THEORY OF ELASTICITY AND PLASTICITY

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METHODS OF SOLVING OF PROBLEMS OF THE LINEAR THEORY OF ELASTICITY

METHODS OF SOLVING OF PROBLEMS OF THE LINEAR ToE

ANALYTICAL METHODS

NUMERICAL METHODS

METHODS OF SOLVING OF PROBLEMS OF THE LINEAR ToE

ANALYTICAL METHODS

NUMERICAL METHODS

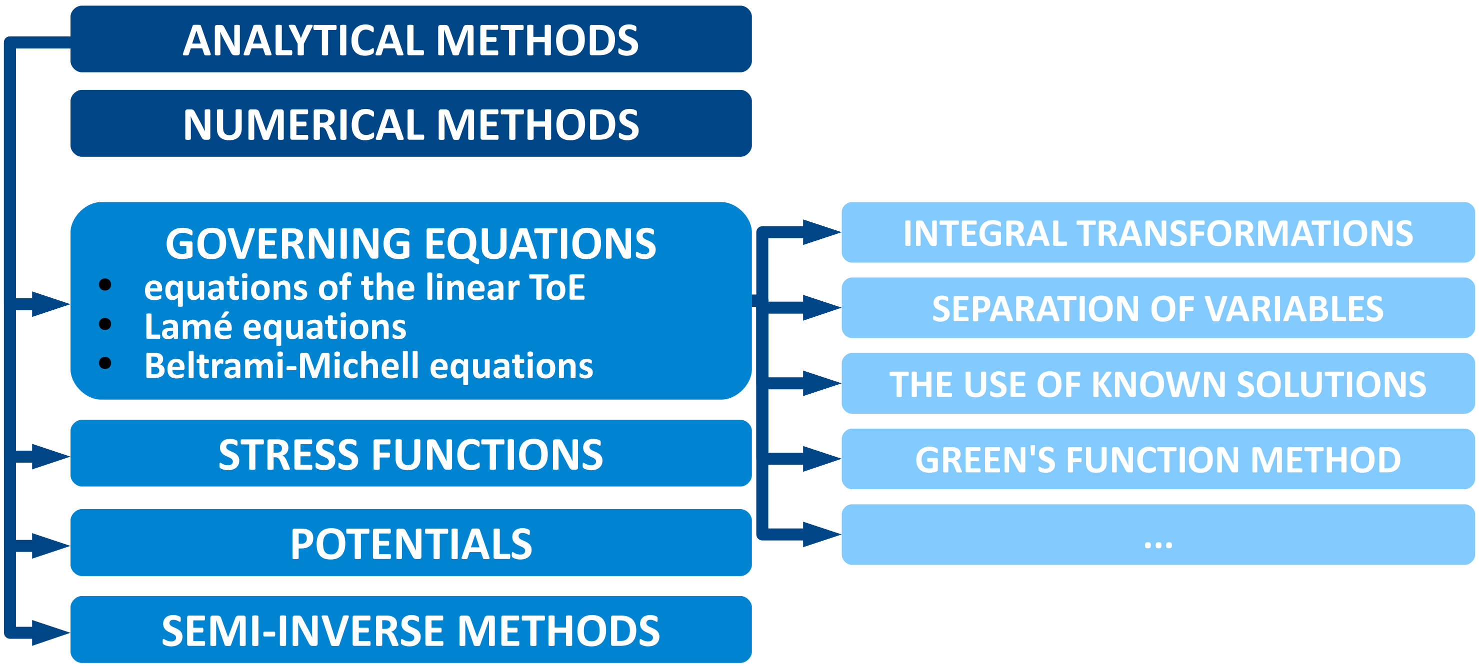
GOVERNING EQUATIONS
- **equations of the linear ToE**
- **Lamé equations**
- **Beltrami-Michell equations**

STRESS FUNCTIONS

POTENTIALS

SEMI-INVERSE METHODS

METHODS OF SOLVING OF PROBLEMS OF THE LINEAR ToE



Finding a solution of the 2nd order PDE is very difficult in a general case. For certain problems it is possible to use the general methods developed within the calculus, e.g.:

- intergal transformations – changing the **differential** equations into **algebraic** equations + **inverse transformation**
- separation of variables – changing **partial** DEs into **ordinary** DEs.
- equations may be transformed, so that they become the equation for which a solution is known.
- If the Green's function of the problem is known, then related problems may be solved by proper integration of this function.

