THEORY OF ELASTICITY AND PLASTICITY

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PLANE AXIS-SYMMETRIC PROBLEMS

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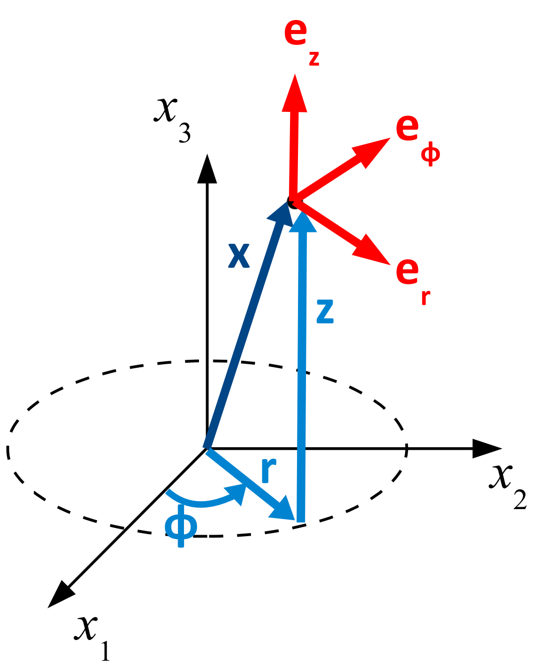
Some problems of the theory of elasticity are characterized by **symmetry**, which makes its **formulation and solution easier**.

A specific case of a symmetry is the **axial symmetry**. Accounting for axial symmetry becomes simple if the problem is formulated in the **cylindrical coordinate system**.

$$\begin{cases} r = \sqrt{x_1^2 + x_2^2} \\ \phi = \operatorname{arctg} \dfrac{x_2}{x_1} \\ z = x_3 \end{cases} \quad \Leftrightarrow \quad \begin{cases} x_1 = r\cos\phi \\ x_2 = r\sin\phi \\ x_3 = z \end{cases}$$



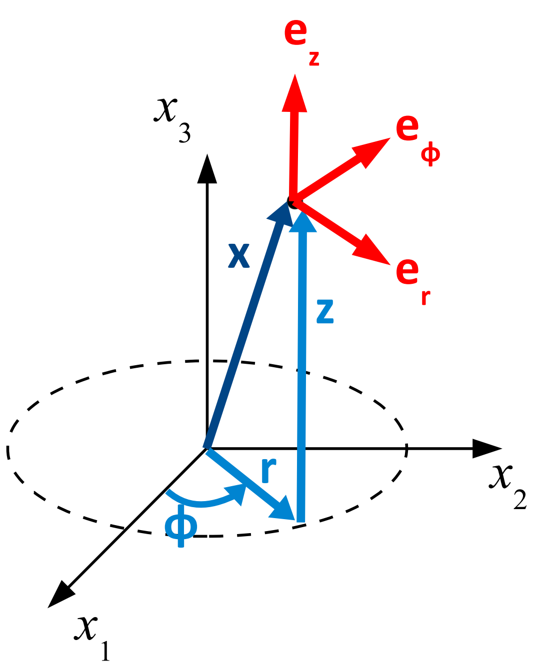
PLANE AXIS-SYMMETRIC PROBLEMS

The cylindrical coordinate system is an example of a curvilinear coordinate system. In such systems the governing equations of the theory of elasticity have different form. This is for the following reasons:

- **Tensorial quantities** in each point of the space are **determined by the reference to the** so called local basis – these are vectors which are **tangent to the lines of cruvilinear coordinates**. **In each point such a local basis is different**.

- Differentiation of tensors **require accounting for both variation of components** of tensors as well as for **variation of the local basis.**

- **Vectors of the local basis change their lengths – the are not constantly normalized.** As a result the magnitude of components of tensors loose its physical interpretation since it is disturbed by the non-unit length of the basis vector. It is necessary to introduce so called physical components, which compensate the change of length of basis vectors. It also influences the form of the governing equations.



PLANE AXIS-SYMMETRIC PROBLEMS

Governing equations of the linear theory of elasticity in the **cylindrical coordinate system**:

Equilibrium equations:

$$\sigma_{rr,r} + \frac{1}{r}\sigma_{r\phi,\phi} + \sigma_{zr,z} + \frac{\sigma_{rr} - \sigma_{\phi\phi}}{r} + b_r = 0$$

$$\sigma_{r\phi,r} + \frac{1}{r}\sigma_{\phi\phi,\phi} + \sigma_{\phi z,z} + \frac{2}{r}\sigma_{r\phi} + b_\phi = 0$$

$$\sigma_{zr,r} + \frac{1}{r}\sigma_{\phi z,\phi} + \sigma_{zz,z} + \frac{1}{r}\sigma_{zr} + b_z = 0$$

Geometric relations:

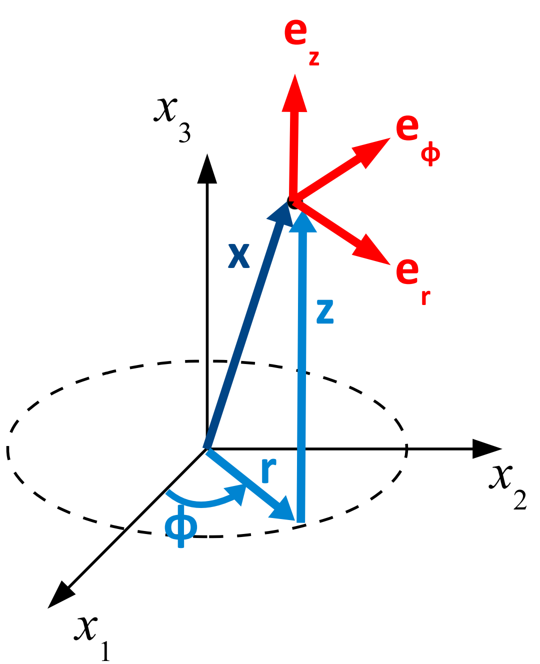
$$\varepsilon_{rr} = u_{r,r} \qquad \varepsilon_{\phi z} = \frac{1}{2}\left(u_{\phi,z} + \frac{1}{r}u_{z,\phi}\right)$$

$$\varepsilon_{\phi\phi} = \frac{1}{r}\left(u_r + u_{\phi,\phi}\right) \qquad \varepsilon_{zr} = \frac{1}{2}\left(u_{z,r} + u_{r,z}\right)$$

$$\varepsilon_{zz} = u_{z,z} \qquad \varepsilon_{r\phi} = \frac{1}{2}\left(\frac{1}{r}u_{r,\phi} + u_{\phi,r} - \frac{u_\phi}{r}\right)$$

Constitutive relations:

$$\sigma_{ij} = 2G\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}\,, \qquad i,j = r,\phi,z$$



PLANE AXIS-SYMMETRIC PROBLEMS

If **all unknown function in the problem do not depend on angle φ**, then the problem is termed and axis-symmetric problem:

$$\frac{\partial}{\partial \phi} \equiv 0$$

Concerning the **axis-symmetric problems** we will also assume that component displacement

$$u_{\phi} = 0$$

This gives us: $$\sigma_{r\phi} = 0, \quad \varepsilon_{r\phi} = 0$$

The problems in which the first condition is fulfilled while $u_{\phi} \neq 0$ will be termed quasi axis-symmetric problems.

