THEORY OF ELASTICITY AND PLASTICITY

Paweł Szeptyński, PhD, Eng.

room: 320 (3rd floor, main building)
Tel. +48 12 628 20 30
e-mail: pszeptynski@pk.edu.pl

WORK, POWER AND ENERGY IN ELASTIC DEFORMATION

WORK, POWER AND ENERGY

The work is intuitively understood as a **product of a force and of a displacement** that the body underwent when the force acted on it:

$$W = F \cdot s$$

In 3-dimensional problems the work is calculated as a **sum of works performed by components of the force vector along parallel displacements**, namely as a **dot product**:

$$W = \mathbf{F} \cdot \mathbf{s} = F_1 s_1 + F_2 s_2 + F_3 s_3$$

In the case of a non-uniform or curvilinear motion in a variable force field, **work performed by a vector field F on displacement of a particle along a curve K** is calculated as a sum of infinitely small increments of work of that force field along infinitely small increments of displacement, namely as an **oriented line integral of a vector field**:

$$W = \int_K \mathbf{F}(\mathbf{x}) \cdot \mathrm{d}\mathbf{x}$$

WORK, POWER AND ENERGY

Power is the **derivative of work with respect to time**:

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} = \mathbf{F}\cdot\mathbf{v}$$

Kinetic energy is a **work** that need to be performed in order to **make a still body of mass m move with velocity $\mathbf{v}$**:

$$E_k = \frac{m}{2}\mathbf{v}\cdot\mathbf{v}$$

In the case of a **potential force field**, namely such that can be determined as a **gradient of a certain scalar-valued function V**, termed the potential:

$$\mathbf{F} = \operatorname{grad} V = \left[\frac{\partial V}{\partial x_1} ; \frac{\partial V}{\partial x_2} ; \frac{\partial V}{\partial x_3}\right]$$

we can speak also of potential energy of such a field – it is a **work which will be performed by forces of that field when a particle is moved from one point in that field to another point**, and this work is equal to the **difference between values of potential in those points**.

$$\Delta E_k = W_{A\rightarrow B} = E_p(A) - E_p(B) = V(B) - V(A)$$

WORK, POWER AND ENERGY

The presented definitions regarded **point masses** and they could be also applied in the description of a motion of a **rigid body**. In the case of **deformable solids** it is necessary to use distinct definitions.

Power of external forces:

$$P \overset{\text{Df.}}{=} \iiint_V b_i v_i \,\mathrm{d}V + \iint_S q_i v_i \,\mathrm{d}S$$

Let's make use of the **tetrahedron's conditions**:

$$P = \iiint_V b_i v_i \,\mathrm{d}V + \iint_S \sigma_{ij} \nu_j v_i \,\mathrm{d}S$$

We use the **Green – Gauss – Ostrogradski theorem**:

$$P = \iiint_V b_i v_i \,\mathrm{d}V + \iiint_V (\sigma_{ij} v_i)_{,j} \,\mathrm{d}V$$

According to the **product rule**:

$$P = \iiint_V b_i v_i \,\mathrm{d}V + \iiint_V (\sigma_{ij,j} v_i + \sigma_{ij} v_{i,j}) \,\mathrm{d}V$$

WORK, POWER AND ENERGY

$$P = \iiint_V b_i v_i \, \mathrm{d}V + \iiint_V \sigma_{ij,j} v_i \, \mathrm{d}V + \iiint_V \sigma_{ij} v_{i,j} \, \mathrm{d}V$$

Let's use the **equations of motion** derived from the Newton's 2nd Law of Motion:

$$P = \iiint_V b_i v_i \, \mathrm{d}V + \iiint_V (\rho a_i - b_i) v_i \, \mathrm{d}V + \iiint_V \sigma_{ij} v_{i,j} \, \mathrm{d}V$$

Due to **additivity of integrals** we obtain:

$$P = \iiint_V \rho a_i v_i \, \mathrm{d}V + \iiint_V \sigma_{ij} v_{i,j} \, \mathrm{d}V$$

The integrand in the first integral may be transformed according to the **product rule**:

$$\rho a_i v_i = \rho \frac{\mathrm{d} v_i}{\mathrm{d} t} v_i = \rho \frac{\mathrm{d}}{\mathrm{d} t} \left(\frac{1}{2} v_i v_i \right)$$

In the **linear theory of elasticity** we assume that density and configuration of a body are approximately constant in time, so differentiation with respect to time may be put outside the integral:

$$P = \frac{\mathrm{d}}{\mathrm{d} t} \iiint_V \frac{\rho}{2} v_i v_i \, \mathrm{d}V + \iiint_V \sigma_{ij} v_{i,j} \, \mathrm{d}V$$

WORK, POWER AND ENERGY

$$P = \frac{\mathrm{d}}{\mathrm{d}t} \iiint\limits_V \frac{\rho}{2} v_i v_i \,\mathrm{d}V + \iiint\limits_V \sigma_{ij} v_{i,j} \,\mathrm{d}V$$

Due to the **symmetry of the stress tensor**, the second integrand may be transformed as follows:

$$\sigma_{ij} v_{i,j} = \frac{1}{2}\left(\sigma_{ij} v_{i,j} + \sigma_{ji} v_{i,j}\right) = \frac{1}{2}\left(\sigma_{ij} v_{i,j} + \sigma_{ij} v_{j,i}\right) = \sigma_{ij} \frac{1}{2}\left(v_{i,j} + v_{j,i}\right)$$