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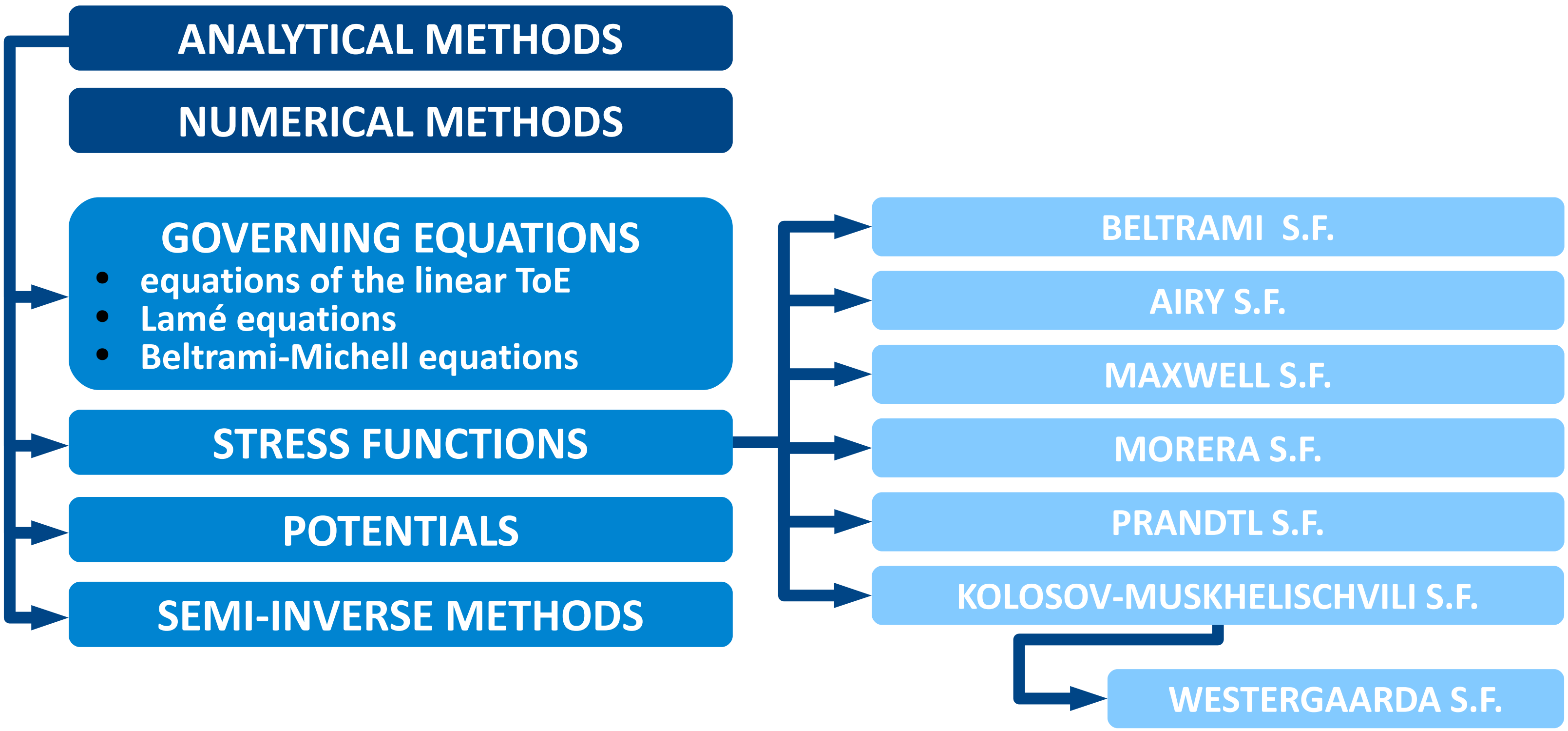
METHODS OF SOLVING OF PROBLEMS OF THE LINEAR ToE



Components of the stress tensor may be expressed as **combinations of max. 6 unknown functions** known as the stress functions.

- stress components expressed in such a way **always satisfy equilibrium equations**.
- they must satisfy also **compatibility conditions** – this yield a **system of PDE for stress functions**.
- for some problems the **number of stress functions may be decreased** – it simplifies the problem.

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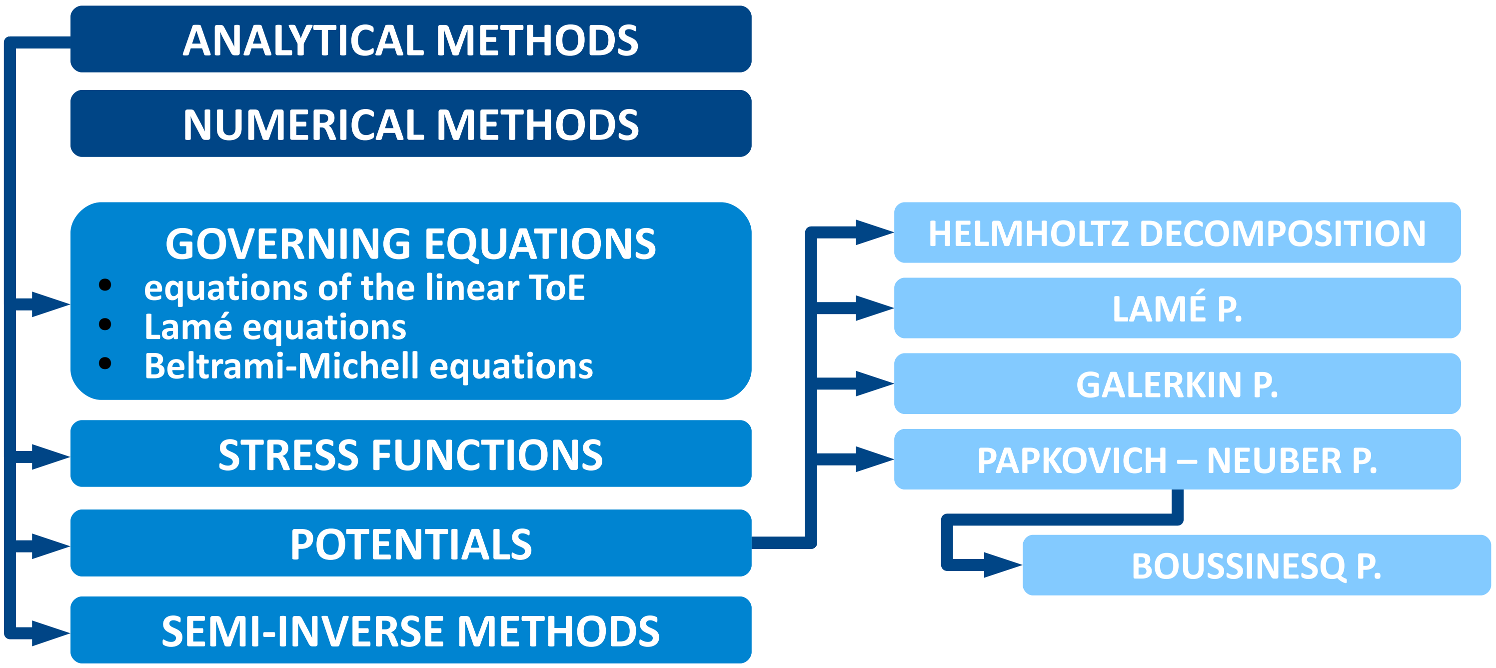
METHODS OF SOLVING OF PROBLEMS OF THE LINEAR ToE



