# THEORY OF ELASTICITY AND PLASTICITY

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**Paweł Szeptyński PhD, Eng.**

# ANISOTROPIC
# LINEAR ELASTIC MATERIALS

# ANISOTROPIC LINEAR ELASTIC MATERIALS

**If response of a material, which is subject to a certain load factor, depend on the direction of action of that factor** (strictly speaking: depend on orientation of that direction with respect to characteristic direction of internal structure of the material) is termed to be anisotropic.

For **linear elastic materials** one may distinguish **8 classes of elastic symmetry** – the correspond to same extent with **crystal systems:**

- Full anisotropy ← triclinic system
- Monoclinic symmetry ← monoclinic system
- Orthotropy ← orthorhombic system
- Trigonal symmetry ← trigonal system
- Tetragonal symmetry ← tetragonal system
- Cylindrical symmetry ← hexagonal system
- Cubic symmetry ← cubic (isometric) system
- Isotropy ← [amorphous solids]

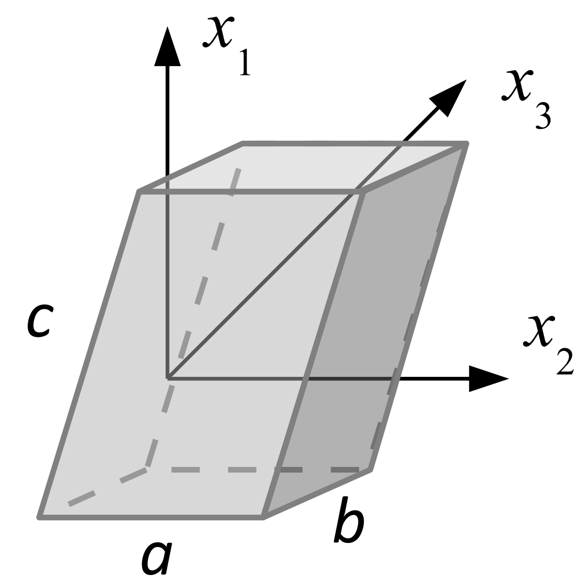
# ANISOTROPIC LINEAR ELASTIC MATERIALS

## TRICLINIC SYMMETRY (FULL ANISOTROPY)

Symmetry elements: - none

Number of independent components: 21

$$\mathbf{S} = \begin{bmatrix} S_{1111} & S_{1122} & S_{1133} & \sqrt{2}S_{1123} & \sqrt{2}S_{1131} & \sqrt{2}S_{1112} \\ & S_{2222} & S_{2233} & \sqrt{2}S_{2223} & \sqrt{2}S_{2231} & \sqrt{2}S_{2212} \\ & & S_{3333} & \sqrt{2}S_{3323} & \sqrt{2}S_{3331} & \sqrt{2}S_{3312} \\ & & & 2S_{2323} & 2S_{2331} & 2S_{2312} \\ & \mathrm{sym} & & & 2S_{3131} & 2S_{3112} \\ & & & & & 2S_{1212} \end{bmatrix}$$



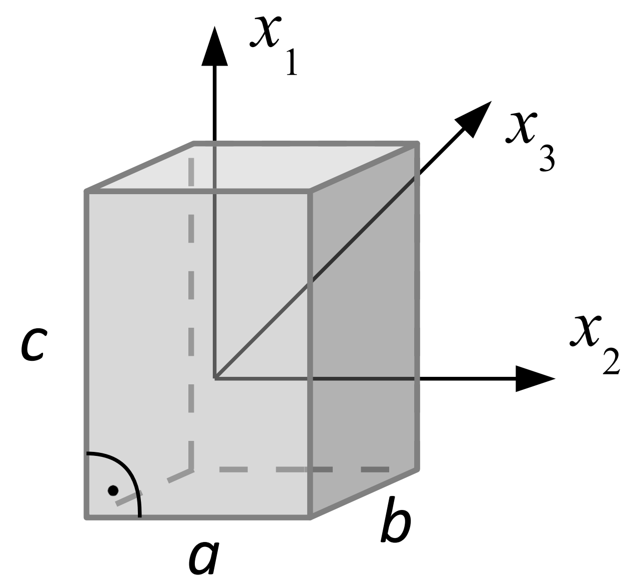
# ANISOTROPIC LINEAR ELASTIC MATERIALS

## MONOCLINIC SYMMETRY

Symmetry elements: - mirror reflection about a plane perpendicular to axis $x_1$

Number of independent components: 13

$$\mathbf{S} = \begin{bmatrix} S_{1111} & S_{1122} & S_{1133} & \sqrt{2}S_{1123} & 0 & 0 \\ & S_{2222} & S_{2233} & \sqrt{2}S_{2223} & 0 & 0 \\ & & S_{3333} & \sqrt{2}S_{3323} & 0 & 0 \\ & & & 2S_{2323} & 0 & 0 \\ & \text{sym} & & & 2S_{3131} & 2S_{3112} \\ & & & & & 2S_{1212} \end{bmatrix}$$



