# THEORY OF ELASTICITY AND PLASTICITY

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# PLANE PROBLEMS

# PLANE PROBLEMS

For certain specific geometries of the problems and prescribed boundary conditions, the stress state may be a tensor field depending only on two independent variables, e.g. $x_1$ i $x_2$, and independent of variable $x_3$. Also all components corresponding with that direction may be equal to 0:

$$\boldsymbol{\sigma}(\mathbf{x}) = \begin{bmatrix} \sigma_{11}(x_1;x_2) & \sigma_{12}(x_1;x_2) & 0 \\ \sigma_{12}(x_1;x_2) & \sigma_{22}(x_1;x_2) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We call such a problem a **plane problem** or a **two-dimensional** one in the plane $(x_1, x_2)$. The stress state as above is termed the plane stress state.

# PLANE PROBLEMS

According to the **generalized Hooke's Law** for a plane stress state we have the following **strain state**:

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \begin{bmatrix} \frac{1}{E}[\sigma_{11}-\nu\sigma_{22}] & \frac{\sigma_{12}}{2G} & 0 \\ \frac{\sigma_{12}}{2G} & \frac{1}{E}[\sigma_{22}-\nu\sigma_{11}] & 0 \\ 0 & 0 & -\frac{\nu}{E}[\sigma_{11}+\sigma_{22}] \end{bmatrix} = \begin{bmatrix} \varepsilon_{11}(x_1;x_2) & \varepsilon_{12}(x_1;x_2) & 0 \\ \varepsilon_{12}(x_1;x_2) & \varepsilon_{22}(x_1;x_2) & 0 \\ 0 & 0 & \varepsilon_{33}(x_1;x_2) \end{bmatrix}$$

- In the plane stress state **also the strain state depends only on two variables** but **it is not a plane state** (it is not given by a two-dimensional tensor represented by a 2 x 2 matrix)

- **The strain state corresponding to the plane stress state** is termed to be an anti-plane state.

# PLANE PROBLEMS

We may also consider a plane strain state:

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \begin{bmatrix} \varepsilon_{11}(x_1;x_2) & \varepsilon_{12}(x_1;x_2) & 0 \\ \varepsilon_{12}(x_1;x_2) & \varepsilon_{22}(x_1;x_2) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then the **stress state** is an anti-plane state:

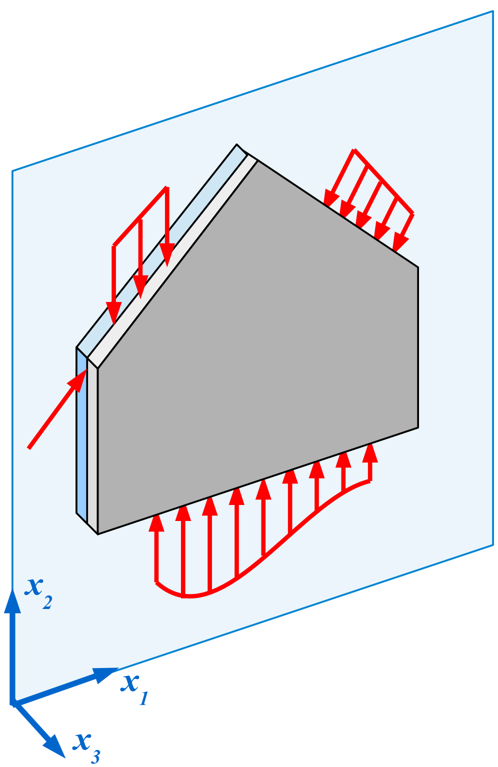
$$\boldsymbol{\sigma}(\mathbf{x}) = \begin{bmatrix} (2G+\lambda)\varepsilon_{11}+\lambda\varepsilon_{22} & 2G\varepsilon_{12} & 0 \\ 2G\varepsilon_{12} & (2G+\lambda)\varepsilon_{22}+\lambda\varepsilon_{11} & 0 \\ 0 & 0 & \lambda(\varepsilon_{11}+\varepsilon_{22}) \end{bmatrix} = \begin{bmatrix} \sigma_{11}(x_1;x_2) & \sigma_{12}(x_1;x_2) & 0 \\ \sigma_{12}(x_1;x_2) & \sigma_{22}(x_1;x_2) & 0 \\ 0 & 0 & \sigma_{33}(x_1;x_2) \end{bmatrix}$$

# PLANE PROBLEMS

**PLANE PROBLEMS**

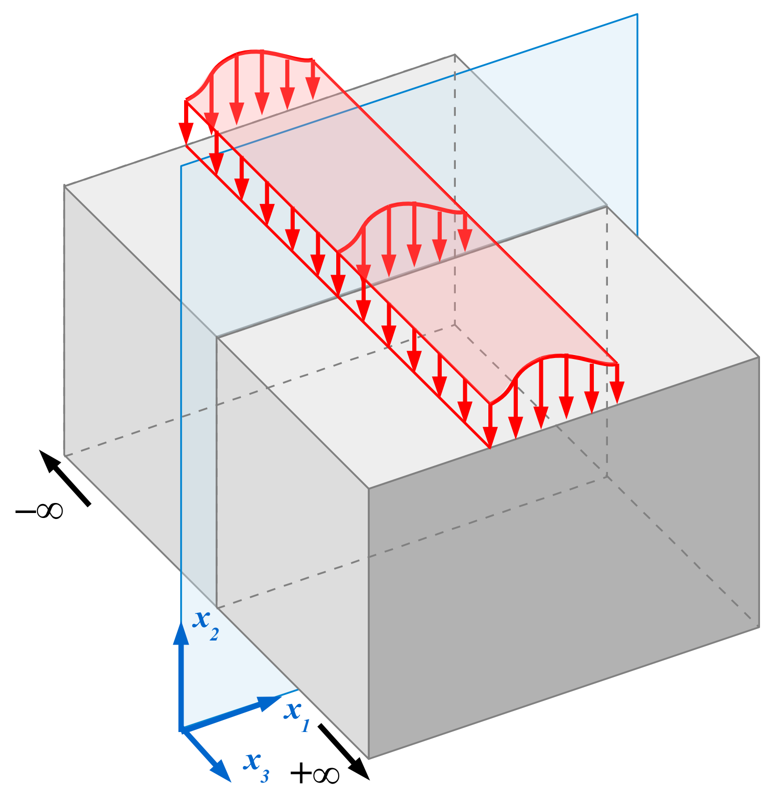
## PLANE STRESS STATE

- load lies within a plane
- unconstrained transverse deformation




## PLANE STRAIN STATE

- transverse strain is constrained
- normal stress perpendicular to the plane of the problem is present