Displacement vector components may be expressed as **derivatives of certain functions –** scalar potentials or **components of** vector potentials**.**

- proper choice of the relation between components of $\mathbf{u}$ and potentials may **guarantee satisfying the governing equations of the linear ToE** if only the potentials satisfy some **simpler equations**, for which the **solution is known**.
- these **solutions must be chosen in suach a way so that boundary conditions are also satisfied**.

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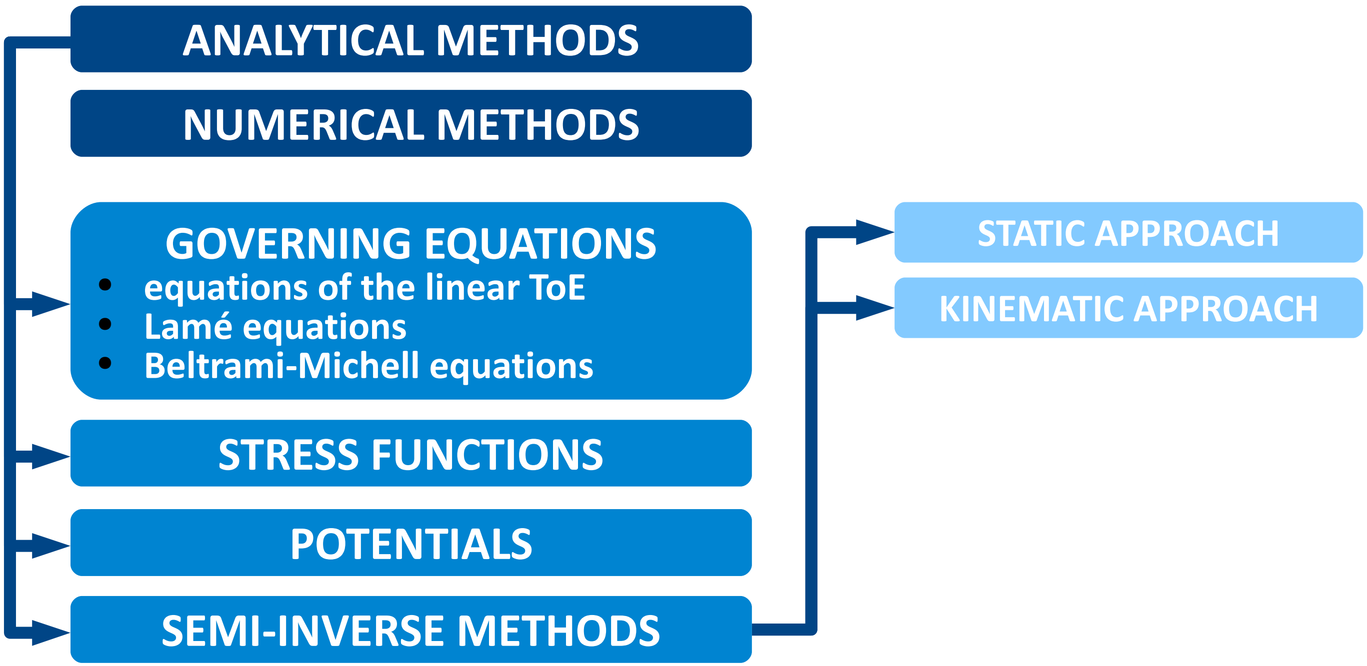
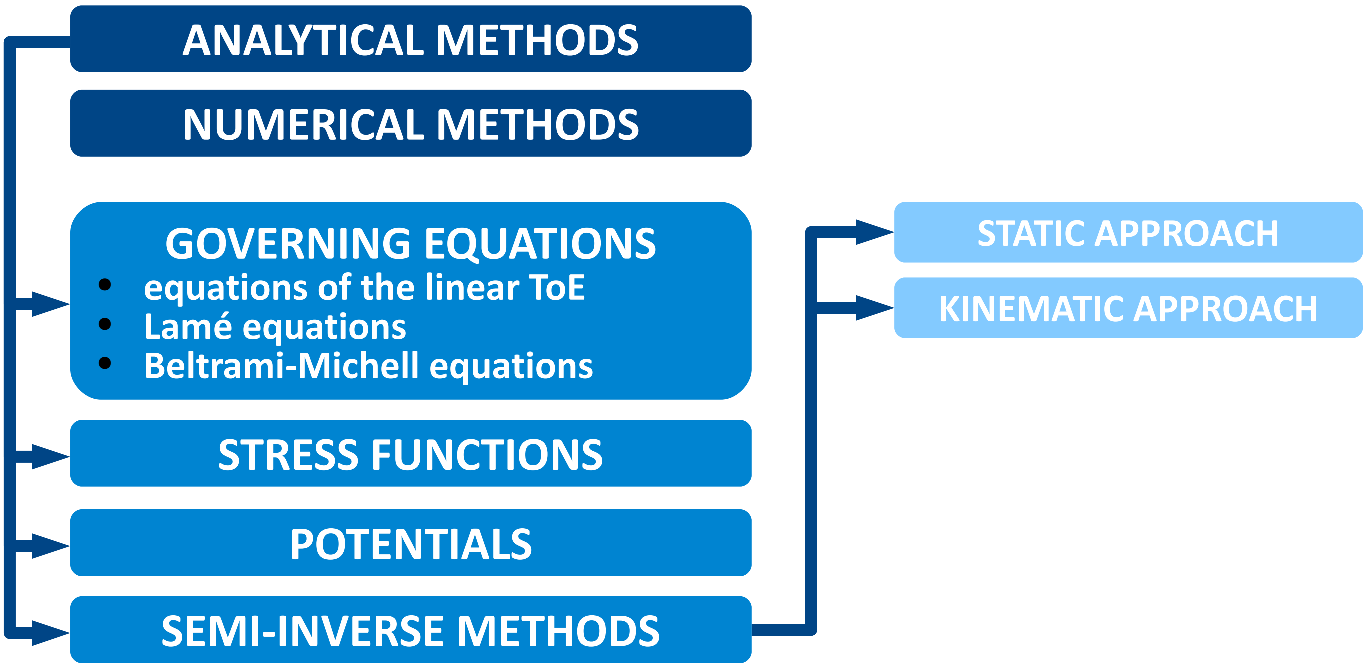
METHODS OF SOLVING OF PROBLEMS OF THE LINEAR ToE