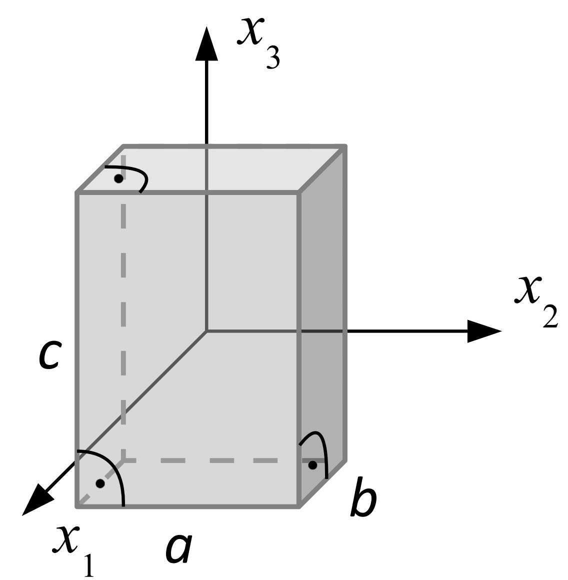
# ANISOTROPIC LINEAR ELASTIC MATERIALS

## ORTHOTROPY

Symmetry elements:
- mirror reflection about planes perpendicular to axes $x_1, x_2, x_3$
- rotation by 180° about axes $x_1, x_2, x_3$

Number of independent components: 9

$$\mathbf{S} = \begin{bmatrix} S_{1111} & S_{1122} & S_{1133} & 0 & 0 & 0 \\ & S_{2222} & S_{2233} & 0 & 0 & 0 \\ & & S_{3333} & 0 & 0 & 0 \\ & & & 2S_{2323} & 0 & 0 \\ & \text{sym} & & & 2S_{3131} & 0 \\ & & & & & 2S_{1212} \end{bmatrix}$$



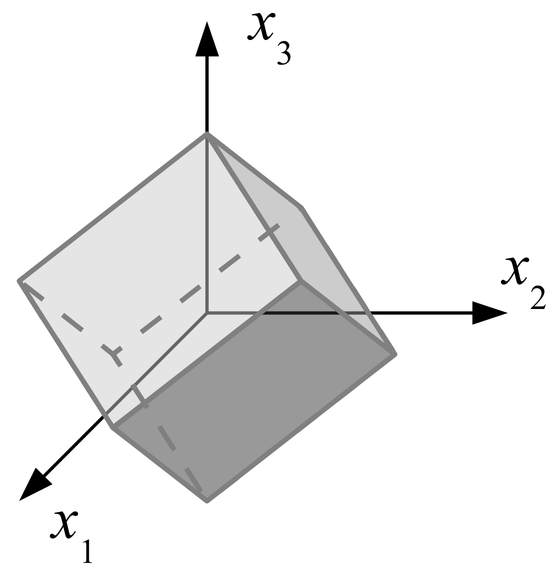
# ANISOTROPIC LINEAR ELASTIC MATERIALS

## TRIGONAL SYMMETRY

Symmetry elements:
- mirror reflection about plane perpendicular to axis $x_1$
- rotation by 120° about axis $x_3$

Number of independent components: 6

$$\mathsf{S} = \begin{bmatrix} \mathsf{S}_{1111} & \mathsf{S}_{1122} & \mathsf{S}_{1133} & \sqrt{2}\,\mathsf{S}_{1123} & 0 & 0 \\ & \mathsf{S}_{1111} & \mathsf{S}_{1133} & -\sqrt{2}\,\mathsf{S}_{1123} & 0 & 0 \\ & & \mathsf{S}_{3333} & 0 & 0 & 0 \\ & & & 2\,\mathsf{S}_{2323} & 0 & 0 \\ & \text{sym} & & & 2\,\mathsf{S}_{2323} & 2\,\mathsf{S}_{1123} \\ & & & & & \mathsf{S}_{1111}-\mathsf{S}_{1122} \end{bmatrix}$$



# ANISOTROPIC LINEAR ELASTIC MATERIALS

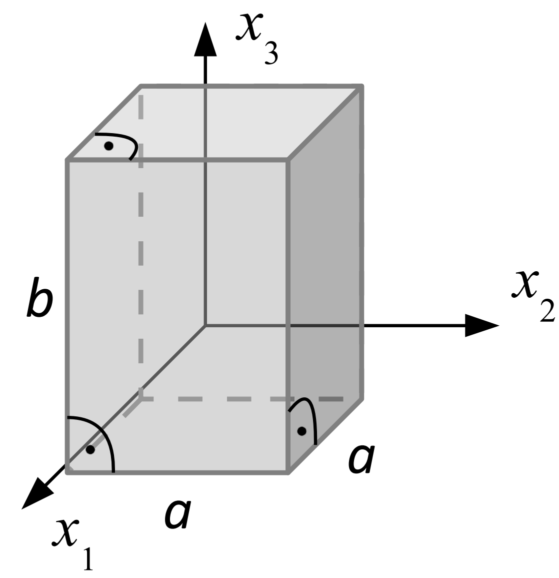
## TETRAGONAL SYMMETRY

Symmetry elements:

- mirror reflection about planes perpendicular to axes $x_1$ , $x_2$ , $x_3$
- rotation by 180° about axes $x_1$ , $x_2$
- rotation by 90° about axis $x_3$