In the **linear theory of elasticity**, in which material and spatial description are equivalent, total differentiation with respect to time in equivalent to partial differentiation, in which the sequence of differentiation may be switched:

$$\sigma_{ij} \frac{1}{2}\left[\left(\frac{\mathrm{d}}{\mathrm{d}t} u_i\right)_{,j} + \left(\frac{\mathrm{d}}{\mathrm{d}t} u_j\right)_{,i}\right] = \sigma_{ij} \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\left(u_{i,j} + u_{j,i}\right)\right] = \sigma_{ij} \dot{\varepsilon}_{ij}$$

WORK, POWER AND ENERGY

Power of external forces may be expressed as follows:

$$P = \frac{\mathrm{d}}{\mathrm{d}t} \iiint\limits_V \frac{\rho}{2} v_i v_i \, \mathrm{d}V + \iiint\limits_V \sigma_{ij} \dot{\varepsilon}_{ij} \, \mathrm{d}V$$

where:

$$E_k = \iiint\limits_V \frac{\rho}{2} v_i v_i \, \mathrm{d}V$$ - kinetic energy

$$\dot{\Phi} = \iiint\limits_V \sigma_{ij} \dot{\varepsilon}_{ij} \, \mathrm{d}V$$ - power of elastic strain

REMARK

- Similarly as in the case of the well known definitions of work and power, the **work of stress along strain**, which is equal to the internal energy of elastic strain, is defined as:

$$\Phi = \frac{1}{2} \iiint\limits_V \sigma_{ij} \varepsilon_{ij} \, \mathrm{d}V$$

WORK, POWER AND ENERGY

Power of external forces may be expressed as follows:

$$P = \frac{\mathrm{d}}{\mathrm{d}\,t} \iiint_V \frac{\rho}{2} v_i v_i \,\mathrm{d}\,V + \iiint_V \sigma_{ij} \dot{\varepsilon}_{ij} \,\mathrm{d}\,V$$

REMARKS:

- The above relation was derived without any reference to any kind of constitutive relations, so it is true for **material of any characteristics**.

- An analogous relation may be derived in a strict way (without any simplifications resulting from linearization) for the **finite strain theory**:

$$P = \frac{\mathrm{d}}{\mathrm{d}\,t} \iiint_V \frac{\rho}{2} v_i v_i \,\mathrm{d}\,V + \iiint_V t_{ij} D_{ij} \,\mathrm{d}\,V$$

where $\mathbf{t}$ is the **Cauchy stress tensor**, and $\mathbf{D}$ is the **stretch rate tensor**:

$$D_{ij} = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right)$$

WORK, POWER AND ENERGY

We will make use of the following quantities

- kinetic energy $$E_k = \iiint\limits_V \frac{\rho}{2} v_i v_i \,\mathrm{d}V$$
- work of body forces $$L_b = \iiint\limits_V b_i u_i \,\mathrm{d}V$$
- work of surface tractions $$L_q = \iint\limits_S q_i u_i \,\mathrm{d}S$$
- work of external forces $$L = L_b + L_q = \iiint\limits_V b_i u_i \,\mathrm{d}V + \iint\limits_S q_i u_i \,\mathrm{d}S$$
- internal energy of elastic strain $$\Phi = \frac{1}{2}\iiint\limits_V \sigma_{ij}\varepsilon_{ij} \,\mathrm{d}V$$
- total potential energy $$\Pi = \Phi - L$$
- total complementary energy $$\Psi = \Phi - L_q$$

PRINCIPLE OF VIRTUAL DISPLACEMENTS

PRINCIPLE OF VIRTUAL DISPLACEMENTS

Let's consider an elastic solid occupying a region V bounded by a surface $S = S_u \cup S_q$. The body is loaded with a system of body forces $\mathbf{b}(\mathbf{x})$, on the boundary S_q it is loaded with a system of surface tractions $\mathbf{q}(\mathbf{x})$ and on the boundary S_u displacements $\mathbf{g}(\mathbf{x})$ are prescribed.

We define the set of kinematically admissible displacement fields, namely, such distributions of the displacement vector that **satisfy the kinematic boundary conditions**:

$$\mathrm{X}_u = \left\{ \breve{\mathbf{u}}: \quad \mathbf{x} \in S_u \quad \Rightarrow \quad \breve{\mathbf{u}}(\mathbf{x}) = \mathbf{g}(\mathbf{x}) \right\}$$

One of them is the true displacement (the **solution of the theory of elasticity**). We shall denote it with:

$$\hat{\mathbf{u}} \in \mathrm{X}_u$$

PRINCIPLE OF VIRTUAL DISPLACEMENTS

All other kinematically admissible displacement fields will be written expressed:

$$\breve{\mathbf{u}} = \hat{\mathbf{u}} + \alpha \boldsymbol{\upsilon}$$

Parameter α is a measure of magnitude of **deviation of a kinematically admissible displacement field from the true one**.