In the semi-inverse methods a general form of the solution is assumed, and then the functions or parameters occurring in that for are chosen in such a way, so that all conditions were satisfied.

- Static approach – **we assume the form of a stress state field that satisfies the boundary conditions and equilibrium equations**. We determine the strain state and if it satisfies the compatibility conditions, we may determine the displacement field. **Specific form of the solution is chosen** in such a way, so that **kinematic boundary conditions were satidfied**.

METHODS OF SOLVING OF PROBLEMS OF THE LINEAR ToE



In the semi-inverse methods a general form of the solution is assumed, and then the functions or parameters occurring in that for are chosen in such a way, so that all conditions were satisfied.

- Kinematic approach – **we assume the form of the displacement field that satisfies the kinematic boundary conditions**. We determine the strain state and stress state. **Specific form of the solution is chosen** in such a way, so that **equilibrium equations and static boudary conditions were satisfied**.

METHODS OF SOLVING OF PROBLEMS OF THE LINEAR ToE

ANALYTICAL METHODS

NUMERICAL METHODS

- FINITE DIFFERENCE METHOD
- FINITE ELEMENT METHOD
- BOUNDARY ELEMENT METHOD
- DISCRETE ELEMENT METHOD
- FINITE VOLUME METHOD
- ...

The majority of most commonly used numerical methods for solving the problems of ToE have the following outline:

- introduction of **algebraic equations** – **satisfying them in a small region is equivalent to satisfying in an approximate way the governing equations** of the problem.
- division of the domain of the problem into small subregions (**discretization**)
- Replacing the system of PDE with a large system of algebraic equations written down for these subregions.

METHODS OF SOLVING OF PROBLEMS OF THE LINEAR ToE

Solutions of the problems of ToE may be presented in the form of:

- closed-form expression – e.g. displacement field in an elastic space loaded with a point force:

$$u_i = \frac{P}{4\pi G}\left[\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\delta_{i1} + \frac{1}{4(1-\nu)}\frac{x_i x_1}{\left[x_1^2+x_2^2+x_3^2\right]^{3/2}}\right]$$

- a series – e.g. deflection of a uniformly loaded rectangular thin plate:

$$w(x,y) = \frac{48 q(1-\nu^2)}{\pi^6 E h^3}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{(1-\cos(m\pi))(1-\cos(n\pi))}{m n\left(\frac{m^2}{L_x^2}+\frac{n^2}{L_y^2}\right)^2}\sin\left(\frac{m\pi x}{L_x}\right)\sin\left(\frac{n\pi y}{L_y}\right)\right]$$

