THEORY OF ELASTICITY AND PLASTICITY

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PRINCIPLE OF VIRTUAL FORCES

PRINCIPLE OF VIRTUAL FORCES

Let's consider an elastic solid which occupies a region V in space, which is bounded by a surface $S = S_u \cup S_q$. The body is loaded on the boundary S_q with a system of surface tractions $\mathbf{q}(\mathbf{x})$ and displacements $\mathbf{g}(\mathbf{x})$ are prescribed on the boundary S_u .

We define the **set of statically admissible stress fields**, namely such stress fields that **satisfy static boundary conditions** and **equilibrium equations**:

$$\mathrm{X}_\sigma = \left\{ \breve{\boldsymbol{\sigma}} : \; \breve{\sigma}_{ij,j} + b_i = 0 \quad \wedge \quad \left(\mathbf{x} \in S_q \quad \Rightarrow \quad \breve{\sigma}_{ij} n_j = q_i \right) \right\}$$

Among them there is one, which is the true stress field (it is the **solution of the problem of theory of elasticity**). Let's denote is with a hat:

$$\hat{\boldsymbol{\sigma}} \in \mathrm{X}_\sigma$$

PRINCIPLE OF VIRTUAL FORCES

All **other statically admissible stress fields** may be expressed as follows:

$$\breve{\boldsymbol{\sigma}} = \hat{\boldsymbol{\sigma}} + \alpha\,\boldsymbol{\tau}$$

Parameter α is a measure of **deviation of a statically admissible stress field from the true one**.

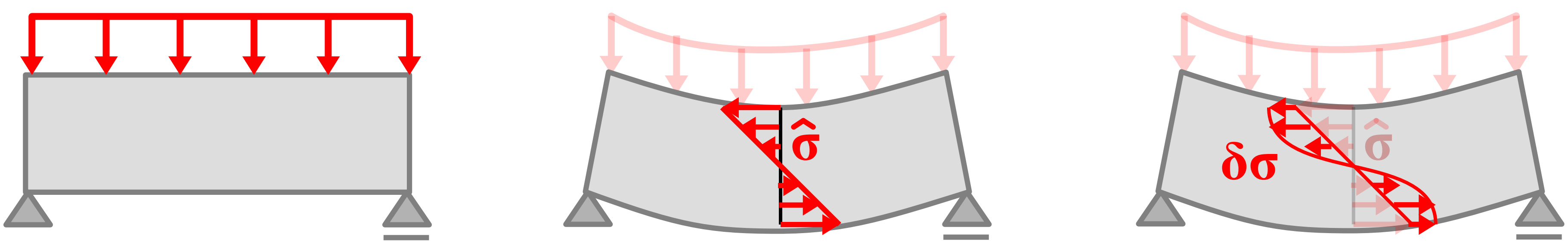
Let's define the virtual stress, as a stress field which is **an infinitely small** (but still not equal to zero) **increment of the true stress field**:

$$\delta\boldsymbol{\sigma} = \frac{\partial \breve{\boldsymbol{\sigma}}}{\partial \alpha}\,\mathrm{d}\alpha = \boldsymbol{\tau}\,\mathrm{d}\alpha\,, \qquad \mathrm{d}\alpha \to 0$$

PRINCIPLE OF VIRTUAL FORCES

true stress field

true stress field
+
virtual stress



reference configuration

actual configuration

PRINCIPLE OF VIRTUAL FORCES

Let's write down the **static boundary conditions** on S_q : $\breve{\sigma}_{ij} n_j = q_i$

$$(\sigma_{ij} + \delta\sigma_{ij}) n_j = q_i$$

$$\underbrace{(\sigma_{ij} n_j)}_{= q_i} + \delta\sigma_{ij} n_j = q_i$$

$$\delta\sigma_{ij} n_j = 0$$

Virtual stress satisfies **uniform static boundary conditions** on S_q .

