# THEORY OF ELASTICITY AND PLASTICITY

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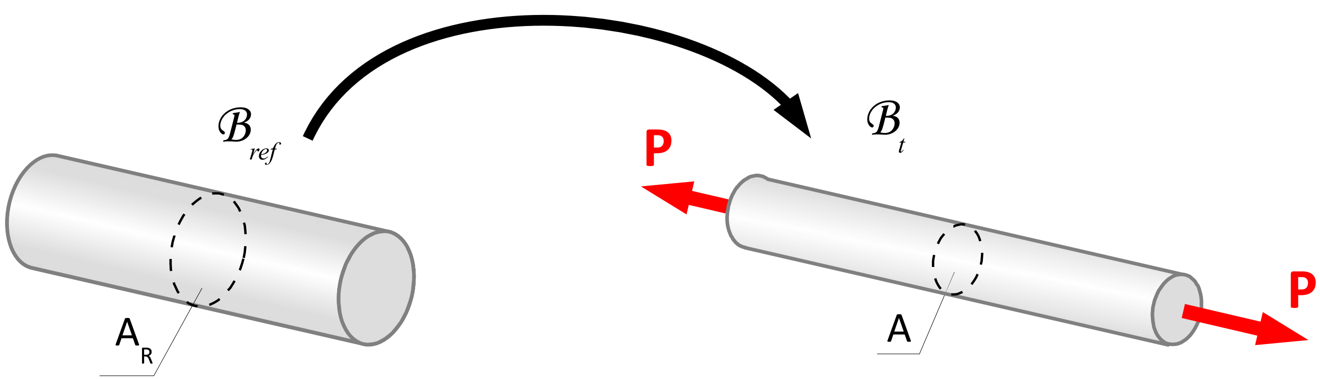
# DYNAMICS IN THE MATERIAL DESCRIPTION

# NOMINAL STRESS

- We would like to have equations of motion in the material description.
- **Equations of motion** describe local **equilibrium between stress, body forces and fictious inertial forces**.
- The Cauchy stress tensor is a measure of true stress – surface density of actual internal surface forces related to the deformed surface area (current configuration). For this reason it is termed also true stress tensor.
- **In the material description** the domain of the spatial variables is the reference configuration. We need such a measure of internal forces, which could be used in such a domain – it is a density of **actual internal surface forces** related to the **original** (**undeformed**) **surface elements' areas**. Such stress is termed a nominal stress.




true stress

$$t = \frac{P}{A}$$

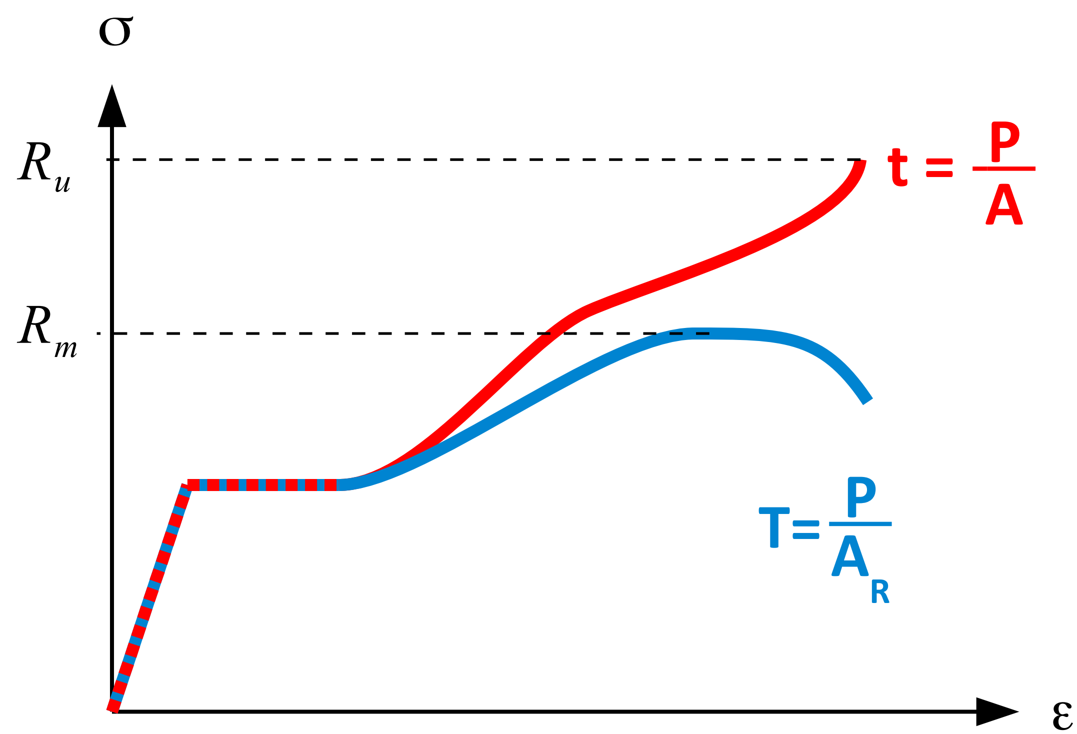
nominal stress

$$T = \frac{P}{A_R}$$

# NOMINAL STRESS

$R_u$ – true rupture stress

$R_m$ – strength of material




**true stress**

$$t = \frac{P}{A}$$

**nominal stress**

$$T = \frac{P}{A_R}$$

REMARKS:

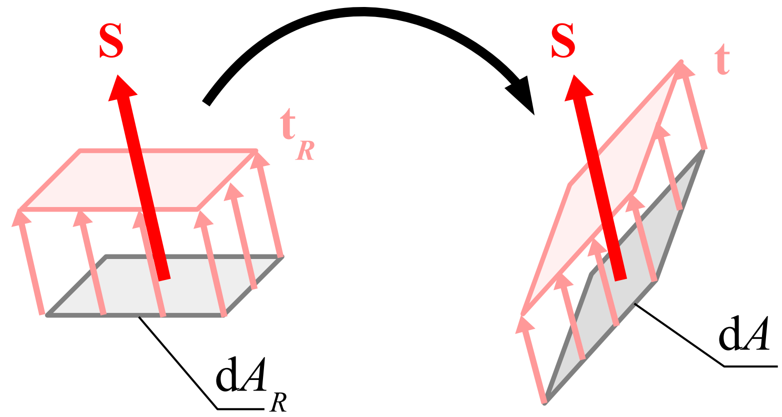
- In the formulae used in the strength of materials, we're not accounting for the fact that area of cross-section or its 2nd moment of area change during the deformation:

$$\sigma = \frac{N}{A} + \frac{M_y}{I_y} z - \frac{M_z}{I_z} y$$

- Similar formulae are used in code recommendations, co construction codes usually use the nominal stress.