PLANE AXIS-SYMMETRIC PROBLEMS

Governing equations of plane axis-symmetric problems:

Equilibrium equation:

$$\sigma_{rr,r} + \frac{1}{r}(\sigma_{rr} - \sigma_{\phi\phi}) = 0$$

Geometric relations:

$$\varepsilon_{rr} = u_{r,r}\,, \qquad \varepsilon_{\phi\phi} = \frac{u_r}{r}\,, \qquad \varepsilon_{r\phi} = 0$$

Constitutive relations:

$$\sigma_{rr} = (2G+\lambda)\varepsilon_{rr} + \lambda\varepsilon_{\phi\phi} = (2G+\lambda)u_{r,r} + \lambda\frac{u_r}{r}$$

$$\sigma_{\phi\phi} = (2G+\lambda)\varepsilon_{\phi\phi} + \lambda\varepsilon_{rr} = (2G+\lambda)\frac{u_r}{r} + \lambda u_{r,r}$$

$$\sigma_{r\phi} = 0$$

Strain compatibility condition:

$$-\frac{1}{r}\varepsilon_{rr,r} + \varepsilon_{\phi\phi,rr} + \frac{2}{r}\varepsilon_{\phi\phi,r} = 0$$

PLANE AXIS-SYMMETRIC PROBLEMS

Displacement equations for **plane axis-symmetric problems.**

Constitutive relations are substituted in the equilibrium equation:

$$\left[(2G+\lambda)u_{r,r}+\lambda\frac{u_r}{r}\right]_{,r}+\frac{1}{r}\left[(2G+\lambda)u_{r,r}+\lambda\frac{u_r}{r}-(2G+\lambda)\frac{u_r}{r}-\lambda u_{r,r}\right]=0$$

After some transformations:

$$\left[(2G+\lambda)u_{r,rr}+\lambda\left(\frac{u_{r,r}}{r}-\frac{u_r}{r^2}\right)\right]+\frac{2G}{r}\left(u_{r,r}-\frac{u_r}{r}\right)=0$$

This will satisfied if only:

$$u_{r,rr}+\frac{u_{r,r}}{r}-\frac{u_r}{r^2}=0$$

Obtained equation is an **ordinary differential equation**. Equations of such type are called the Euler's equation. **Its solution is known**

$$u_r(r)=C_1 r+\frac{C_2}{r}$$

Constants of integration are determined according to the **boundary conditions**.

PLANE AXIS-SYMMETRIC PROBLEMS

Plane problems (not only axis-symmetric ones) in **polar coordinates** may be also solved with the use of the Airy stress function.

Biharmonic equation **in polar coordinates** has the following form:

$$\nabla^4 F = F_{,rrrr} + \frac{2}{r^2} F_{,rr\phi\phi} + \frac{1}{r^4} F_{,\phi\phi\phi\phi} + \frac{2}{r} F_{,rrr} - \frac{2}{r^3} F_{,r\phi\phi} - \frac{1}{r^2} F_{,rr} + \frac{4}{r^4} F_{,\phi\phi} + \frac{1}{r^3} F_{,r} = 0$$

The relation between the Airy stress function and **components of the stress tensor** are as follows

$$\sigma_{rr} = \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \phi^2}, \qquad \sigma_{\phi\phi} = \frac{\partial^2 F}{\partial r^2}, \qquad \sigma_{r\phi} = \frac{1}{r^2}\frac{\partial F}{\partial \phi} - \frac{1}{r}\frac{\partial^2 F}{\partial r \partial \phi}$$

PLANE AXIS-SYMMETRIC PROBLEMS

There is a formula for a **general solution of a biharmonic equation in polar coordinates** – the so called Michell solution is as follows:

$$\begin{aligned} F(r,\phi) = {} & A_{01} r^2 + A_{02} r^2 \ln r + A_{03} \ln r + A_{04} \phi + \\ & + \left(A_{11} r^3 + A_{12} r \ln r + A_{14} r^{-1} \right) \cos\phi + A_{13} r \phi \sin\phi \\ & + \left(B_{11} r^3 + B_{12} r \ln r + B_{14} r^{-1} \right) \sin\phi + B_{13} r \phi \cos\phi \\ & + \sum_{n=2}^{\infty} \left[\left(A_{n1} r^{n+2} + A_{n2} r^{-n+2} + A_{n3} r^{n} + A_{n4} r^{-n} \right) \cos(n\phi) \right] \\ & + \sum_{n=2}^{\infty} \left[\left(B_{n1} r^{n+2} + B_{n2} r^{-n+2} + B_{n3} r^{n} + B_{n4} r^{-n} \right) \sin(n\phi) \right] \end{aligned}$$

Unknown coefficients in the above expression are determined according to the **boundary conditions**.

PLANE AXIS-SYMMETRIC PROBLEMS

For **plane axis-symmetric problems**, the biharmonic equation takes even simpler form:

$$\nabla^4 F = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial F}{\partial r}\right]\right]\right] = 0$$

Its solution can be found by direct integration:

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial F}{\partial r}\right]\right]\right] = 0 \quad \Rightarrow \quad r\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial F}{\partial r}\right]\right] = C_1$$

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial F}{\partial r}\right]\right] = \frac{C_1}{r} \quad \Rightarrow \quad \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial F}{\partial r}\right] = C_1 \ln r + C_2$$

$$\frac{\partial}{\partial r}\left[r\frac{\partial F}{\partial r}\right] = C_1 r \ln r + C_2 r \quad \Rightarrow \quad r\frac{\partial F}{\partial r} = \frac{C_1}{4}r^2\left(2\ln r - 1\right) + \frac{C_2}{2}r^2 + C_3$$

$$\frac{\partial F}{\partial r} = \frac{C_1}{4}r\left(2\ln r - 1\right) + \frac{C_2}{2}r + \frac{C_3}{r} \quad \Rightarrow \quad F = \frac{C_1}{4}r^2\left(\ln r - 1\right) + \frac{C_2}{4}r^2 + C_3 \ln r + C_4$$

PLANE AXIS-SYMMETRIC PROBLEMS

Finally, **solution of the biharmonic equation** for **plane axis-symmetric problems** has the following form:

$$F(r) = A_{00} + A_{01} r^2 + A_{02} r^2 \ln r + A_{03} \ln r$$

Components of the **stress tensor**:

$$\sigma_{rr} = 2A_{01} + A_{02}(2\ln r + 1) + \frac{A_{03}}{r^2}$$

$$\sigma_{\phi\phi} = 2A_{01} + A_{02}(2\ln r + 3) - \frac{A_{03}}{r^2}$$

$$\sigma_{r\phi} = 0$$

Constant of integration are determined according to the **boundary conditions**.

FLAMANT SOLUTION

FLAMANT SOLUTION

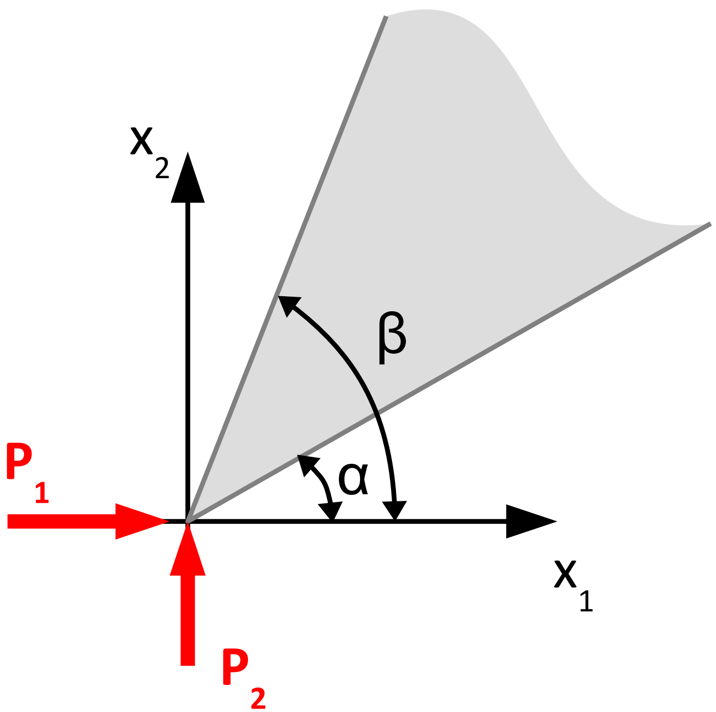
In the broader sense, the Flamant problem is the problem of an **elastic wedge loaded with a point force at its tip**. In the narrower sense, the Flamant problem is a problem of **an elastic half-space** (wedge angle equal 180°) **loaded with a point force**.

Problem of an elastic wedge

There is **an infinitely long wedge** enclosed by surfaces, which are inclined one to another at angle (β-α). It is made of **linear elastic material** (Hooke's material) characterized by elastic constants G, λ (respective for plane stress or plane strain state) and it is **loaded at its tip with a point force $\mathbf{P}$**.