- an integral – e.g. displacement field in an elastic half-plane loaded at its edge:

$$\mathbf{u}(\mathbf{x}) = \begin{cases} u_1 = \dfrac{1}{\pi E}\displaystyle\int_{-L}^{L} p(\xi)\left[(1+\hat{\nu})\frac{(x_1-\xi)x_2}{(x_1-\xi)^2+x_2^2} - (1-\hat{\nu})\,\mathrm{arctg}\frac{x_1-\xi}{x_2}\right]\mathrm{d}\xi \\ u_2 = -\dfrac{1}{\pi E}\displaystyle\int_{-L}^{L} p(\xi)\left[\ln\left[(x_1-\xi)^2+x_2^2\right] + (1+\hat{\nu})\frac{(x_1-\xi)^2}{(x_1-\xi)^2+x_2^2}\right]\mathrm{d}\xi \end{cases}$$

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THE BELTRAMI – MICHELL STRESS COMPATIBILITY EQUATIONS

BELTRAMI-MICHELL EQUATIONS

Let's consider the strain compatibility conditions ($i, j, k, l = 1{,}2{,}3$):

$$\nabla\times(\nabla\times\boldsymbol{\varepsilon}) = \mathbf{0} \qquad \Leftrightarrow \qquad \varepsilon_{ik,jl}-\varepsilon_{jk,il}-\varepsilon_{il,jk}+\varepsilon_{jl,ik}=0$$

Let's substitute the **constitutive relations**:

$$\varepsilon_{ij}=\frac{1+\nu}{E}\sigma_{ij}-\frac{\nu}{E}\delta_{ij}\sigma_{nn}$$

We obtain:

$$\frac{1+\nu}{E}\left[\sigma_{ik,jl}-\sigma_{jk,il}-\sigma_{il,jk}+\sigma_{jl,ik}\right]-\frac{\nu}{E}\left[\delta_{ik}\sigma_{nn,jl}-\delta_{jk}\sigma_{nn,il}-\delta_{il}\sigma_{nn,jk}+\delta_{jl}\sigma_{nn,ik}\right]=0$$

Let's perform the **contraction** (summation) **with respect to indices** i and k:

$$\frac{1+\nu}{E}\left[\sigma_{ii,jl}-\sigma_{ji,il}-\sigma_{il,ji}+\sigma_{jl,ii}\right]-\frac{\nu}{E}\left[\delta_{ii}\sigma_{nn,jl}-\delta_{ji}\sigma_{nn,il}-\delta_{il}\sigma_{nn,ji}+\delta_{jl}\sigma_{nn,ii}\right]=0$$

BELTRAMI-MICHELL EQUATIONS

Let's account for the properties of the **Kronecker's delta**:

$$\frac{1+\nu}{E}\left[\sigma_{ii,jl}-\sigma_{ji,il}-\sigma_{il,ji}+\sigma_{jl,ii}\right]-\frac{\nu}{E}\left[3\sigma_{nn,jl}-\sigma_{nn,jl}-\sigma_{nn,jl}+\delta_{jl}\sigma_{nn,ii}\right]=0$$

The above expression may be rearranged:

$$\sigma_{jl,ii}+\left[1-\frac{\nu}{1+\nu}\right]\sigma_{nn,jl}-\sigma_{ji,il}-\sigma_{li,ij}-\frac{\nu}{1+\nu}\delta_{jl}\sigma_{nn,ii}=0$$

BELTRAMI-MICHELL EQUATIONS

Let's deal now with the equilibrium equations:

$$\sigma_{ji,i} + b_j = 0$$

We may differentiate this relation with respect to variable x_l, and the write this expression down with switched free indices l and j. We obtain:

$$-\sigma_{ji,il} = b_{j,l} \qquad -\sigma_{li,ij} = b_{l,j}$$

The **strain compatibility conditions** have already given us:

$$\sigma_{jl,ii} + \left[1 - \frac{\nu}{1+\nu}\right]\sigma_{nn,jl} - \sigma_{ji,il} - \sigma_{li,ij} - \frac{\nu}{1+\nu}\delta_{jl}\sigma_{nn,ii} = 0$$

Let's substitute the transformed **equilibrium equations**:

$$\sigma_{jl,ii} + \left[1 - \frac{\nu}{1+\nu}\right]\sigma_{nn,jl} + \left(b_{j,l} + b_{l,j}\right) - \frac{\nu}{1+\nu}\delta_{jl}\sigma_{nn,ii} = 0$$

BELTRAMI-MICHELL EQUATIONS

Let's introduce another **auxiliary relation**. We will perform **contraction** (summation) **with respect to indices** l and j :

$$\sigma_{kk,ii} + \left[1 - \frac{\nu}{1+\nu}\right]\sigma_{nn,kk} + \left(b_{k,k} + b_{k,k}\right) - \frac{\nu}{1+\nu}\delta_{kk}\sigma_{nn,ii} = 0$$

Dummy indices (wrt which we are performing summation), **may be changed**:

$$\sigma_{nn,kk} + \left[1 - \frac{\nu}{1+\nu}\right]\sigma_{nn,kk} + 2\,b_{k,k} - \frac{3\nu}{1+\nu}\sigma_{nn,kk} = 0$$

This may be simplified, so that following relation is obtained:

$$\sigma_{nn,kk} = -\frac{1+\nu}{1-\nu}b_{k,k}$$

This result will be now substituted in the equation in which we have just performed contranction with respect to l and j .

