THEORY OF ELASTICITY AND PLASTICITY

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FUNDAMENTAL SOLUTIONS IN LINEAR SYSTEMS

SUPERPOSITION PRINCIPLE

Governing equations of the linear theory of elasticity constitute a system of linear equations (differential and algebraic equations)

In linear system the superposition principle is valid – according to this principle:

> **An effect of a combination of causes is a respective combination of effects of all causes accounted for separately**
>
> $$\left.\begin{aligned} L(x_1) &= y_1 \\ L(x_2) &= y_2 \\ &\dots \\ L(x_p) &= y_p \end{aligned}\right\} \quad \Rightarrow \quad L(\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_p x_p) = \alpha_1 y_1 + \alpha_2 y_2 + \ldots + \alpha_p y_p$$

x_i - causes – **external forces** (surface tractions, body forces), **kinematic excitations** (boundary conditions or inhomogeneous parts of equations)

y_i - effects – **displacement**, **strain**, **stress**

L - linear operator representing the mathematical structure of the problem – **governing equations**

SUPERPOSITION PRINCIPLE

PROBLEM (1)

$$\begin{cases} \sigma_{ij,j}^{(1)} + b_i^{(1)} = 0 \\ \varepsilon_{ij}^{(1)} = \frac{1}{2}\left(u_{i,j}^{(1)} + u_{j,i}^{(1)}\right) \\ \sigma_{ij}^{(1)} = 2G\,\varepsilon_{ij}^{(1)} + \lambda\,\delta_{ij}\,\varepsilon_{kk}^{(1)} \end{cases} \qquad \begin{cases} \sigma_{ij}^{(1)} n_j = q_i^{(1)} & \text{na} \quad S_q \\ u_i^{(1)} = \hat{u}_i^{(1)} & \text{na} \quad S_u \end{cases}$$

PROBLEM(2)

$$\begin{cases} \sigma_{ij,j}^{(2)} + b_i^{(2)} = 0 \\ \varepsilon_{ij}^{(2)} = \frac{1}{2}\left(u_{i,j}^{(2)} + u_{j,i}^{(2)}\right) \\ \sigma_{ij}^{(2)} = 2G\,\varepsilon_{ij}^{(2)} + \lambda\,\delta_{ij}\,\varepsilon_{kk}^{(2)} \end{cases} \qquad \begin{cases} \sigma_{ij}^{(2)} n_j = q_i^{(2)} & \text{na} \quad S_q \\ u_i^{(2)} = \hat{u}_i^{(2)} & \text{na} \quad S_u \end{cases}$$

PROBLEM (1+2) = PROBLEM (1) + PROBLEM (2)

$$\begin{cases} u_i^{(1+2)} = u_i^{(1)} + u_i^{(2)} \\ \varepsilon_{ij}^{(1+2)} = \varepsilon_{ij}^{(1)} + \sigma_{ij}^{(2)} \\ \sigma_{ij}^{(1+2)} = \sigma_{ij}^{(1)} + \sigma_{ij}^{(2)} \\ \\ b_i^{(1+2)} = b_i^{(1)} + b_i^{(2)} \\ \hat{u}_i^{(1+2)} = \hat{u}_i^{(1)} + \hat{u}_i^{(2)} \\ q_i^{(1+2)} = q_i^{(1)} + q_i^{(2)} \end{cases} \quad \Rightarrow \quad \begin{cases} \sigma_{ij,j}^{(1+2)} + b_i^{(1+2)} = 0 \\ \varepsilon_{ij}^{(1+2)} = \frac{1}{2}\left(u_{i,j}^{(1+2)} + u_{j,i}^{(1+2)}\right) \\ \sigma_{ij}^{(1+2)} = 2G\,\varepsilon_{ij}^{(1+2)} + \lambda\,\delta_{ij}\,\varepsilon_{kk}^{(1+2)} \end{cases} \qquad \begin{cases} \sigma_{ij}^{(1+2)} n_j = q_i^{(1+2)} & \text{na} \quad S_q \\ u_i^{(1+2)} = \hat{u}_i^{(1+2)} & \text{na} \quad S_u \end{cases}$$

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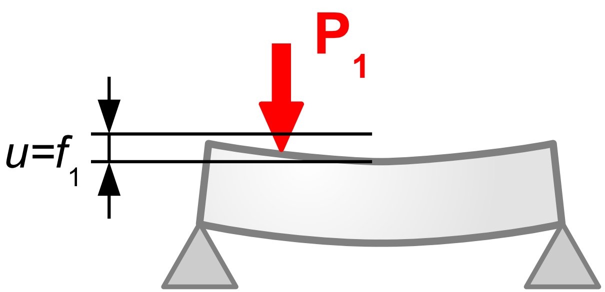
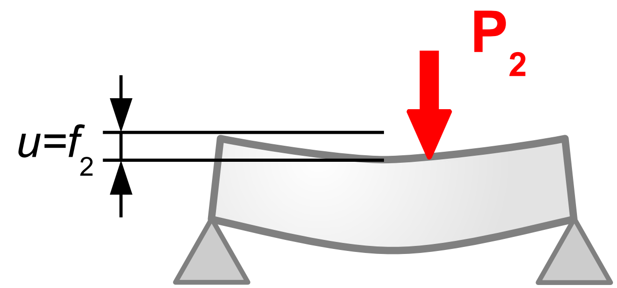
SUPERPOSITION PRINCIPLE

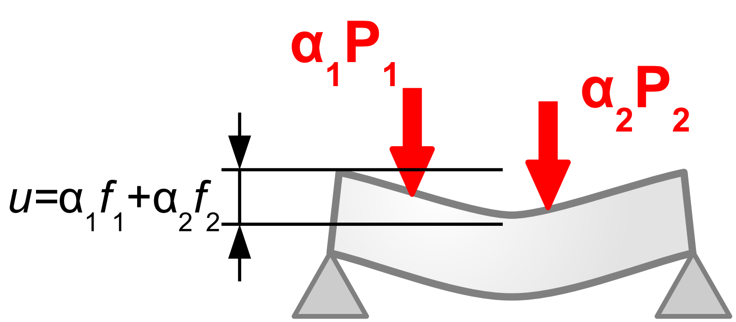
PROBLEM (1)



PROBLEM(2)

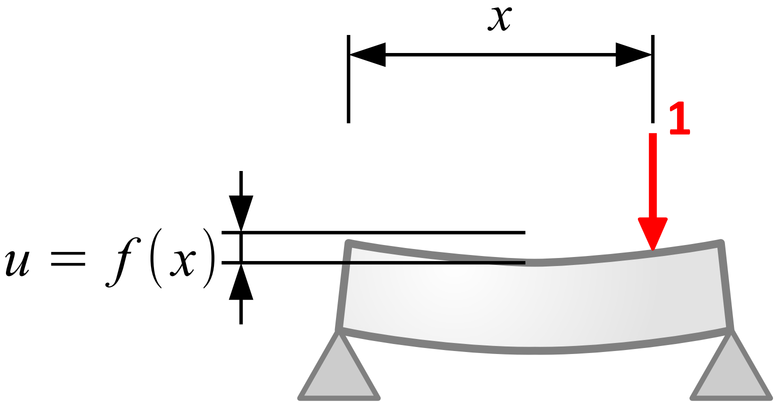


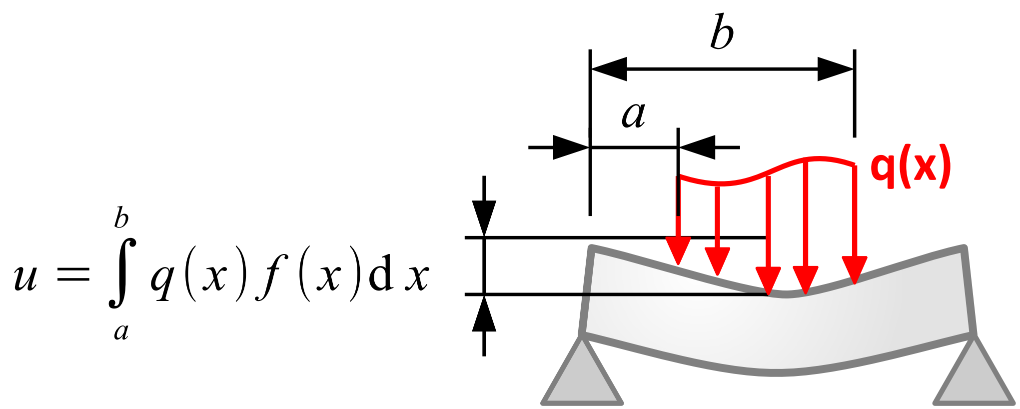
PROBLEM (α_1 ·„1” + α_2·„2”) = α_1 · PROBLEM („1”) + α_1 · PROBLEM („2”)