Number of independent components: 6

$$\mathsf{S} = \begin{bmatrix} \mathsf{S}_{1111} & \mathsf{S}_{1122} & \mathsf{S}_{1133} & 0 & 0 & 0 \\ & \mathsf{S}_{1111} & \mathsf{S}_{1133} & 0 & 0 & 0 \\ & & \mathsf{S}_{3333} & 0 & 0 & 0 \\ & & & 2\mathsf{S}_{2323} & 0 & 0 \\ & \text{sym} & & & 2\mathsf{S}_{2323} & 0 \\ & & & & & 2\mathsf{S}_{1212} \end{bmatrix}$$



# ANISOTROPIC LINEAR ELASTIC MATERIALS

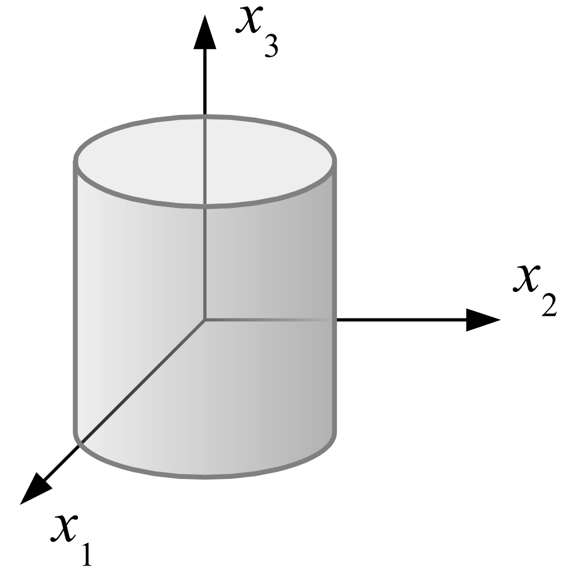
## CYLINDRICAL SYMMETRY

Symmetry elements:

- mirror reflection about planes perpendicular to axes $x_1$ , $x_2$ , $x_3$
- rotation by 180° about axes $x_1$ , $x_2$
- rotation by any angle about axis $x_3$

Number of independent components: 5

$$\mathbf{S} = \begin{bmatrix} S_{1111} & S_{1122} & S_{1133} & 0 & 0 & 0 \\ & S_{1111} & S_{1133} & 0 & 0 & 0 \\ & & S_{3333} & 0 & 0 & 0 \\ & & & 2S_{2323} & 0 & 0 \\ & \text{sym} & & & 2S_{2323} & 0 \\ & & & & & S_{1111}-S_{1122} \end{bmatrix}$$



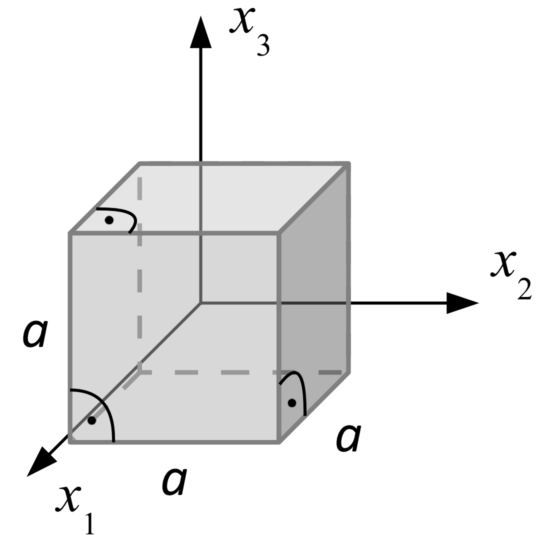
# ANISOTROPIC LINEAR ELASTIC MATERIALS

## CUBIC SYMMETRY

Symmetry elements:
- mirror reflections abut planes perpendicular to axes $x_1, x_2, x_3$
- rotation by 90° about axes $x_1, x_2, x_3$

Number of independent components: 3

$$\mathbf{S} = \begin{bmatrix} S_{1111} & S_{1122} & S_{1122} & 0 & 0 & 0 \\ & S_{1111} & S_{1122} & 0 & 0 & 0 \\ & & S_{1111} & 0 & 0 & 0 \\ & & & 2S_{2323} & 0 & 0 \\ & \text{sym} & & & 2S_{2323} & 0 \\ & & & & & 2S_{2323} \end{bmatrix}$$



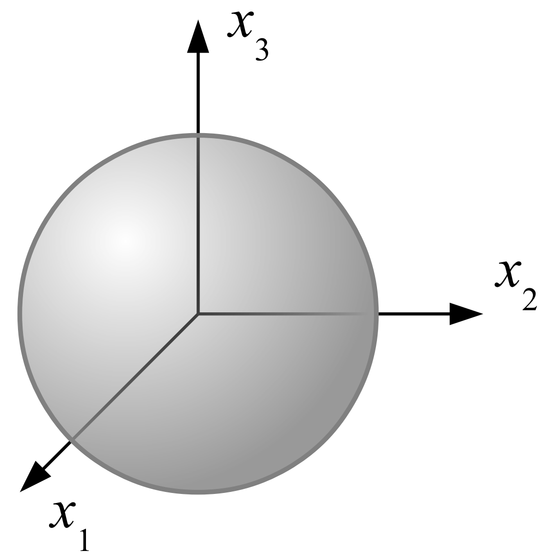
# ANISOTROPIC LINEAR ELASTIC MATERIALS

## ISOTROPY

Symmetry elements: - any rotations and reflections

Number of independent components: 2

$$\mathbf{S} = \begin{bmatrix} S_{1111} & S_{1122} & S_{1122} & 0 & 0 & 0 \\ & S_{1111} & S_{1122} & 0 & 0 & 0 \\ & & S_{1111} & 0 & 0 & 0 \\ & & & S_{1111}-S_{1122} & 0 & 0 \\ & \text{sym} & & & S_{1111}-S_{1122} & 0 \\ & & & & & S_{1111}-S_{1122} \end{bmatrix}$$



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# ISOTROPIC
# LINEAR ELASTIC MATERIALS

# ISOTROPIC HOOKE'S MATERIALS

**If the response of a material for a given load factor is independent of the direction of action of that factor,** then the material is termed isotropic.