Let's define virtual surface tractions as:

$$\delta q_i = \delta\sigma_{ij} n_j$$

We have then:

$$\delta q_i = 0 \ \text{ na } \ S_q$$

PRINCIPLE OF VIRTUAL FORCES

Let's write down **equilibrium equations** for **statically admissible stress field**:

$$\breve{\sigma}_{ij,j} + b_i = 0$$

$$(\sigma_{ij} + \delta\sigma_{ij})_{,j} + b_i = 0$$

For **true stress** field **equilibrium equations** must be satisfied:

$$\underbrace{(\sigma_{ij,j} + b_i)}_{=0} + \delta\sigma_{ij,j} = 0$$

$$\boxed{\delta\sigma_{ij,j} = 0}$$

Virtual stress satisfies **homogeneous equilibrium equations** if only **variation of body forces is equal to 0:**

$$\delta b_i = 0$$

PRINCIPLE OF VIRTUAL FORCES

Let's consider **equilibrium equations** for **virtual stress:**

$$\delta\sigma_{ij,j}=0$$

Let's calculate a **dot product** with **true displacement field:**

$$\delta\sigma_{ij,j}u_i=0$$

Let's **integrate** it over configuration of the body:

$$\iiint_V \delta\sigma_{ij,j}u_i\,\mathrm{d}V=0$$

According to the **product rule**:

$$\iiint_V (\delta\sigma_{ij}u_i)_{,j}-\delta\sigma_{ij}u_{i,j}\,\mathrm{d}V=0$$

According to the **Green – Gauss – Ostrogradski theorem**:

$$\iint_S \delta\sigma_{ij}n_j u_i\,\mathrm{d}S-\iiint_V \delta\sigma_{ij}u_{i,j}\,\mathrm{d}V=0$$

According to the **definition of virtual surface tractions**:

$$\iint_S \delta q_i u_i\,\mathrm{d}S-\iiint_V \delta\sigma_{ij}u_{i,j}\,\mathrm{d}V=0$$

Due to **symmetry of the stress tensor**:

$$\iint_S \delta q_i u_i\,\mathrm{d}S-\iiint_V \delta\sigma_{ij}\frac{u_{i,j}+u_{j,i}}{2}\,\mathrm{d}V=0$$

PRINCIPLE OF VIRTUAL FORCES

Since we are considering the **true displacement field**, the **kinematic relations must also be satisfied**:

$$\iint_S \delta q_i u_i \,\mathrm{d}S - \iiint_V \delta\sigma_{ij} \underbrace{\frac{u_{i,j}+u_{j,i}}{2}}_{\varepsilon_{ij}} \mathrm{d}V = 0$$

We obtain:

$$\iint_S \delta q_i u_i \,\mathrm{d}S = \iiint_V \delta\sigma_{ij}\varepsilon_{ij}\,\mathrm{d}V \qquad \forall \delta\sigma_{ij}$$

This means:

Work of arbitrary chosen **virtual stress along true strains** is **equal** to the **work of virtual surface tractions** (corresponding with considered virtual stress) **along true displacements**.

Equality of virtual works is a necessary condition which must be **satisfied by true displacements** and **corresponding strains** for **any virtual stress and corresponding virtual surface tractions**.

PRINCIPLE OF VIRTUAL FORCES

Let's check if it is also a sufficient condition:

We assume that **for any** $\delta\sigma_{ij}$ virtual works are the same:

$$\iint\limits_S \delta q_i u_i \,\mathrm{d}S = \iiint\limits_V \delta\sigma_{ij}\varepsilon_{ij}\,\mathrm{d}V$$

According to the **definition of virtual surface traction**:

$$\iint\limits_S \delta\sigma_{ij} n_j u_i \,\mathrm{d}S = \iiint\limits_V \delta\sigma_{ij}\varepsilon_{ij}\,\mathrm{d}V$$

According to the **Green – Gauss – Ostrogradski theorem**:

$$\iiint\limits_V (\delta\sigma_{ij} u_i)_{,j} \,\mathrm{d}V = \iiint\limits_V \delta\sigma_{ij}\varepsilon_{ij}\,\mathrm{d}V$$

According to the **product rule**:

$$\iiint\limits_V (\delta\sigma_{ij,j} u_i + \delta\sigma_{ij} u_{i,j}) \,\mathrm{d}V = \iiint\limits_V \delta\sigma_{ij}\varepsilon_{ij}\,\mathrm{d}V$$

Virtual stress satisfies equilibrium equations: $\delta\sigma_{ij,j} = 0$

We obtain:

$$\iiint\limits_V \delta\sigma_{ij} u_{i,j} \,\mathrm{d}V - \iiint\limits_V \delta\sigma_{ij}\varepsilon_{ij}\,\mathrm{d}V = 0$$