The origin of the coordinate system is assumed in the tip of the wedge. Bottom surface of the wedge is inclined at angle α to the horizontal axis. Then:

$$\mathbf{P} = [P_1; P_2]$$



FLAMANT SOLUTION

Polar coordinate system is introduced:

$$\begin{cases} r = \sqrt{x_1^2+x_2^2} \\ \phi = \operatorname{arctg}\dfrac{x_2}{x_1} \end{cases} \Leftrightarrow \begin{cases} x_1 = r\cos\phi \\ x_2 = r\sin\phi \end{cases}$$

Equilibrium equations:

$$\sigma_{rr,r} + \frac{1}{r}\sigma_{r\phi,\phi} + \frac{1}{r}(\sigma_{rr}-\sigma_{\phi\phi}) = 0$$

$$\sigma_{r\phi,r} + \frac{1}{r}\sigma_{\phi\phi,\phi} + \frac{2}{r}\sigma_{r\phi} = 0$$

Geometric relations:

$$\varepsilon_{rr} = u_{r,r}$$

$$\varepsilon_{\phi\phi} = \frac{1}{r}u_{\phi,\phi} + \frac{u_r}{r}$$

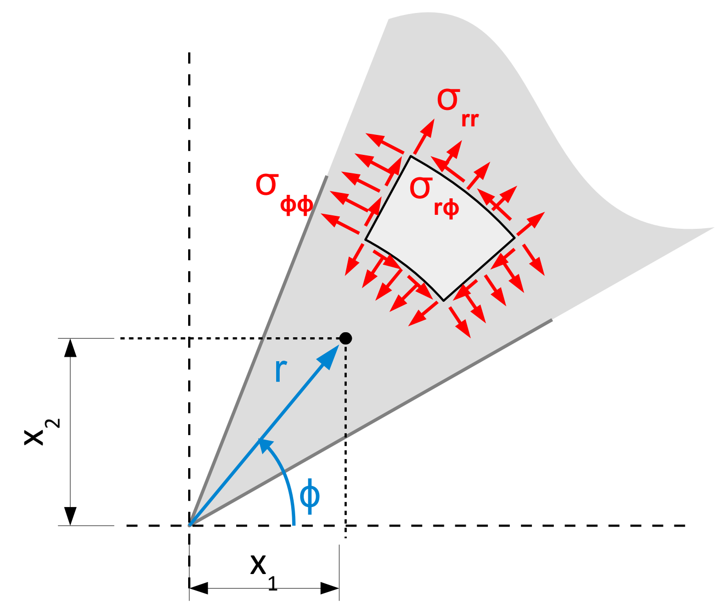
$$\varepsilon_{r\phi} = \frac{1}{2}\left(u_{\phi,r} - \frac{u_\phi}{r} + \frac{1}{r}u_{r,\phi}\right)$$

Constitutive relations:

$$\sigma_{rr} = (2G+\lambda)\varepsilon_{rr} + \lambda\varepsilon_{\phi\phi}$$

$$\sigma_{\phi\phi} = (2G+\lambda)\varepsilon_{\phi\phi} + \lambda\varepsilon_{rr}$$

$$\sigma_{r\phi} = 2G\varepsilon_{r\phi}$$

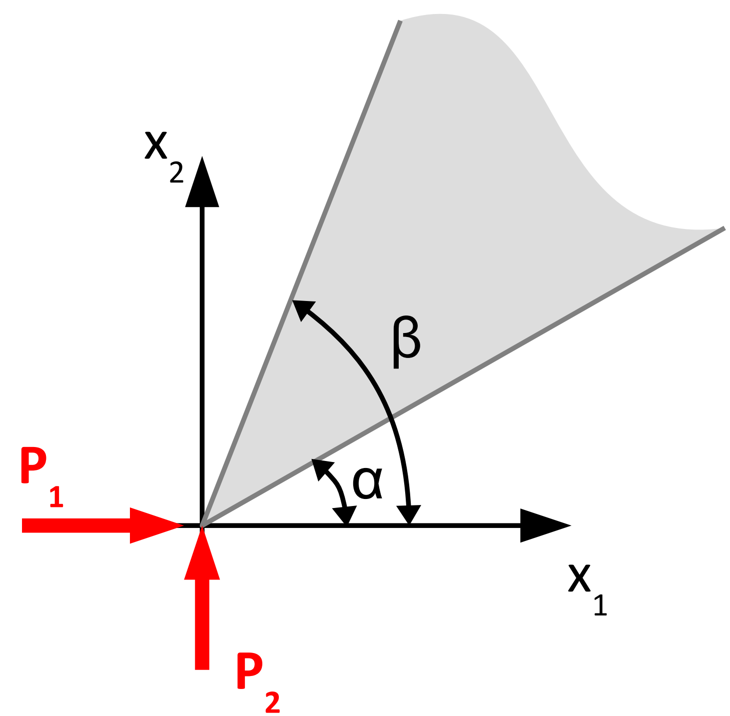


FLAMANT SOLUTION

Boundary conditions:

- **Bottom boundary** ($\phi = \alpha$, r – arbitrary but not equal to 0)
  - no normal traction $\sigma_{\phi\phi}(r,\alpha) = 0$
  - no tangent traction $\sigma_{r\phi}(r,\alpha) = 0$

- **Top boundary** ($\phi = \beta$, r – arbitrary but not equal to 0)
  - no normal traction $\sigma_{\phi\phi}(r,\beta) = 0$
  - no tangent traction $\sigma_{r\phi}(r,\beta) = 0$



All the above conditions will be satisfied if:

$$\sigma_{\phi\phi}(r,\phi) \equiv 0 \qquad \sigma_{r\phi}(r,\phi) \equiv 0$$

Let's assume that the above relations are true.

FLAMANT SOLUTION

REMARK:

- We haven't prescribed the **boundary condition at the tip of the wedge**. We cannot do it with the use of formula:

$$\sigma_{ij} n_j = q_j$$

since the external **unit normal vector is not defined at the tip**. We will prescribe this condition later in another way.

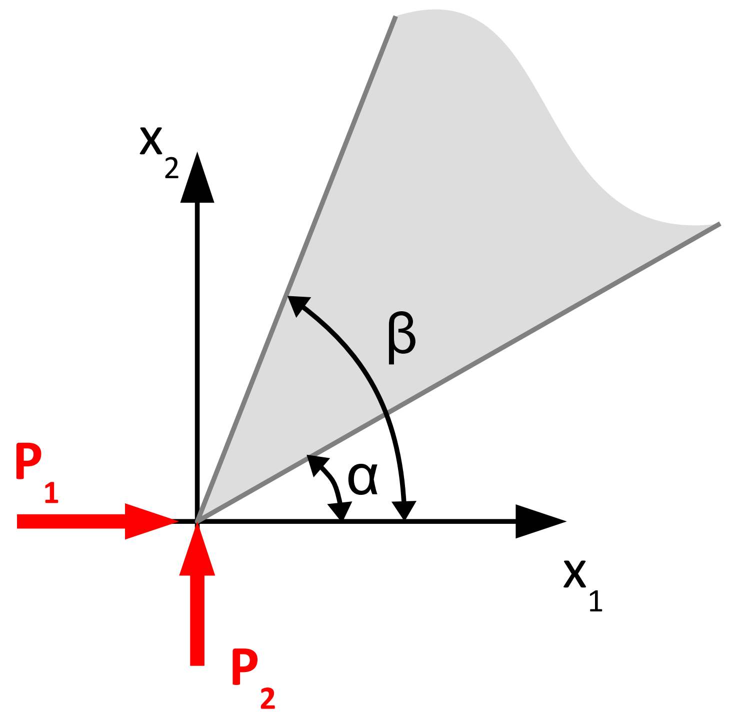
Assuming that the component σ_{rr} is the only non-zero component of the stress tensor, the **equilibrium equations** are reduced to a single equation:

$$\sigma_{rr,r} + \frac{\sigma_{rr}}{r} = 0$$

Its solution is:

$$\sigma_{rr}(r;\phi) = \frac{C(\phi)}{r}$$

Constant of integration C is a function of ϕ.