**The constitutive relations** must also be **isotropic** – **isotropic functions** of tensorial argument may be defined in terms of invariants of that argument. In particular, the elastic potential should be such a function:

$$W(\mathbf{E}) = W\left(I_1(\mathbf{E}); I_2(\mathbf{E}); I_3(\mathbf{E})\right)$$

where:

- 1st invariant $$I_1(\mathbf{E}) = \mathrm{tr}(\mathbf{E}) = \mathbf{1}\cdot\mathbf{E} = E_{11} + E_{22} + E_{33}$$
- 2nd invariant $$I_2(\mathbf{E}) = \frac{1}{2}\left[\left(\mathrm{tr}(\mathbf{E})\right)^2 - \mathrm{tr}\left(\mathbf{E}^2\right)\right] = \frac{1}{2}\left[\left(\mathbf{1}\cdot\mathbf{E}\right)^2 - \mathbf{1}\cdot\left(\mathbf{E}^2\right)\right] =$$ $$= \left(E_{22}E_{33} - E_{23}^2\right) + \left(E_{33}E_{11} - E_{31}^2\right) + \left(E_{11}E_{22} - E_{12}^2\right)$$
- 3rd invariant $$I_3(\mathbf{E}) = \det(\mathbf{E}) = E_{11}E_{22}E_{33} + 2E_{23}E_{31}E_{12} - E_{11}E_{23}^2 - E_{22}E_{31}^2 - E_{33}E_{12}^2$$

# ISOTROPIC HOOKE'S MATERIALS

**Elastic potential** in the linear elastic material (Hooke's material) must be a homogeneous quadratic functions of components of the strain tensor:

- **1st invariant** is a function of the **1st degree** wrt component of $\mathbf{E}$. It must be accounted for as a **square.**
- **2nd invariant** is a function of the **2nd degree** wrt component of $\mathbf{E}$. It must be accounted for in the **1st power.**
- **3rd invariant** is a function of the **3rd degree** wrt component of $\mathbf{E}$. It cannot be accounted for.

$$W(\mathbf{E}) = W(I_1; I_2) = \alpha (I_1)^2 + \beta I_2$$

Constitutive relation may be expressed in the following way:

$$\mathbf{T}_S = \frac{\partial W}{\partial \mathbf{E}} = \frac{\partial W}{\partial I_1}\frac{\partial I_1}{\partial \mathbf{E}} + \frac{\partial W}{\partial I_2}\frac{\partial I_2}{\partial \mathbf{E}}$$

$$\mathbf{T}_S = \alpha\left(2 I_1 \mathbf{1}\right) + \beta\left(I_1 \mathbf{1} - \mathbf{E}\right)$$

# ISOTROPIC HOOKE'S MATERIALS

**Constitutive relation** for isotropic Hooke's material:

$$\mathbf{T}_S = (2\alpha+\beta) I_1 \mathbf{1} - \beta \mathbf{E}$$

Let's introduce the following parameters:

- $\lambda = (2\alpha+\beta)$ - the 1st Lamé's parameter
- $\mu = -\frac{1}{2}\beta$ - the 2nd Lamé's parameter

$$\mathbf{T}_S = 2\mu \mathbf{E} + \lambda \operatorname{tr}(\mathbf{E}) \mathbf{1} \quad \Leftrightarrow \quad S_{ij} = 2\mu E_{ij} + \lambda E_{kk} \delta_{ij}$$

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ & S_{22} & S_{23} \\ \text{sym} & & S_{33} \end{bmatrix} = 2\mu \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ & E_{22} & E_{23} \\ \text{sym} & & E_{33} \end{bmatrix} + \lambda (E_{11}+E_{22}+E_{33}) \begin{bmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ \text{sym} & & 1 \end{bmatrix}$$

# ISOTROPIC HOOKE'S MATERIALS

**Constitutive relation** for isotropic Hooke's material:

$$\mathbf{T}_S = 2\mu\,\mathbf{E} + \lambda\,\mathrm{tr}(\mathbf{E})\,\mathbf{1} \qquad \Leftrightarrow \qquad S_{ij} = 2\mu\,E_{ij} + \lambda\,E_{kk}\,\delta_{ij}$$

Explicitly written:

$$\begin{cases} S_{11} = (2\mu+\lambda)E_{11} + \lambda(E_{22}+E_{33}) \\ S_{22} = (2\mu+\lambda)E_{22} + \lambda(E_{33}+E_{11}) \\ S_{33} = (2\mu+\lambda)E_{33} + \lambda(E_{11}+E_{22}) \\ S_{23} = 2\mu\,E_{23} \\ S_{31} = 2\mu\,E_{31} \\ S_{12} = 2\mu\,E_{12} \end{cases}$$

The above relations are sometimes referred to as the 1st form of the generalized Hooke's Law.