# PLANE PROBLEMS

Transverse linear strain perpendicular to the plane of a plane stress state depend on in-plane strains:

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \begin{bmatrix} \frac{1}{E}[\sigma_{11}-\nu\sigma_{22}] & \frac{\sigma_{12}}{2G} & 0 \\ \frac{\sigma_{12}}{2G} & \frac{1}{E}[\sigma_{22}-\nu\sigma_{11}] & 0 \\ 0 & 0 & -\frac{\nu}{E}[\sigma_{11}+\sigma_{22}] \end{bmatrix} = \begin{bmatrix} \varepsilon_{11}(x_1;x_2) & \varepsilon_{12}(x_1;x_2) & 0 \\ \varepsilon_{12}(x_1;x_2) & \varepsilon_{22}(x_1;x_2) & 0 \\ 0 & 0 & \varepsilon_{33}(x_1;x_2) \end{bmatrix}$$

$$\begin{cases} \varepsilon_{11} = \frac{1}{E}[\sigma_{11}-\nu\sigma_{22}] \\ \varepsilon_{22} = \frac{1}{E}[\sigma_{22}-\nu\sigma_{11}] \end{cases} \Rightarrow \begin{cases} \sigma_{11} = \frac{E}{1-\nu^2}[\varepsilon_{11}+\nu\varepsilon_{22}] \\ \sigma_{22} = \frac{E}{1-\nu^2}[\varepsilon_{22}+\nu\varepsilon_{11}] \end{cases} \Rightarrow \boxed{\varepsilon_{33} = -\frac{\nu}{E}[\sigma_{11}+\sigma_{22}] = -\frac{\nu}{1-\nu}(\varepsilon_{11}+\varepsilon_{22})}$$

- If in a plane stress state we know the in-plane components of the strain state, then we know the whole strain state.
- The problem becomes in fact two-dimensional.

# PLANE PROBLEMS

Transverse normal stress perpendicular to the plane of a plane strain state depend on in-plane stresses:

$$\boldsymbol{\sigma}(\mathbf{x}) = \begin{bmatrix} (2G+\lambda)\varepsilon_{11}+\lambda\varepsilon_{22} & 2G\varepsilon_{12} & 0 \\ 2G\varepsilon_{12} & (2G+\lambda)\varepsilon_{22}+\lambda\varepsilon_{11} & 0 \\ 0 & 0 & \lambda(\varepsilon_{11}+\varepsilon_{22}) \end{bmatrix} = \begin{bmatrix} \sigma_{11}(x_1;x_2) & \sigma_{12}(x_1;x_2) & 0 \\ \sigma_{12}(x_1;x_2) & \sigma_{22}(x_1;x_2) & 0 \\ 0 & 0 & \sigma_{33}(x_1;x_2) \end{bmatrix}$$

$$\begin{cases} \sigma_{11} = (2G+\lambda)\varepsilon_{11} + \lambda\varepsilon_{22} \\ \sigma_{22} = (2G+\lambda)\varepsilon_{22} + \lambda\varepsilon_{11} \end{cases} \Rightarrow \begin{cases} \varepsilon_{11} = \dfrac{(2G+\lambda)\sigma_{11} - \lambda\sigma_{22}}{4G(G+\lambda)} \\ \varepsilon_{22} = \dfrac{(2G+\lambda)\sigma_{22} - \lambda\sigma_{11}}{4G(G+\lambda)} \end{cases} \Rightarrow$$

$$\Rightarrow \quad \boxed{\sigma_{33} = \lambda[\varepsilon_{11}+\varepsilon_{22}] = \frac{\lambda}{2(G+\lambda)}(\sigma_{11}+\sigma_{22}) = \nu(\sigma_{11}+\sigma_{22})}$$

- If in a plane strain state we know the in-plane components of the stress state, then we know the whole stress state.
- The problem becomes in fact two-dimensional.

# PLANE PROBLEMS

- **All components of the stress and strain tensors depend on only two independent variables.**

- **Diagonal components of those tensors corresponding with the direction, which is perpendicular to the plane of the problem, are known** – they are either equal to 0, or can be determined knowing the components of the plane parts of those tensors.

- **Components of the displacement vector** can be calculated by an **integration of strains**.

- In order to check the dimensionality of the problem we should analyse the **equilibrium equations**:

plane stress

$$\sigma_{13}=\sigma_{23}=\sigma_{33}=0 \quad \Rightarrow \quad \begin{cases} \sigma_{11,1}+\sigma_{12,2}+b_1=0 \\ \sigma_{12,1}+\sigma_{22,2}+b_2=0 \\ b_3=0 \end{cases} \quad \Rightarrow \quad b_3=0$$

plane strain

$$\sigma_{13}=\sigma_{23}=0\ , \quad \sigma_{33,3}=0 \quad \Rightarrow \quad \begin{cases} \sigma_{11,1}+\sigma_{12,2}+b_1=0 \\ \sigma_{12,1}+\sigma_{22,2}+b_2=0 \\ b_3=0 \end{cases} \quad \Rightarrow \quad b_3=0$$

- If only the body forces vector is of the form $\mathbf{b}=[b_1;b_2;0]$, then we have only two equilibrium equations.