FLAMANT SOLUTION

Also geometric relations or the **strain compatibility condition** must be satisfied – the latter one in case of a plane problem is reduced to a single equation of the form:

$$\Delta(\sigma_{11}+\sigma_{22}) = 0$$

In **polar coordinates** it has the following form:

$$\Delta\sigma_{rr} = \sigma_{rr,rr} + \frac{1}{r}\sigma_{rr,r} + \frac{1}{r^2}\sigma_{rr,\phi\phi} = 0$$

If the just determined solution of the equilibrium equation is substituted in the above condition then – after some simple transformations – we will obtain an equation in which the function $C(\phi)$ is an unknown:

$$\frac{d^2 C}{d\phi^2} + C = 0$$

This is a simple ordinary differential equation. Its solution is as follows:

$$C(\phi) = C_1 \cos\phi + C_2 \sin\phi$$

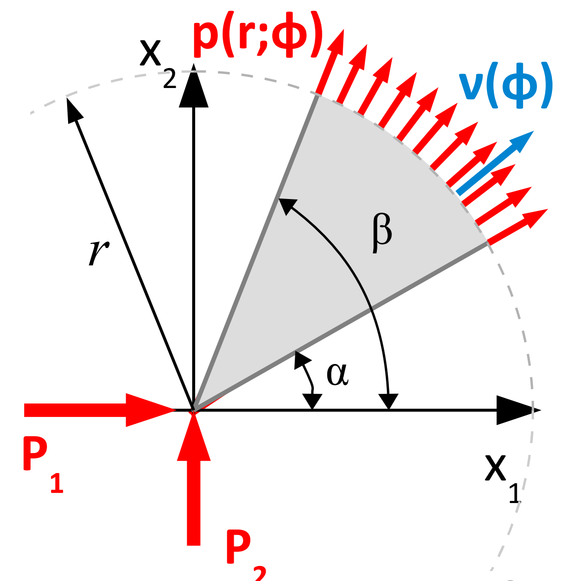
FLAMANT SOLUTION

Finally, the distribution of radial normal stress is given by the following formula:

$$\sigma_{rr}(r;\phi) = \frac{C_1}{r}\cos\phi + \frac{C_2}{r}\sin\phi$$

Constant of integration are determined according to the **boundary condition prescribed for the tip of the wedge**. It is formulated in the following way:

- We **cut out the tip of the wedge with a cylindrical surface** of radius r and with its axis passing through the tip and being perpendicular to the plane of the problem.
- We write down the **equilibrium condition** for the system of forces acting on such a part of the body. This system consists of the two **components of the point force** and of the **system of normal stresses applied to the cutting surface**.
- This condition must be fulfilled for all possible values of r.



FLAMANT SOLUTION

Stresses at the cutting surface:

$$\sigma_{ij}\nu_j = p_i \qquad \Rightarrow \qquad \mathbf{p}(r;\phi) = \sigma_{rr}(r;\phi)\cdot\boldsymbol{\nu}(\phi) = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \sigma_{rr}\cos\phi \\ \sigma_{rr}\sin\phi \end{bmatrix}$$

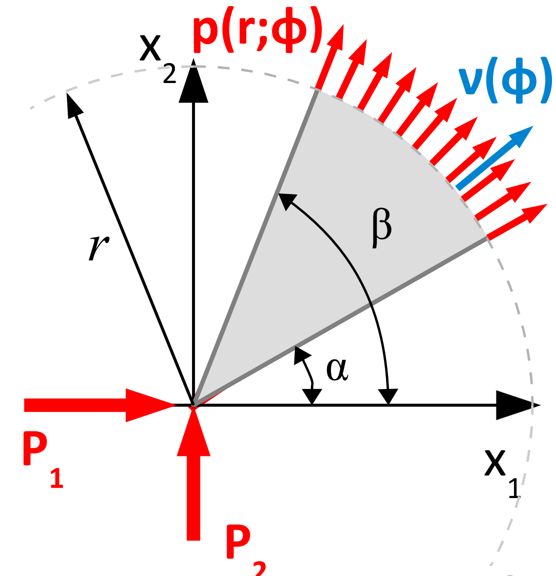
Sum of the system of forces – component along x_1 direction:

$$S_1 = P_1 + \int_K p_1\,\mathrm{d}s = P_1 + \int_{\phi=\alpha}^{\beta} \sigma_{rr}\cos\phi\, r\,\mathrm{d}\phi =$$

$$= P_1 + \int_{\alpha}^{\beta} r\cos\phi\left[\frac{C_1}{r}\cos\phi\,\mathrm{d}\phi + \frac{C_2}{r}\sin\phi\right]\mathrm{d}\phi =$$

$$= P_1 + C_1\int_{\alpha}^{\beta}\cos^2\phi\,\mathrm{d}\phi + C_2\int_{\alpha}^{\beta}\sin\phi\cos\phi\,\mathrm{d}\phi =$$

$$= P_1 + \frac{C_1}{4}\left[2(\beta-\alpha) + (\sin 2\beta - \sin 2\alpha)\right] + \frac{C_2}{4}\left[\cos 2\alpha - \cos 2\beta\right]$$



FLAMANT SOLUTION

Stresses at the cutting surface:

$$\sigma_{ij}\nu_j = p_i \qquad \Rightarrow \qquad \mathbf{p}(r;\phi) = \sigma_{rr}(r;\phi)\cdot\boldsymbol{\nu}(\phi) = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \sigma_{rr}\cos\phi \\ \sigma_{rr}\sin\phi \end{bmatrix}$$

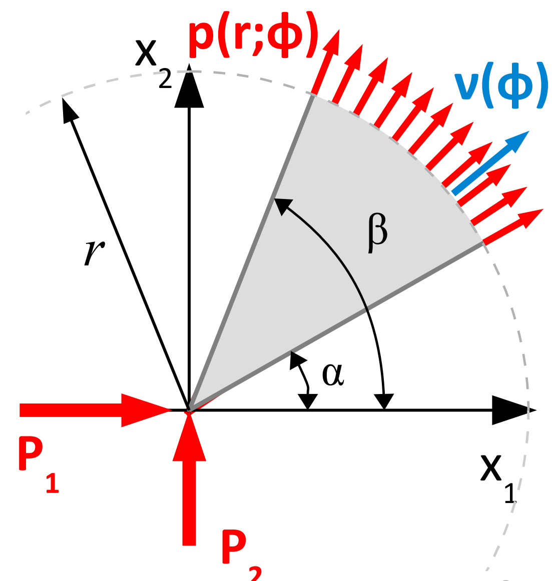
Sum of the system of forces – component along x_2 direction:

$$S_2 = P_2 + \int_K p_2 \,\mathrm{d}s = P_2 + \int_{\phi=\alpha}^{\beta} \sigma_{rr}\sin\phi\, r\,\mathrm{d}\phi =$$

$$= P_2 + \int_{\alpha}^{\beta} r\sin\phi\left[\frac{C_1}{r}\cos\phi\,\mathrm{d}\phi + \frac{C_2}{r}\sin\phi\right]\mathrm{d}\phi =$$

$$= P_2 + C_1\int_{\alpha}^{\beta}\cos\phi\sin\phi\,\mathrm{d}\phi + C_2\int_{\alpha}^{\beta}\sin^2\phi\,\mathrm{d}\phi =$$

$$= P_2 + \frac{C_1}{4}\left[\cos 2\alpha - \cos 2\beta\right] + \frac{C_2}{4}\left[2(\beta-\alpha) + (\sin 2\alpha - \sin 2\beta)\right]$$



FLAMANT SOLUTION

A concurrent system of forces is in equilibrium if and only if its **sum is equal to zero**

$$\begin{cases} P_1 + \dfrac{C_1}{4}\left[2(\beta-\alpha) + (\sin 2\beta - \sin 2\alpha)\right] + \dfrac{C_2}{4}\left[\cos 2\alpha - \cos 2\beta\right] = 0 \\ P_2 + \dfrac{C_1}{4}\left[\cos 2\alpha - \cos 2\beta\right] + \dfrac{C_2}{4}\left[2(\beta-\alpha) + (\sin 2\alpha - \sin 2\beta)\right] = 0 \end{cases}$$

We obtain:

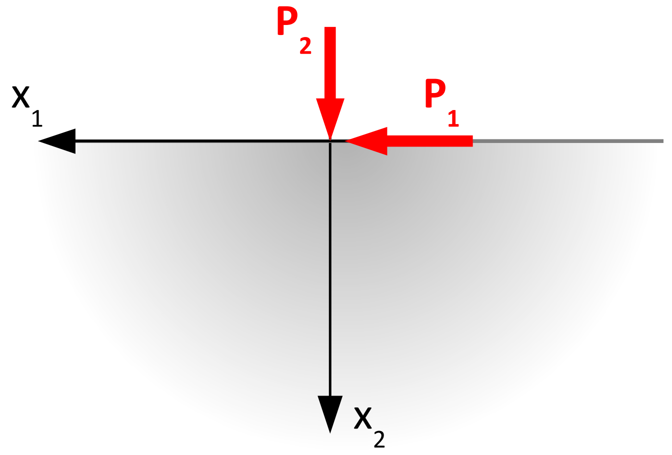
$$\begin{cases} C_1 = \dfrac{2P_1\left[\sin 2\beta - \sin 2\alpha - 2(\beta-\alpha)\right] + 2P_2\left[\cos 2\alpha - \cos 2\beta\right]}{\cos(2\beta - 2\alpha) + 2(\beta-\alpha)^2 - 1} \\ C_2 = \dfrac{2P_1\left[\cos 2\beta - \cos 2\alpha\right] + 2P_2\left[\sin 2\alpha - \sin 2\beta - 2(\beta-\alpha)\right]}{\cos(2\beta - 2\alpha) + 2(\beta-\alpha)^2 - 1} \end{cases}$$