PRINCIPLE OF VIRTUAL FORCES

Due to **symmetry of the stress tensor**:

$$\iiint\limits_V \delta\sigma_{ij} \frac{u_{i,j} + u_{j,i}}{2} \mathrm{d}V - \iiint\limits_V \delta\sigma_{ij} \varepsilon_{ij} \mathrm{d}V = 0$$

$$\iiint\limits_V \delta\sigma_{ij} \left[\frac{1}{2} \left(u_{i,j} + u_{j,i} \right) - \varepsilon_{ij} \right] \mathrm{d}V = 0$$

Expression at the left hand-side must be identically equal to 0. **An integral over a region of positive measure is identically equal to 0 if and only if the integrand is identically equal to 0**. Since this relation must be true for **any virtual stress**, we conclude that the above equation will be satisfied if and only if:

$$\varepsilon_{ij} = \frac{1}{2} \left(u_{i,j} + u_{j,i} \right)$$

This means that Cauchy's geometric relation **are satisfied** – **consequence of equality of virtual works is satisfying the geometric relations**.

PRINCIPLE OF VIRTUAL FORCES

Let's consider again equation:

$$\iint_S \delta\sigma_{ij} n_j u_i \,\mathrm{d}S = \iiint_V \delta\sigma_{ij} u_{i,j} \,\mathrm{d}V$$

Virtual stress satisfies homogeneous static boundary conditions, $\delta\sigma_{ij} n_j = 0$ **on boundary** S_q. Integration over S_q gives us zero, so only integration over boundary S_u - where **displacements** $\mathbf{g}(\mathbf{x})$ **are prescribed** - will give us any non-zero value:

$$\iint_{S_u} \delta\sigma_{ij} n_j g_i \,\mathrm{d}S = \iiint_V \delta\sigma_{ij} u_{i,j} \,\mathrm{d}V$$

Product rule is applied to the integral on the left hand-side:

$$\iint_{S_u} \delta\sigma_{ij} n_j g_i \,\mathrm{d}S = \iiint_V \left[\left(\delta\sigma_{ij} u_i\right)_{,j} - \delta\sigma_{ij,j} u_i \right] \mathrm{d}V$$

We're accounting for the fact that **virtual stress satisfies equilibrium equations**:

$$\iint_{S_u} \delta\sigma_{ij} n_j g_i \,\mathrm{d}S = \iiint_V \left(\delta\sigma_{ij} u_i\right)_{,j} \mathrm{d}V$$

PRINCIPLE OF VIRTUAL FORCES

Accordin to the **Green – Gauss – Ostrogradski theorem**:

$$\iint_{S_u} \delta\sigma_{ij} n_j g_i \, \mathrm{d}S = \iint_{S} \delta\sigma_{ij} u_i n_j \, \mathrm{d}V$$

In the integral at the right hand-side **integration over** S_q **gives us zero** due to **homogeneous static boundary conditions for virtual stress**. Only integration over S_u gives us any non-zero value:

$$\iint_{S_u} \delta\sigma_{ij} n_j g_i \, \mathrm{d}S = \iint_{S_u} \delta\sigma_{ij} u_i n_j \, \mathrm{d}V$$

$$\iint_{S_u} \delta\sigma_{ij} n_j (g_i - u_i) \, \mathrm{d}S = 0$$

The above relation must hold true for **any virtual stress** – we may conclude that:

$$u_i = g_i \quad \text{on the boundary } S_u$$

namely, that static boundary conditions **are satisfied**.

PRINCIPLE OF VIRTUAL FORCES

We may formulate following theorem:

PRINCIPLE OF VIRTUAL FORCES

A necessary and sufficient condition for a certain **statically admissible stress field** to be a **true stress field** is that the **work of virtual surface tractions along true displacements is equal to the work of virtual stress along true strains** for any virtual stress and corresponding virtual surface tractions:

$$\iint_S \delta q_i u_i \, \mathrm{d}S = \iiint_V \delta\sigma_{ij} \varepsilon_{ij} \, \mathrm{d}V \qquad \forall \, \delta\boldsymbol{\sigma}$$

$$\delta_{\boldsymbol{\sigma}} L_q = \delta_{\boldsymbol{\sigma}} \Phi \qquad \forall \, \delta\boldsymbol{\sigma}$$

PRINCIPLE OF VIRTUAL FORCES

$$\iint_S \delta q_i u_i \,\mathrm{d}S = \iiint_V \delta\sigma_{ij}\varepsilon_{ij}\,\mathrm{d}V \qquad \forall\,\delta\boldsymbol{\sigma}$$

REMARKS:

- This theorem is sometimes termed a variant of **Principle of Virtual Works**.
- It is **true only for small strain theory**.
- It is **true for any constitutive relations**.