Since both $\breve{\mathbf{u}}$ as well as $\hat{\mathbf{u}}$ **satisfy the kinematic boundary conditions**, so it must be:

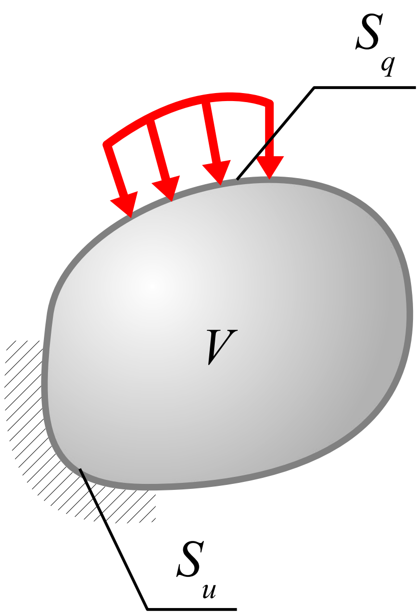
$$\boldsymbol{\upsilon} = \mathbf{0} \quad \text{on boundary } S_u$$

We define the virtual displacement, as a **displacement field which deviates from the true one only in an infinitesimal extent**:

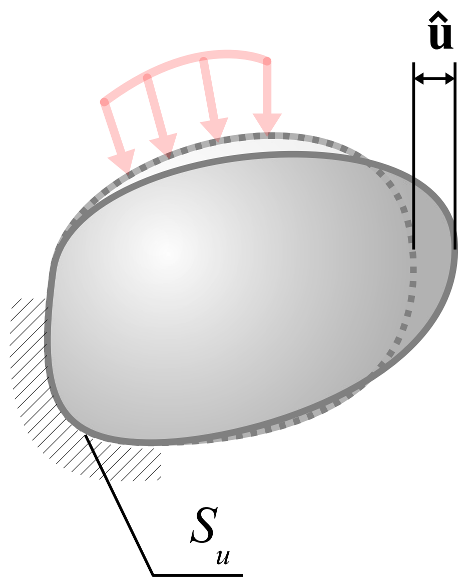
$$\delta \mathbf{u} = \frac{\partial \breve{\mathbf{u}}}{\partial \alpha} \mathrm{d}\alpha = \boldsymbol{\upsilon}\, \mathrm{d}\alpha \,, \qquad \mathrm{d}\alpha \to 0$$

Virtual displacement field satisfies **homogeneous kinematic boundary condition** on S_u.

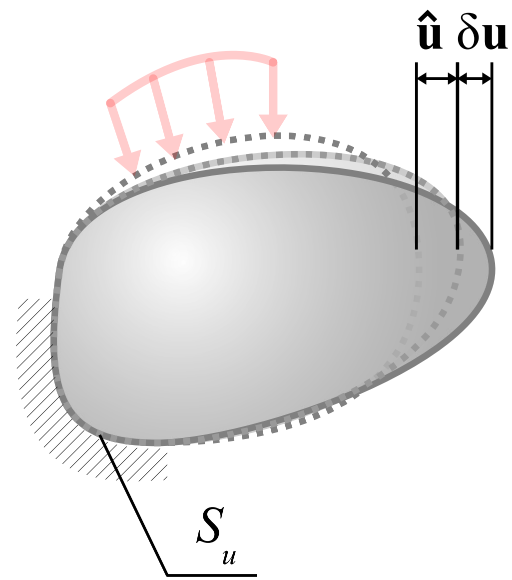
PRINCIPLE OF VIRTUAL DISPLACEMENTS



undeformed body
(reference configuration)



true displacement



true displacement
+
virtual displacement

PRINCIPLE OF VIRTUAL DISPLACEMENTS

Let's consider the equilibrium equations: $$\sigma_{ij,j} + b_i = 0$$

Let's make a **dot product** of both its sides with a **virtual displacement**: $$\sigma_{ij,j}\delta u_i + b_i \delta u_i = 0$$

Now we **integrate** the obtained result: $$\iiint\limits_V \left[\sigma_{ij,j}\delta u_i + b_i \delta u_i\right] \mathrm{d}V = 0$$

According to the **product rule**:

$$(\sigma_{ij}\delta u_i)_{,j} = \sigma_{ij,j}\delta u_i + \sigma_{ij}\delta u_{i,j} \quad \Rightarrow \quad \sigma_{ij,j}\delta u_i = (\sigma_{ij}\delta u_i)_{,j} - \sigma_{ij}\delta u_{i,j}$$

we may write: $$\iiint\limits_V \left[(\sigma_{ij}\delta u_i)_{,j} - \sigma_{ij}\delta u_{i,j} + b_i \delta u_i\right] \mathrm{d}V = 0$$

Due to **additivity** of integrals: $$\iiint\limits_V (\sigma_{ij}\delta u_i)_{,j} \mathrm{d}v + \iiint\limits_V b_i \delta u_i \mathrm{d}V = \iiint\limits_V \sigma_{ij}\delta u_{i,j} \mathrm{d}V$$

PRINCIPLE OF VIRTUAL DISPLACEMENTS

Let's make us of the **Green – Gauss – Ostrogradski theorem**:

$$\iint\limits_S \sigma_{ij} n_j \delta u_i \,\mathrm{d}S + \iiint\limits_V b_i \delta u_i \,\mathrm{d}V = \iiint\limits_V \sigma_{ij} \delta u_{i,j} \,\mathrm{d}V$$

Since $\delta \mathbf{u} = \mathbf{0}$ on boundary S_u, so integration over the boundary will give us a non-zero value only for the part of boundary S_q on which static boundary conditions are satisfied: $\boldsymbol{\sigma}\,\mathbf{n} = \mathbf{q}$

$$\iint\limits_{S_q} q_i \delta u_i \,\mathrm{d}S + \iiint\limits_V b_i \delta u_i \,\mathrm{d}V = \iiint\limits_V \sigma_{ij} \delta u_{i,j} \,\mathrm{d}V$$

Due to **symmetry of the stress tensor**, we may write:

$$\sigma_{ij} \delta u_{i,j} = \frac{1}{2}(\sigma_{ij} \delta u_{i,j} + \sigma_{ji} \delta u_{i,j}) = \frac{1}{2}(\sigma_{ij} \delta u_{i,j} + \sigma_{ij} \delta u_{j,i}) = \sigma_{ij} \underbrace{\frac{\delta u_{i,j} + \delta u_{j,i}}{2}}_{\delta \varepsilon_{ij}} = \sigma_{ij}\, \delta \varepsilon_{ij}$$

Tensor $\delta \boldsymbol{\varepsilon}$ is a **strain tensor corresponding with the virtual displacement field** $\delta \mathbf{u}$.