SUPERPOSITION PRINCIPLE

In particular, we may consider a continuous (**infinite** and **uncountable**) set of causes – the effect of such a set of causes will be a **sum of an infinite and uncountable number of causes**, namely: an integral:





GREEN'S FUNCTIONS

If a problem is given by a linear differential operator L:

$$L[\, u(x)\,]=q(x)$$

then the function G being a **kernel of an integral operator which is an inverse of the given differential operator** will be termed the Green's functions. We may state that::

- Green's function **is a solution** of a differential problem in which the **inhomogeneous part is given by a** Dirac's delta distribution ("impulse function"):

$$L[G(x,\xi)]=\delta(\xi-x)$$

- Solution of a problem with an arbitrary inhomogeneous term may be determined as a convolution (an integral of a product of function – integral operator) of that inhomogeneous term with the Green's function (kernel of this operator):

$$u(x)=\int G(x,\xi)q(\xi)\,\mathrm{d}\xi$$

KELVIN'S SOLUTION

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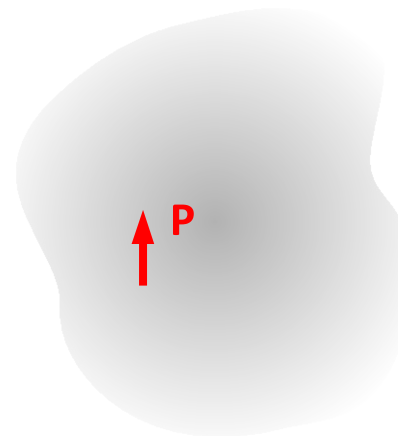
KELVIN'S SOLUTION

Elastic space loaded with a point force.



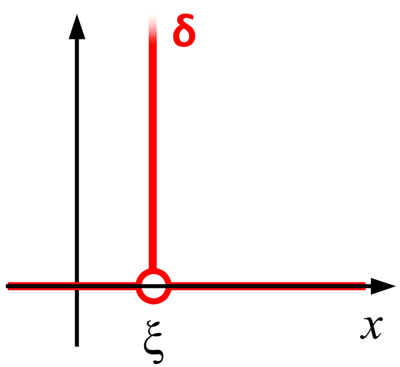
- whole 3-dimensional space is considered an **infinitely large elastic body**. This space may be interpreted as a **neighbourhood of an internal point which is far away from the boundary** and the **neighbourhood is much smaller than the body** as a whole.

- The body is made of a **homogeneous isotropic Hooke's material** characterized by elastic constants $\lambda,\ G$

- **External load** is given by a **point force**, which is modelled by a **Dirac's delta distribution**:

$$\delta(x,\xi)=\begin{cases}0 & \Leftrightarrow \quad x\neq\xi \\ \infty & \Leftrightarrow \quad x=\xi\end{cases},\quad \int_{-\infty}^{\infty}\delta(x,\xi)\,\mathrm{d}x=1 \quad \Rightarrow$$

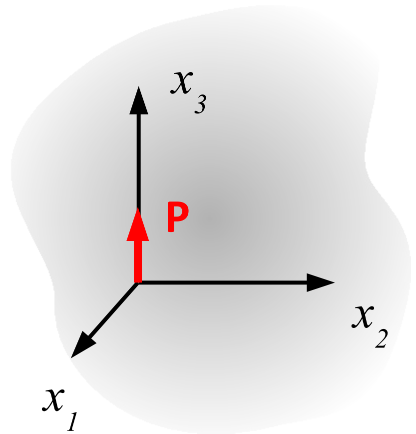
$$\Rightarrow \quad \int_{-\infty}^{\infty}\delta(x,\xi)f(\xi)\,\mathrm{d}\xi=f(x)$$



- **Boundary conditions** are prescribed **in infinity** – we state that **displacements, strain and stresses tend to 0 in infinity.**

KELVIN'S SOLUTION

Elastic space loaded with a point force.



- We assume a Cartesian coordinate system with its origin in the point of application of the point force, axis x_3 is oriented in the same way as the force.

- Let's make use of the Lamé's displacement equations:

$$G\nabla^2 u_i + (\lambda + G) u_{k,ik} + b_i = 0$$

- Body forces vector:

$$\mathbf{b} = [0; 0; P\delta_{\mathbf{0}}]$$

- Boundary conditions:

$$\lim_{R\to\infty} u_i = 0, \qquad i=1,2,3$$

$$\lim_{R\to\infty} u_{i,j} = 0, \qquad i,j=1,2,3 \qquad \text{where} \qquad R = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

KELVIN'S SOLUTION

We will solve the problem with the use of the Fourier integral transform.

In the case of a function $f(x)$ of single independent variable, the **transform** is given by the following equation:

$$\mathscr{F}\{f(x)\}(\omega)=\hat{f}(\omega)=\int_{-\infty}^{\infty} f(x)e^{-\mathrm{i}\omega x}\,\mathrm{d}x$$

Inverse transform:

$$\mathscr{F}^{-1}\{\hat{f}(\omega)\}(x)=f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{\mathrm{i}\omega x}\,\mathrm{d}x$$

Function $\hat{f}(\omega)$ is termed the transform of function $f(x)$, and function $f(x)$ is termed original function or inverse transform of $\hat{f}(\omega)$. Original function is a function of independent variable x, while its transform is a function of variable ω, which has an inverse physical dimension (here: 1/m).

REMARK:

- It is a matter of convention weather sign „-” is placed in the exponent in the definition of the transform or inverse transform. It is also a matter of convention how the factor $(2\pi)^{-1}$ is accounted for in those definitions. Various conventions are used.

KELVIN'S SOLUTION

In **3-dimensional** case the transform may be performed with respect to each independent variable. Each time we need to introduce a new variable in the domain of transforms. We will transform the components of the **displacement vector**.

$$\mathscr{F}\left\{u_i\right\} = \hat{u}_i = \int\limits_{x_1=-\infty}^{\infty} \left[\int\limits_{x_2=-\infty}^{\infty} \left[\int\limits_{x_3=-\infty}^{\infty} u_i e^{-\mathrm{i}\,\omega_3 x_3} \mathrm{d}\,x_3 \right] e^{-\mathrm{i}\,\omega_2 x_2} \mathrm{d}\,x_2 \right] e^{-\mathrm{i}\,\omega_1 x_1} \mathrm{d}\,x_1 =$$

$$= \iiint\limits_{-\infty}^{\infty} u_i e^{-\mathrm{i}(x_1\omega_1 + x_2\omega_2 + x_3\omega_3)} \mathrm{d}\,x_1 \mathrm{d}\,x_2 \mathrm{d}\,x_3$$

Inverse transform:

$$u_i = \mathscr{F}^{-1}\left\{\hat{u}_i\right\} = \frac{1}{(2\pi)^3} \iiint\limits_{-\infty}^{\infty} \hat{u}_i e^{\mathrm{i}(x_1\omega_1 + x_2\omega_2 + x_3\omega_3)} \mathrm{d}\,\omega_1 \mathrm{d}\,\omega_2 \mathrm{d}\,\omega_3$$

KELVIN'S SOLUTION

Integral operator used in the definition of the Fourier transform is linear, so:

$$\mathscr{F}\{\alpha_1 u_1 + \alpha_2 u_2\} = \alpha_1 \mathscr{F}\{u_1\} + \alpha_2 \mathscr{F}\{u_2\}$$

The use of the Fourier transform **with the** Lamé's displacement equations gives us:

$$\mathscr{F}\{G\nabla^2 u_i + (\lambda + G) u_{k,ik} + b_i\} = G\mathscr{F}\{\nabla^2 u_i\} + (\lambda + G)\mathscr{F}\{u_{k,ik}\} + \mathscr{F}\{b_i\} = 0$$

We need to find:

- **Transforms of the body forces:** $\mathscr{F}\{b_i\} = ?$
- **Transforms of the derivatives of displacements:** $\mathscr{F}\{u_{i,jk}\} = ?$