# PLANE PROBLEMS

**Constitutive relations** in the **plane stress problems**:

$$\begin{cases} \varepsilon_{11} = \frac{1}{E}\left[(1+\nu)\sigma_{11} - \nu(\sigma_{11}+\sigma_{22}+\sigma_{33})\right] \\ \varepsilon_{22} = \frac{1}{E}\left[(1+\nu)\sigma_{22} - \nu(\sigma_{11}+\sigma_{22}+\sigma_{33})\right] \\ \varepsilon_{12} = \frac{1+\nu}{E}\sigma_{12} \end{cases}$$

$\sigma_{33} = 0$ →

$$\begin{cases} \varepsilon_{11} = \frac{1}{E}\left[\sigma_{11} - \nu\sigma_{22}\right] \\ \varepsilon_{22} = \frac{1}{E}\left[\sigma_{22} - \nu\sigma_{11}\right] \\ \varepsilon_{12} = \frac{1+\nu}{E}\sigma_{12} \end{cases}$$

# PLANE PROBLEMS

**Constitutive relations** in the **plane strain problems**:

$$\begin{cases} \varepsilon_{11} = \dfrac{1}{E}\left[(1+\nu)\sigma_{11} - \nu(\sigma_{11}+\sigma_{22}+\sigma_{33})\right] \\ \varepsilon_{22} = \dfrac{1}{E}\left[(1+\nu)\sigma_{22} - \nu(\sigma_{11}+\sigma_{22}+\sigma_{33})\right] \\ \varepsilon_{12} = \dfrac{1+\nu}{E}\sigma_{12} \end{cases}$$

$$\sigma_{33} = \nu(\sigma_{11}+\sigma_{22}) \quad \longrightarrow$$

$$\begin{cases} \varepsilon_{11} = \dfrac{1+\nu}{E}\left[\sigma_{11} - \dfrac{\nu}{1-\nu}\sigma_{22}\right] \\ \varepsilon_{22} = \dfrac{1+\nu}{E}\left[\sigma_{22} - \dfrac{\nu}{1-\nu}\sigma_{11}\right] \\ \varepsilon_{12} = \dfrac{1+\nu}{E}\sigma_{12} \end{cases}$$

If we introduce the following material constants $\hat{E} = \dfrac{E}{1-\nu^2}$ , $\hat{\nu} = \dfrac{\nu}{1-\nu}$ , then:

$$G = \frac{E}{2(1+\nu)} = \frac{\hat{E}}{2(1+\hat{\nu})} = \hat{G}$$

$$\begin{cases} \varepsilon_{11} = \dfrac{1}{\hat{E}}\left[\sigma_{11} - \hat{\nu}\sigma_{22}\right] \\ \varepsilon_{22} = \dfrac{1}{\hat{E}}\left[\sigma_{22} - \hat{\nu}\sigma_{11}\right] \\ \varepsilon_{12} = \dfrac{1+\hat{\nu}}{\hat{E}}\sigma_{12} \end{cases}$$

# PLANE PROBLEMS

Constitutive relations have an analogous form in both plane stress and plane strain problems.

$$\begin{cases} \varepsilon_{11} = \dfrac{1}{\hat{E}}[\sigma_{11} - \hat{\nu}\sigma_{22}] \\ \varepsilon_{22} = \dfrac{1}{\hat{E}}[\sigma_{22} - \hat{\nu}\sigma_{11}] \\ \varepsilon_{12} = \dfrac{1+\hat{\nu}}{\hat{E}}\sigma_{12} \end{cases}$$

where:

$$\hat{E} = \begin{cases} E & \Leftrightarrow \quad \text{PSN} \\ \dfrac{16\,E}{(7-\kappa)(\kappa+1)} = \dfrac{E}{1-\nu^2} & \Leftrightarrow \quad \text{PSO} \end{cases}$$

$$\hat{\nu} = \frac{3-\kappa}{\kappa+1} = \begin{cases} \nu & \Leftrightarrow \quad \text{PSN} \\ \dfrac{\nu}{1-\nu} & \Leftrightarrow \quad \text{PSO} \end{cases}$$

where

$$\begin{cases} \kappa = \dfrac{3-\nu}{1+\nu} & \Leftrightarrow \quad \text{PSN} \\ \kappa = 3-4\nu & \Leftrightarrow \quad \text{PSO} \end{cases}$$

# PLANE PROBLEMS

Governing equations of the plane problems:

- Equilibrium equations:

$$\begin{cases} \sigma_{11,1}+\sigma_{12,2}+b_1=0 \\ \sigma_{12,1}+\sigma_{22,2}+b_2=0 \end{cases}$$

- Constitutive relations:

$$\begin{cases} \varepsilon_{11}=\frac{1}{\hat{E}}[\sigma_{11}-\hat{\nu}\sigma_{22}] \\ \varepsilon_{22}=\frac{1}{\hat{E}}[\sigma_{22}-\hat{\nu}\sigma_{11}] \\ \varepsilon_{12}=\frac{1+\hat{\nu}}{\hat{E}}\sigma_{12} \end{cases}$$

- Kinematic relations:

$$\varepsilon_{11}=\frac{\partial u_1}{\partial x_1} \qquad \varepsilon_{22}=\frac{\partial u_2}{\partial x_2} \qquad \varepsilon_{33}=\frac{\partial u_3}{\partial x_3} \qquad \varepsilon_{12}=\frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)$$

If we account for the fact that $\frac{\partial \varepsilon_{ij}}{\partial x_3}=0$, then the strain compatibility equations are always automatically satisfied except for the following one:

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2}-2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2}+\frac{\partial^2 \varepsilon_{22}}{\partial x_1^2}=0$$