Obtained solution is independent of r, despite the fact that the cutting surface changed with any change of r. These are indeed constants which determine the **distribution of stress in the Flamant solution**:

$$\sigma_{rr}(r;\phi) = \frac{C_1}{r}\cos\phi + \frac{C_2}{r}\sin\phi$$

FLAMANT SOLUTION

In case of an elastic half-space: $\alpha = 0^\circ\ ,\quad \beta = 180^\circ$



$$C_1 = -\frac{2P_1}{\pi}\ ,\qquad C_2 = -\frac{2P_2}{\pi}$$

$$\sigma_{rr}(r,\phi) = -\frac{2}{\pi r}\left[P_1\cos\phi + P_2\sin\phi\right]$$

The solution may be expressed with the use of Cartesian coordinates with the use of relations:

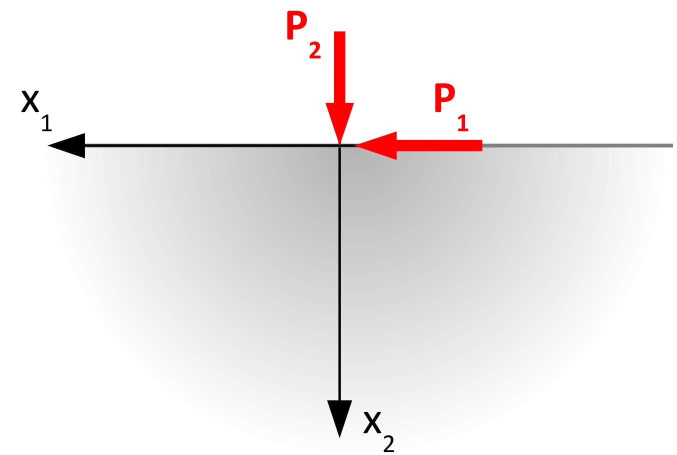
$$\cos\phi = \frac{x_1}{r} = \frac{x_1}{\sqrt{x_1^2+x_2^2}} \qquad \sin\phi = \frac{x_2}{r} = \frac{x_2}{\sqrt{x_1^2+x_2^2}}$$

and with the use of the **transformation rule**:

$$\sigma_{11} = \sigma_{rr}\cos^2\phi$$
$$\sigma_{22} = \sigma_{rr}\sin^2\phi$$
$$\sigma_{12} = \sigma_{rr}\sin\phi\cos\phi$$

FLAMANT SOLUTION

In case of an elastic half-space: $\alpha = 0^\circ\ ,\ \ \beta = 180^\circ$



Stress state:

$$\sigma_{11} = -\frac{2\,x_1^2}{\pi\,(x_1^2+x_2^2)^2}\left[P_1 x_1 + P_2 x_2\right]$$

$$\sigma_{22} = -\frac{2\,x_2^2}{\pi\,(x_1^2+x_2^2)^2}\left[P_1 x_1 + P_2 x_2\right]$$

$$\sigma_{12} = -\frac{2\,x_1 x_2}{\pi\,(x_1^2+x_2^2)^2}\left[P_1 x_1 + P_2 x_2\right]$$

Strain state:

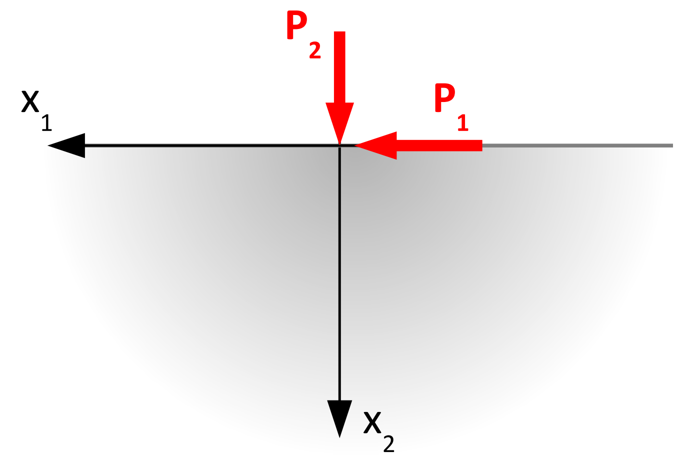
$$\varepsilon_{11} = -\frac{2\,(P_1 x_1+P_2 x_2)}{\pi\,\hat{E}\,(x_1^2+x_2^2)^2}\left[(1+\hat{\nu})\,x_1^2 - \hat{\nu}\,(x_1^2+x_2^2)\right]$$

$$\varepsilon_{22} = -\frac{2\,(P_1 x_1+P_2 x_2)}{\pi\,\hat{E}\,(x_1^2+x_2^2)^2}\left[(1+\hat{\nu})\,x_2^2 - \hat{\nu}\,(x_1^2+x_2^2)\right]$$

$$\varepsilon_{12} = -\frac{2\,(1+\hat{\nu})\,x_1 x_2\,(P_1 x_1+P_2 x_2)}{\pi\,\hat{E}\,(x_1^2+x_2^2)^2}$$

FLAMANT SOLUTION

It can be verified that the strain compatibility condition is satisfied:



$$\varepsilon_{11,22}-2\varepsilon_{12,12}+\varepsilon_{22,11}=0$$

It is then possible to integrate the geometric relations, what gives us the displacement field:

$$u_1=\int\varepsilon_{11}\,\mathrm{d}x_1+D_1(x_2)=$$

$$=-\frac{1}{\pi\hat{E}(x_1^2+x_2^2)}\left[\left((x_1^2+x_2^2)\ln(x_1^2+x_2^2)+(1+\hat{\nu})x_2^2\right)P_1-\left((1+\hat{\nu})x_1x_2-(1-\hat{\nu})(x_1^2+x_2^2)\operatorname{arctg}\frac{x_1}{x_2}\right)P_2\right]+D_1(x_2)$$

$$u_2=\int\varepsilon_{22}\,\mathrm{d}x_2+D_2(x_1)=$$

$$=-\frac{1}{\pi\hat{E}(x_1^2+x_2^2)}\left[\left((x_1^2+x_2^2)\ln(x_1^2+x_2^2)+(1+\hat{\nu})x_1^2\right)P_2-\left((1+\hat{\nu})x_1x_2-(1-\hat{\nu})(x_1^2+x_2^2)\operatorname{arctg}\frac{x_2}{x_1}\right)P_1\right]+D_2(x_1)$$

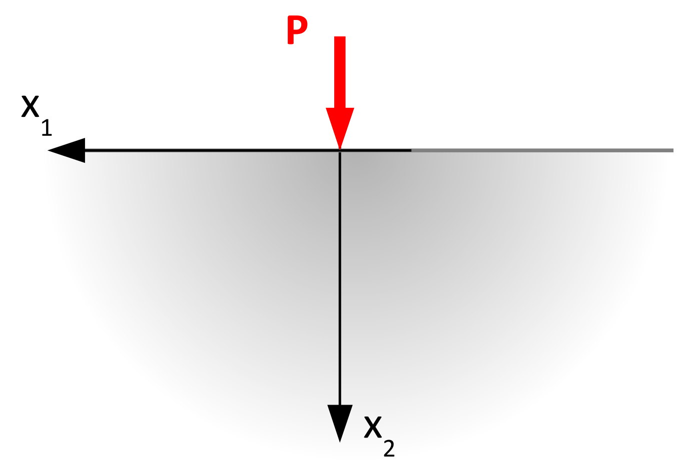
$$D_1(x_2)=R\,x_2+U_1\,,\quad D_2(x_1)=-R\,x_1+U_2\qquad R,\,U_1,\,U_2=\mathrm{const}$$

GENERAL SOLUTION OF HOMOGENEOUS GEOMETRIC RELATIONS
(RIGID MOTION = ROTATION + TRANSLATION)

Paweł Szeptyński PhD, Eng.