BELTRAMI-MICHELL EQUATIONS

We obtain a system of the Beltrami – Michell equations (**stress compatibility conditions**)

$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{kk,ij} + \left(b_{i,j} + b_{j,i}\right) + \frac{\nu}{1-\nu}\delta_{ij} b_{k,k} = 0$$

Written explicitly:

$$\begin{cases}
\nabla^2\sigma_{11} + \frac{1}{1+\nu}\left(\sigma_{11,11}+\sigma_{22,11}+\sigma_{33,11}\right) + 2b_{1,1} + \frac{\nu}{1-\nu}\left(b_{1,1}+b_{2,2}+b_{3,3}\right) = 0 \\
\nabla^2\sigma_{22} + \frac{1}{1+\nu}\left(\sigma_{11,22}+\sigma_{22,22}+\sigma_{33,22}\right) + 2b_{2,2} + \frac{\nu}{1-\nu}\left(b_{1,1}+b_{2,2}+b_{3,3}\right) = 0 \\
\nabla^2\sigma_{33} + \frac{1}{1+\nu}\left(\sigma_{11,33}+\sigma_{22,33}+\sigma_{33,33}\right) + 2b_{3,3} + \frac{\nu}{1-\nu}\left(b_{1,1}+b_{2,2}+b_{3,3}\right) = 0 \\
\nabla^2\sigma_{23} + \frac{1}{1+\nu}\left(\sigma_{11,23}+\sigma_{22,23}+\sigma_{33,23}\right) + \left(b_{2,3} + b_{3,2}\right) = 0 \\
\nabla^2\sigma_{31} + \frac{1}{1+\nu}\left(\sigma_{11,31}+\sigma_{22,31}+\sigma_{33,31}\right) + \left(b_{3,1} + b_{1,3}\right) = 0 \\
\nabla^2\sigma_{12} + \frac{1}{1+\nu}\left(\sigma_{11,12}+\sigma_{22,12}+\sigma_{33,12}\right) + \left(b_{1,2} + b_{2,1}\right) = 0
\end{cases}$$

BELTRAMI-MICHELL EQUATIONS

If the **body forces are uniformly distributed**, the Beltrami-Michell equations take the following form:

$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{kk,ij} = 0$$

Written explicitly:

$$\begin{cases} \nabla^2\sigma_{11} + \frac{1}{1+\nu}\left(\sigma_{11,11}+\sigma_{22,11}+\sigma_{33,11}\right) = 0 \\ \nabla^2\sigma_{22} + \frac{1}{1+\nu}\left(\sigma_{11,22}+\sigma_{22,22}+\sigma_{33,22}\right) = 0 \\ \nabla^2\sigma_{33} + \frac{1}{1+\nu}\left(\sigma_{11,33}+\sigma_{22,33}+\sigma_{33,33}\right) = 0 \\ \nabla^2\sigma_{23} + \frac{1}{1+\nu}\left(\sigma_{11,23}+\sigma_{22,23}+\sigma_{33,23}\right) = 0 \\ \nabla^2\sigma_{31} + \frac{1}{1+\nu}\left(\sigma_{11,31}+\sigma_{22,31}+\sigma_{33,31}\right) = 0 \\ \nabla^2\sigma_{12} + \frac{1}{1+\nu}\left(\sigma_{11,12}+\sigma_{22,12}+\sigma_{33,12}\right) = 0 \end{cases}$$

BELTRAMI-MICHELL EQUATIONS

REMARKS:

- Laplacian is defined as follows:

$$\nabla^2 = \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

- **The Beltrami – Michell equations** were derived
  - from the strain compatibility conditions
  - Assuming that equilibrium equations **are satisfied**.

- **Not every solution of the Beltrami – Michell equations is a solution of a problem of linear ToE.** Such a **solution must also satisfy the equilibrium equations**.

- The Beltrami – Michell equations are in fact **strain compatibility conditions expressed in terms of stress**. For this reason they are termed also **stress compatibility conditions**. Satisfying those equations (together with equilibrium equations) guarantee that **kinematic relations are integrable**, so:
  - If a stress field satisfies this system of equations, then according to the Hooke's Law we may calculate strain field.
  - Since strain compatibility conditions are satisfied, then according to the kinematic relations it is possible to determine the displacement field from the obtained strain field.

STRESS FUNCTIONS

STRESS FUNCTIONS

Let's consider a **3-dimensional symmetric 2nd order tensor field** $\mathbf{\Phi} = \mathbf{\Phi}(\mathbf{x})$, which will be termed the Beltrami stress tensor. We assume that the components $\Phi_{ij}(\mathbf{x})$ of this tensor field are at least twice differentiable. Let's determine the stress tensor field according to the following relation::

$$\sigma_{ij} = \epsilon_{ikm}\epsilon_{jln}\Phi_{kl,mn}$$

Let's **substitute the stress state** expressed in the above way into the Navier equilibrium equations, **neglecting the body forces**.

$$\sigma_{ij,j} = 0$$

$$\epsilon_{ikm}\epsilon_{jln}\Phi_{kl,mnj} = 0$$

After making the use of the properties of the Levi-Civita symbol it may be shown that the above relations is identically satisfied for any symmetric tensor field $\Phi_{ij}(\mathbf{x})$ of the C^2 class, so the Beltrami stress tensor always satisfies the equilibrium equations (**without body forces**).