PRINCIPLE OF VIRTUAL DISPLACEMENTS

We may write:

$$\iint\limits_{S_q} q_i \delta u_i \, \mathrm{d}S + \iiint\limits_{V} b_i \delta u_i \, \mathrm{d}V = \iiint\limits_{V} \sigma_{ij} \delta \varepsilon_{ij} \, \mathrm{d}V$$

This means:

work of true external forces along virtual displacements
is equal to the work of the true stresses along virtual strains

This relation is **true for any virtual displacement field.**

It is a necessary condition which must be **satisfied by a true stress field and true external forces** (satisfying equilibrium equations and static boundary conditions).

Let's check if this is also a sufficient condition.

PRINCIPLE OF VIRTUAL DISPLACEMENTS

We **assume** that:

$$\iint_{S_q} q_i \delta u_i \, \mathrm{d}S + \iiint_V b_i \delta u_i \, \mathrm{d}V = \iiint_V \sigma_{ij} \delta u_{i,j} \, \mathrm{d}V$$

According to the **product rule**:

$$\iint_{S_q} q_i \delta u_i \, \mathrm{d}S + \iiint_V b_i \delta u_i \, \mathrm{d}V = \iiint_V (\sigma_{ij} \delta u_i)_{,j} - \sigma_{ij,j} \delta u_i \, \mathrm{d}V$$

According to the Green – Gauss – Ostrogradski theorem and **additivity** of integration:

$$\iint_{S_q} q_i \delta u_i \, \mathrm{d}S + \iiint_V b_i \delta u_i \, \mathrm{d}V = \iint_S \sigma_{ij} n_j \delta u_i \, \mathrm{d}S - \iiint_V \sigma_{ij,j} \delta u_i \, \mathrm{d}V$$

In the first integral on the right-hand side, integration over S_u gives us 0, since on that part $\delta \mathbf{u} = \mathbf{0}$.
Making an account for **additivity** of integration again enable us to write:

$$\iint_{S_q} (q_i - \sigma_{ij} n_j) \delta u_i \, \mathrm{d}S + \iiint_V (\sigma_{ij,j} + b_i) \delta u_i \, \mathrm{d}V = 0$$

and this relations must be **true for any** $\delta \mathbf{u}$.

PRINCIPLE OF VIRTUAL DISPLACEMENTS

It will be so if and only if the both integrands are identically equal to 0:

$$\iint\limits_{S_q} \underbrace{(q_i - \sigma_{ij} n_j)}_{=0} \delta u_i \, \mathrm{d} S + \iiint\limits_{V} \underbrace{(\sigma_{ij,j} + b_i)}_{=0} \delta u_i \, \mathrm{d} V = 0$$

which means that:

- equilibrium equations are satisfied: $$\sigma_{ij,j} + b_i = 0$$
- static boundary conditions are satisfied: $$\sigma_{ij} n_j = q_i$$

Equality between work of external forces on virtual displacements and work of stresses along respective virtual strains imply that **equilibrium equations** and **static boundary conditions are satisfied**.

Equality of virtual works is a necessary and sufficient condition
for satisfying the equilibrium equations and static boundary conditions.

Paweł Szeptyński PhD, Eng.

PRINCIPLE OF VIRTUAL DISPLACEMENTS

REMARKS:

- If **virtual work of external forces is equal to the virtual work of internal forces**, then equilibrium equations and static boundary conditions **are satisfied**.
- If the **stress is determined according to a certain kinematically admissible displacement** field, then – by definition also kinematic boundary **conditions are satisfied**.
- Strains corresponding with a kinematically admissible displacement field are determined according to the **kinematic relations** – we assume that they are also **satisfied**.

CONCLUSIONS:

- If certain **kinematically admissible displacement field satisfy the condition of equality of virtual works** of internal and external forces, the it also satisfy all **governing equations of the theory of elasticity** – it is the solution of this theory.

- **Inverse theorem is also true**.

PRINCIPLE OF VIRTUAL DISPLACEMENTS

PRINCIPLE OF VIRTUAL DISPLACEMENTS

A **necessary and sufficient condition** for a certain **kinematically admissible displacement field** to be the **true displacement** is that **work of external forces along virtual displacements is equal to the work of stresses along respective virtual strains** for any virtual displacement and corresponding virtual strain:

$$\iint_{S_q} q_i \,\delta u_i \,\mathrm{d}S + \iiint_V b_i \,\delta u_i \,\mathrm{d}V = \iiint_V \sigma_{ij} \,\delta \varepsilon_{ij} \,\mathrm{d}V \qquad \forall\, \delta \mathbf{u}$$

$$\delta_{\mathbf{u}} L = \delta_{\mathbf{u}} \Phi \qquad \forall\, \delta \mathbf{u}$$

PRINCIPLE OF VIRTUAL DISPLACEMENTS

$$\iint\limits_{S_q} q_i \delta u_i \,\mathrm{d}S + \iiint\limits_V b_i \delta u_i \,\mathrm{d}V = \iiint\limits_V \sigma_{ij} \delta \varepsilon_{ij} \,\mathrm{d}V \qquad \forall \delta \mathbf{u}$$

REMARKS:

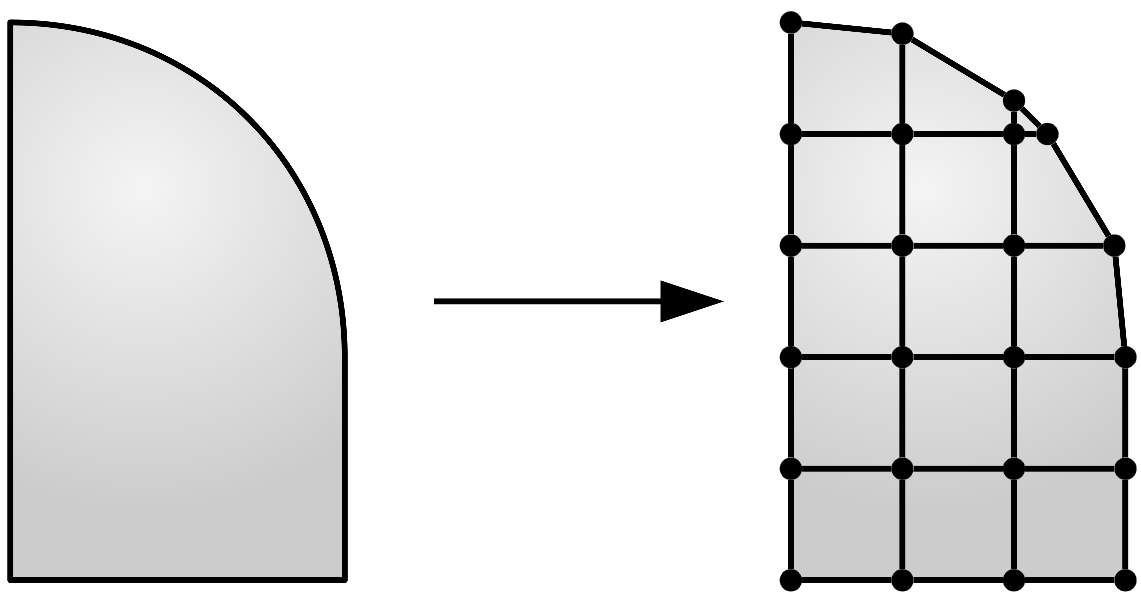
- The above theorem is also termed the **Principle of Virtual Works**.
- PVD is true only for the small strain theory.
- It concerns only static problems.
- It is true for any constitutive relations.
- Formulation of the problem with the use of PVD is an **integral formulation** (so called **global** or **weak** formulation) not a **differential** (**local**, **strong**) one, as in the case of governing equations of theory of elasticity. **Both are equivalent**. The difference is that in integral formulation the solution may be **not differentiable** or even **discontinuous** – such a solution is termed the weak solution.

FINITE ELEMENT METHOD

FINITE ELEMENT METHOD

Principle of Virtual Displacements is fundamental for formulation of a **numerical method** of solving the problems of linear theory of elasticity in its **weak formulation**, namely of the Finite Element Method (FEM). In this method:

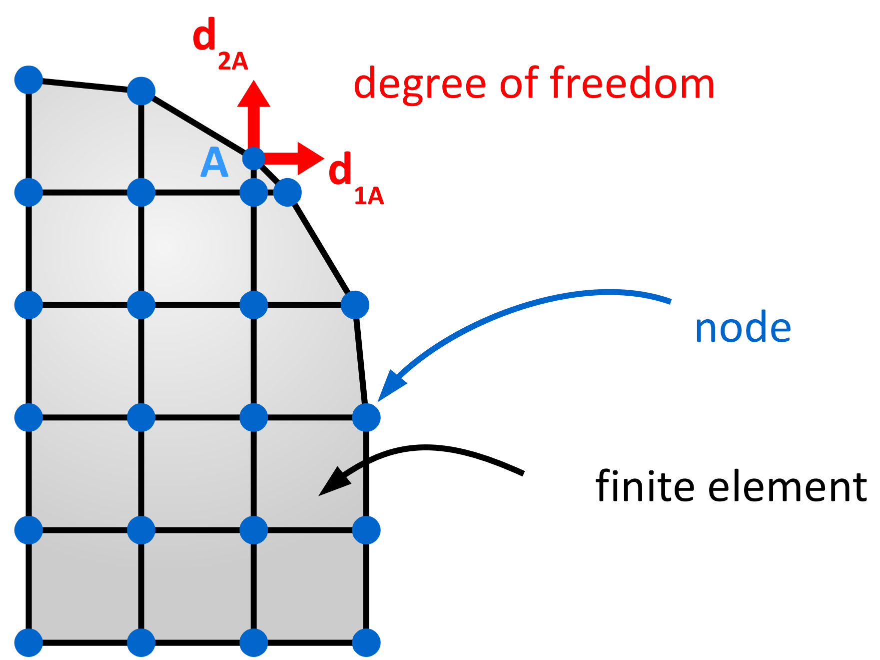
- **We distinguish in the configuration of a body a finite number of points** – nodes (it is co called discretization). Nodal displacements will be unknowns in the newly formulated problem.



FINITE ELEMENT METHOD

Principle of Virtual Displacements is fundamental for formulation of a **numerical method** of solving the problems of linear theory of elasticity in its **weak formulation**, namely of the Finite Element Method (FEM). In this method:

- Degrees of freedom (DOF) are assigned to each **node** – these are **directions of admissible displacements** or **rotations**.
- Areas between neighbouring nodes are termed finite elements.