# NOMINAL STRESS

Let's introduce the nominal stress vector $\mathbf{t}_{\boldsymbol{R}}$ defined in such a way that the **sum of true stresses integrated over an infinitely small deformed surface element** was the same as the **sum of nominal stresses integrated over the same surface element before deformation**:




$$\mathbf{S} = \mathbf{t}\,\mathrm{d}A = \mathbf{t}_{\boldsymbol{R}}\,\mathrm{d}A_R$$

Let's express the **true stress vector** in terms of **Cauchy stress tensor**:

$$(\mathbf{T}_{\boldsymbol{\sigma}})^{\mathrm{T}}\mathbf{n}\,\mathrm{d}A = \mathbf{t}_{\boldsymbol{R}}\,\mathrm{d}A_R$$

The vector $\mathbf{n}\,\mathrm{d}A$ is transformed according to the **Nanson formula**:

$$\mathbf{n}\,\mathrm{d}A = J\,\mathbf{N}\mathbf{F}^{-1}\mathrm{d}A_R \quad \Leftrightarrow \quad n_i\mathrm{d}A = J\,N_j f_{ji}\mathrm{d}A_R$$

$$\mathbf{n}\,\mathrm{d}A = J\,(\mathbf{F}^{-1})^{\mathrm{T}}\mathbf{N}\,\mathrm{d}A_R \quad \Leftrightarrow \quad n_i\mathrm{d}A = J\,f_{ji}N_j\mathrm{d}A_R$$

# NOMINAL STRESS

As a result, we obtain:

$$J\,\mathbf{T}_{\boldsymbol{\sigma}}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}\mathbf{N}\,\mathrm{d}A_R = \mathbf{t}_{\boldsymbol{R}}\,\mathrm{d}A_R$$

This relation must hold true for any surface element $\mathrm{d}A_R$, so:

$$J\,\mathbf{T}_{\boldsymbol{\sigma}}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}\mathbf{N} = \mathbf{t}_{\boldsymbol{R}}$$

In an analogous way as in case of relation of the Cauchy stress tensor and true stress vector, making use of the **symmetry of the Cauchy stress tensor** $\mathbf{T}_{\boldsymbol{\sigma}}^{\mathrm{T}} = \mathbf{T}_{\boldsymbol{\sigma}}$, we may write:

| | | |
|---|---|---|
| $\mathbf{T}_{\boldsymbol{R}}\mathbf{N} = \mathbf{t}_{\boldsymbol{R}}$ | where | $\mathbf{T}_{\boldsymbol{R}} = J\,\mathbf{T}_{\boldsymbol{\sigma}}\mathbf{F}^{-\mathrm{T}}$ |
| $T_{ij}N_j = T_i$ | where | $T_{ij} = J\,t_{ik}\,f_{jk}$ |

The tensor $\mathbf{T}_{\boldsymbol{R}}$ is termed the Piola – Kirchhoff stress tensor of the 1st kind or the nominal stress tensor. Since the tensor $\mathbf{F}^{-1}$ is **not symmetric**, so also the **PK1 stress tensor is not symmetric**.

# EQUATIONS OF MOTION IN THE MATERIAL DESCRIPTION

Let's consider a small sub region of the reference configuration of a continuum.

- Interaction of this subregion with the rest of the body is provided by the stress, whichin reference configuration is described by a nominal stress vector.
- It is easy to finde a **relation between body forces in the reference configuration and in the current configuration**. Sum of the forces must be the same in both configurations:

$$\iiint_V b_i \,\mathrm{d}V = \iiint_{V_R} B_i \,\mathrm{d}V_R$$

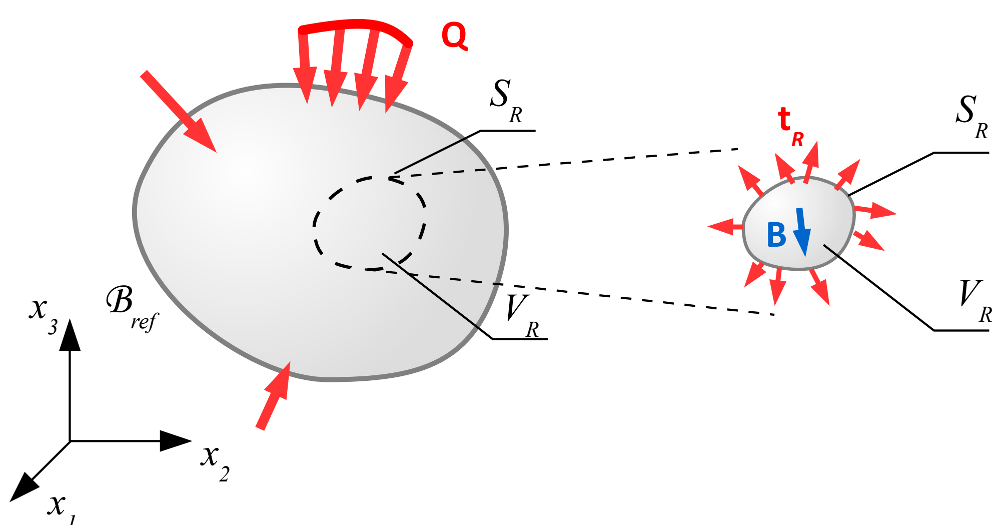
$$\iiint_{V_R} J\, b_i \,\mathrm{d}V_R = \iiint_{V_R} B_i \,\mathrm{d}V_R \quad \forall_{V} R$$

$$J\, b_i = B_i$$

In case of a load due to gravity:

$$b_i = \rho\, g_i$$

$$B_i = J\, b_i = J\,\rho\, g_i = \rho_R\, g_i$$



# EQUATIONS OF MOTION IN THE MATERIAL DESCRIPTION

Let's write down the principle of momentum:

$$\dot{\mathbf{P}} = \mathbf{S} \quad \Leftrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V_R} \rho_R v_i \,\mathrm{d}V_R = \iiint_{V_R} B_i \,\mathrm{d}V_R + \iint_{S_R} T_i \,\mathrm{d}S_R \qquad i=1,2,3$$

Domain of integration is time-independent, so we may put the time derivative symbol inside the integral.

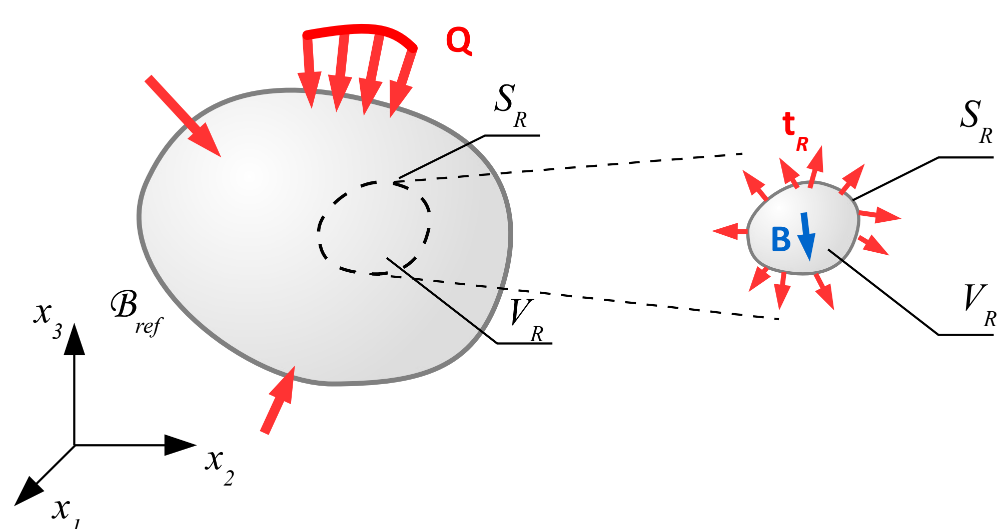
$$\iiint_{V_R} \frac{\mathrm{d}}{\mathrm{d}t}\rho_R v_i \,\mathrm{d}V_R = \iiint_{V_R} B_i \,\mathrm{d}V_R + \iint_{S_R} T_i \,\mathrm{d}S_R$$