FLAMANT SOLUTION

In case of an elastic half-space loaded only with a normal force:



Displacement field:

$$u_1 = \frac{P_2}{\pi \hat{E}}\left[(1+\hat{\nu})\frac{x_1 x_2}{x_1^2+x_2^2}-(1-\hat{\nu})\,\mathrm{arctg}\,\frac{x_1}{x_2}\right]+R x_2+U_1$$

$$u_2 = -\frac{P_2}{\pi \hat{E}}\left[\ln(x_1^2+x_2^2)+(1+\hat{\nu})\frac{x_1^2}{x_1^2+x_2^2}\right]-R x_1+U_2$$

Strain state:

$$\varepsilon_{11} = -\frac{2 P x_2}{\pi \hat{E}(x_1^2+x_2^2)^2}\left[(1+\hat{\nu})x_1^2-\hat{\nu}(x_1^2+x_2^2)\right] \qquad \varepsilon_{22} = -\frac{2 P x_2}{\pi \hat{E}(x_1^2+x_2^2)^2}\left[(1+\hat{\nu})x_2^2-\hat{\nu}(x_1^2+x_2^2)\right]$$

$$\varepsilon_{12} = -\frac{2 P(1+\hat{\nu})x_1 x_2^2}{\pi \hat{E}(x_1^2+x_2^2)^2}$$

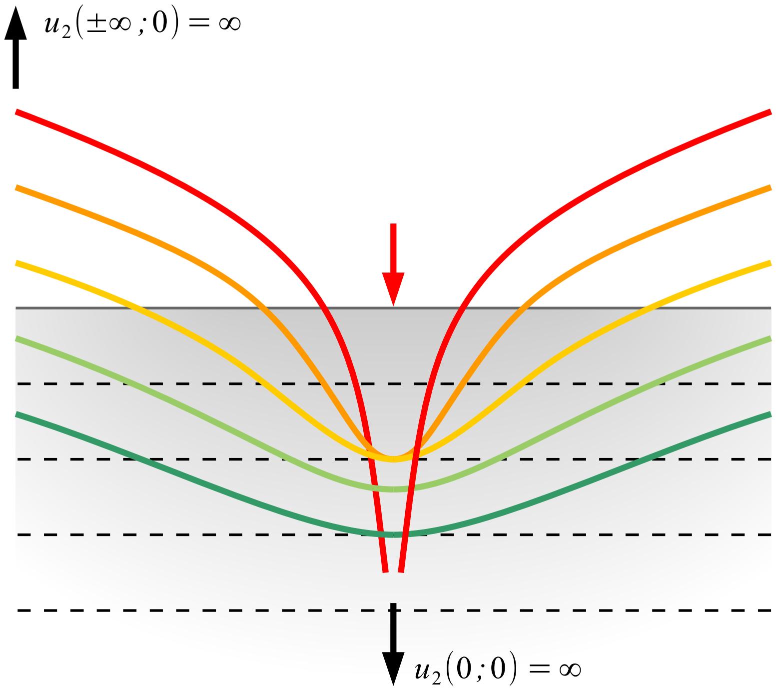
Stress state:

$$\sigma_{11} = -\frac{2 P_2 x_2 x_1^2}{\pi(x_1^2+x_2^2)^2} \qquad \sigma_{22} = -\frac{2 P_2 x_2^3}{\pi(x_1^2+x_2^2)^2} \qquad \sigma_{12} = -\frac{2 P_2 x_1 x_2^2}{\pi(x_1^2+x_2^2)^2}$$

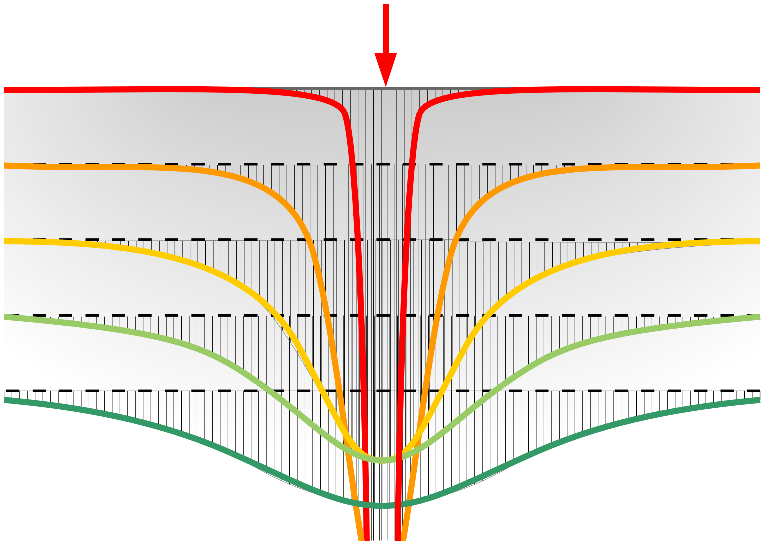
FLAMANT SOLUTION

REMARKS:

- Similarly as in the case of the Kelvin solution as well as the Boussinesq solution, the **Flamant solution has a singularity in the point of application of the point force** – values of the distribution of displacement, strain and stress tend to infinity.



distribution of displacement u_2

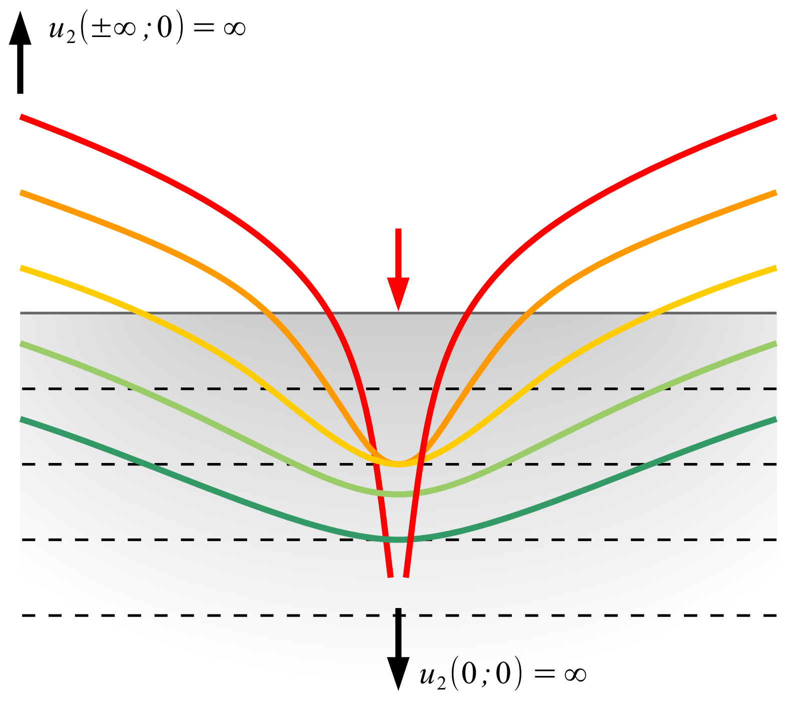


distribution of stress σ_{22}

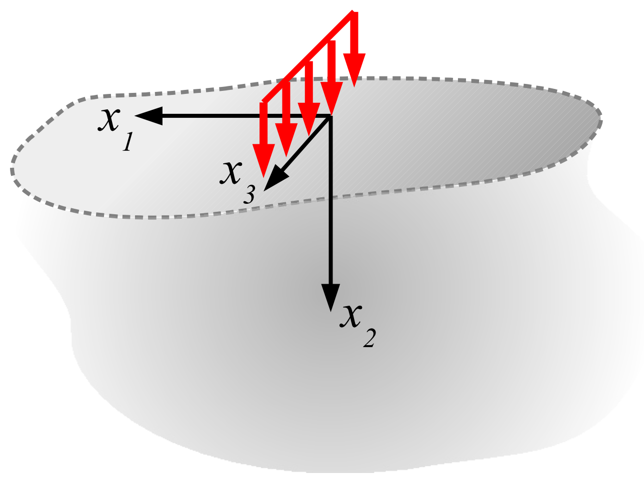
FLAMANT SOLUTION

REMARKS:

- **Displacements in point which are infinitely distant from the load** also **tend to infinity**. In fact, we are considering only a "cross-section" of a problem in which an infinitely large load is applied – therefore also the displacements should be infinitely large.



distribution of displacement u_2

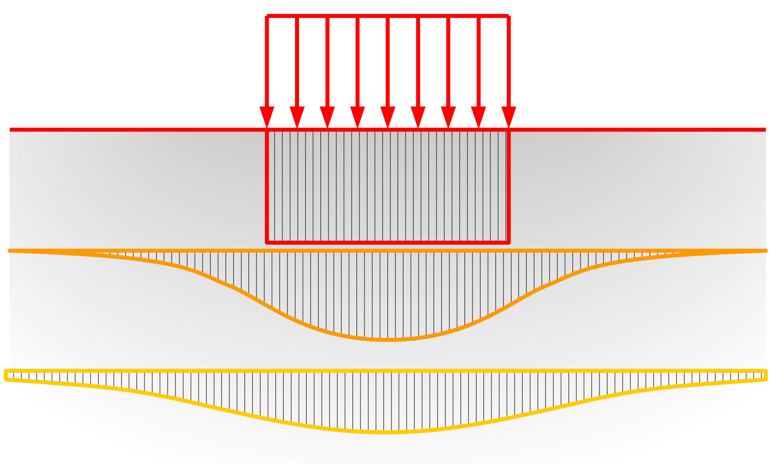


FLAMANT SOLUTION

REMARKS:

- The **Flamant solution** may be used as a Green's function for related problems.