The above relations may be considered the system of linear algebraic equations for the strain tensor components (unknowns). Such a system may be solved.

# ISOTROPIC HOOKE'S MATERIALS

**Constitutive relation** for isotropic Hooke's material:

$$\mathbf{E} = \frac{1}{E}\left[(1+\nu)\mathbf{T}_s - \nu\,\mathrm{tr}(\mathbf{T}_s)\mathbf{1}\right] \quad \Leftrightarrow \quad E_{ij} = \left[(1+\nu)S_{ij} - \nu S_{kk}\delta_{ij}\right]$$

Explicitly written:

$$\begin{cases} E_{11} = \frac{1}{E}\left[S_{11} - \nu(S_{22}+S_{33})\right] \\ E_{22} = \frac{1}{E}\left[S_{22} - \nu(S_{33}+S_{11})\right] \\ E_{33} = \frac{1}{E}\left[S_{33} - \nu(S_{11}+S_{22})\right] \\ E_{23} = \frac{1}{2G}S_{23} \\ E_{31} = \frac{1}{2G}S_{31} \\ E_{12} = \frac{1}{2G}S_{12} \end{cases}$$

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$$

– Young's modulus
(longitudinal stiffness modulus)

$$\nu = \frac{\lambda}{2(\lambda+\mu)}$$

– Poisson's ratio
(lateral expansion coefficient)

$$G = \frac{E}{2(1+\nu)} = \mu$$

– Kirchhoff's modulus
(transverse stiffness modulus,
shear modulus, modulus of rigidity)

The above relations are sometimes referred to as the $2^{nd}$ form of the generalized Hooke's Law.

# ISOTROPIC HOOKE'S MATERIALS

**Each 2nd rank tensor** may be **decomposed** into a sum of an **isotropic tensor** and **traceless tensor** (deviator). In particular:

- Isotropic stress tensor:

$$\mathbf{A}_T = \frac{1}{3}\mathrm{tr}(\mathbf{T}_S)\mathbf{1} = \begin{bmatrix} p & 0 & 0 \\ & p & 0 \\ \mathrm{sym} & & p \end{bmatrix} \qquad \underbrace{p = \frac{1}{3}(S_{11}+S_{22}+S_{33})}_{\text{hydrostatic pressure}}$$

- Stress deviator tensor:

$$\mathbf{D}_T = \mathbf{T}_S - \mathbf{A}_T$$

- Isotropic strain tensor:

$$\mathbf{A}_E = \frac{1}{3}\mathrm{tr}(\mathbf{E})\mathbf{1} = \frac{1}{3}\begin{bmatrix} \theta & 0 & 0 \\ & \theta & 0 \\ \mathrm{sym} & & \theta \end{bmatrix} \qquad \underbrace{\theta = E_{11}+E_{22}+E_{33}}_{\text{dilatation (relative volume change)}}$$

- Strain deviator tensor:

$$\mathbf{D}_E = \mathbf{E} - \mathbf{A}_E$$

# ISOTROPIC HOOKE'S MATERIALS

Let's substitute this decomposition into the **1st form of the Hooke's Law:**

$$\mathbf{A}_T + \mathbf{D}_T = 2\mu(\mathbf{A}_E + \mathbf{D}_E) + \lambda\,\mathrm{tr}(\mathbf{E})\mathbf{1}$$

$$\underbrace{\mathbf{A}_T}_{\text{isotropic}} + \underbrace{\mathbf{D}_T}_{\text{traceless}} = \underbrace{(2\mu + 3\lambda)\mathbf{A}_E}_{\text{isotropic}} + \underbrace{2\mu\mathbf{D}_E}_{\text{traceless}}$$

Two tensors are equal if and only if their isotropic and traceless parts are equal respectively:

$$\begin{cases} \mathbf{A}_T = 3K\,\mathbf{A}_E & \text{– change of volume} \\ \mathbf{D}_T = 2G\,\mathbf{D}_E & \text{– change of shape} \end{cases}$$

$$K = \lambda + \frac{2}{3}\mu$$ – Helmholtz modulus (volumetric stiffness modulus, bulk modulus)

$$G = \mu$$ – Kirchhoff's modulus (transverse stiffness modulus, shear modulus, modulus of rigidity)

The above relations are sometimes referred to as the 3rd form of the generalized Hooke's Law.