Reference density is also time-independent:

$$\iiint_{V_R} \rho_R a_i \,\mathrm{d}V_R = \iiint_{V_R} B_i \,\mathrm{d}V_R + \iint_{S_R} T_i \,\mathrm{d}S_R$$

Nominal stress is expressed in terms of the PK1 tensor:

$$\iiint_{V_R} \rho_R a_i \,\mathrm{d}V_R = \iiint_{V_R} B_i \,\mathrm{d}V_R + \iint_{S_R} T_{ij} N_j \,\mathrm{d}S_R$$



# EQUATIONS OF MOTION IN THE MATERIAL DESCRIPTION

We're using the **Green – Gauss – Ostrogradski theorem**:

$$\iiint_{V_R} \rho_R a_i \,\mathrm{d}V_R = \iiint_{V_R} B_i \,\mathrm{d}V_R + \iiint_{V_R} T_{ij,j} \,\mathrm{d}V_R$$

Due to additivity of integrals:

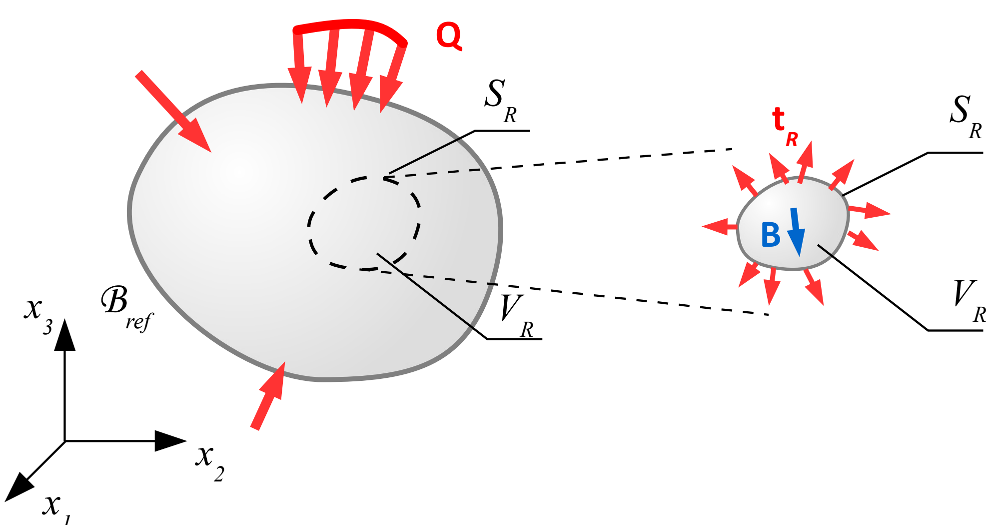
$$\iiint_{V_R} \left(\rho_R a_i - B_i - T_{ij,j}\right) \mathrm{d}V_R = 0$$

This relation **must hold true for any subregion** $V_R$ :

$$T_{ij,j} + B_i = \rho_R \ddot{u}_i \qquad i=1,2,3$$

We've obtained equations of motion **of a continuum** in the material description.

Still, there is a tiny problem...



# EQUATIONS OF MOTION IN THE MATERIAL DESCRIPTION

- **Equations of motion** guarantee satisfying the **principle of momentum**.
- In the **spatial description satisfying of the principle of angular momentum** was provided by the **symmetry of the true stress tensor**.
- In the **material description**, the **nominal stress tensor** that we're using is **not symmetric**. It has 9 independent components and we have only 3 equations of motion.
- Satisfying the principle of angular momentum will be provided by writing down the symmetry relations of the true stress tensor expressed in terms of the nominal stress tensor:

$$\mathbf{T}_R = J\,\mathbf{T}_\sigma \mathbf{F}^{-\mathrm{T}} \quad \Leftrightarrow \quad \mathbf{T}_\sigma = \frac{1}{J}\mathbf{T}_R \mathbf{F}^{\mathrm{T}}$$

According to the principle of angular momentum:

$$\mathbf{T}_\sigma = \mathbf{T}_\sigma^{\mathrm{T}} \quad \Rightarrow \quad \frac{1}{J}\mathbf{T}_R \mathbf{F}^{\mathrm{T}} = \frac{1}{J}\left(\mathbf{T}_R \mathbf{F}^{\mathrm{T}}\right)^{\mathrm{T}} \quad \Rightarrow \quad \mathbf{T}_R \mathbf{F}^{\mathrm{T}} = \mathbf{F}\,\mathbf{T}_R^{\mathrm{T}}$$

$$T_{ik}\frac{\partial x_j}{\partial X_k} = \frac{\partial x_i}{\partial X_k} T_{jk} \qquad i,j=1,2,3 \quad i \neq j$$

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# EQUATIONS OF MOTION IN THE MATERIAL DESCRIPTION

System of equations of motion in the material description:

$$\begin{cases} T_{ij,j} + B_i = \rho_R \ddot{u}_i & i = 1,2,3 \\[2ex] T_{ik} \dfrac{\partial x_j}{\partial X_k} = \dfrac{\partial x_i}{\partial X_k} T_{jk} & \begin{matrix} i,j = 1,2,3 \\ i \neq j \end{matrix} \end{cases}$$

Initial conditions:

- Initial position: $\mathbf{u}(\mathbf{X}, t_0) = \mathbf{u_0}(\mathbf{X})$
- Initial velocity: $\dot{\mathbf{u}}(\mathbf{X}, t_0) = \mathbf{v_0}(\mathbf{X})$

Boundary conditions:

- Kinematic boundary conditions: $\mathbf{u}(\mathbf{X}, t) = \hat{\mathbf{u}}_0(\mathbf{X}, t)$ for $\mathbf{X} \in S_u$
- Static boundary conditions: $\mathbf{Q} = \mathbf{T}_R \mathbf{N}$ for $\mathbf{X} \in S_q$

# EQUATIONS OF MOTION IN THE MATERIAL DESCRIPTION

The number of unknowns in the equations of motion may be decreased by introducing a new measure of stress.