CASTIGLIANO'S THEOREM

CASTIGLIANO'S THEOREM

We define the total complementary energy as follows:

$$\Psi = \Phi - L_q = \frac{1}{2}\iiint_V \sigma_{ij}\varepsilon_{ij}\,\mathrm{d}V - \iint_S q_i u_i\,\mathrm{d}S$$

This quantity may be considered a **functional** depending on the **function of distribution of the stress tensor** (Castigliano's functional):

$$\Psi[\boldsymbol{\sigma}] = \iiint_V \frac{1}{2}\mathsf{C}_{ijkl}\sigma_{ij}\sigma_{kl}\,\mathrm{d}V - \iint_S \sigma_{ij}n_j u_i\,\mathrm{d}S$$

Its **first variation** (with respect to the stress tensor distribution) is equal:

$$\delta\Psi[\boldsymbol{\sigma}] = \frac{\mathrm{d}}{\mathrm{d}\alpha}\Psi[\boldsymbol{\sigma}+\alpha\boldsymbol{\delta}\sigma]\Big|_{\alpha=0} = \iiint_V \delta\sigma_{ij}\varepsilon_{ij}\,\mathrm{d}V - \iint_S \delta q_i u_i\,\mathrm{d}S$$

CASTIGLIANO'S THEOREM

According to the **Principle of Virtual Forces**:

$$\iiint_V \delta\sigma_{ij}\varepsilon_{ij}\,\mathrm{d}V - \iint_S \delta q_i u_i\,\mathrm{d}S = 0 \qquad \Rightarrow \qquad \delta\Psi[\boldsymbol{\sigma}] = 0$$

CONCLUSIONS:

- **the first variation of the Castigliano's functional is equal to zero**
- functional has a **stationary value**
- functional satisfies the **necessary condition for a local extremum**.

The second variation of the **Castigliano's functional** is equal:

$$\delta^2\Psi[\boldsymbol{\sigma}] = \frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}\Psi[\boldsymbol{\sigma}+\alpha\boldsymbol{\delta}\sigma]\Big|_{\alpha=0} = \iiint_V \mathsf{C}_{ijkl}\,\delta\sigma_{ij}\,\delta\sigma_{kl}\,\mathrm{d}V$$

Its value must be **positive** (due to **positive-definiteness of the compliance tensor**, which is a consequence of the 2nd **Law of Thermodynamics**), so the **for the true stress tensor distribution the Castigliano's functional** has a minimum value.

CASTIGLIANO'S THEOREM

We may formulate:

> CASTIGLIANO'S THEOREM
>
> Among all **statically admissible stress fields** in a linear-elastic body, the **true** one is the one for which **total complementary energy** (Castigliano's functional) has a minimum **value**.

REMARKS:

- The theorem is true only within the **linear theory of elasticity**:
  - geometric linearity – **small strain theory**
  - physical linearity – **linear constitutive relation** of the generalized Hooke's Law.

BETTI – MAXWELL
RECIPROCAL WORK THEOREM

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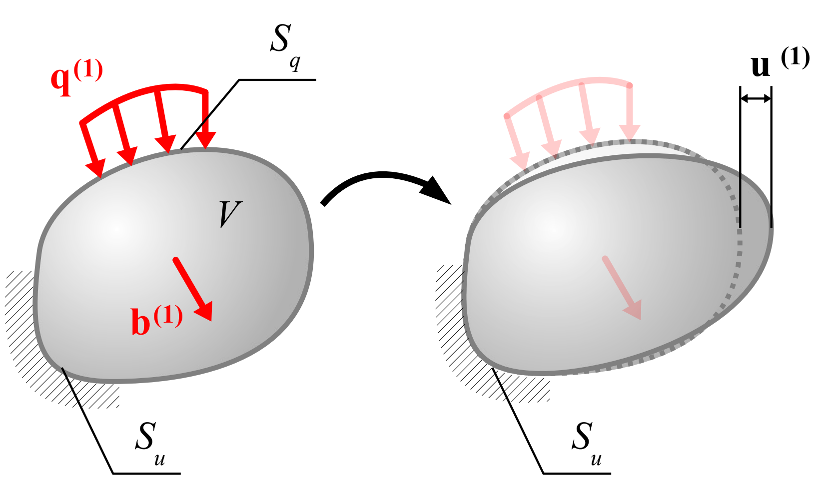
BETTI – MAXWELL RECIPROCAL WORK THEOREM