**Law of change of volume** may be written down in the form of a **scalar equation**: $p = K\theta$

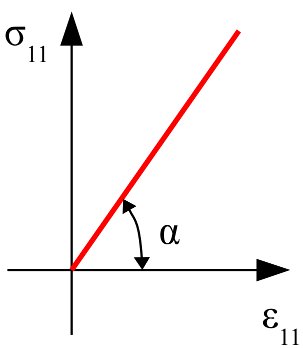
# ISOTROPIC HOOKE'S MATERIALS

| Material | Young's modulus [GPa] | Poisson's ratio [-] | Kirchhoff's modulus [GPa] |
|---|---|---|---|
| Iron | 200 | 0,28 | 65 |
| Copper | 120 | 0,36 | 45 |
| Aluminium | 70 | 0,33 | 26 |
| Tin | 50 | 0,33 | 18 |
| Zinc | 108 | 0,33 | 43 |
| Lead | 16 | 0,43 | 5,6 |
| Structural steel | 200 | 0,30 | 80 |
| Gray cast iron | 120 | 0,21 | 50 |
| Bronze | 100 | 0,34 | 45 |
| Brass | 100 | 0,33 | 40 |
| Spruce wood | 12 | - | - |
| Granite | 75 | 0,25 | 28 |
| Marble | 60 | 0,30 | 23 |
| Concrete | 30 | 0,20 | 12,5 |
| PCV | 1,5 | 0,42 | 0,6 |
| PTFE | 0,5 | 0,46 | 0,2 |
| PE | 0,14 | 0,40 | 0,05 |
| PS | 3,8 | 0,40 | 1 |
| Rubber | 0,05 | 0,50 | 0,0005 |

# ISOTROPIC HOOKE'S MATERIALS

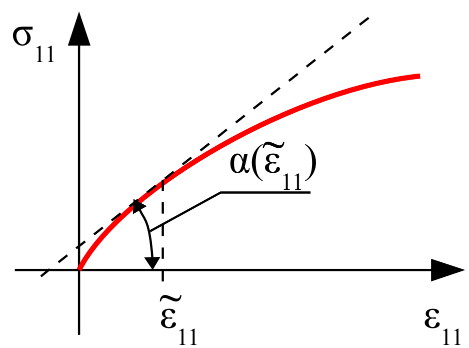
## Uniaxial stress state (pure tension)

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \rightarrow \quad \boldsymbol{\varepsilon} = \begin{bmatrix} \frac{\sigma}{E} & 0 & 0 \\ 0 & -\nu\frac{\sigma}{E} & 0 \\ 0 & 0 & -\nu\frac{\sigma}{E} \end{bmatrix}$$



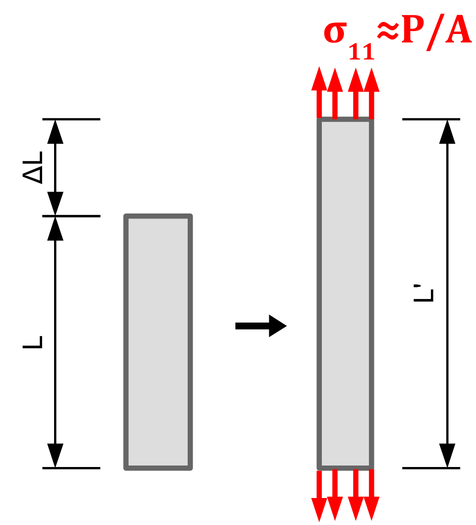

$$E = \frac{\sigma_{11}}{\varepsilon_{11}} = \operatorname{tg}\alpha$$

**Young's modulus**
for **linear elastic materials**




$$E(\tilde{\varepsilon}_{11}) = \left.\frac{d\sigma_{11}}{d\varepsilon_{11}}\right|_{\tilde{\varepsilon}_{11}} = \operatorname{tg}\alpha(\tilde{\varepsilon}_{11})$$

**Tengent Young's modulus**
for **non-linear materials**




$$\varepsilon_{11} \approx \frac{\Delta L}{L}$$

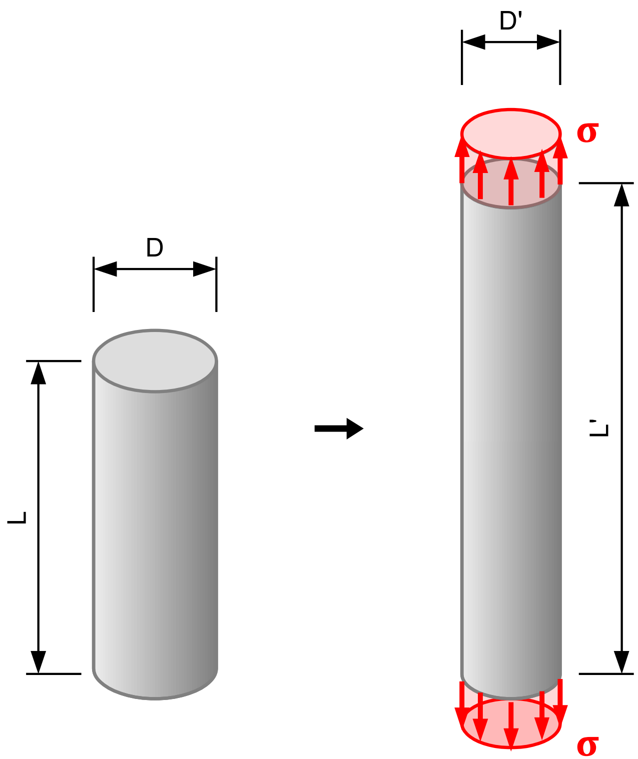
$\sigma_{11} \approx P/A$

$$E = \frac{\sigma_{11}}{\varepsilon_{11}} \approx \frac{P}{A} \cdot \frac{L}{\Delta L}$$

# ISOTROPIC HOOKE'S MATERIALS

Uniaxial stress state (pure tension)