We're defining the Piola – Kirchhoff stress tensor of the 2nd kind (material stress tensor) as follows:

$$\mathbf{T}_S = \mathbf{F}^{-1}\mathbf{T}_R = J\,\mathbf{F}^{-1}\mathbf{T}_\sigma\mathbf{F}^{-T} \quad \Leftrightarrow \quad S_{ij} = f_{ik}T_{kj} = J\,f_{ik}t_{kl}f_{jl}$$

If we express the PK2 stress tensor in terms of the Cauchy (true) stress tensor, we will obtain:

$$(\mathbf{T}_S)^T = J(\mathbf{F}^{-1}\mathbf{T}_\sigma\mathbf{F}^{-T})^T = J[\mathbf{F}^{-1}(\mathbf{T}_\sigma\mathbf{F}^{-T})]^T = J(\mathbf{T}_\sigma\mathbf{F}^{-T})^T\mathbf{F}^{-T} =$$
$$= J(\mathbf{F}^{-T})^T\mathbf{T}_\sigma^{\ T}\mathbf{F}^{-T} = J\,\mathbf{F}^{-1}\mathbf{T}_\sigma\mathbf{F}^{-T} = \mathbf{T}_S$$

The tensor is thus **symmetric**. If we express the PK1 stress tensor in terms of the PK2 stress tensor and substitute this relation in the relations derived from the principle of angular momentum, then:

$$\left.\begin{array}{r}\mathbf{T}_R\mathbf{F}^T = \mathbf{F}\,\mathbf{T}_R^T \\ \mathbf{T}_R = \mathbf{F}\,\mathbf{T}_S\end{array}\right\} \Rightarrow \quad \mathbf{F}\,\mathbf{T}_S\mathbf{F}^T = \mathbf{F}(\mathbf{F}\,\mathbf{T}_S)^T \quad \Rightarrow \quad \mathbf{F}\,\mathbf{T}_S\mathbf{F}^T = \mathbf{F}\,\mathbf{T}_S^T\mathbf{F}^T \quad \Rightarrow \quad \mathbf{F}\,\mathbf{T}_S\mathbf{F}^T = \mathbf{F}\,\mathbf{T}_S\mathbf{F}^T$$

We have obtained an **identity**. The relation is **always true** and the **principle of angular momentum is satisfied**.

# EQUATIONS OF MOTION IN THE MATERIAL DESCRIPTION

The relation between PK1 and PK2 stress tensor in an index notation is as follows:

$$T_{ij} = F_{ik} S_{kj} = x_{i,k} S_{kj} = (X_i + u_i)_{,k} S_{kj} = (X_{i,k} + u_{i,k}) S_{kj} = (\delta_{ik} + u_{i,k}) S_{kj} = S_{ij} + u_{i,k} S_{kj}$$

Equations of motion may be now expressed in terms of the PK2 stress tensor:

$$T_{ij,j} + B_i = \rho_R \ddot{u}_i$$

$$\left[ S_{ij} + u_{i,k} S_{kj} \right]_{,j} + B_i = \rho_R \ddot{u}_i$$

$$S_{ij,j} + u_{i,kj} S_{kj} + u_{i,k} S_{kj,j} + B_i = \rho_R \ddot{u}_i$$

- The equations are much more complicated (and **nonlinear**!)
- These are equations fo 3 unknown displacements and 6 unknown stress tensor components
- They need also **initial** and **boundary conditions**.
- Static boundary conditions:

$$\mathbf{Q} = \mathbf{T}_R \mathbf{N} = \mathbf{F} \mathbf{T}_S \mathbf{N} \quad \text{for} \quad \mathbf{X} \in S_q$$

Still, there is a tiny problem...

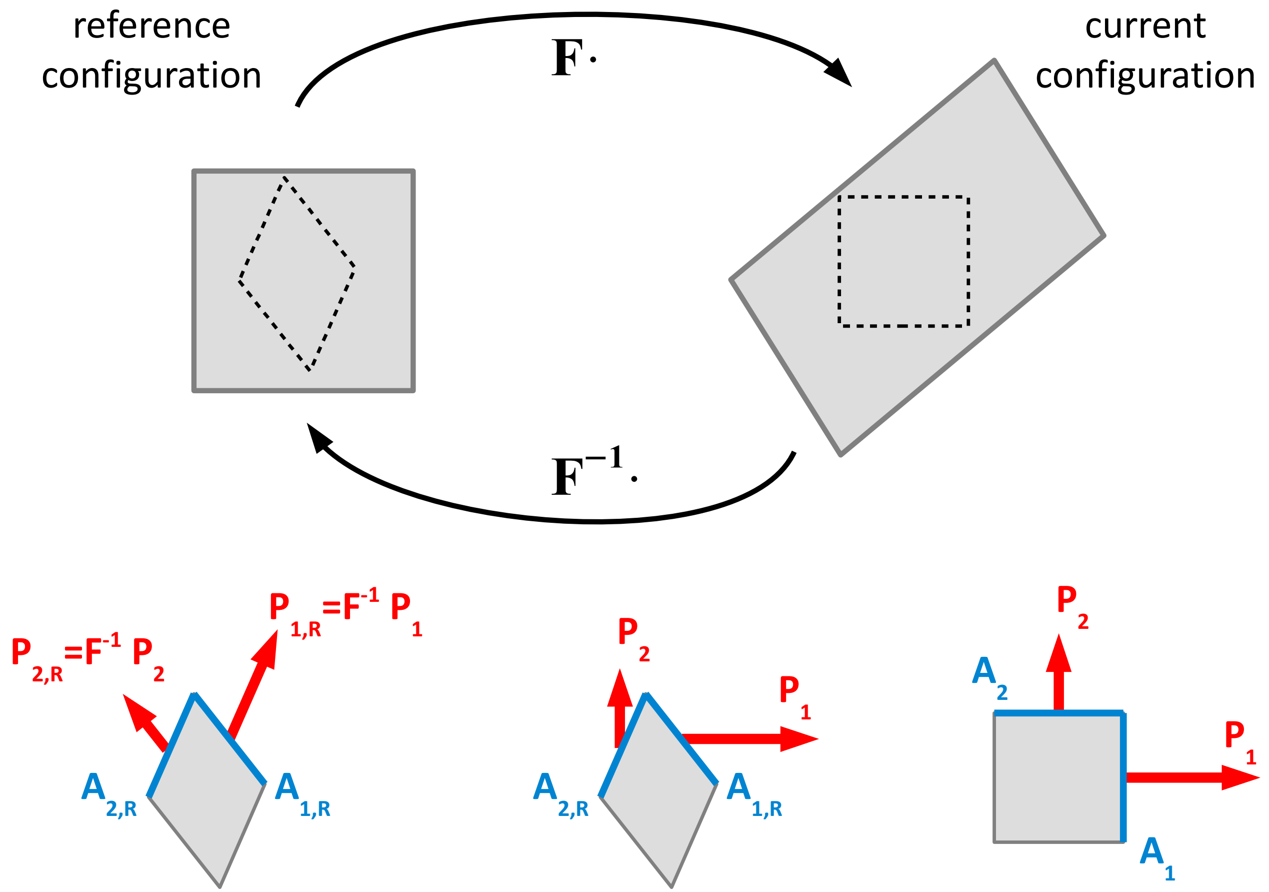
# EQUATIONS OF MOTION IN THE MATERIAL DESCRIPTION

The PK2 stress tensor **lacks a clear, direct physical interpretation**.