# AIRY STRESS FUNCTION

# AIRY STRESS FUNCTION

Let's consider the only strain compatibility equation which govern the **plane problems**:

$$\varepsilon_{11,22}-2\varepsilon_{12,12}+\varepsilon_{22,11}=0$$

Let's express it in terms of the stress tensor components:

$$\frac{1}{\hat{E}}[\sigma_{11}-\hat{\nu}\sigma_{22}]_{,22}-2\frac{\sigma_{12,12}}{2G}+\frac{1}{\hat{E}}[\sigma_{22}-\hat{\nu}\sigma_{11}]_{,11}=0$$

Wa maymultiply both sides of the equations by $\hat{E}$ . After some algebra:

$$\sigma_{11,22}-\hat{\nu}\sigma_{22,22}+\sigma_{22,11}-\hat{\nu}\sigma_{11,11}-2(1+\hat{\nu})\sigma_{12,12}=0$$

# AIRY STRESS FUNCTION

It can be rewritten as follows:

$$\nabla^2\sigma_{11}-\sigma_{11,11}-\hat{\nu}\sigma_{11,11}+\nabla^2\sigma_{22}-\sigma_{22,22}-\hat{\nu}\sigma_{22,22}-2(1+\hat{\nu})\sigma_{12,12}=0$$

where: $\nabla^2=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}$

Further transformations give us:

$$\nabla^2\sigma_{11}-(1+\hat{\nu})\sigma_{11,11}+\nabla^2\sigma_{22}-(1+\hat{\nu})\sigma_{22,22}-2(1+\hat{\nu})\sigma_{12,12}=0$$

$$\nabla^2(\sigma_{11}+\sigma_{22})-(1+\hat{\nu})\left[\sigma_{11,11}+\sigma_{22,22}+2\sigma_{12,12}\right]=0$$

$$\nabla^2(\sigma_{11}+\sigma_{22})-(1+\hat{\nu})\left[(\sigma_{11,1}+\sigma_{12,2})_{,1}+(\sigma_{12,1}+\sigma_{22,2})_{,2}\right]=0$$

# AIRY STRESS FUNCTION

In the equation

$$\nabla^2(\sigma_{11}+\sigma_{22})-(1+\hat{\nu})\left[(\sigma_{11,1}+\sigma_{12,2})_{,1}+(\sigma_{12,1}+\sigma_{22,2})_{,2}\right]=0$$

We may account for the equilibrium equations:

$$\begin{cases}\sigma_{11,1}+\sigma_{12,2}+b_1=0\\ \sigma_{12,1}+\sigma_{22,2}+b_2=0\end{cases} \quad\Rightarrow\quad \begin{cases}(\sigma_{11,1}+\sigma_{12,2})_{,1}=-b_{1,1}\\ (\sigma_{12,1}+\sigma_{22,2})_{,2}=-b_{2,2}\end{cases}$$

We obtain:

$$\boxed{\nabla^2(\sigma_{11}+\sigma_{22})+(1+\hat{\nu})\left[b_{1,1}+b_{2,2}\right]=0}$$

# AIRY STRESS FUNCTION

Let's introduce the Airy stress function which is defined as follows:

$$F(x_1; x_2): \quad \begin{cases} F_{,11} = \sigma_{22} \\ F_{,22} = \sigma_{11} \\ F_{,12} = -\sigma_{12} - b_1 x_2 - b_2 x_1 \end{cases}$$

If we write down the **equilibrium equations** with the use of the above definitions, they will take the following form:

$$\begin{cases} b_{1,2} x_2 + b_{2,2} x_1 = 0 \\ b_{1,1} x_2 + b_{2,1} x_1 = 0 \end{cases}$$

**For a uniform distribution of the body forces** (e.g. uniform gravity, constant density) **equilibrium equations are identically satisfied.** For uniform body forces the stress compatibility **equation** is as follows:

$$\boxed{\nabla^2 (\sigma_{11} + \sigma_{22}) = 0}$$

# AIRY STRESS FUNCTION

The governing equation for the plane problem is:

$$\nabla^2(\sigma_{11}+\sigma_{22}) = 0$$

In the **plane stress state**, when $\sigma_{33} = 0$ , the expression in the bracket is proportional to the hydrostatic stress. In such a case the **distribution of the** hydrostatic stress is given by a **harmonic function** (namely, it satisfies the **Laplace equations** – in some cases we know the solution).

Let's express the stress components with the use of the Airy stress function:

$$\nabla^2(F_{,22}+F_{,11}) = \nabla^2(\nabla^2 F) = \nabla^4 F = 0$$

So the Airy stress function, the derivatives of which determine the such a **stress state in plane problems**, **which satisfies the equilibrium conditions for uniform body forces**, must be a biharmonic function (namely it satisfies the **biharmonic equation** – in some cases we know the solution):

$$\nabla^4 F = F_{,1111}+2F_{,1122}+F_{,2222} = 0$$

# AIRY STRESS FUNCTION

REMARKS:

- **Biharmonic equation** is a **4th order partial differential equation** with respect to the Airy stress function.

- In order to determine the solution uniquely we need **boundary conditions for the unknown function itself** (not only for its derivatives).

- **Stress ans strain state** components may be expressed in terms of the **2nd order derivatives** of the Airy stress function, while the **displacement** components with the **1st order derivatives.**

- **The solution is ambiguous.** There is an infinite number of Airy stress functions which satisfy the equilibrium equations, compatibility equations and prescribed static and kinematic boundary conditions.

# AIRY STRESS FUNCTION

REMARKS:

- **The governing equation for stress components** in the plane problem **does not depend on material constants** as well as the static **boundary conditions**.