KELVIN'S SOLUTION

Transforms of the components of the body forces vector:

$$\mathscr{F}\{b_1\} = \mathscr{F}\{b_2\} = \mathscr{F}\{0\} = \iiint_{-\infty}^{\infty} 0 \cdot e^{-\mathrm{i}(x_1\omega_1 + x_2\omega_2 + x_3\omega_3)} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 = 0$$

$$\mathscr{F}\{b_3\} = \mathscr{F}\{P\delta_{\mathbf{0}}\} = P\iiint_{-\infty}^{\infty} \delta_{\mathbf{0}} e^{-\mathrm{i}(x_1\omega_1 + x_2\omega_2 + x_3\omega_3)} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 = P e^{-\mathrm{i}(x_1\omega_1 + x_2\omega_2 + x_3\omega_3)}\Big|_{\mathbf{x}=\mathbf{0}} = P$$

CONCLUSIONS:

- Fourier transform of a **zero function** is a **zero function**
- Fourier transform of a **delta distribution** is a **unit function**

KELVIN'S SOLUTION

Transforms of the derivatives of the components of the displacement vector:

Let's calculate for example $\mathscr{F}\left\{u_{i,12}\right\}$

$$\mathscr{F}\left\{u_{i,12}\right\} = \iiint\limits_{-\infty}^{\infty} \frac{\partial u_i}{\partial x_1 \partial x_2} e^{-\mathrm{i}(x_1\omega_1 + x_2\omega_2 + x_3\omega_3)} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3$$

Let's integrate by parts with respect to x_1

$$\mathscr{F}\left\{u_{i,12}\right\} = \iint\limits_{-\infty}^{\infty} \left[\left[\frac{\partial u_i}{\partial x_2} e^{-\mathrm{i}(x_1\omega_1 + x_2\omega_2 + x_3\omega_3)} \right]_{x_1=-\infty}^{\infty} - \int\limits_{x_1=-\infty}^{\infty} \frac{\partial u_i}{\partial x_2} \frac{\partial}{\partial x_1} e^{-\mathrm{i}(x_1\omega_1 + x_2\omega_2 + x_3\omega_3)} \mathrm{d}x_1 \right] \mathrm{d}x_2 \mathrm{d}x_3$$

Let's deal now with the **boundary term** (it would be great to get rid of it):

$$\left[\frac{\partial u_i}{\partial x_2} e^{-\mathrm{i}(x_1\omega_1 + x_2\omega_2 + x_3\omega_3)} \right]_{x_1=-\infty}^{\infty}$$

KELVIN'S SOLUTION

Transforms of the derivatives of the components of the displacement vector:

$$\left[\frac{\partial u_i}{\partial x_2} e^{-\mathrm{i}(x_1\omega_1+x_2\omega_2+x_3\omega_3)}\right]_{x_1=-\infty}^{\infty}$$

- according to the **Euler's formula** $e^{\mathrm{i}\phi} = \cos\phi + \mathrm{i}\sin\phi$ where $\phi = -(x_1\omega_1 + x_2\omega_2 + x_3\omega_3)$
- values of **trigonometric functions** are within interval $\langle -1\,;1\rangle$
- as a result the term $e^{-\mathrm{i}(x_1\omega_1+x_2\omega_2+x_3\omega_3)}$ has surely **finite value** for $\phi \rightarrow \pm\infty$
- according to the assumed **boundary conditions** we have $\lim\limits_{R\rightarrow\infty} u_{i,j} = 0$
- **a product of a finite value and 0 gives always 0.**
- values of the boundary term for both limit cases is zero.
- finally:

$$\left[\frac{\partial u_i}{\partial x_2} e^{-\mathrm{i}(x_1\omega_1+x_2\omega_2+x_3\omega_3)}\right]_{x_1=-\infty}^{\infty} = 0$$

KELVIN'S SOLUTION

Transforms of the derivatives of the components of the displacement vector:

We have then:

$$\mathscr{F}\{u_{i,12}\} = \iint_{-\infty}^{\infty}\left[-(-\mathrm{i}\,\omega_1)\int_{x_1=-\infty}^{\infty}\frac{\partial u_i}{\partial x_2}e^{-\mathrm{i}(x_1\omega_1+x_2\omega_2+x_3\omega_3)}\mathrm{d}x_1\right]\mathrm{d}x_2\,\mathrm{d}x_3$$

$$\mathscr{F}\{u_{i,12}\} = \mathrm{i}\,\omega_1\iiint_{-\infty}^{\infty}\frac{\partial u_i}{\partial x_2}e^{-\mathrm{i}(x_1\omega_1+x_2\omega_2+x_3\omega_3)}\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3$$

Let's **integrate by parts again** – this time with respect to variable x_2

The whole procedure and corresponding considerations is repeated. As a result we obtain:

$$\mathscr{F}\{u_{i,12}\} = (\mathrm{i}\,\omega_1)(\mathrm{i}\,\omega_2)\underbrace{\iiint_{-\infty}^{\infty}u_i e^{-\mathrm{i}(x_1\omega_1+x_2\omega_2+x_3\omega_3)}\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3}_{\mathscr{F}\{u_i\}=\hat{u}_i} = \mathrm{i}^2\omega_1\omega_2\hat{u}_i = -\omega_1\omega_2\hat{u}_i$$

In general:

$$\boxed{\mathscr{F}\{u_{i,jk}\} = -\omega_j\omega_k\hat{u}_i}$$

KELVIN'S SOLUTION

The system of Lamé's displacement equations which is governing the Kelvin problem:

$$\begin{cases} G\left(u_{1,11}+u_{1,22}+u_{1,33}\right)+(\lambda+G)\left(u_{1,11}+u_{2,21}+u_{3,31}\right)=0 \\ G\left(u_{2,11}+u_{2,22}+u_{2,33}\right)+(\lambda+G)\left(u_{1,12}+u_{2,22}+u_{3,32}\right)=0 \\ G\left(u_{3,11}+u_{3,22}+u_{3,33}\right)+(\lambda+G)\left(u_{1,13}+u_{2,23}+u_{3,33}\right)+P\,\delta_{\mathbf{0}}=0 \end{cases}$$

It is a **system of** linear partial differential equations of the 2nd order.

In the space of **transforms** the considered system takes the following form:

$$\begin{cases} -G\left(\omega_1^2+\omega_2^2+\omega_3^2\right)\hat{u}_1-\omega_1(\lambda+G)\left(\omega_1\hat{u}_1+\omega_2\hat{u}_2+\omega_3\hat{u}_3\right)=0 \\ -G\left(\omega_1^2+\omega_2^2+\omega_3^2\right)\hat{u}_2-\omega_2(\lambda+G)\left(\omega_1\hat{u}_1+\omega_2\hat{u}_2+\omega_3\hat{u}_3\right)=0 \\ -G\left(\omega_1^2+\omega_2^2+\omega_3^2\right)\hat{u}_3-\omega_3(\lambda+G)\left(\omega_1\hat{u}_1+\omega_2\hat{u}_2+\omega_3\hat{u}_3\right)=-P \end{cases}$$

It is a **system of** linear algebraic equations. We solve it easily.

KELVIN'S SOLUTION

The solutions of the linear system for transforms:

$$\hat{u}_1 = -\frac{P(\lambda+G)}{G(\lambda+2G)}\frac{\omega_1\omega_3}{(\omega_1^2+\omega_2^2+\omega_3^2)^2}$$

$$\hat{u}_2 = -\frac{P(\lambda+G)}{G(\lambda+2G)}\frac{\omega_2\omega_3}{(\omega_1^2+\omega_2^2+\omega_3^2)^2}$$

$$\hat{u}_3 = \frac{P}{G}\frac{1}{\omega_1^2+\omega_2^2+\omega_3^2} - \frac{P(\lambda+G)}{G(\lambda+2G)}\frac{\omega_3^2}{(\omega_1^2+\omega_2^2+\omega_3^2)^2}$$

- we must now find **inverse transforms**.
- we're looking for $f_1 = \mathscr{F}^{-1}\left\{\frac{1}{\omega_1^2+\omega_2^2+\omega_3^2}\right\}$
- if we find $f_2 = \mathscr{F}^{-1}\left\{\frac{1}{(\omega_1^2+\omega_2^2+\omega_3^2)^2}\right\}$ then $f_{2,ij} = \mathscr{F}^{-1}\left\{-\frac{\omega_i\omega_j}{\omega_1^2+\omega_2^2+\omega_3^2}\right\}$

KELVIN'S SOLUTION

There is a formula for inverse transform, however, calculating the resulting integrals is pretty much difficult. We may use the tables of originals and transforms:

	Function	Fourier transform unitary, ordinary frequency	Fourier transform unitary, angular frequency	Fourier transform non-unitary, angular frequency	Remarks
500	$f(\mathbf{x})$	$\hat{f}(\boldsymbol{\xi}) = \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-2\pi i \mathbf{x}\cdot\boldsymbol{\xi}}\, d\mathbf{x}$	$\hat{f}(\boldsymbol{\omega}) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-i\boldsymbol{\omega}\cdot\mathbf{x}}\, d\mathbf{x}$	$\hat{f}(\boldsymbol{\nu}) = \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-i\mathbf{x}\cdot\boldsymbol{\nu}}\, d\mathbf{x}$	
501	$\chi_{[0,1]}(\lvert\mathbf{x}\rvert)\left(1-\lvert\mathbf{x}\rvert^2\right)^{\delta}$	$\frac{\Gamma(\delta+1)}{\pi^{\delta}\,\lvert\boldsymbol{\xi}\rvert^{\frac{n}{2}+\delta}} J_{\frac{n}{2}+\delta}(2\pi\lvert\boldsymbol{\xi}\rvert)$	$2^{\delta}\,\frac{\Gamma(\delta+1)}{\lvert\boldsymbol{\omega}\rvert^{\frac{n}{2}+\delta}} J_{\frac{n}{2}+\delta}(\lvert\boldsymbol{\omega}\rvert)$	$\frac{\Gamma(\delta+1)}{\pi^{\delta}} \left\lvert\frac{\boldsymbol{\nu}}{2\pi}\right\rvert^{-\frac{n}{2}-\delta} J_{\frac{n}{2}+\delta}(\lvert\boldsymbol{\nu}\rvert)$	The function $\chi_{[0,1]}$ is the indicator function of the interval $[0,1]$. The function $\Gamma(x)$ is the gamma function. The function $J_{\frac{n}{2}+\delta}$ is a Bessel function of the first kind, with order $\frac{n}{2}+\delta$. Taking $n=2$ and $\delta=0$ produces 402.[52]
502	$\lvert\mathbf{x}\rvert^{-\alpha}, \quad 0<\operatorname{Re}\alpha<n.$	$\frac{(2\pi)^{\alpha}}{c_{n,\alpha}} \lvert\boldsymbol{\xi}\rvert^{-(n-\alpha)}$	$\frac{(2\pi)^{\frac{n}{2}}}{c_{n,\alpha}} \lvert\boldsymbol{\omega}\rvert^{-(n-\alpha)}$	$\frac{(2\pi)^{n}}{c_{n,\alpha}} \lvert\boldsymbol{\nu}\rvert^{-(n-\alpha)}$	See Riesz potential where the constant is given by $c_{n,\alpha} = \pi^{\frac{n}{2}} 2^{\alpha} \frac{\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}.$ The formula also holds for all $\alpha \neq -n, -n-1, \ldots$ by analytic continuation, but then the function and its Fourier transforms need to be understood as suitably regularized tempered distributions. See homogeneous distribution.[remark 6]
503	$\frac{1}{\lvert\boldsymbol{\sigma}\rvert\,(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}\mathbf{x}^{\mathrm{T}}\boldsymbol{\sigma}^{-\mathrm{T}}\boldsymbol{\sigma}^{-1}\mathbf{x}}$	$e^{-2\pi^2\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\sigma}\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\xi}}$	$(2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2}\boldsymbol{\omega}^{\mathrm{T}}\boldsymbol{\sigma}\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\omega}}$	$e^{-\frac{1}{2}\boldsymbol{\nu}^{\mathrm{T}}\boldsymbol{\sigma}\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\nu}}$	This is the formula for a multivariate normal distribution normalized to 1 with a mean of 0. Bold variables are vectors or matrices. Following the notation of the aforementioned page, $\boldsymbol{\Sigma} = \boldsymbol{\sigma}\,\boldsymbol{\sigma}^{\mathrm{T}}$ and $\boldsymbol{\Sigma}^{-1} = \boldsymbol{\sigma}^{-\mathrm{T}}\,\boldsymbol{\sigma}^{-1}$
504	$e^{-2\pi\alpha\lvert\mathbf{x}\rvert}$	$\frac{c_n\,\alpha}{\left(\alpha^2+\lvert\boldsymbol{\xi}\rvert^2\right)^{\frac{n+1}{2}}}$	$\frac{c_n\,(2\pi)^{\frac{n+2}{2}}\,\alpha}{\left(4\pi^2\alpha^2+\lvert\boldsymbol{\omega}\rvert^2\right)^{\frac{n+1}{2}}}$	$\frac{c_n\,(2\pi)^{n+1}\,\alpha}{\left(4\pi^2\alpha^2+\lvert\boldsymbol{\nu}\rvert^2\right)^{\frac{n+1}{2}}}$	Here[53] $c_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{[illegible]}$, $\operatorname{Re}(\alpha)>0$

$$f_1 = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} \frac{e^{i(\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3)}}{\omega_1^2 + \omega_2^2 + \omega_3^2}\, d\omega_1\, d\omega_2\, d\omega_3 = \frac{1}{4\pi} \frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

$$f_2 = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} \frac{e^{i(\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3)}}{(\omega_1^2 + \omega_2^2 + \omega_3^2)^2}\, d\omega_1\, d\omega_2\, d\omega_3 = -\frac{1}{8\pi} \sqrt{x_1^2 + x_2^2 + x_3^2}$$

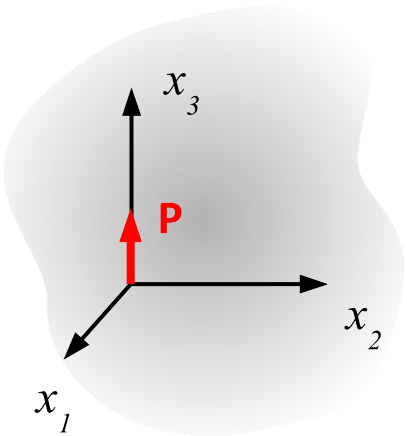
KELVIN'S SOLUTION

Finally, the Kelvin's solution is given by the formulas:

$$u_1 = \frac{P(\lambda+G)}{8\pi G(\lambda+2G)} \frac{x_1 x_3}{\left[x_1^2+x_2^2+x_3^2\right]^{3/2}}$$

$$u_2 = \frac{P(\lambda+G)}{8\pi G(\lambda+2G)} \frac{x_2 x_3}{\left[x_1^2+x_2^2+x_3^2\right]^{3/2}}$$

$$u_3 = \frac{P}{4\pi G} \frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}} + \frac{P(\lambda+G)}{8\pi G(\lambda+2G)} \frac{x_3^2}{\left[x_1^2+x_2^2+x_3^2\right]^{3/2}}$$



- Distribution of strain is determined according the **above distribution of displacement** and kinematic relations.
- Distribution of stress is determined according the **found distribution of strain** and constitutive relations.