Let's determine the **material stress vector** in a similar way as in cases of other stress vectors:

$$\mathbf{t}_S = \mathbf{T}_S \cdot \mathbf{N} = \mathbf{F}^{-1} \mathbf{T}_R \mathbf{N} = \mathbf{F}^{-1} \mathbf{t}_R$$

- We know that deformation gradient and its inverse relate infinitely small material fibres – deformed one and the undeformed one – one with another.
- Stress is a vector measure – its direction and sense is interpreted as a directions and sense of a force referred to the surface areas, and its length is equal the ratio of the magnitude of force and considered area.
- Vectors $\mathbf{t}_R$ and $\mathbf{t}_S$ are referred to the same surface are. A map between them may be interpreted as a map between two forces.
- If we considered stress and force vectors as if they were material fibres, then the vector $\mathbf{t}_R$ would be a "deformed" vector $\mathbf{t}_S$ – or, alternatively – vector $\mathbf{t}_S$ would be the **nominal stress vector which had been "pulled back" to the reference configuration**. Such a stress is termed the material stress.




$$S_{11} = \frac{P_{1,R}}{A_{1,R}}$$

material stress

$$T_{11} = \frac{P_{1}}{A_{1,R}}$$

nominal stress

$$t_{11} = \frac{P_{1}}{A_{1}}$$

true stress

# CONSTITUTIVE RELATIONS

# CONSTITUTIVE RELATIONS

In order to describe the continuum we use:

- **3 components of the** displacement vector
- **6 components** of the **symmetric** strain tensor
- **6 components** (9 components)of the **symmetric** (non-symmetric) stress tensor

We have in total 15 (18) unknown functions. We have formulated the following relations which should enable us to determine those unknowns:

- **6** kinematic relations or **6** strain compatibility relations – relations between **strain** and **displacement**:
  - If the kinematic relations are satisfied, then strain compatibility conditions are also identically satisfied.
  - If the strain compatibility relations are satisfied, then the kinematic relations are integrable and they have a unique solution.

- **3** equations of motion derived from the **principle of momentum** – relation between stress and acceleration.
  - **Additional 3 relations** biding components of a **non-symmetric stress tensor**, derived for the principle of angular momentum. Symmetric stress tensors (Cauchy, PK2) satisfy those relation identically.

We have 15 (18) unknowns and 9 (12) equations. We're lacking 6 relations between stress and strain components.

# CONSTITUTIVE RELATIONS

Relations between components of the stress tensor and of the strain tensor are termed constitutive relations.

The postulates are formulated – so called material postulated – that are required to be satisfied when formulating the constitutive relations:

- PRINCIPLE OF DETERMINISM
- PRINCIPLE OF LOCALITY
- PRINCIPLE OF MATERIAL OBJECTIVITY

# MATERIAL POSTULATES

## PRINCIPLE OF DETERMINISM

The stress state in the given particle $\mathbf{X}$ and in the given instant of time $t$ is **determined** by the **choice of the particle** and by a **history of motion of all other particles** belonging to the body.

$$\mathbf{T}(\mathbf{X};t) = f\left(\mathbf{X}, \mathbf{x}(\boldsymbol{\xi}, t-\tau)\right), \quad \boldsymbol{\xi} \in B_{ref}, \quad \tau \in \langle 0 ; \infty)$$

It means that:

- If we know the process (history) of deformation of the body, then the constitutive relation must be such that this knowledge is enough to determine the stress state.
- We assume that the current stress state is not influenced by other factors than past deformation.

**REMARK**: $\mathbf{T}$ denotes a certain (chosen by us) measure of stress.

# MATERIAL POSTULATES

## PRINCIPLE OF LOCALITY

The stress state in the given particle $\mathbf{X}$ and in the given instant of time $t$ depends on the **history of motion of particles which are in an arbitrary small neighbourhood of that particle**:

$$\mathbf{T}(\mathbf{X};t) = f\left(\mathbf{X}, \mathbf{x}(\boldsymbol{\xi}, t-\tau)\right), \quad \boldsymbol{\xi} \in B_{ref}, \quad \tau \in \langle 0 ; \infty)$$

$$|\boldsymbol{\xi} - \mathbf{X}| < \varepsilon \to 0$$

It means that:

- The stress state in the particle depend directly only on the motion of neighbouring particles – other particles influence it only indirectly (by a “chain” of such direct actions between neighbouring particles).
- Constitutive relation cannot relate the stress state in a given particle with the deformation of some other distant particles.

# MATERIAL POSTULATES

### PRINCIPLE OF MATERIAL OBJECTIVITY

Constitutive relations determining internal properties of a physical system and relations between parts of that system must be independent of the choice of the frame of reference.

It means that:

- Constitutive relation must be mathematically formulated in such a way, that a certain local history of deformation is always associated with the same state of stress, independently of what frame of reference we are using, namele independently of the choice of the point of reference, choice of measure of length and directions of distance measurements as well as of the choice of the time lapse rate.

# HOMOGENEITY AND ISOTROPY

Among typical constructional materials we may distinguish homogeneous and inhomogeneous materials as well as isotropic and anisotropic materials.

- HOMOGENEITY – a homogeneous material is such that its mechanical properties are the same in all points (e.g. metals, alloys, clay, polymers).
- INHOMOGENEITY – in inhomogeneous materials, **mechanical properties of the material depend on the choice of the particle** (they change within the configuration of a body). These are all composite materials (i.e. concrete) or those, which exhibit internal microstructure (i.e. timber).
  - Inhomogeneous materials (e.g. timber, concrete, composites) may be still described with the model of homogeneous material if only the size of particles of various component materials are considerably smaller than the overall dimensions of a body.
- ISOTROPY – in isotropic materials **mechanical properties of a body do not depend on the direction of testing them**, e.g. stiffness or strength of the material is the same regardless th direction of the applied force (e.g. metals, alloys, concrete, clay, polymers).
- ANISOTROPY – in anisotropic materials **the response of the material on the loading factor depends on the mutual orientation of direction of action of that factor and of axes or planes of symmetry of internal structure of the material** (e.g. crystals, timber, composites).

# FIRST GRADIENT THEORY

According to the locality principle, we can state that a function determining the stress state in particle $\mathbf{X}$ will depend only on the history of deformation of neighbouring particles $\boldsymbol{\xi} = \mathbf{X} + \mathrm{d}\,\mathbf{X}$.

Position of such a neighbouring particle $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ can be approximated by expanding the deformation relation into the Taylor series. For the sake of readability we will neglect the notation of dependency on time:

$$x_i(\mathbf{X}+\mathrm{d}\,\mathbf{X}) = x_i(\mathbf{X}) + \frac{1}{1!}\left.\frac{\partial x_i}{\partial X_k}\right|_{\mathbf{X}} \mathrm{d}\,X_k + \frac{1}{2!}\left.\frac{\partial^2 x_i}{\partial X_k \partial X_l}\right|_{\mathbf{X}} \mathrm{d}\,X_k \mathrm{d}\,X_l + \frac{1}{3!}\frac{\partial^3 x_i}{\partial X_k \partial X_l \partial X_m} \mathrm{d}\,X_k \mathrm{d}\,X_l \mathrm{d}\,X_m + \ldots$$

$$\mathbf{x}(\mathbf{X}+\mathrm{d}\,\mathbf{X}) = \mathbf{x}(\mathbf{X}) + \frac{1}{1!}\nabla\mathbf{x}\;\mathrm{d}\,\mathbf{X} + \frac{1}{2!}\nabla^2\mathbf{x}\;\mathrm{d}\,\mathbf{X}\;\mathrm{d}\,\mathbf{X} + \frac{1}{3!}\nabla^3\mathbf{x}\;\mathrm{d}\,\mathbf{X}\;\mathrm{d}\,\mathbf{X}\;\mathrm{d}\,\mathbf{X} + \ldots$$