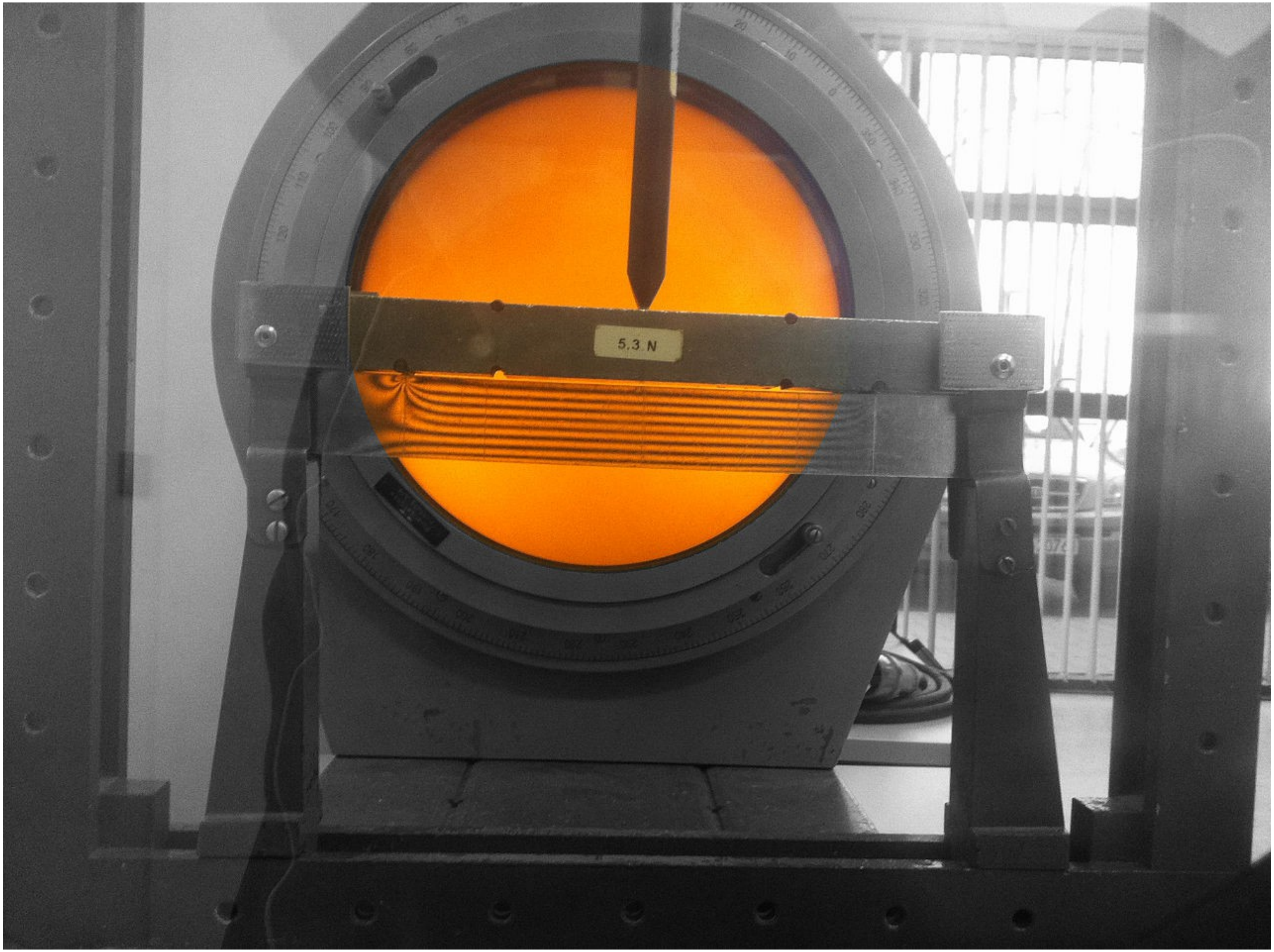
- Kinematic boundary conditions depend on displacements, which may be determined only if the material constants are known.

- We may formulate the **Lévy's theorem:**

> **If the boundary conditions are only of static type, then the distribution of stress in an isotropic elastic material in plane stress or plane strain problem is independent of the material that the material is made of.**

- We may perform a model of a physical object which meets the requirements of the above theorem. The stresses in that model will be the same as in thr true object made of a different material. **The model may be made of transparent plastics or crystals** – in such materials the **polarized light passes through them in a different way depending on the magnitude and orientation of principal stresses**. This constitutes the foundations of an analytical method termed the photoelasticity.

# AIRY STRESS FUNCTION

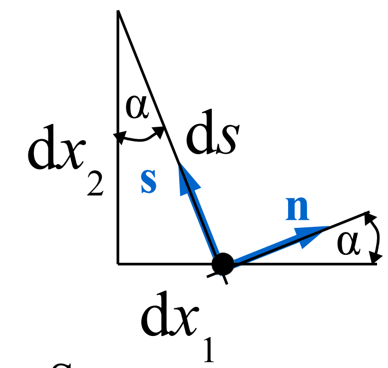


# BOUNDARY CONDITIONS FOR THE AIRY STRESS FUNCTION

- Let's consider a plane elastic membrane of thickness $h$, . The **stress state is plane**.

- The boundary of that membrane is given by a plane curve.

- **In every point of that boundary** we may determine a unit normal vector $\mathbf{n}$ and a unit tangent vector $\mathbf{s}$ :