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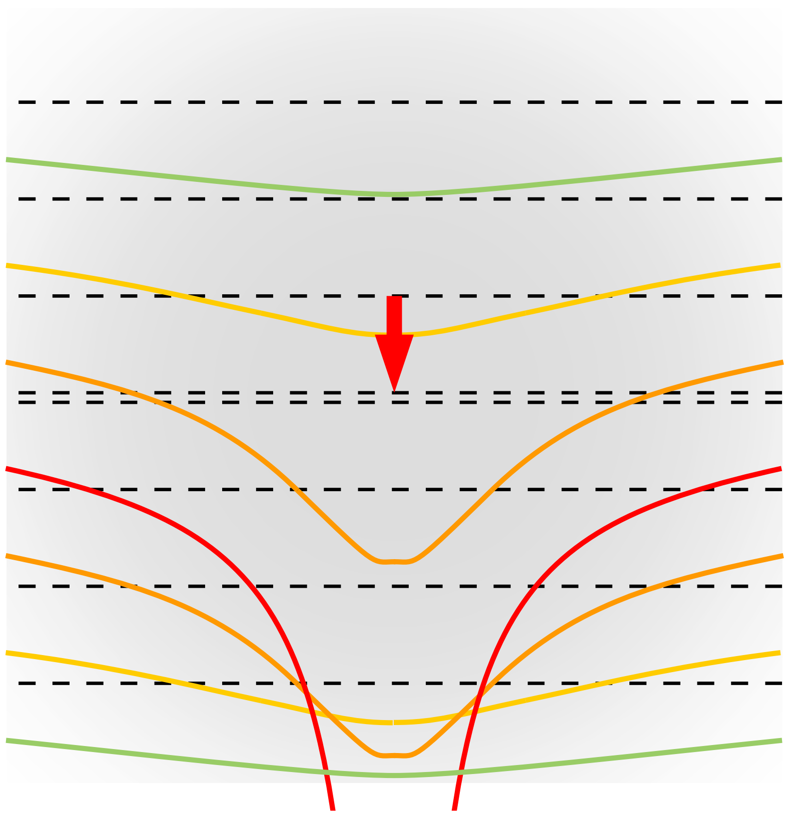
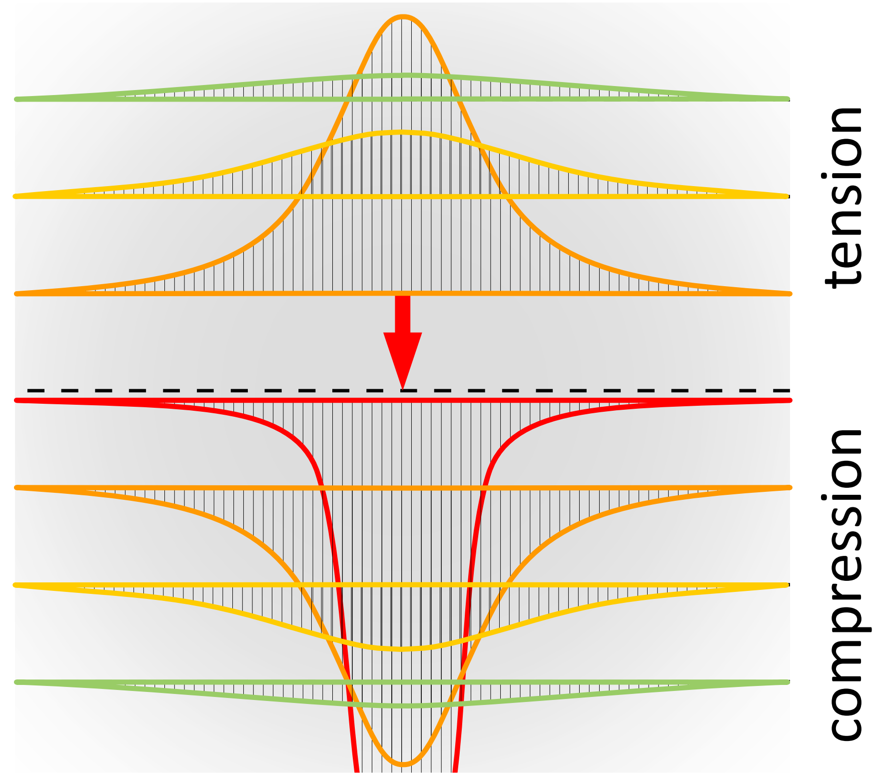
KELVIN'S SOLUTION

Kelvin's solution has a singularity in the point of application of the point force, namely values of the components of the displacement, strain and stress field diverge to infinity in this point:



distribution of displacement u_3



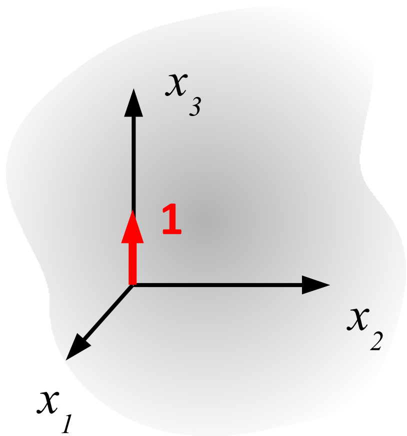
distribution of stress σ_{33}

KELVIN'S SOLUTION

Kelvin's solution may be considered a Green's function for all problems of an elastic space loaded with any system of body forces (of given direction):

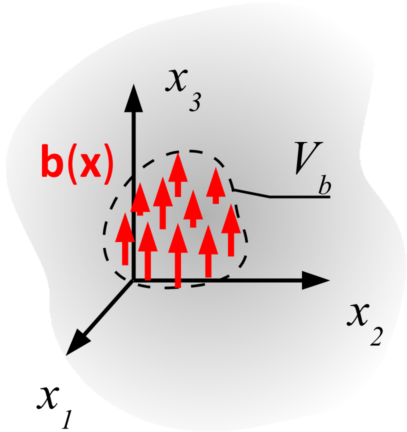
Kelvin's solution (**Green's function**):

$$\tilde{u}_i = \frac{1}{4\pi G}\left[\frac{1}{\left[(x_n-\xi_n)(x_n-\xi_n)\right]^{1/2}}\delta_{i3} + \frac{1}{4(1-\nu)}\frac{(x_i-\xi_i)(x_3-\xi_3)}{\left[(x_n-\xi_n)(x_n-\xi_n)\right]^{3/2}}\right]$$



Integral of Green's function: $\mathbf{u} = \int \left[\mathbf{G}(\mathbf{x}-\boldsymbol{\xi})\, q(\boldsymbol{\xi})\right] \mathrm{d}\boldsymbol{\xi}$

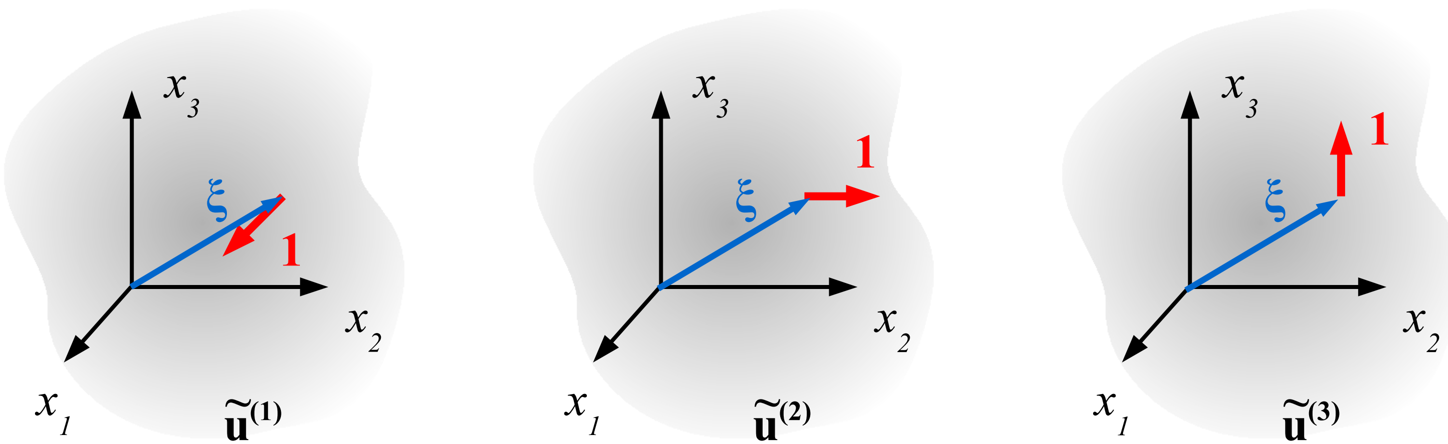
$$u_i(\mathbf{x}) = \iiint\limits_{V_b} \left[\tilde{u}_i(\mathbf{x}-\boldsymbol{\xi})\, b_3(\boldsymbol{\xi})\right] \mathrm{d}\boldsymbol{\xi}$$



KELVIN'S SOLUTION

- For a body forces field of varying orientation, the field is divided into 3 fields – each one of that fields has only one non-zero component, namely the one corresponding with the chosen axis of the coordinate system.

- Those fields are integrated separately and then they are added according to the **superposition principle**.



$$\tilde{u}_i^{(k)} = \frac{1}{4\pi G}\left[\frac{1}{\left[(x_n-\xi_n)(x_n-\xi_n)\right]^{1/2}}\delta_{ik} + \frac{1}{4(1-\nu)}\frac{(x_i-\xi_i)(x_k-\xi_k)}{\left[(x_n-\xi_n)(x_n-\xi_n)\right]^{3/2}}\right]$$

KELVIN'S SOLUTION

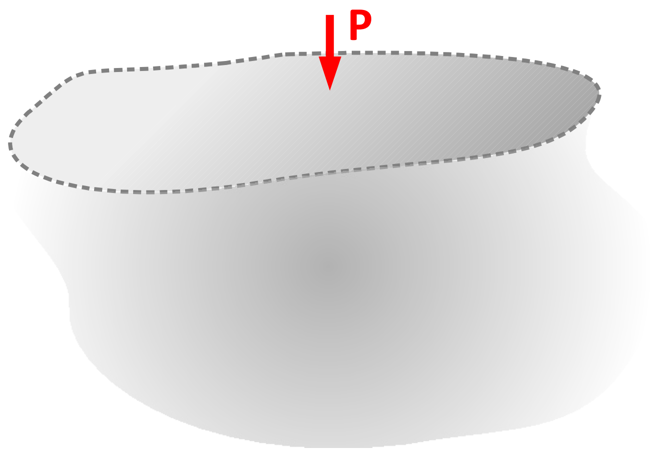
REMARKS:

- **Kelvin's solution** may be considered a **Green's function** for all problems of an elastic space loaded with any system of body forces. In such cases **integration of the Kelvin's solution makes the singularity vanish**.
    - Solution of a problem with point load (Dirac distribution) has a singularity.
    - Solution of a problem with a continuous load doesn't have a singularity.
- For this reason this solution is termed the fundamental solution of the linear theory of elasticity.
- It is also used in the so called Somigliana's formula, which in turn is used in solving the problems concerning the **elastic bodies of finite dimensions and arbitrary shapes**.
- Somigliana's formula provides also a basis for a numerical method of solving the problems of the linear theory of elasticity, namely for the Boundary Element Method.