Coefficients in the second term are the **components** of the material deformation gradient $\nabla\mathbf{x}=\mathbf{F}$. Coefficients by further terms are the gradients of higher orders (e.g. the 2$^{\text{nd}}$ deformation gradient if a gradient of the deformation gradient – a rank 3 tensor). Local constitutive relation could be then written in the general form:

$$\begin{aligned}\mathbf{T}(\mathbf{X};t) &= f\left(\mathbf{X}\;;\;\mathbf{x}(\mathbf{X}+\mathrm{d}\,\mathbf{X};t-\tau)\right)\\ &= f\left(\mathbf{X}\;;\;\mathbf{x}(\mathbf{X};t-\tau)\;;\;\nabla\mathbf{x}(\mathbf{X};t-\tau)\;;\;\nabla^2\mathbf{x}(\mathbf{X};t-\tau);\ldots\right)\end{aligned}$$

# FIRST GRADIENT THEORY

We have:

$$\mathbf{T}(\mathbf{X};t) = f\left(t-\tau, \mathbf{X}\ ;\ \mathbf{x}\ ;\ \nabla\mathbf{x}\ ;\ \nabla^2\mathbf{x};\ldots\right)$$

According to the material objectivity principle, **function $f$ cannot depend on $\mathbf{x}$**. If it was so, then in case of the a the same given deformations (determined by gradients) but different positions, the stress state could be different, namely, the stress state would depend on the choice of the frame of reference, on position of a body in the space or on its translation – this is against intuition and experimental results. We will assume that:

$$\mathbf{T}(\mathbf{X};t) = f\left(t-\tau, \mathbf{X}\ ;\ \nabla\mathbf{x}\ ;\ \nabla^2\mathbf{x};\ldots\right)$$

A wide class of elastic materials can be described with the use of functions depending only on **position $\mathbf{X}$**, **time** and on the **1st deformation gradient**:

$$\mathbf{T}(\mathbf{X};t) = f\left(t-\tau, \mathbf{X}, \mathbf{F}(\mathbf{X})\right)$$

Theories using such relations are termed the 1st gradient theories.

# FIRST GRADIENT THEORY

According to the polar decomposition theorem, **deformation gradient** may be expressed in terms of the **stretch tensor**:

$$\mathbf{F} = \mathbf{R}\,\mathbf{U} = \mathbf{V}\,\mathbf{R}$$

Deformation gradient is also related with the **deformation tenso**r:

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$$

Deformation gradient is related with the **strain tensor**:

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{C} - \mathbf{1}\right)$$

- Fuction $f$ could account for any of those alternative measures of deformation.
- Similarly, various measures of the stress may be chosen for formulation of constitutive relations.

WHAT MEASURES OF STRESS AND STRAIN SHOULD BE RELATED ONE TO ANOTHER WITHIN THE CONSTITUTIVE RELATION?

# FIRST GRADIENT THEORY

There exists a certain reason concerning a proper choice of such a pair of tensors. It emerges that certain **pairs of stress and strain tensor** have such a property, that **their scalar product is equal the density of power of elastic strain**. Such pair are termed energetic conjugates. These are i.e.:

Cauchy stress tensor ⇄ symmetric part spatial displacement gradient

$$\mathbf{T}_{\boldsymbol{\sigma}} \qquad \boldsymbol{\eta} = \frac{1}{2}\left[\mathbf{h} + \mathbf{h}^{\mathrm{T}}\right]$$

Piola – Kirchhoff stress tensor of the 1st kind ⇄ material deformation gradient

$$\mathbf{T}_R = J\,\mathbf{T}_{\boldsymbol{\sigma}}\mathbf{F}^{-\mathrm{T}} \qquad \mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}$$

Piola – Kirchhoff stress tensor of the 2nd kind ⇄ material strain tensor

$$\mathbf{T}_S = J\,\mathbf{F}^{-1}\mathbf{T}_{\boldsymbol{\sigma}}\mathbf{F}^{-\mathrm{T}} \qquad \mathbf{E} = \frac{1}{2}\left(\mathbf{C} - \mathbf{1}\right)$$

# FIRST GRADIENT THEORY

We will deal with the relation between **Piola – Kirchhoff stress tensor of the 2nd kind** and **material strain tensor**

- Both are appropriate for the material description
- Both are symmetric

$$\mathbf{T}_S(t, \mathbf{X}) = f\left(t - \tau, \mathbf{X}, \mathbf{E}(\mathbf{X}, t)\right)$$

REMARK:

- If the material is homogeneous, then the constitutive relation **does not depend explicitly on the choice of the particle**:

$$\mathbf{T}_S(t, \mathbf{X}) = f\left(t - \tau, \mathbf{E}(\mathbf{X}, t)\right)$$

The dependency on the choice of the particle is indirect, via the dependency of the strain state on the choice of the particle.

# ELASTIC MATERIALS

A material is termed elastic (**in the sense of Cauchy**) if:

- Current stress state depends only on the current strain state and it **does not depend on the history of deformation** (strain path):

$$\mathbf{T}_S(t,\mathbf{X}) = f\left(\mathbf{X},\mathbf{E}(\mathbf{X},t)\right)$$

- **Constitutive relation is** invertible:

$$\exists f^{-1}: \qquad \mathbf{E}(t,\mathbf{X}) = f^{-1}\left(\mathbf{X},\mathbf{T}_S(\mathbf{X},t)\right)$$

# ELASTIC MATERIALS

**An elastic material** (in the sense of Cauchy) is termed hyperelastic (**elastic in the sense of Green**) if:

- **There exists a scalar-valued function** $W$ **of a tensorial argument,** termed the elastic potential, such that:

$$\mathbf{T}_S = \frac{\partial W}{\partial \mathbf{E}} \quad \Leftrightarrow \quad S_{ij} = \frac{\partial W}{\partial E_{ij}} \qquad i,j=1,2,3$$

# ELASTIC MATERIALS

Let's consider a hyperelastic material. If the considered strain is small (strain tensor is close to a zero tensor $\mathbf{E} \approx \mathbf{0}$), then the **elastic potential** can be expanded into a **Taylor series** in the neighbourhood of $\mathbf{0}$:

$$W(\mathbf{E}) \approx W(\mathbf{0}) + \frac{1}{1!}\frac{\partial W}{\partial E_{ij}}\bigg|_{\mathbf{0}} E_{ij} + \frac{1}{2!}\frac{\partial^2 W}{\partial E_{ij} E_{kl}}\bigg|_{\mathbf{0}} E_{ij} E_{kl} + \frac{1}{3!}\frac{\partial^3 W}{\partial E_{ij} E_{kl} E_{mn}}\bigg|_{\mathbf{0}} E_{ij} E_{kl} E_{mn} + \dots$$

Stress state is determined as a derivative of the elastic potential with respect to the strain state:

$$S_{ij} = \frac{\partial W}{\partial E_{ij}} \approx \frac{\partial W}{\partial E_{ij}}\bigg|_{\mathbf{0}} + \frac{\partial^2 W}{\partial E_{ij} E_{kl}}\bigg|_{\mathbf{0}} E_{kl} + \dots$$