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \rightarrow \quad \boldsymbol{\varepsilon} = \begin{bmatrix} \frac{\sigma}{E} & 0 & 0 \\ 0 & -\nu\frac{\sigma}{E} & 0 \\ 0 & 0 & -\nu\frac{\sigma}{E} \end{bmatrix}$$




$$\varepsilon_{11} \approx \frac{\Delta L}{L} = \frac{L'-L}{L}$$

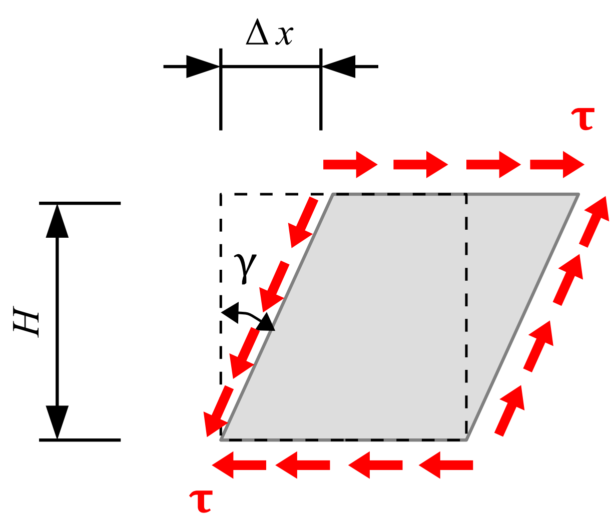
$$\varepsilon_{22} \approx \frac{\Delta D}{D} = \frac{D'-D}{D}$$

$$\nu = -\frac{\varepsilon_{22}}{\varepsilon_{11}} \approx -\frac{\Delta D}{D} \cdot \frac{L}{\Delta L}$$

# ISOTROPIC HOOKE'S MATERIALS

Simple shear state

$$\boldsymbol{\sigma} = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \rightarrow \quad \boldsymbol{\varepsilon} = \begin{bmatrix} 0 & \frac{\tau}{2G} & 0 \\ \frac{\tau}{2G} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$




$$\gamma = 2\varepsilon_{12} = \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) \approx \frac{\Delta x}{H}$$

$$\gamma \approx \mathrm{tg}\,\gamma = \frac{\Delta x}{H}$$

$$\tau = G\gamma \approx G\frac{\Delta x}{H}$$

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# LINEAR THEORY OF ELASTICITY

# LINEARIZATION OF THE THEORY OF ELASTICITY

A complete set of governing equation of non-linear theory of elasticity in the material description for a **hyperelastic material**:

- **3** equations of motion: $\left[S_{ij}+S_{kj}u_{i,k}\right]_{,j}+B_i=\rho_R\ddot{u}_i$
- **6** constitutive relations: $S_{ij}=\dfrac{\partial W}{\partial E_{ij}}$
- **6** kinematic relations: $E_{ij}=\dfrac{1}{2}\left(u_{i,j}+u_{j,i}+u_{k,i}u_{k,j}\right)$

  or (alternatively) **6** strain compatibility conditions: $\epsilon_{pqi}F_{jq,p}=0$

**15 equations for 15 unknowns**:

- **3 components** of the displacement vector: $u_1, u_2, u_3$
- **6 components** of the strain tensor: $E_{11},\ E_{22},\ E_{33},\ E_{23},\ E_{31},\ E_{12}$
- **6 components** of the stress tensor: $S_{11},\ S_{22},\ S_{33},\ S_{23},\ S_{31},\ S_{12}$

# LINEARIZATION OF THE THEORY OF ELASTICITY

For many **linear problems** it is possible to prove that:

- The **solution** of the problem exists
- The **solution** of the problem is unique

For complex **non-linear problems** such theorems usually are not true, namely:

- The **solution** of the problem **may not exist at all**
- There may exist **more than one possible solution**

Each source of non-linearity in the theory may cause difficulties:

- Equation of motion in spatial description are linear.
- **Relations between components of the PK1 stress tensor** resulting from principle of angular **momentum** are non-linear.
- Equations of motion in material description formulated with the use of the **PK2 stress tensor** are non-linear.
- Kinematic relations in the **finite strain theory** are non-linear.
- Constitutive relations for a **hyperelasicmaterial** may be in general non-linear.

# LINEARIZATION OF THE THEORY OF ELASTICITY

## SMALL STRAIN

$$u_{i,m} \ll 1$$

- Kinematic relation are approximately linear

$$E_{ij} \approx \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) = \varepsilon_{ij}$$

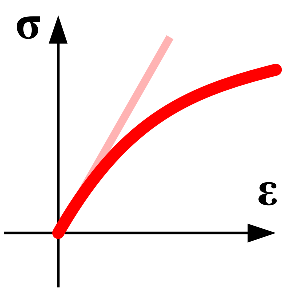
- Deformed elements are approximately the same as undeformed ones

$$\mathbf{E} \approx \boldsymbol{\varepsilon}$$

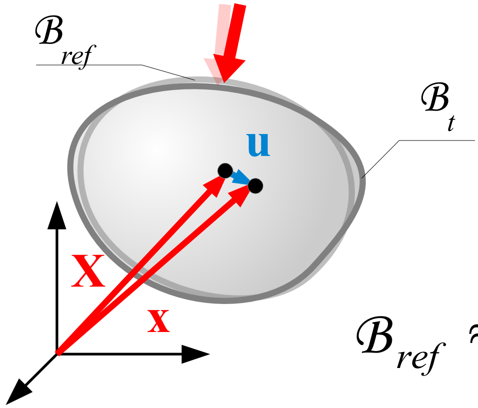
$$\mathbf{T}_{\sigma} \approx \mathbf{T}_{R} \approx \mathbf{T}_{S}$$