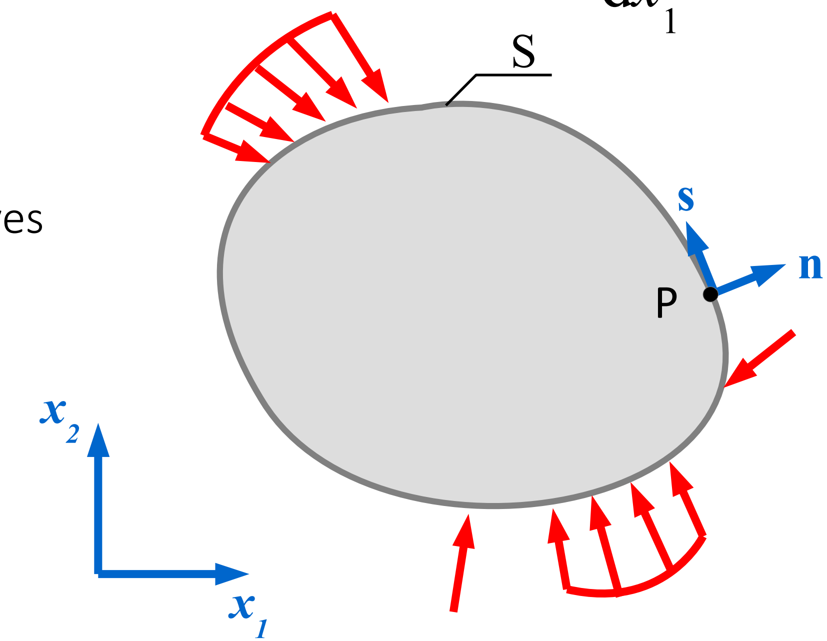

$$\mathbf{n} = [n_1 \,; n_2] = [\cos\alpha \;;\; \sin\alpha]$$

$$\mathbf{s} = [s_1 \,; s_2] = [-\sin\alpha \;;\; \cos\alpha]$$

- Trigonometric functions of the angle of inclination of that vector may be expressed in terms of the following derivatives

$$\cos\alpha = \frac{\mathrm{d}\,x_2}{\mathrm{d}\,s} \qquad \sin\alpha = -\frac{\mathrm{d}\,x_1}{\mathrm{d}\,s}$$

where $s$ is the natural parameter of the curve.



# BOUNDARY CONDITIONS FOR THE AIRY STRESS FUNCTION

- Let's write down the static boundary conditions:

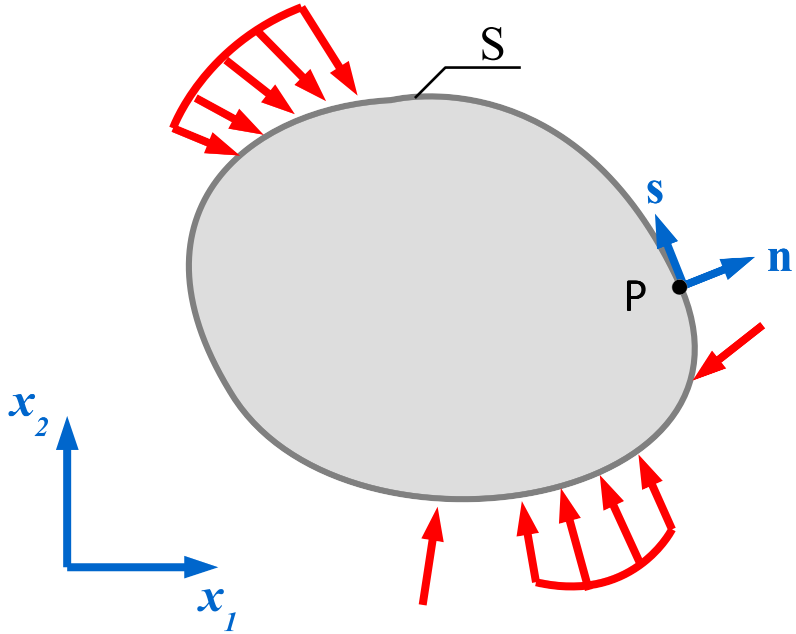
$$\sigma_{ij} n_j = q_i \quad \Rightarrow \quad \begin{cases} \sigma_{11} n_1 + \sigma_{12} n_2 = q_1 \\ \sigma_{21} n_1 + \sigma_{22} n_2 = q_2 \end{cases}$$

- **Stress tensor components** are expressed in terms of **Airy stress function**. Components of the unit **normal** are expressed by the **derivatives**.

$$\begin{cases} \dfrac{\partial^2 F}{\partial x_2^2} \dfrac{d x_2}{d s} + \left( - \dfrac{\partial^2 F}{\partial x_1 \partial x_2} \right) \left( - \dfrac{d x_1}{d s} \right) = q_1 \\ \left( - \dfrac{\partial^2 F}{\partial x_1 \partial x_2} \right) \dfrac{d x_2}{d s} + \dfrac{\partial^2 F}{\partial x_1^2} \left( - \dfrac{d x_1}{d s} \right) = q_2 \end{cases}$$

- We make an account for the **chain rule**:

$$\begin{cases} q_1 = \left[ \dfrac{d x_1}{d s} \dfrac{\partial}{\partial x_1} + \dfrac{d x_2}{d s} \dfrac{\partial}{\partial x_2} \right] \left( \dfrac{\partial F}{\partial x_2} \right) = \dfrac{d}{d s} \left( \dfrac{\partial F}{\partial x_2} \right) \\ q_2 = - \left[ \dfrac{d x_1}{d s} \dfrac{\partial}{\partial x_1} + \dfrac{d x_2}{d s} \dfrac{\partial}{\partial x_2} \right] \left( \dfrac{\partial F}{\partial x_1} \right) = - \dfrac{d}{d s} \left( \dfrac{\partial F}{\partial x_1} \right) \end{cases}$$