A non-zero value of the first term in that expansion would suggest that even in an undeformed body ( $E_{kl}=0$ ) there is always a non-zero stress state. This is against the experimental results. If we in turn neglect all terms of degree 3 or more, then the constitutive relation is of the following form:

$$S_{ij} \approx \mathsf{S}_{ijkl} E_{kl} \qquad \text{where} \qquad \mathsf{S}_{ijkl} = \frac{\partial^2 W}{\partial E_{ij} \partial E_{kl}}\bigg|_{\mathbf{E}=\mathbf{0}}$$

# ELASTIC MATERIALS

**A hyperelastic material** (**elastic in the sense of Green**) is termed a Hooke's material or a linear elastic material if:

- **Elastic potential is a homogeneous quadratic function of the strain state,** namely it is a combination of squares of components of the strain tensor and of products of two first powers of two different components of the strain tensor

$$W = \frac{1}{2}\mathsf{S}_{ijkl} E_{ij} E_{kl} =$$

$$= \frac{1}{2}\left[\mathsf{S}_{1111} E_{11} E_{11} + \mathsf{S}_{1112} E_{11} E_{12} + \mathsf{S}_{1113} E_{11} E_{13} + \mathsf{S}_{1121} E_{11} E_{21} + \ldots + \mathsf{S}_{3333} E_{33} E_{33}\right]$$

$$\mathbf{T}_S = \frac{\partial W}{\partial \mathbf{E}} = \mathsf{\mathbf{S}}\mathbf{E} \quad \Leftrightarrow \quad S_{ij} = \frac{\partial W}{\partial E_{ij}} = \mathsf{S}_{ijkl} E_{kl}$$

**REMARK**: If the **strain is sufficiently small**, then **all materials can be described in an approximate way** with the use of the **Hooke's material** model.

# ELASTIC MATERIALS

The constitutive relation for linear elastic materials is termed the generalized Hooke's Law and it can be written explicitly as follows:

$$\begin{aligned} S_{11} = {} & \mathsf{S}_{1111} E_{11} + \tfrac{1}{2}\left(\mathsf{S}_{1112} + \mathsf{S}_{1211}\right) E_{12} + \tfrac{1}{2}\left(\mathsf{S}_{1113} + \mathsf{S}_{1211}\right) E_{13} + \\ & + \tfrac{1}{2}\left(\mathsf{S}_{1121} + \mathsf{S}_{2111}\right) E_{21} + \mathsf{S}_{1122} E_{22} + \tfrac{1}{2}\left(\mathsf{S}_{1123} + \mathsf{S}_{2311}\right) E_{23} + \\ & + \tfrac{1}{2}\left(\mathsf{S}_{1131} + \mathsf{S}_{3111}\right) E_{31} + \tfrac{1}{2}\left(\mathsf{S}_{1132} + \mathsf{S}_{3211}\right) E_{32} + \mathsf{S}_{1133} E_{33} \end{aligned}$$

$$\begin{aligned} S_{12} = {} & \mathsf{S}_{1112} E_{11} + \mathsf{S}_{1212} E_{12} + \tfrac{1}{2}\left(\mathsf{S}_{1213} + \mathsf{S}_{1312}\right) E_{13} + \\ & + \tfrac{1}{2}\left(\mathsf{S}_{1221} + \mathsf{S}_{2112}\right) E_{21} + \mathsf{S}_{1222} E_{22} + \tfrac{1}{2}\left(\mathsf{S}_{1223} + \mathsf{S}_{2312}\right) E_{23} + \\ & + \tfrac{1}{2}\left(\mathsf{S}_{1231} + \mathsf{S}_{3112}\right) E_{31} + \tfrac{1}{2}\left(\mathsf{S}_{1232} + \mathsf{S}_{3212}\right) E_{32} + \mathsf{S}_{1233} E_{33} \end{aligned}$$

$$\ldots$$

**Each component of the stress tensor** is calculated as a **linear combination of all components of the strain tensor** – coefficients of such a combination are termed elastic constants.

**Paweł Szeptyński PhD, Eng.**

# ELASTIC MATERIALS

- The matrix $\mathsf{S}_{ijkl}$ is a matrix of a linear map between two tensors – it is thus a representation matrix of a **4th rank tensor**. Tensor $\mathsf{S}$ is termed the stiffness tensor.

- The stiffness tensor will be **invertible** if and only if **its determinant is not equal 0**.

- **Inverse of the stiffness tensor** is termed the compliance tensor and it is denoted with **C**:

$$\mathbf{T}_S = \mathsf{S} \cdot \mathbf{E} \quad \Leftrightarrow \quad S_{ij} = \mathsf{S}_{ijkl} E_{kl}$$

$$\mathbf{E} = \mathsf{C} \cdot \mathbf{T}_S \quad \Leftrightarrow \quad E_{ij} = \mathsf{C}_{ijkl} S_{kl}$$

- Stiffness tensors and compliance tensors are termed elasticity tensors or Hooke's tensors. These are 3-dimensional 4th rank tensors, so they have in general **$3^4$=81 component**, which are termed elastic constants.

- Elastic constants are not truly constant...
  - In inhomogeneous materials, each component of elasticity tensors (elastic constant) is in general a function of particle, namely $\mathsf{S}_{ijkl} = \mathsf{S}_{ijkl}(\mathbf{X})$
  - Even in homogeneous materials any change of the coordinate system will in general lead to transformation of components of the tensor. Elastic constants are then not "constant" in the sense that they are not invariant.

# ELASTIC MATERIALS

- Not all components of the elasticity tensors are independent one of another:

$$S_{ij} = \mathsf{S}_{ijkl} E_{kl}$$

  - Due to **symmetry of the stress tensor**: $S_{ij} = S_{ji} \quad \Rightarrow \quad \mathsf{S}_{ijkl} = \mathsf{S}_{jikl}$
  - Due to **symmetry of the strain tensor**: $E_{kl} = E_{lk} \quad \Rightarrow \quad \mathsf{S}_{ijkl} = \mathsf{S}_{ijlk}$
  - According to the definition of the components of the stress tensor being **derivatives of the elastic potential** and due to **symmetry of differentiation**:

$$\mathsf{S}_{ijkl} = \left.\frac{\partial}{\partial E_{ij}} \frac{\partial W}{\partial E_{kl}}\right|_{\mathbf{0}} = \left.\frac{\partial}{\partial E_{kl}} \frac{\partial W}{\partial E_{ij}}\right|_{\mathbf{0}} = \mathsf{S}_{klij}$$

- Elasticity tensors are characterized by the following internal symmetries:

$$\mathsf{S}_{ijkl} = \mathsf{S}_{jikl} = \mathsf{S}_{ijlk} = \mathsf{S}_{klij}$$
$$\mathsf{C}_{ijkl} = \mathsf{C}_{jikl} = \mathsf{C}_{ijlk} = \mathsf{C}_{klij}$$

# ELASTIC MATERIALS

- Finally, elasticity tensors of fully anisotropic material have 21 independent components.
- Among these 21 parameters we may distinguish:
  - 6 invariants of physical dimension of stress (Pa) – these are **eigenvalues of the stiffness tensor**, and they are termed the Kelvin moduli. They are proportionality coefficients between stress states and strain states which are eigenvectors of the elasticity tensors. **Eigenvalues of the compliance tensor are inverses of the Kelvin moduli**.
  - 12 non-dimensional invariants termed stiffness distributors, which determine the form of eigenvectors of elasticity tensors, namely: **the form of the stress** (strain) **state being a response for a given strain** (stress) **state**.
  - 3 non-dimensional parameters, which determine in an unambiguous way the orientation **of the internal structure of considered material with respect to the assumed frame of reference** (e.g. Euler angles, components of 3 mutually orthogonal versors).
- Only 18 parameters characterize the mechanical properties of the material without any reference to its orientation in space.
- If the internal structure of material is characterized by any planes of axes of symmetry (external symmetries) then the total number of independent components is decreased.