- If the strain is sufficiently small any constitutive relation may be approximated with a linear one

$$S_{ij} = \frac{\partial W}{\partial E_{ij}} \approx \approx \underbrace{\frac{\partial^2 W}{\partial E_{ij} \partial E_{kl}}}_{S_{ijkl}} E_{kl}$$




## SMALL DISPLACEMENT




$$\mathcal{B}_{ref} \approx \mathcal{B}_t$$

- Current configuration is approximately the same as the original one
- Spatial and material coordinates are approximately the same
- material description ≈ spatial description
- Linear equations of motion

**LINEAR THEORY OF ELASTICITY**

# LINEARIZATION OF THE THEORY OF ELASTICITY

Complete set of governing equations of the linear theory of elasticity:

- 3 equations of motion: $\sigma_{ij,j} + b_i = \rho \ddot{u}_i$
- 6 constitutive relations: $\sigma_{ij} = \mathsf{S}_{ijkl} \varepsilon_{kl}$
- 6 kinematic relations: $\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$

  or (alternatively) **6** strain compatibility conditions: $\epsilon_{pri} \epsilon_{qsj} \varepsilon_{pq,rs} = 0$

**15 equations for 15 unknowns**:

- **3 components** of the displacement vector: $u_1, u_2, u_3$
- **6 components** of the strain tensor: $\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{23}, \varepsilon_{31}, \varepsilon_{12}$
- **6 components** of the stress tensor: $\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{31}, \sigma_{12}$

# LINEARIZATION OF THE THEORY OF ELASTICITY

REMARKS:

- Small strain tensor $\boldsymbol{\varepsilon}$ is considered to be identical with the **material strain tensor** $\mathbf{E}$ .
- Stress tensor $\boldsymbol{\sigma}$ is considered to be identical with the true (Cauchy) stress tensor $\mathbf{T}_{\boldsymbol{\sigma}}$ .
- Following theorem may be proved:

KIRCHHOFF'S THEOREM

***If the boundary conditions are given over the whole boundary surface enclosing the configuration of the body and these conditions are either entirely static*** *(prescribed tractions)* ***or entirely kinematic*** *(prescribed displacements)* ***then the solution of the problem of the linear theory of elasticity in the static case is unique.***

# METHODS OF SOLVING THE PROBLEMS OF THE LINEAR THEORY OF ELASTICITY

# LAMÉ'S DISPLACEMENT EQUATIONS

Let's consider the equations of motion: $\sigma_{ij,j} + b_i = \rho \ddot{u}_i$

Let's make use of the constitutive relations: $\sigma_{ij} = 2G\varepsilon_{ij} + \lambda \varepsilon_{kk}\delta_{ij}$

We will get:

$$\left[2G\varepsilon_{ij} + \lambda\delta_{ij}\varepsilon_{kk}\right]_{,j} + b_i = \rho\ddot{u}_i$$

$$2G\varepsilon_{ij,j} + \lambda\delta_{ij}\varepsilon_{kk,j} + b_i = \rho\ddot{u}_i$$

Strains may be expressed in terms of displacements according to kinematic relations: $\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$

$$2G\cdot\frac{1}{2}\left(u_{i,jj} + u_{j,ij}\right) + \lambda\delta_{ij}\frac{1}{2}\left(u_{k,kj} + u_{k,kj}\right) + b_i = \rho\ddot{u}_i$$

$$G\left(u_{i,jj} + u_{j,ij}\right) + \lambda\delta_{ij}u_{k,kj} + b_i = \rho\ddot{u}_i$$

$$G\left(u_{i,jj} + u_{j,ij}\right) + \lambda u_{j,ji} + b_i = \rho\ddot{u}_i$$

# LAMÉ'S DISPLACEMENT EQUATIONS

We obtain:

$$G u_{i,jj} + (G+\lambda) u_{j,ji} + b_i = \rho \ddot{u}_i \,, \qquad i=1,2,3$$

Explicitly written:

$$\begin{cases} G\nabla^2 u_1 + (G+\lambda)(u_{1,11}+u_{2,21}+u_{3,31}) + b_1 = \rho \ddot{u}_1 \\ G\nabla^2 u_2 + (G+\lambda)(u_{1,12}+u_{2,22}+u_{3,32}) + b_2 = \rho \ddot{u}_2 \\ G\nabla^2 u_3 + (G+\lambda)(u_{1,13}+u_{2,23}+u_{3,33}) + b_3 = \rho \ddot{u}_3 \end{cases}$$

In absolute notation:

$$G \Delta \mathbf{u} + (G+\lambda)\left[\nabla(\mathbf{u}\cdot\nabla)\right] + \mathbf{b} = \rho \ddot{\mathbf{u}}$$

The  above relations are referred to as the Lamé's displacement equations or Navier – Cauchy equations.

- It is a **system of 3 linear partial differential equations od the 2nd order**.
- **Initial conditions** and **kinematic boundary condition** are the conditions for **3 unknown displacements**.
- **Static boundary conditions** must be expressed in terms of **derivatives of displacements**.

## THANK YOU FOR YOUR ATTENTION