BOUSSINESQ SOLUTION

BOUSSINESQ SOLUTION

Elastic half-space loaded with a point force.



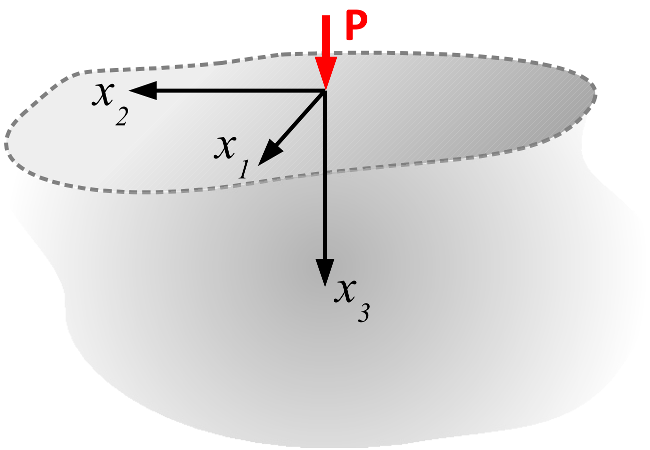
- Whole **elastic half-space** is considered **an infinitely large elastic body**. The half-space may be considered a **neighbourhood of a point on the boundary** of that body and the **size of that neighbourhood is much smaller than the whole body**.

- The elastic body is made of homogeneous isotropic Hooke's material characterized by elastic constants $\lambda,\ G$

- **External load** is given by a **point force** modelled by a **Dirac's delta distribution**.

BOUSSINESQ SOLUTION

Elastic half-space loaded with a point force.



- we assume a **Cartesian coordinate** system with its **origin in the point of application of the point force**, axis x_3 is oriented in the same way as the force, towards interior of the half-space.

- we will use the Lamé's displacement equations:

$$G\nabla^2 u_i + (\lambda + G) u_{k,ik} + b_i = 0$$

- body forces vector:

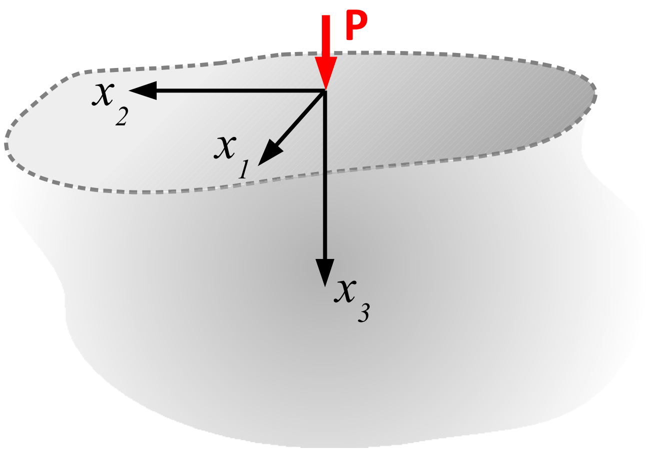
$$\mathbf{b} = [0;0;0]$$

- surface tractions vector:

$$\mathbf{q} = [0;0;P\delta_{\mathbf{0}}]$$

BOUSSINESQ SOLUTION

Elastic half-space loaded with a point force.



- Static boundary conditions:

  in plane $x_3 = 0$ of unit normal: $\mathbf{n} = [0\,;0\,;-1]$

  - in the **point of application of the force**: $x_1 = x_2 = x_3 = 0$

$$\sigma_{ij} n_j = q_i \qquad \Rightarrow \qquad \sigma_{33} = -P\,\delta_{0,} \quad \sigma_{13} = \sigma_{23} = 0$$

  - otherwise: $x_3 = 0, \quad x_1 \neq 0, \quad x_2 \neq 0$

$$\sigma_{ij} n_j = q_i \qquad \Rightarrow \qquad \sigma_{13} = \sigma_{23} = \sigma_{33} = 0$$

- kinematic boundary conditions:

$$\lim_{R\to\infty} u_i = 0, \quad \lim_{R\to\infty} u_{i,j} = 0, \qquad i,j = 1,2,3 \qquad \text{where} \qquad R = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

BOUSSINESQ SOLUTION

We will look for a solution in the form of a combination of **potentials**.

Let's consider the Boussinesq potentials A and B:

- POTENTIAL A:
  - **displacement** field: $u_i^A = \frac{1}{2G}\phi_{,i}$ where $\nabla^2\phi = 0$
  - **strain** field: $\varepsilon_{ij}^A = \frac{1}{2G}\phi_{,ij}$
  - **stress** field: $\sigma_{ij}^A = \phi_{,ij}$

- POTENTIAL B:
  - **displacement** field: $u_i^B = \frac{1}{2G}\left[\psi_{,i}x_3 - (3-4\nu)\psi\delta_{i3}\right]$ where $\nabla^2\psi = 0$
  - **strain** field: $\varepsilon_{ij}^B = \frac{1}{2G}\left[\psi_{,ij}x_3 - (1-2\nu)(\psi_{,j}\delta_{i3} + \psi_{,i}\delta_{j3})\right]$
  - **stress** field: $\sigma_{ij}^B = \psi_{,ij}x_3 - (1-2\nu)(\psi_{,j}\delta_{i3} + \psi_{,i}\delta_{j3}) - 2\nu\psi_{,3}\delta_{ij}$

We assume that the **solution** is in the following form:

$$u_i(\mathbf{x}) = A\, u_i^A(\mathbf{x}) + B\, u_i^B(\mathbf{x})$$

BOUSSINESQ SOLUTION

We assume that the solution is in the following form:

$$u_i(\mathbf{x}) = A\, u_i^A(\mathbf{x}) + B\, u_i^B(\mathbf{x})$$

REMARKS:

- we must find now such **harmonic functions** (satisfying the Laplace equations) $\phi(\mathbf{x})$ and $\psi(\mathbf{x})$ and such **constants** A and B, that all **boundary conditions** were satisfied.
- We're not sure if the true solutions can be represented in such a form. We cannot reject a possibility that even if such a solution is found, also other more general solutions of that problem may still exist.
- An additional assumption is that functions $\phi(\mathbf{x})$ and $\psi(\mathbf{x})$ are related by the following equation:

$$\psi = \phi_{,3}$$

BOUSSINESQ SOLUTION

According to the all above stated assumption, the **stress state** may be expressed as follows:

$$\sigma_{13} = A\,\phi_{,13} + B\left(\phi_{,331}\,x_3 - (1-2\nu)\,\phi_{,13}\right)$$

$$\sigma_{23} = A\,\phi_{,23} + B\left(\phi_{,332}\,x_3 - (1-2\nu)\,\phi_{,23}\right)$$

$$\sigma_{33} = A\,\phi_{,33} + B\left(\phi_{,333}\,x_3 - 2(1-\nu)\,\phi_{,33}\right)$$

Constants A and B are chosen in such a way that in every point of the plane $x_3 = 0$ the stress was such that:

$$\begin{cases} \sigma_{13}(x_3{=}0) = A\,\phi_{13} - B(1-2\nu)\,\phi_{13} = 0 \\ \sigma_{23}(x_3{=}0) = A\,\phi_{,23} - B(1-2\nu)\,\phi_{,23} = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} A = (1-2\nu) \\ B = 1 \end{cases}$$