# ELASTIC MATERIALS

Making an account for internal symmetries, the generalized Hooke's Law may be written down in the following way:

$$S_{11} = \mathsf{S}_{1111} E_{11} + \mathsf{S}_{1122} E_{22} + \mathsf{S}_{1133} E_{33} + 2\mathsf{S}_{1123} E_{23} + 2\mathsf{S}_{1131} E_{31} + 2\mathsf{S}_{1112} E_{12}$$

$$S_{22} = \mathsf{S}_{2211} E_{11} + \mathsf{S}_{2222} E_{22} + \mathsf{S}_{2233} E_{33} + 2\mathsf{S}_{2223} E_{23} + 2\mathsf{S}_{2231} E_{31} + 2\mathsf{S}_{2212} E_{12}$$

$$S_{33} = \mathsf{S}_{3311} E_{11} + \mathsf{S}_{3322} E_{22} + \mathsf{S}_{3333} E_{33} + 2\mathsf{S}_{3323} E_{23} + 2\mathsf{S}_{3331} E_{31} + 2\mathsf{S}_{3312} E_{12}$$

$$S_{23} = \mathsf{S}_{2311} E_{11} + \mathsf{S}_{2322} E_{22} + \mathsf{S}_{2333} E_{33} + 2\mathsf{S}_{2323} E_{23} + 2\mathsf{S}_{2331} E_{31} + 2\mathsf{S}_{2312} E_{12}$$

$$S_{31} = \mathsf{S}_{3111} E_{11} + \mathsf{S}_{3122} E_{22} + \mathsf{S}_{3133} E_{33} + 2\mathsf{S}_{3123} E_{23} + 2\mathsf{S}_{3131} E_{31} + 2\mathsf{S}_{3112} E_{12}$$

$$S_{12} = \mathsf{S}_{1211} E_{11} + \mathsf{S}_{1222} E_{22} + \mathsf{S}_{1233} E_{33} + 2\mathsf{S}_{1223} E_{23} + 2\mathsf{S}_{1231} E_{31} + 2\mathsf{S}_{1212} E_{12}$$

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# ELASTIC MATERIALS

The generalized Hooke's Law may be written down in a matrix form:

$$\begin{bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ S_{23} \\ S_{31} \\ S_{12} \end{bmatrix} = \begin{bmatrix} \mathsf{S}_{1111} & \mathsf{S}_{1122} & \mathsf{S}_{1133} & \mathsf{S}_{1123} & \mathsf{S}_{1131} & \mathsf{S}_{1112} \\ & \mathsf{S}_{2222} & \mathsf{S}_{2233} & \mathsf{S}_{2223} & \mathsf{S}_{2231} & \mathsf{S}_{2212} \\ & & \mathsf{S}_{3333} & \mathsf{S}_{3323} & \mathsf{S}_{3331} & \mathsf{S}_{3312} \\ & & & \mathsf{S}_{2323} & \mathsf{S}_{2331} & \mathsf{S}_{2312} \\ & \text{sym} & & & \mathsf{S}_{3131} & \mathsf{S}_{3112} \\ & & & & & \mathsf{S}_{1212} \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{23} \\ 2E_{31} \\ 2E_{12} \end{bmatrix}$$

# ELASTIC MATERIALS

However, so called Mandel's notation in considered more appropriate now:

$$\begin{bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ \sqrt{2}S_{23} \\ \sqrt{2}S_{31} \\ \sqrt{2}S_{12} \end{bmatrix} = \begin{bmatrix} \mathsf{S}_{1111} & \mathsf{S}_{1122} & \mathsf{S}_{1133} & \sqrt{2}\mathsf{S}_{1123} & \sqrt{2}\mathsf{S}_{1131} & \sqrt{2}\mathsf{S}_{1112} \\ & \mathsf{S}_{2222} & \mathsf{S}_{2233} & \sqrt{2}\mathsf{S}_{2223} & \sqrt{2}\mathsf{S}_{2231} & \sqrt{2}\mathsf{S}_{2212} \\ & & \mathsf{S}_{3333} & \sqrt{2}\mathsf{S}_{3323} & \sqrt{2}\mathsf{S}_{3331} & \sqrt{2}\mathsf{S}_{3312} \\ & & & 2\mathsf{S}_{2323} & 2\mathsf{S}_{2331} & 2\mathsf{S}_{2312} \\ & \text{sym} & & & 2\mathsf{S}_{3131} & 2\mathsf{S}_{3112} \\ & & & & & 2\mathsf{S}_{1212} \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ \sqrt{2}E_{23} \\ \sqrt{2}E_{31} \\ \sqrt{2}E_{12} \end{bmatrix}$$

$$\begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ \sqrt{2}E_{23} \\ \sqrt{2}E_{31} \\ \sqrt{2}E_{12} \end{bmatrix} = \begin{bmatrix} \mathsf{C}_{1111} & \mathsf{C}_{1122} & \mathsf{C}_{1133} & \sqrt{2}\mathsf{C}_{1123} & \sqrt{2}\mathsf{C}_{1131} & \sqrt{2}\mathsf{C}_{1112} \\ & \mathsf{C}_{2222} & \mathsf{C}_{2233} & \sqrt{2}\mathsf{C}_{2223} & \sqrt{2}\mathsf{C}_{2231} & \sqrt{2}\mathsf{C}_{2212} \\ & & \mathsf{C}_{3333} & \sqrt{2}\mathsf{C}_{3323} & \sqrt{2}\mathsf{C}_{3331} & \sqrt{2}\mathsf{C}_{3312} \\ & & & 2\mathsf{C}_{2323} & 2\mathsf{C}_{2331} & 2\mathsf{C}_{2312} \\ & \text{sym} & & & 2\mathsf{C}_{3131} & 2\mathsf{C}_{3112} \\ & & & & & 2\mathsf{C}_{1212} \end{bmatrix} \begin{bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ \sqrt{2}S_{23} \\ \sqrt{2}S_{31} \\ \sqrt{2}S_{12} \end{bmatrix}$$

Such a notation is symmetric and after proper defining of the rotation operator, the norms of tensors as well as rotation of tensors are realized within the matrix calculus in the same way as calculating lengths of vectors and rotations of vectors in 3-dimensional space.

**THANK YOU FOR YOUR ATTENTION**