- For example, in case of a half-space load uniformly on the area of length $2L$:



distribution of stress σ_{22}

$$\sigma_{22} = \int_{-L}^{L} q(\xi)\hat{\sigma}_{22}(x_1-\xi, x_2)\,d\xi =$$

$$= -\frac{2q}{\pi}\int_{-L}^{L} \frac{x_2^3}{((x_1-\xi)^2+x_2^2)^2}\,d\xi =$$

$$= -\frac{q}{\pi}\left[\frac{(x_1+L)x_2}{(x_1+L)^2+x_2^2} - \frac{(x_1-L)x_2}{(x_1-L)^2+x_2^2} + \operatorname{arctg}\frac{x_1+L}{x_2} - \operatorname{arctg}\frac{x_1-L}{x_2}\right]$$

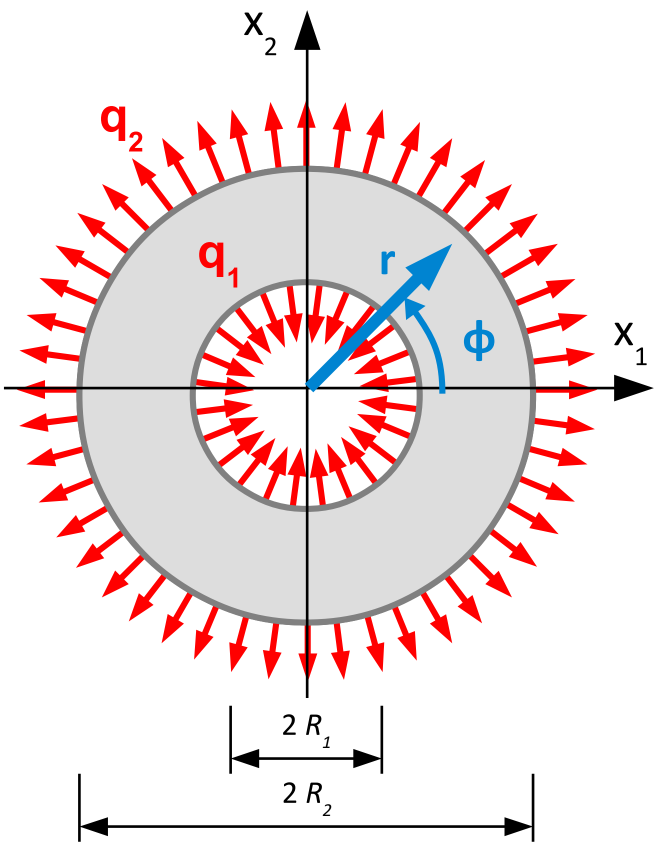
LAMÉ SOLUTION

LAMÉ SOLUTION

The Lamé problem is a problem of a **thick-walled tube loaded at both internal and external surface with a uniform pressure**. The tube is made of a linear elastic material (Hooke's material) characterized by elastic constants G, λ. Internal radius is R_1 , external radius is R_2 .

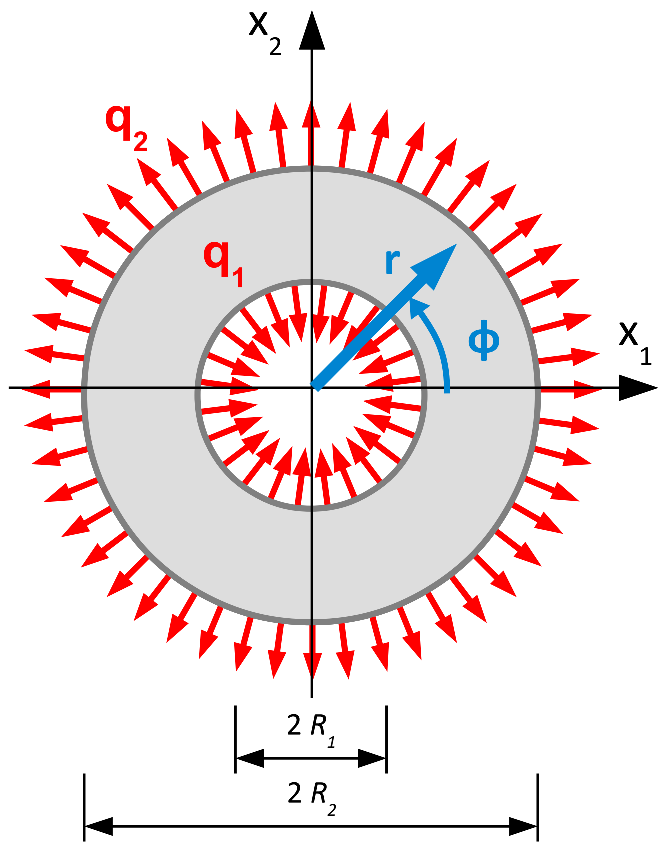
The problems is an axis-symmetric one.

It will be convenient to use the **polar coordinates** to find the solution.



LAMÉ SOLUTION

In case of a **plane axis-symmetric problem** we have: $u_\phi \equiv 0$



Equilibrium equation:

$$\sigma_{rr,r} + \frac{\sigma_{rr} - \sigma_{\phi\phi}}{r} = 0$$

Geometric relations:

$$\varepsilon_{rr} = u_{r,r}$$

$$\varepsilon_{\phi\phi} = \frac{u_r}{r}$$

$$\varepsilon_{r\phi} = 0$$

Constitutive relations:

$$\sigma_{rr} = (2G + \lambda)\varepsilon_{rr} + \lambda\varepsilon_{\phi\phi}$$

$$\sigma_{\phi\phi} = (2G + \lambda)\varepsilon_{\phi\phi} + \lambda\varepsilon_{rr}$$

$$\sigma_{r\phi} = 0$$

Boundary conditions:

internal surface: $\sigma_{rr}(R_1) = q_1$

external surface: $\sigma_{rr}(R_2) = q_2$

LAMÉ SOLUTION

The problem will be solved with the use of displacement equation for **axis-symmetric problems**:

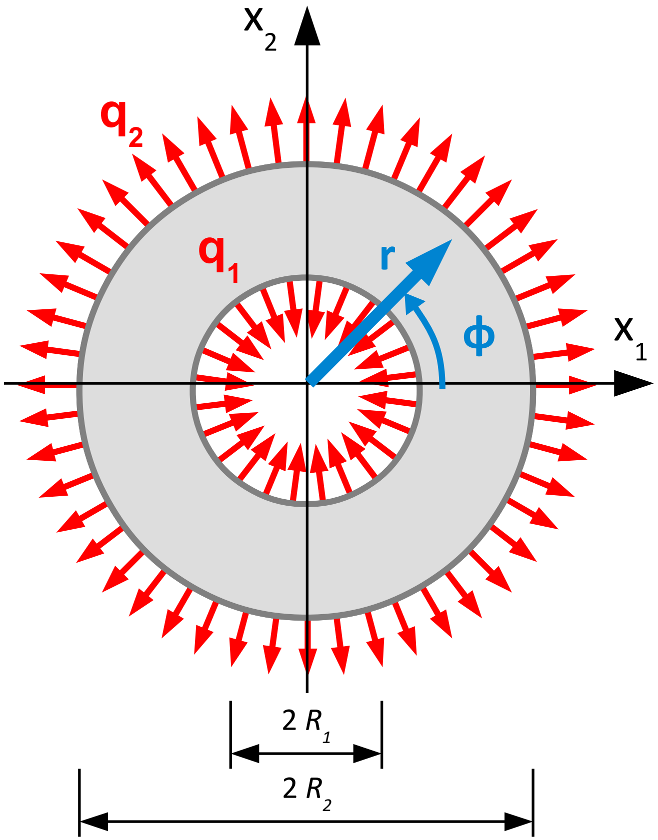
$$u_{r,rr}+\frac{u_{r,r}}{r}-\frac{u_r}{r^2}=0$$