Then:

$$\sigma_{13} = \phi_{,331}\,x_3$$

$$\sigma_{23} = \phi_{,332}\,x_3$$

$$\sigma_{33} = \phi_{,333}\,x_3 - \phi_{,33}$$

BOUSSINESQ SOLUTION

The boundary conditions for the shear stress are satisfied. We need to chose the harmonic functions $\phi(\mathbf{x})$ in such a way that also the **boundary conditions for the normal stress** were satisfied, namely:

in plane: $x_3=0$

$$\sigma_{33}(x_3=0) = -\phi_{,33} = \begin{cases} 0 & \Leftrightarrow \quad x_1\neq 0, x_2\neq 0 \\ -P\delta & \Leftrightarrow \quad x_1=x_2=0 \end{cases}$$

The above conditions provide the boundary conditions for the Laplace equation which must be satisfied by the potential:

$$\nabla^2\phi = \phi_{,11}+\phi_{,22}+\phi_{,33}=0$$

We may perform the Fourier transform with respect to the variables $x_1,\ x_2$

$$\mathscr{F}\{\phi_{,11}\} = -\omega_1^2\hat{\phi}$$

$$\mathscr{F}\{\phi_{22}\} = -\omega_2^2\hat{\phi} \quad \Rightarrow \quad \mathscr{F}\{\nabla^2\phi\} = \frac{\partial^2}{\partial x_3^2}\hat{\phi} - (\omega_1^2+\omega_2^2)\hat{\phi} = 0$$

$$\mathscr{F}\{\phi_{33}\} = \frac{\partial^2}{\partial x_3^2}\hat{\phi}$$

where

$$\phi=\phi(x_1,\ x_2,\ x_3)$$
$$\hat{\phi}=\hat{\phi}(\omega_1,\ \omega_2,\ x_3)$$

BOUSSINESQ SOLUTION

Obtained equation is an ordinary differential equation with respect to variable x_3 - it is a **2nd order linear equation** with **constant coefficients**:

$$\frac{\partial^2}{\partial x_3^2}\hat{\phi} - (\omega_1^2+\omega_2^2)\hat{\phi} = 0$$

The **solution** can be found easily:

$$\hat{\phi}(\omega_1, \omega_2, x_3) = C_1 e^{(\omega_1^2+\omega_2^2) x_3} + C_2 e^{-(\omega_1^2+\omega_2^2) x_3}$$

Constants of integration are chosen in such a way that the **boundary conditions** were satisfied:

- one of those conditions is that $\lim\limits_{R\to\infty} u_i = 0$, in particular $\lim\limits_{x_3\to\infty} u_3 = 0,$

$$u_3 = \frac{1-2\nu}{2G}\phi_{,3} = \frac{1-2\nu}{2G}\left[C_1(\omega_1^2+\omega_2^2)e^{(\omega_1^2+\omega_2^2)x_3} - C_2(\omega_1^2+\omega_2^2)e^{-(\omega_1^2+\omega_2^2)x_3}\right] = 0$$

- it is possible only for $C_1 = 0$

BOUSSINESQ SOLUTION

The other constant will be determined according to the **static boundary condition in the point of application of the point force**:

$$\sigma_{33}\big|_{x_3=0} = -\phi_{,33}\big|_{x_3=0} = -P\,\delta_{\mathbf{0}}$$

In the **space of transforms** (with respect to variables x_1 , x_2)

$$\mathscr{F}\left\{\phi_{,33}\big|_{x_3=0}\right\} = P\,\mathscr{F}\left\{\delta_{\mathbf{0}}\right\} \quad \Rightarrow \quad \left.\frac{\partial^2 \hat{\phi}}{\partial x_3^2}\right|_{x_3=0} = P$$

$$\left.\frac{\partial^2 \hat{\phi}}{\partial x_3^2}\right|_{x_3=0} = \left.\frac{\partial^2}{\partial x_3^2}\left[C_2 e^{-(\omega_1^2+\omega_2^2)x_3}\right]\right|_{x_3=0} = C_2(\omega_1^2+\omega_2^2)^2 e^{-(\omega_1^2+\omega_2^2)x_3}\Big|_{x_3=0} = C_2(\omega_1^2+\omega_2^2)^2 = P$$

$$C_2 = \frac{P}{(\omega_1^2+\omega_2^2)^2}$$

BOUSSINESQ SOLUTION

Transform of the potential:

$$\hat{\phi} = \frac{P}{(\omega_1^2+\omega_2^2)^2} e^{-(\omega_1^2+\omega_2^2) x_3}$$

Inverse transform must be found now:

$$\phi = \frac{P}{(2\pi)^2} \iint_{-\infty}^{\infty} \frac{e^{(\omega_1^2+\omega_2^2) x_3} \cdot e^{i(\omega_1 x_1+\omega_2 x_2)}}{(\omega_1^2+\omega_2^2)^2} d\omega_1 d\omega_2$$

It's again worth using the tables of originals and transforms. **Original of the** potential is equal:

$$\phi = -\frac{P}{2\pi} \ln\left(\sqrt{x_1^2+x_2^2+x_3^2}+x_3\right)$$

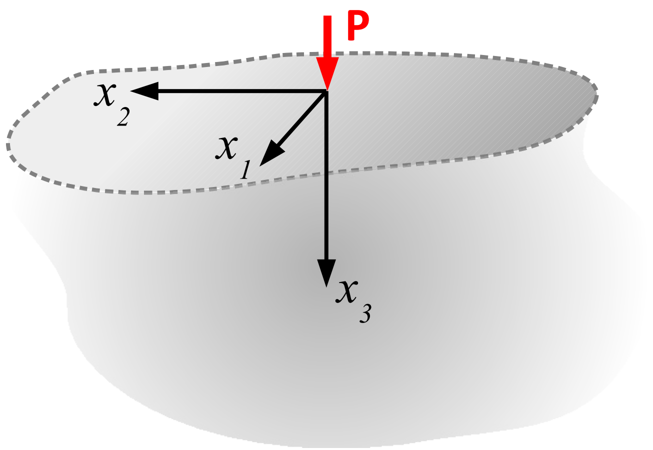
BOUSSINESQ SOLUTION

Distribution of **displacements** given by the Boussinesq solution:

$$u_1 = \frac{P x_1}{4\pi G}\left[\frac{x_3}{R^3} - \frac{1-2\nu}{R(R+x_3)}\right]$$

$$u_2 = \frac{P x_2}{4\pi G}\left[\frac{x_3}{R^3} - \frac{1-2\nu}{R(R+x_3)}\right]$$

$$u_3 = \frac{P}{4\pi G}\left[\frac{x_3^2}{R^3} + (3-4\nu)\frac{1}{R} - \frac{1-2\nu}{R+x_3}\left(\frac{x_3}{R}+1\right)\right]$$

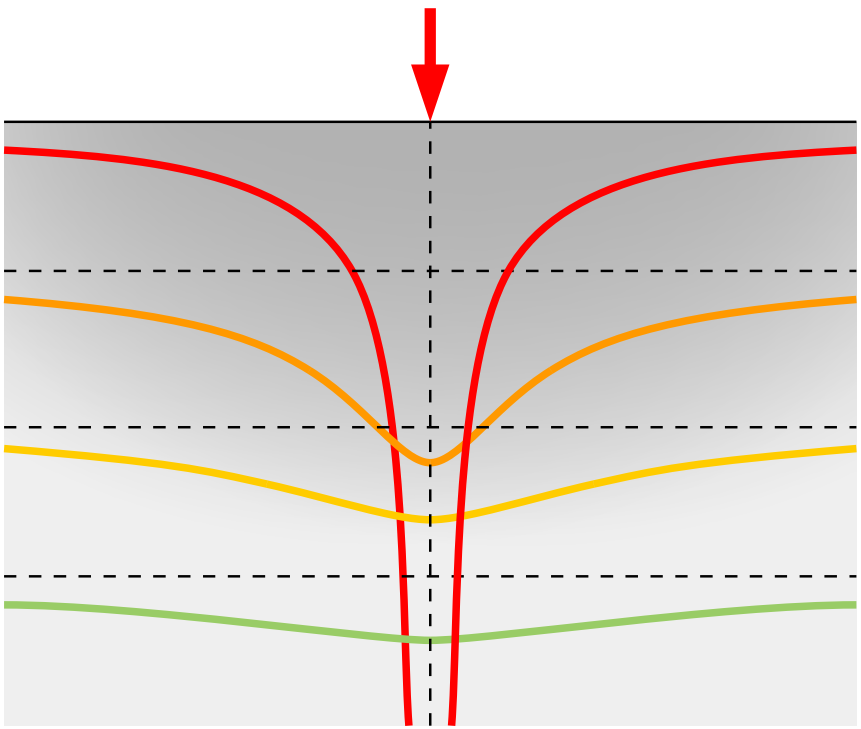


In general: $u_i = \frac{1}{4\pi G}\left[\frac{x_3 x_i}{R^3} + (3-4\nu)\frac{\delta_{i3}}{R} - \frac{1-2\nu}{R+x_3}\left(\frac{x_i}{R}+\delta_{i3}\right)\right]$

