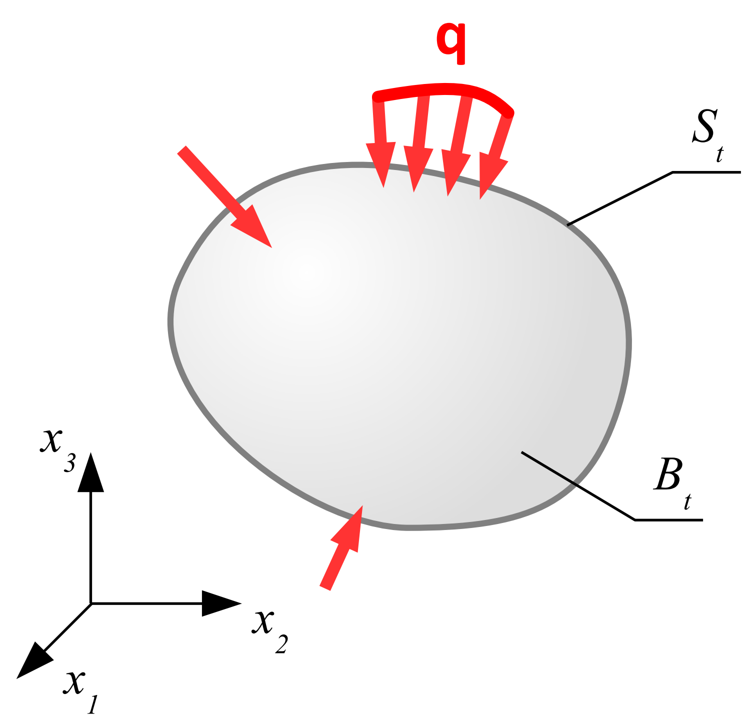
$$t_{ji}\nu_j - q = 0$$

Finally, we obtain:

STATIC BOUNDARY CONDITIONS

- Prescribed surface traction at the boundary S_t:

$$\mathbf{T}_{\boldsymbol{\sigma}}(\mathbf{x})^{\mathrm{T}}\cdot\boldsymbol{\nu}(\mathbf{x}) = \mathbf{q}(\mathbf{x}) \qquad \mathbf{x}\in S_t$$



BOUNDARY AND INITIAL CONDITIONS

REMARKS:

- Boundary of any continuum may be divided into two parts:
  - S_u part, on which **displacements are prescribed** in each instant of time (supports or kinematic excitation)
  - S_q part, on which **surface tractions are prescribed** in each instant of time
- If "nothing happens" at certain part of the boundary (free boundary) then this is in fact a boundary for which zero surface tractions are prescribed – they correspond with **homogeneous** (**zero**) **static boundary conditions**.

THANK YOU FOR YOUR ATTENTION