This is the **Euler's equation**. Its **solution** is as follows:

$$u_r(r)=C_1 r+\frac{C_2}{r}$$

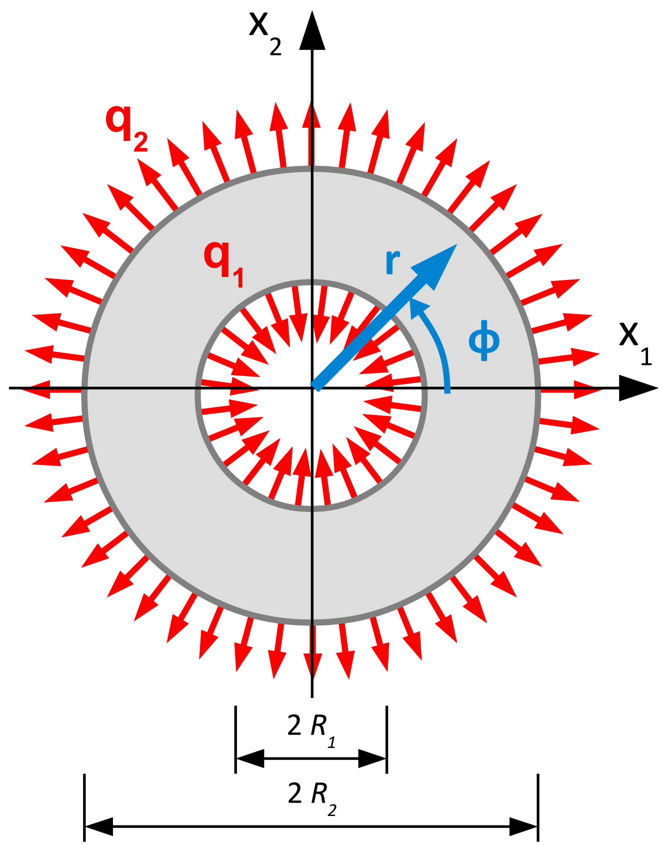
Constants of integration are found according to the **boundary conditions**:

$$\begin{cases}\sigma_{rr}(R_1)=2C_1(G+\lambda)-\dfrac{2C_2G}{R_1^2}=q_1\\ \sigma_{rr}(R_2)=2C_1(G+\lambda)-\dfrac{2C_2G}{R_2^2}=q_2\end{cases}\Rightarrow\begin{cases}C_1=\dfrac{q_2R_2^2-q_1R_1^2}{(2G+\lambda)(R_2^2-R_1^2)}\\ C_2=\dfrac{R_1^2R_2^2(q_2-q_1)}{2G(R_2^2-R_1^2)}\end{cases}$$



LAMÉ SOLUTION

The Lamé solutions is as follows:



Displacement field:

$$u_r(r) = \frac{q_2 R_2^2 - q_1 R_1^2}{(2G+\lambda)(R_2^2 - R_1^2)} \cdot r + \frac{R_1^2 R_2^2 (q_2 - q_1)}{2G(R_2^2 - R_1^2)} \cdot \frac{1}{r}$$

Strain state:

$$\varepsilon_{rr}(r) = \frac{q_2 R_2^2 - q_1 R_1^2}{(2G+\lambda)(R_2^2 - R_1^2)} - \frac{R_1^2 R_2^2 (q_2 - q_1)}{2G(R_2^2 - R_1^2)} \cdot \frac{1}{r^2}$$

$$\varepsilon_{\phi\phi}(r) = \frac{q_2 R_2^2 - q_1 R_1^2}{(2G+\lambda)(R_2^2 - R_1^2)} + \frac{R_1^2 R_2^2 (q_2 - q_1)}{2G(R_2^2 - R_1^2)} \cdot \frac{1}{r^2}$$

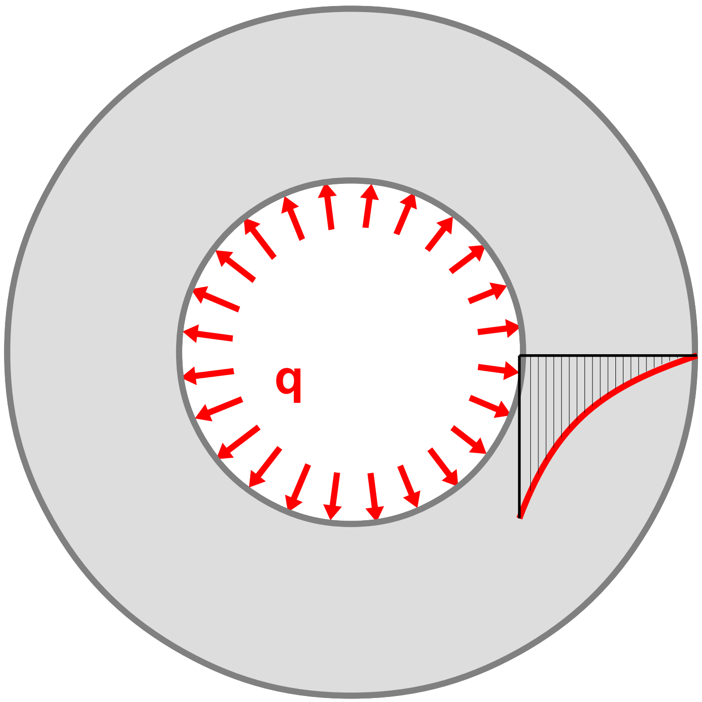
Stress state:

$$\sigma_{rr}(r) = \frac{q_2 R_2^2 - q_1 R_1^2}{R_2^2 - R_1^2} - \frac{(q_2 - q_1) R_1^2 R_2^2}{R_2^2 - R_1^2} \cdot \frac{1}{r^2}$$

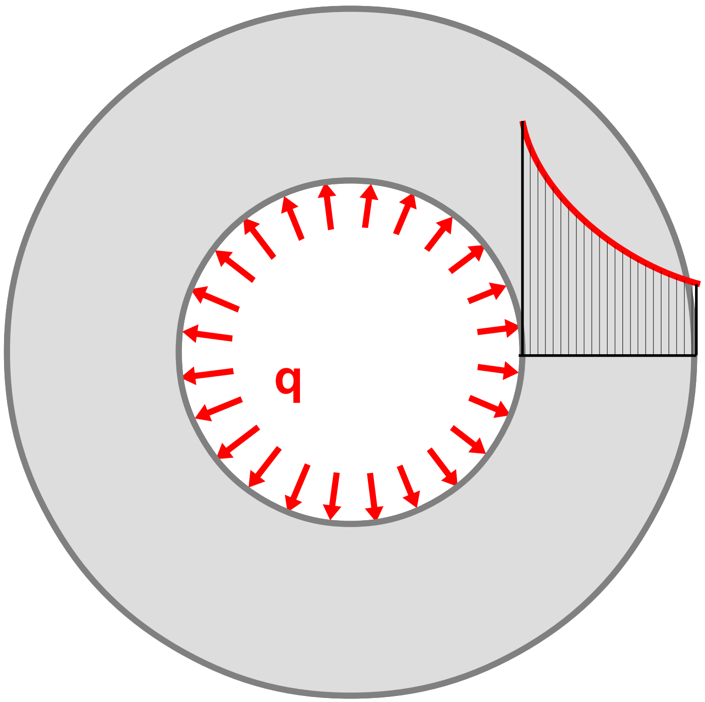
$$\sigma_{\phi\phi}(r) = \frac{q_2 R_2^2 - q_1 R_1^2}{R_2^2 - R_1^2} + \frac{(q_2 - q_1) R_1^2 R_2^2}{R_2^2 - R_1^2} \cdot \frac{1}{r^2}$$

LAMÉ SOLUTION

In case of a tube loaded only with internal pressure:



distribution of radial normal stress σ_{rr}



distribution of circumferential normal stress $\sigma_{\phi\phi}$

REMARK:

- The **Lamé solution** is used in the **design of pipelines**.

THANK YOU FOR YOUR ATTENTION