In case, **when it also satisfies the Beltrami – Michell equations**, it will be the **solution of the problem of linear ToE**.

STRESS FUNCTIONS

In order to simplify the notation, the Beltrami stress tensor will be presented in the following way:

$$\mathbf{\Phi} = \begin{bmatrix} A & B & C \\ B & D & E \\ C & E & F \end{bmatrix}$$

Stress tensor components are calculated according to the relation: $\sigma_{ij} = \epsilon_{ikm}\epsilon_{jln}\Phi_{kl,mn}$

$$\sigma_{11} = \frac{\partial^2 D}{\partial x_3^2} - 2\frac{\partial^2 E}{\partial x_2 \partial x_3} + \frac{\partial^2 F}{\partial x_2^2} \qquad \sigma_{23} = \frac{\partial^2 C}{\partial x_1 \partial x_2} - \frac{\partial^2 E}{\partial x_1^2} - \frac{\partial^2 A}{\partial x_2 \partial x_3} + \frac{\partial^2 B}{\partial x_3 \partial x_1}$$

$$\sigma_{22} = \frac{\partial^2 F}{\partial x_1^2} - 2\frac{\partial^2 C}{\partial x_3 \partial x_1} + \frac{\partial^2 A}{\partial x_3^2} \qquad \sigma_{31} = \frac{\partial^2 B}{\partial x_2 \partial x_3} - \frac{\partial^2 C}{\partial x_2^2} - \frac{\partial^2 D}{\partial x_3 \partial x_1} + \frac{\partial^2 E}{\partial x_1 \partial x_2}$$

$$\sigma_{33} = \frac{\partial^2 A}{\partial x_2^2} - 2\frac{\partial^2 B}{\partial x_1 \partial x_2} + \frac{\partial^2 D}{\partial x_1^2} \qquad \sigma_{12} = \frac{\partial^2 E}{\partial x_3 \partial x_1} - \frac{\partial^2 B}{\partial x_3^2} - \frac{\partial^2 F}{\partial x_1 \partial x_2} + \frac{\partial^2 C}{\partial x_2 \partial x_3}$$

Components A, B, … are functions of $\mathbf{x}$. We will call them the Beltrami stress functions. The Beltrami – Michell equations constitute a **system of partial differential equation of the 4th order** for these functions. These equations are quite complicated, but they are **linear** and have **constant coefficients**.

STRESS FUNCTIONS

In the cases of some problems of the theory of elasticity it is possible to **account for some properties of symmetry** of the problem resulting from its **geometry** and **boundary conditions**.

In such situations it is possible to express the solution of the problem with the use of **simplified form of the Beltrami stress tensor**, namely such that some of its components are equal to 0:

EXAMPLE:

Maxwell stress function:

$$\mathbf{\Phi} = \begin{bmatrix} A & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & F \end{bmatrix}$$

$$\sigma_{11} = \frac{\partial^2 D}{\partial x_3^2} + \frac{\partial^2 F}{\partial x_2^2} \qquad \sigma_{23} = -\frac{\partial^2 A}{\partial x_2 \partial x_3}$$

$$\sigma_{22} = \frac{\partial^2 F}{\partial x_1^2} + \frac{\partial^2 A}{\partial x_3^2} \qquad \sigma_{31} = -\frac{\partial^2 D}{\partial x_3 \partial x_1}$$

$$\sigma_{33} = \frac{\partial^2 A}{\partial x_2^2} + \frac{\partial^2 D}{\partial x_1^2} \qquad \sigma_{12} = -\frac{\partial^2 F}{\partial x_1 \partial x_2}$$

STRESS FUNCTIONS

The most commonly used stress function is the Airy stress function. We assume that it depends only on two independent variables (e.g. x_1, x_2 – **the problem of plane stress state**)

Airy stress function:

$$\mathbf{\Phi} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & F(x_1, x_2) \end{bmatrix}$$

$$\sigma_{11} = \frac{\partial^2 F}{\partial x_2^2} \qquad \sigma_{22} = \frac{\partial^2 F}{\partial x_1^2} \qquad \sigma_{12} = -\frac{\partial^2 F}{\partial x_1 \partial x_2}$$

$$\sigma_{33} = 0 \qquad \sigma_{23} = 0 \qquad \sigma_{31} = 0$$

If we add the 1st and the 2nd equation of the Beltrami – Michell system written down for the stress state expressed in the terms of the Airy stress function, then such a relation will be satisfied if the **Airy stress function is a** biharmonic function (**it satisfies the biharmonic equation**):

$$\nabla^4 F = \frac{\partial^4 F}{\partial x_1^4} + 2\frac{\partial^4 F}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 F}{\partial x_2^4} = 0$$

POTENTIALS

POTENTIALS

The name „potential" is related to a similar object used in the electrostatics. **Electric potential** V is a **scalar-valued function**, the **gradient of which** (a vector) is an opposite vector to the **electric field vector** $\mathbf{E}$.

$$\nabla V = -\mathbf{E}$$

According to the **Gauss Law** (for electric field) the **potential satisfies the Poisson's equation**:

$$\Delta V = \frac{\rho}{\epsilon}$$

whre ρ is a volume density of an electric charge and ϵ is the permittivity of considered medium.