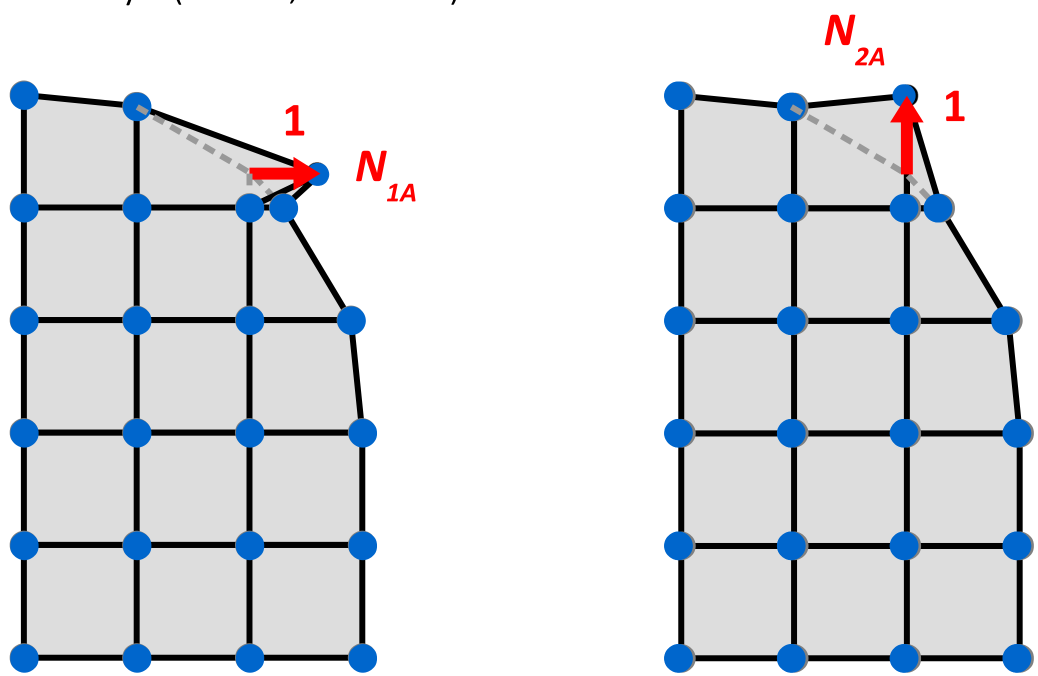


Paweł Szeptyński PhD, Eng.

FINITE ELEMENT METHOD

Principle of Virtual Displacements is fundamental for formulation of a **numerical method** of solving the problems of linear theory of elasticity in its **weak formulation**, namely of the Finite Element Method (FEM). In this method:

- **Displacements between nodes are approximated** with the use of the so called shape functions – they can be chosen by us. They are often of such kind that the have a **unit value in a single node and zero in all other nodes**. The shape of the function between this specified node and neighbouring nodes is chosen in a number of ways (linear, cubic...)



FINITE ELEMENT METHOD

- Displacement field is determined as a **linear combination of shape functions**.
- If the nodal displacements in the shape functions have unit values, then the **coefficients** of that combination are simply the **displacements along appropriate degrees of freedom**.

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \phi_1^{[1]} & \cdots & \phi_1^{[n]} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \phi_2^{[1]} & \cdots & \phi_2^{[n]} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \phi_3^{[1]} & \cdots & \phi_3^{[n]} \end{bmatrix}}_{N_{iA}} \underbrace{\begin{bmatrix} d_{11} \\ d_{12} \\ \vdots \\ d_{3n} \end{bmatrix}}_{d_A}$$

$$u_i = \sum_{A=1}^{3n} N_{iA} d_A , \qquad i=1,2,3$$

$u_i(\mathbf{x})$ – i-th component of the displacement vector

$N_{iA}(\mathbf{x})$ – shape function corresponding with i-th component of displacement corresponding with A-th degree of freedom

d_A – generalized displacement corresponding with A-th degree of freedom

FINITE ELEMENT METHOD

- actual strain: $$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) = \frac{1}{2}\sum_{A=1}^{3N}\left(N_{iA,j} + N_{jA,i}\right)d_A$$

- actual stress: $$\sigma_{ij} = S_{ijpq}\varepsilon_{pq} = = \frac{1}{2}S_{ijpq}\sum_{A=1}^{3N}\left(N_{pA,q} + N_{qA,p}\right)d_A$$

- virtual displacement: $$\delta u_i = \sum_{B=1}^{3N} N_{iB}\,\delta d_B$$

- virtual strain: $$\delta\varepsilon_{ij} = \frac{1}{2}\left(\delta u_{i,j} + \delta u_{j,i}\right) = \frac{1}{2}\sum_{B=1}^{3N}\left(N_{iB,j} + N_{jB,i}\right)\delta d_B$$

Paweł Szeptyński PhD, Eng.

FINITE ELEMENT METHOD

Let's write down the Principle of Virtual Displacements: $\iint\limits_{S_q} q_i \delta u_i \,\mathrm{d}S + \iiint\limits_V b_i \delta u_i \,\mathrm{d}V = \iiint\limits_V \sigma_{ij} \delta \varepsilon_{ij} \,\mathrm{d}V$

We substitute the expressions for actual stress and virtual displacement and virtual strain:

$$\iint\limits_{S_q} \left[q_i \sum_{B=1}^{3N} N_{iB} \delta d_B \right] \mathrm{d}S + \iiint\limits_V \left[b_i \sum_{B=1}^{3N} N_{iB} \delta d_B \right] \mathrm{d}V = \iiint\limits_V \frac{1}{2} S_{ijpq} \sum_{A=1}^{3N} \left(N_{pA,q} + N_{A,p} \right) d_A \cdot \frac{1}{2} \sum_{B=1}^{3N} \left(N_{iB,j} + N_{jB,i} \right) \delta d_B \,\mathrm{d}V$$