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# BOUNDARY CONDITIONS FOR THE AIRY STRESS FUNCTION

- We've obtained the following relations

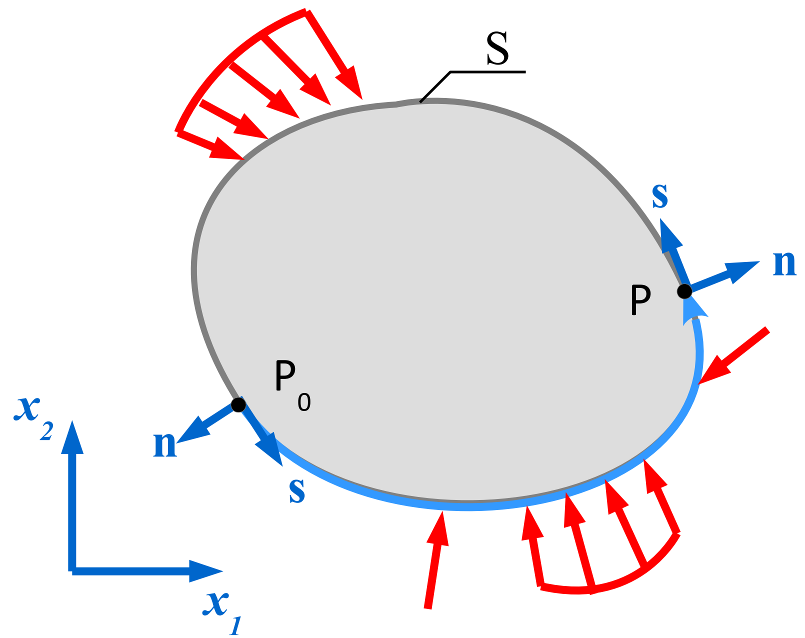
$$q_1 = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial F}{\partial x_2}\right) \qquad q_2 = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial F}{\partial x_1}\right)$$

- Let's choose a point on the boundary: $P_0=(x_1^{P_0};x_2^{P_0})$
  We consider this point **fixed** (immovable).

- Let's choose another point on the boundary: $P=(x_1^{P};x_2^{P})$
  Its position is considered variable.

- Let's consider a **section of a boundary** which is **oriented from the point $P_0$ to the point $P$** in such a way that an observer oriented in the same way as this section will have the **interior of the membrane on his left-hand side**.

- Let's sum (integrate) the surface tractions applied to that section of boundary.



# BOUNDARY CONDITIONS FOR THE AIRY STRESS FUNCTION

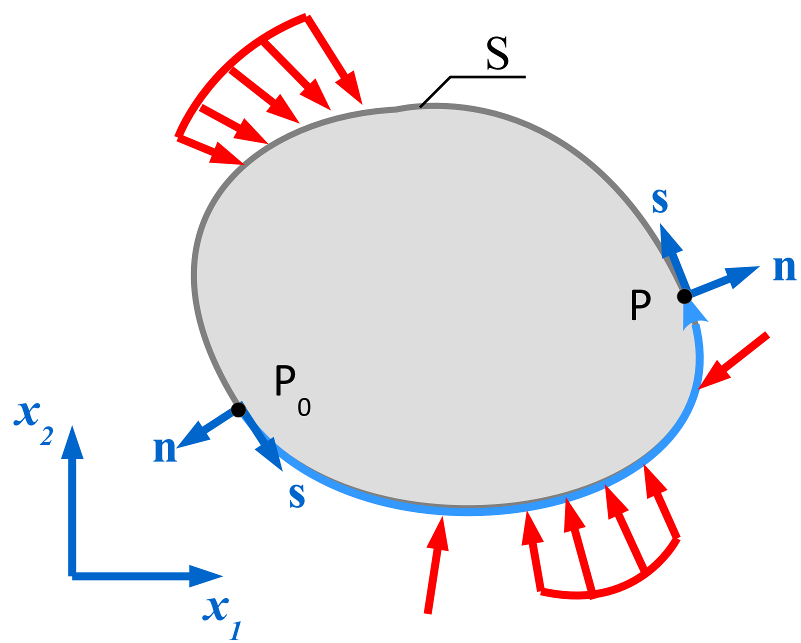
- Sum of tractions:

$$\begin{cases} Q_1 = \iint\limits_A q_1 \mathrm{d}A = h\int\limits_{P_0}^{P} q_1 \mathrm{d}s = h\int\limits_{P_0}^{P} \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial F}{\partial x_2}\right)\mathrm{d}s = h\frac{\partial F}{\partial x_2}\Big|_{P_0}^{P} \\ Q_2 = \iint\limits_A q_2 \mathrm{d}A = h\int\limits_{P_0}^{P} q_2 \mathrm{d}s = -h\int\limits_{P_0}^{P} \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial F}{\partial x_1}\right)\mathrm{d}s = -h\frac{\partial F}{\partial x_1}\Big|_{P_0}^{P} \end{cases}$$

REMARK:

- Surface tractions: $[q]=\mathrm{N/m^2}$.
- Sum of tractions: $[Q]=\mathrm{N}$

- We integrate over a whole side surface of the membrane, so also wrt to variable $x_3$. Distribution of traction along the thickness is uniform. For this reason the integral is a double (surface) one and the result of integration along the thickness is $h$.



# BOUNDARY CONDITIONS FOR THE AIRY STRESS FUNCTION

- Moment of tractions applied to the section of boundary $P_0 P$ about a point $P$:

$$M = h\int_{P_0}^{P}\left[q_1(x_2^P - x_2) - q_2(x_1^P - x_1)\right]\mathrm{d}s = h\left[x_2^P\int_{P_0}^{P} q_1\,\mathrm{d}s - x_1^P\int_{P_0}^{P} q_2\,\mathrm{d}s + \int_{P_0}^{P}(q_2 x_1 - q_1 x_2)\,\mathrm{d}s\right]$$

- Components of the traction are expressed by derivatives: $q_1 = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial F}{\partial x_2}\right) \qquad q_2 = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial F}{\partial x_1}\right)$

$$M = h\left[x_2^P\left.\frac{\partial F}{\partial x_2}\right|_{P_0}^{P} + x_1^P\left.\frac{\partial F}{\partial x_1}\right|_{P_0}^{P} - \int_{P_0}^{P}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial F}{\partial x_1}\right)x_1\,\mathrm{d}s - \int_{P_0}^{P}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial F}{\partial x_2}\right)x_2\,\mathrm{d}s\right]$$

- Integrands above are products of a function and of a derivative – we may **integrate them by parts**.