Generally, a potential is a (scalar or vector valued) function such that:

- An **operation of a certain differential operator on that function** gives the **solution of the stated mathematical problem**.
- It s**atisfies a certain differential equation**, the solution of which is usually known.

POTENTIALS

EXAMPLES:

- Lamé strain potential:
  - displacement field: $$u_i = \frac{1}{G}\phi_{,i} \quad \Leftrightarrow \quad \mathbf{u} = \frac{1}{G}\nabla\phi = \left[\frac{1}{G}\frac{\partial\phi}{\partial x_1}\,;\,\frac{1}{G}\frac{\partial\phi}{\partial x_2}\,;\,\frac{1}{G}\frac{\partial\phi}{\partial x_3}\right]$$
  - potential satisfies the **Poisson equation**: $\nabla^2\phi = C = const.$
  - **solution** of the Poisson equation is known – e.g. for an elastic space:

$$\phi(x_1, x_2) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[C\ln\left(\frac{1}{\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2}}\right)\right]\mathrm{d}\xi_1\,\mathrm{d}\xi_2$$

REMARKS:

- if the **potential satisfies the Poisson equation**, then the displacement vector $\mathbf{u}$ given by this potential **satisfies the** Lamé equations **for** statics neglecting the body forces.
- this **potential cannot describe all possible solutions**, but only those for which the **small rotation tensor is zero**!

POTENTIALS

EXAMPLES:

- Galerkin vector potential:
  - displacement field:
    $$u_i = \frac{1}{G} g_{i,jj} - \frac{\lambda + G}{G(\lambda + 2G)} g_{j,ji}$$
  - the potential satisfied the **inhomogeneous biharmonic equation**: $\nabla^4 g_i = -b_i$

REMARKS:

- For some domains (configurations of a body) and boundary conditions the solution of this equation is known.
- if the **Gelerkin vector potential satisfies the inhomogeneous biharmonic equation**, then the displacement vector $\mathbf{u}$ given by this potential **satisfies the** Lamé equations **for statics**.

POTENTIALS

EXAMPLES:

- Papkovich – Neuber potential:
  - displacement field: $$u_i = \frac{1}{G}\left[\psi_i + \frac{1}{4(1-\nu)}\left(\phi - \psi_j x_j\right)_{,i}\right]$$
  - **vector potential** satisfies the **Poisson equation**: $$\nabla^2 \psi_i = -b_i$$
  - **scalar potential** satisfies the **Poisson equation**: $$\nabla^2 \phi = -b_k x_k$$

REMARKS:

- For some domains and boundary conditions the solution of this equation is known.
- if the **Papkovich-Neuber potentials satisfy proper equations**, then the displacement vector $\mathbf{u}$ given by this potential **satisfies the Lamé equations for statics**.
- **Every displacement field, which is a solution of a problem of linear ToE, may be expressed by a proper Papkovich – Neuber potential.**

POTENTIALS

EXAMPLES:

- Boussinesq potentials – these are **special cases** of the **Papkovich – Neuber potentials:**
  - **potential A** $\boldsymbol{\psi} = [0\,;0\,;0], \qquad \phi = 2(1-\nu)\phi_A$ where $\nabla^2 \phi_A = 0$
  - **potential B** $\boldsymbol{\psi} = -2(1-\nu)[0\,;0\,;\phi_B] \qquad \phi = 0$ where $\nabla^2 \phi_B = 0$
  - **potential C** $\boldsymbol{\psi} = -2(1-\nu)[0\,;\phi_C\,;0] \qquad \phi = 0$ where $\nabla^2 \phi_C = 0$
  - **potential D** $\boldsymbol{\psi} = -2(1-\nu)[\phi_D\,;0\,;0] \qquad \phi = 0$ where $\nabla^2 \phi_D = 0$
  - **potential E** $\boldsymbol{\psi} = \left[\frac{1}{2}\frac{\partial \phi_E}{\partial x_2}\,;-\frac{1}{2}\frac{\partial \phi_E}{\partial x_1}\,;0\right] \qquad \phi = \frac{1}{2}\left(\frac{\partial \phi_E}{\partial x_2}x_1 - \frac{\partial \phi_E}{\partial x_1}x_2\right)$ where $\nabla^2 \phi_E = 0$
  - **potential F** $\boldsymbol{\psi} = \left[-\frac{1}{2}\frac{\partial \phi_F}{\partial x_3}\,;0\,;\frac{1}{2}\frac{\partial \phi_F}{\partial x_1}\right] \qquad \phi = \frac{1}{2}\left(\frac{\partial \phi_F}{\partial x_1}x_3 - \frac{\partial \phi_F}{\partial x_3}x_1\right)$ where $\nabla^2 \phi_F = 0$
  - **potential G** $\boldsymbol{\psi} = \left[0\,;\frac{1}{2}\frac{\partial \phi_G}{\partial x_3}\,;-\frac{1}{2}\frac{\partial \phi_G}{\partial x_2}\right] \qquad \phi = \frac{1}{2}\left(\frac{\partial \phi_G}{\partial x_3}x_2 - \frac{\partial \phi_G}{\partial x_2}x_3\right)$ where $\nabla^2 \phi_G = 0$

REMARK: Boussinesq potentials describe the solutions of problems in which **body forces are neglected.**

THANK YOU FOR YOUR ATTENTION