After some transformation:

$$\sum_{B=1}^{3N} \underbrace{\left(\iint\limits_{S_q} q_i N_{iB} \,\mathrm{d}S + \iiint\limits_V b_i N_{iB} \,\mathrm{d}V \right)}_{f_B} \delta d_B = \sum_{A=1}^{3N} \sum_{B=1}^{3N} \underbrace{\left(\iiint\limits_V \frac{1}{4} S_{ijpq} \left(N_{pA,q} + N_{qA,p} \right) \left(N_{iB,j} + N_{jB,i} \right) \mathrm{d}V \right)}_{K_{BA}} d_A \delta d_B$$

Finally:

$$\sum_{A=1}^{3N} K_{BA} \cdot d_A \cdot \delta d_B = f_B \cdot \delta d_B \qquad \forall \delta d_B$$

FINITE ELEMENT METHOD

Obtained relation must hold true for any values of δd_B , so we must have:

$$\sum_{A=1}^{3N} K_{BA} \cdot d_A = f_B \qquad B = 1, \dots, 3N$$

where:

- stiffness matrix:

$$K_{BA} = \sum_{B=1}^{3N} \left[\left(\iiint_V \frac{1}{4} \mathsf{S}_{ijpq} \left(N_{pA,q} + N_{qA,p} \right) \left(N_{iB,j} + N_{jB,i} \right) \mathrm{d} V \right) \right]$$

- nodal load vector:

$$f_B = \sum_{B=1}^{3N} \left[\left(\iint_{S_q} q_i N_{iB} \mathrm{d} S + \iiint_V b_i N_{iB} \mathrm{d} V \right) \right]$$

Obtained relation constitutes a **system of linear algebraic equations** for generalized displacements along chosen degrees of freedom.

FINITE ELEMENT METHOD

REMARKS:

- **Efficient and fast algorithms solving** large systems of **linear algebraic equations** are known.
- The **greater number** of **nodes** (and elements) and **degrees of freedom** in a node, the **greater precision** of the solution and the **greater is the system** to be solved.
- Integrals determining the entries of the stiffness matrix and load vector are also calculated numerically. If the domain of integration (shape of the finite element) or integrand (shape function) are strongly irregular, then the solution may be flawed.
- Problems of the non-linear theory of elasticity are solved with the use of FEM in an incremental approach
  - Total increment of load is divided into a large number of small increments.
  - A small increment of load is applied to the reference configuration. Geometric and constitutive relations are approximated with linear relations.
  - Obtained actual configuration becomes a new reference configuration for a next step of loading, and the linearization of equations of theory of elasticity is performed for new distributions of stress, strain and displacements. The whole outline is then repeated.

LAGRANGE THEOREM

LAGRANGE THEOREM

We define the total potential energy Π of the system as a **difference between elastic strain energy** and **work of external loads**:

$$\Pi = \Phi - L = \frac{1}{2}\iiint_V \sigma_{ij}\varepsilon_{ij}\,\mathrm{d}V - \left[\iiint_V b_i u_i\,\mathrm{d}V + \iint_{S_q} q_i u_i\,\mathrm{d}S\right]$$

This quantity may be considered a **functional** depending on the **distribution of the displacement field**. It is termed the Lagrange functional:

$$\Pi[\mathbf{u}] = \iiint_V \frac{1}{8}\mathsf{S}_{ijkl}\left(u_{k,l}+u_{l,k}\right)\left(u_{i,j}+u_{j,i}\right)\mathrm{d}V - \left[\iiint_V b_i u_i\,\mathrm{d}V + \iint_{S_q} q_i u_i\,\mathrm{d}S\right]$$

Variation of the above functional may be determined as follows:

$$\delta\Pi = \frac{\mathrm{d}}{\mathrm{d}\alpha}\Pi[\mathbf{u}+\alpha\,\delta\mathbf{u}]\Big|_{\alpha=0}$$

LAGRANGE THEOREM

Variation of the total potential energy:

$$\delta \Pi = \frac{\mathrm{d}}{\mathrm{d}\alpha} \Pi[\mathbf{u} + \alpha \delta \mathbf{u}] \Big|_{\alpha=0} =$$

$$= \frac{\mathrm{d}}{\mathrm{d}\alpha} \frac{1}{2} \iiint_V \frac{1}{4} \mathsf{S}_{ijkl} \left[(u_k + \alpha \delta u_k)_{,l} + (u_l + \alpha \delta u_l)_{,k} \right] \left[(u_i + \alpha \delta u_i)_{,j} + (u_j + \alpha \delta u_j)_{,i} \right] \mathrm{d}V \Big|_{\alpha=0} -$$

$$- \frac{\mathrm{d}}{\mathrm{d}\alpha} \left[\iiint_V b_i (u_i + \alpha \delta u_i) \mathrm{d}V + \iint_{S_q} q_i (u_i + \alpha \delta u_i) \mathrm{d}S \right] \Big|_{\alpha=0}$$

Let's perform the differentiation with respect to the independent variables and rearrange the expression

$$\delta \Pi = \frac{\mathrm{d}}{\mathrm{d}\alpha} \frac{1}{2} \iiint_V \frac{1}{4} \mathsf{S}_{ijkl} \left[(u_{k,l} + u_{l,k}) + \alpha (\delta u_{k,l} + \delta u_{l,k}) \right] \left[(u_{i,j} + u_{j,i}) + \alpha (\delta u_{i,j} + \delta u_{j,i}) \right] \mathrm{d}V \Big|_{\alpha=0} -$$

$$- \frac{\mathrm{d}}{\mathrm{d}\alpha} \left[\iiint_V (b_i u_i + \alpha b_i \delta u_i) \mathrm{d}V + \iint_{S_q} (q_i u_i + \alpha q_i \delta u_i) \mathrm{d}S \right] \Big|_{\alpha=0}$$