# BOUNDARY CONDITIONS FOR THE AIRY STRESS FUNCTION

$$M = h\left[x_2^P\left(\left.\frac{\partial F}{\partial x_2}\right|_P - \left.\frac{\partial F}{\partial x_2}\right|_{P_0}\right) + x_1^P\left(\left.\frac{\partial F}{\partial x_1}\right|_P - \left.\frac{\partial F}{\partial x_1}\right|_{P_0}\right) - \int_{P_0}^{P}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial F}{\partial x_1}\right)x_1\,\mathrm{d}s - \int_{P_0}^{P}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial F}{\partial x_2}\right)x_2\,\mathrm{d}s\right]$$

Integration by parts: $\int_a^b (f'g)\,\mathrm{d}x = [fg]_a^b - \int_a^b (fg')\,\mathrm{d}x$

the 1[st] integral $$\int_{P_0}^{P}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial F}{\partial x_1}\right)x_1\,\mathrm{d}s = \left(x_1^P\left.\frac{\partial F}{\partial x_1}\right|_P - x_1^{P_0}\left.\frac{\partial F}{\partial x_1}\right|_{P_0}\right) - \int_{P_0}^{P}\frac{\partial F}{\partial x_1}\frac{\mathrm{d}x_1}{\mathrm{d}s}\,\mathrm{d}s$$

the 2[nd] integral $$\int_{P_0}^{P}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial F}{\partial x_2}\right)x_2\,\mathrm{d}s = \left(x_2^P\left.\frac{\partial F}{\partial x_2}\right|_P - x_2^{P_0}\left.\frac{\partial F}{\partial x_2}\right|_{P_0}\right) - \int_{P_0}^{P}\frac{\partial F}{\partial x_2}\frac{\mathrm{d}x_2}{\mathrm{d}s}\,\mathrm{d}s$$

After substituting the above result we obtain:

$$M = h\left[(x_2^P - x_2^{P_0})\left.\frac{\partial F}{\partial x_2}\right|_{P_0} + (x_1^P - x_1^{P_0})\left.\frac{\partial F}{\partial x_1}\right|_{P_0} + \int_{P_0}^{P}\left(\frac{\partial F}{\partial x_1}\frac{\mathrm{d}x_1}{\mathrm{d}s} + \frac{\partial F}{\partial x_2}\frac{\mathrm{d}x_2}{\mathrm{d}s}\right)\mathrm{d}s\right]$$

# BOUNDARY CONDITIONS FOR THE AIRY STRESS FUNCTION

The integrand is the **total differential of the Airy stress function with respect to the natural parameter**:

$$\frac{\partial F}{\partial x_1}\frac{\mathrm{d}x_1}{\mathrm{d}s}+\frac{\partial F}{\partial x_2}\frac{\mathrm{d}x_2}{\mathrm{d}s}=\frac{\mathrm{d}F}{\mathrm{d}s} \qquad \Rightarrow \qquad \int\limits_{P_0}^{P}\left(\frac{\partial F}{\partial x_1}\frac{\mathrm{d}x_1}{\mathrm{d}s}+\frac{\partial F}{\partial x_2}\frac{\mathrm{d}x_2}{\mathrm{d}s}\right)\mathrm{d}s=\int\limits_{P_0}^{P}\frac{\mathrm{d}F}{\mathrm{d}s}\mathrm{d}s=F(P)-F(P_0)$$

We have then:

$$M=h\left[\left(x_2^P-x_2^{P_0}\right)\left.\frac{\partial F}{\partial x_2}\right|_{P_0}+\left(x_1^P-x_1^{P_0}\right)\left.\frac{\partial F}{\partial x_1}\right|_{P_0}+F(P)-F(P_0)\right]$$

It was already mentioned that there is **an infinite number of Airy stress functions** which are solutions of the stated problem. They all differ one from another with a constant terms and linear functions of independent variables, so **to any stress function an expression** $a_0+a_1x_1+a_2x_2$ **may be added**, the **coefficients of which may be chosen arbitrary**. They may be chosen in such a way, that:

$$\left.\frac{\partial F}{\partial x_1}\right|_{P_0}=0\ , \qquad \left.\frac{\partial F}{\partial x_2}\right|_{P_0}=0\ , \qquad F|_{P_0}=0$$

# BOUNDARY CONDITIONS FOR THE AIRY STRESS FUNCTION

We obtain:

$$F(P)=\frac{1}{h}M(P)\,,\qquad \left.\frac{\partial F}{\partial x_1}\right|_P=-\frac{1}{h}Q_2(P)\,,\qquad \left.\frac{\partial F}{\partial x_2}\right|_P=\frac{1}{h}Q_1(P)$$

Boundary values of the Airy stress function and of its derivatives may be determined solely by the surface tractions (its integral – forces, and integral of forces – moments). In such a way **surface tractions determine the boundary conditions for the Airy stress function**.

In particular, the considered coordinate system may be the system, the axes of which are parallel to the unit normal and unit tangent vector in point P. In such a situation the boundary conditions may be written down as follows:

$$F(P)=\frac{1}{h}M(P)\,,\qquad \left.\frac{\partial F}{\partial n}\right|_P=-\frac{1}{h}Q_s(P)\,,\qquad \left.\frac{\partial F}{\partial s}\right|_P=\frac{1}{h}Q_n(P)$$

# BOUNDARY CONDITIONS FOR THE AIRY STRESS FUNCTION

REMARKS:

- This 2nd formulation leads to more important conclusions. We may prescribe the boundary values of
  - **the function itself –** these are the **boundary conditions of Dirichlet type**.
  - **The directional derivative along the external unit normal –** these are the **boundary conditions of Neumann type**..

- Solutions of the biharmonic euqations for such boundary conditions are known in some cases.

- If we know the distribution of the Airy stress function along the boundary (moment distribution) then direct differentiation gives us directional derivative along a tangent direction (which is equal to the transverse forces). If only we are able to formulate a condition of the type:

$$F(P) = \frac{1}{h} M(P)$$

then the conditions

$$\left. \frac{\partial F}{\partial s} \right|_P = \frac{1}{h} Q_n(P)$$

may be neglected (it is a dependent one)

# THANK YOU FOR YOUR ATTENTION