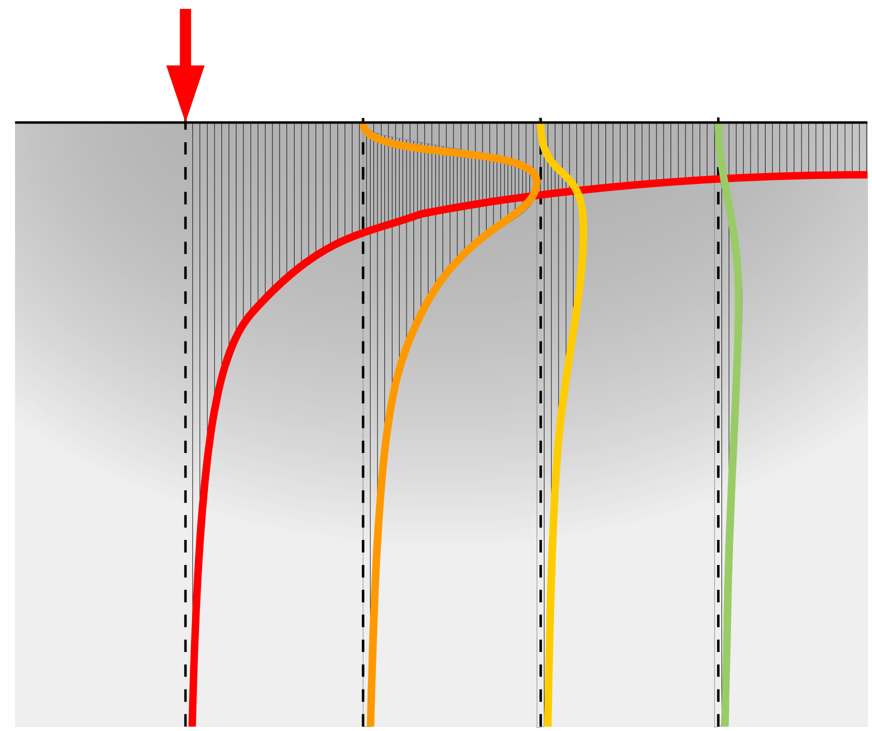
Strain is found according to kinematic relations, and stress is found according to constitutive relations.

BOUSSINESQ SOLUTION

The Boussinesq solution has a singularity **in the point of application of the point force**, namely values of the components of the displacement, strain and stress field diverge to infinity in this point:



distribution of displacement u_3



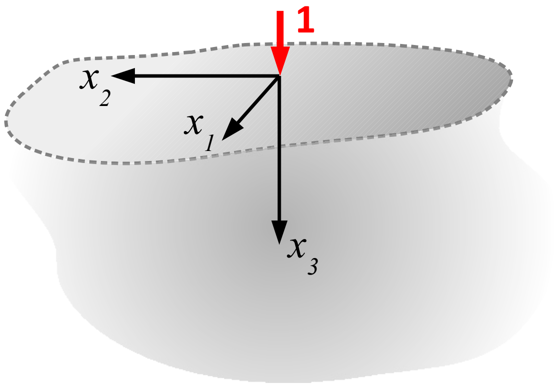
distribution of stress σ_{33}

BOUSSINESQ SOLUTION

Boussinesq solution may be used as a Green's function in order to find displacements, strains and stresses in an **elastic half-space loaded with any continuous surface tractions field** which is perpendicular to the plane $x_3 = 0$. For **continuous** and **bounded** loads there is **no singularity**.

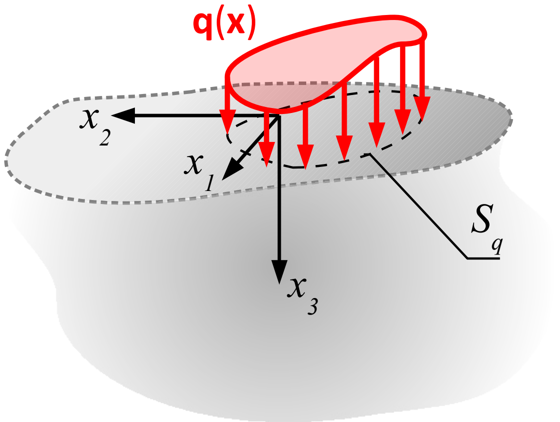
Boussinesq solution (**Green's function**):

$$\tilde{u}_i(x_1, x_2, x_3) = \frac{1}{4\pi G}\left[\frac{x_3 x_i}{R^3} + (3-4\nu)\frac{\delta_{i3}}{R} - \frac{1-2\nu}{R+x_3}\left(\frac{x_i}{R} + \delta_{i3}\right)\right]$$



Integral of the Green's function: $\mathbf{u} = \int \left[\mathbf{G}(\mathbf{x}-\boldsymbol{\xi})\, q(\boldsymbol{\xi})\right] \mathrm{d}\boldsymbol{\xi}$

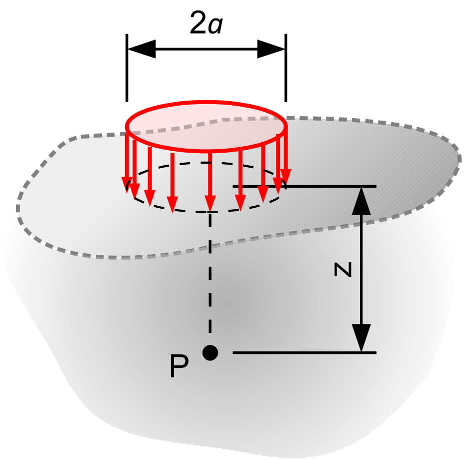
$$u_i = \iint_{S_q} \left[\tilde{u}_i(x_1-\xi_1, x_2-\xi_2, x_3)\, q(\xi_1, \xi_2)\right] \mathrm{d}\xi_1\, \mathrm{d}\xi_2$$



BOUSSINESQ SOLUTION

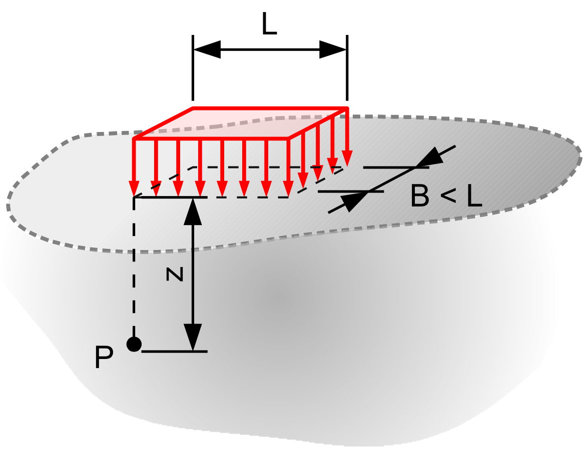
Stress distribution **under the centre of a uniformly loaded circular area**:

$$\sigma_{zz}(z) = q\left[\frac{1}{\left(1+\left(\frac{a}{z}\right)^2\right)^{3/2}} - 1\right]$$



Stress distribution **under the corner of a uniformly loaded rectangular area**:

$$\sigma_{zz} = \frac{q}{4\pi}\left[\frac{2\frac{BL}{z^2}\sqrt{\left(\frac{B}{z}\right)^2+\left(\frac{L}{z}\right)^2+1}}{\left(\frac{B}{z}\right)^2+\left(\frac{L}{z}\right)^2+\left(\frac{BL}{z^2}\right)^2+1}\cdot\frac{\left(\frac{B}{z}\right)^2+\left(\frac{L}{z}\right)^2+2}{\left(\frac{B}{z}\right)^2+\left(\frac{L}{z}\right)^2+1} + \operatorname{arctg}\left(\frac{2\frac{BL}{z^2}\sqrt{\left(\frac{B}{z}\right)^2+\left(\frac{L}{z}\right)^2+1}}{\left(\frac{B}{z}\right)^2+\left(\frac{L}{z}\right)^2-\left(\frac{BL}{z^2}\right)^2+1}\right)\right]$$



Boussinesq solution is used in **soil mechanics** and consequently also in **geotechnical standards**.

THANK YOU FOR YOUR ATTENTION