LAGRANGE THEOREM

Let's make use of the geometric relations (definition of the small strain tensor):

$$\delta\Pi = \frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{1}{2}\iiint_V S_{ijkl}\left[\varepsilon_{kl}+\alpha\,\delta\varepsilon_{kl}\right]\left[\varepsilon_{ij}+\alpha\,\delta\varepsilon_{ij}\right]\mathrm{d}V - $$

$$-\left[\iiint_V \left(b_i u_i+\alpha\, b_i\,\delta u_i\right)\mathrm{d}V+\iint_{S_q}\left(q_i u_i+\alpha\, q_i\,\delta u_i\right)\mathrm{d}S\right]\Bigg|_{\alpha=0}$$

Let's multiply the expressions in the first integral:

$$\delta\Pi = \frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{1}{2}\iiint_V S_{ijkl}\left[\varepsilon_{kl}\varepsilon_{ij}+\alpha\left(\delta\varepsilon_{kl}\varepsilon_{ij}+\varepsilon_{kl}\delta\varepsilon_{ij}\right)+\alpha^2\delta\varepsilon_{kl}\delta\varepsilon_{ij}\right]\mathrm{d}V - $$

$$-\left[\iiint_V \left(b_i u_i+\alpha\, b_i\,\delta u_i\right)\mathrm{d}V+\iint_{S_q}\left(q_i u_i+\alpha\, q_i\,\delta u_i\right)\mathrm{d}S\right]\Bigg|_{\alpha=0}$$

LAGRANGE THEOREM

Let's differentiate with respect to α:

$$\delta\Pi = \frac{1}{2}\iiint_V \mathsf{S}_{ijkl}\left[\left(\delta\varepsilon_{kl}\varepsilon_{ij}+\varepsilon_{kl}\delta\varepsilon_{ij}\right)+2\alpha\delta\varepsilon_{kl}\delta\varepsilon_{ij}\right]\mathrm{d}V - \left[\iiint_V b_i\delta u_i\,\mathrm{d}V + \iint_{S_q} q_i\delta u_i\,\mathrm{d}S\right]\Bigg|_{\alpha=0}$$

After substituting $\alpha=0$ we obtain:

$$\delta\Pi = \frac{1}{2}\iiint_V \mathsf{S}_{ijkl}\left(\delta\varepsilon_{kl}\varepsilon_{ij}+\varepsilon_{kl}\delta\varepsilon_{ij}\right)\mathrm{d}V - \left[\iiint_V b_i\delta u_i\,\mathrm{d}V + \iint_{S_q} q_i\delta u_i\,\mathrm{d}S\right]$$

Let's make use of the **symmetry of the stiffness tensor** (we make use of the constitutive relations for linear-elastic material**!**):

$$\delta\Pi = \iiint_V \mathsf{S}_{ijkl}\varepsilon_{kl}\delta\varepsilon_{ij}\,\mathrm{d}V - \left[\iiint_V b_i\delta u_i\,\mathrm{d}V + \iint_{S_q} q_i\delta u_i\,\mathrm{d}S\right]$$

LAGRANGE THEOREM

For **linear-elastic** material:

$$\delta \Pi = \iiint_V \sigma_{ij} \delta \varepsilon_{ij} \, \mathrm{d} V - \left[\iiint_V b_i \delta u_i \, \mathrm{d} V + \iint_{S_q} q_i \delta u_i \, \mathrm{d} S \right]$$

According to the Principle of Virtual Displacements:

$$\iiint_V \sigma_{ij} \delta \varepsilon_{ij} \, \mathrm{d} V - \left[\iiint_V b_i \delta u_i \, \mathrm{d} V + \iint_{S_q} q_i \delta u_i \, \mathrm{d} S \right] = 0 \qquad \forall \delta \mathbf{u}$$

This means that:

- **first variation of the total potential energy**
- **total potential energy has stationary value**
- For **actual displacement** total potential energy satisfies the **necessary conditions for a local extremum**

If it is a minimum or maximum depends on the sign of the second variation.

LAGRANGE THEOREM

The second variation of the total potential energy:

$$\delta^2 \Pi = \frac{d^2}{d\alpha^2} \Pi[\mathbf{u} + \alpha \delta \mathbf{u}]\Big|_{\alpha=0} = \iiint_V S_{ijkl} \delta \varepsilon_{ij} \delta \varepsilon_{kl} \, dV > 0$$

Positiveness of the above expression is due to fact that **stiffness tensor is positive determinate** – this is a consequence of the **2nd Law of Thermodynamics**. If it was not so, than the material could have a "negative stiffness" (negative Kelvin modulus) – for a certain stress state, the body would be strained in an opposite direction (spring loaded with tensile force would be contracted – loaded with compressive force would be stretched). We may formulate:

LAGRANGE THEOREM

Among all **kinematically admissible displacement fields** in a **linear-elastic body**, the **true one** is this and only this, for which the **total potential energy of the system** (Lagrange functional) takes the minimum **value**.

LAGRANGE THEOREM

REMARKS:

- **Lagrange Theorem is true only within the linear theory of elasticity.**

- In the case of the finite strain theory or in the case of a material of non-linear characteristics this theorem do not have to be true.

Paweł Szeptyński PhD, Eng.

THANK YOU FOR YOUR ATTENTION