Let's consider an elastic solid occupying region V in space, which is bounded by a surface $S = S_u \cup S_q$. Kinematic boundary conditions are prescribed on boundary S_u and static boundary conditions are prescribed on boundary S_q (in particular it may be a free boundary).

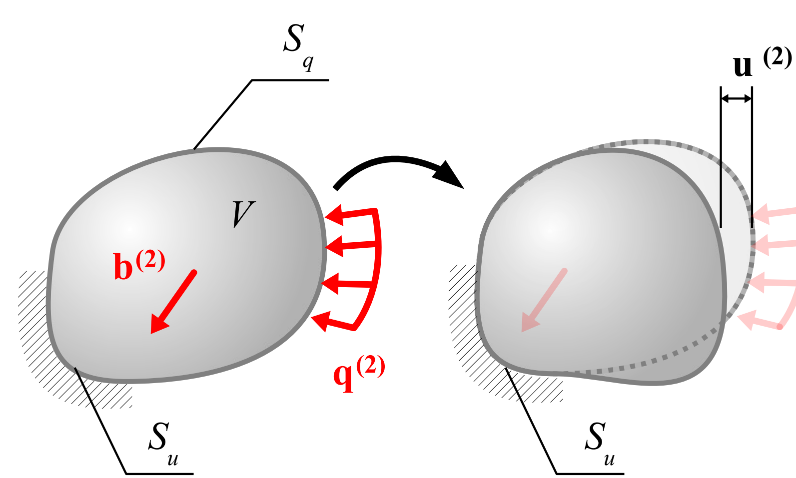
We consider **two systems of external loads** and **corresponding displacement fields**:

1st SYSTEM OF LOADS: $q_i^{(1)}\ ,\ b_i^{(1)}$



reference configuration — actual configuration

2nd SYSTEM OF LOAD : $q_i^{(2)}\ ,\ b_i^{(2)}$



reference configuration — actual configuration

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BETTI – MAXWELL RECIPROCAL WORK THEOREM

Let's calculate a **dot product** of the **equilibrium equation** for the **1st system of loads** with the **displacement field corresponding with the 2nd system of loads** and *vice versa*, and then let's **integrate** those expressions:

$$\iiint_V \left(\sigma_{ij,j}^{(1)} u_i^{(2)} + b_i^{(1)} u_i^{(2)}\right) \mathrm{d}V = 0 \qquad \iiint_V \left(\sigma_{ij,j}^{(2)} u_i^{(1)} + b_i^{(2)} u_i^{(1)}\right) \mathrm{d}V = 0$$

Right hand-sides of both expressions are the same – we may write the following equation:

$$\iiint_V \left(\sigma_{ij,j}^{(1)} u_i^{(2)} + b_i^{(1)} u_i^{(2)}\right) \mathrm{d}V = \iiint_V \left(\sigma_{ij,j}^{(2)} u_i^{(1)} + b_i^{(2)} u_i^{(1)}\right) \mathrm{d}V$$

According to the **product rule**:

$$\iiint_V \left[\left(\sigma_{ij}^{(1)} u_i^{(2)}\right)_{,j} - \sigma_{ij}^{(1)} u_{i,j}^{(2)} + b_i^{(1)} u_i^{(2)}\right] \mathrm{d}V = \iiint_V \left[\left(\sigma_{ij}^{(2)} u_i^{(1)}\right)_{,j} - \sigma_{ij}^{(2)} u_{i,j}^{(1)} + b_i^{(2)} u_i^{(1)}\right] \mathrm{d}V$$

BETTI – MAXWELL RECIPROCAL WORK THEOREM

According to the **Green – Gauss – Ostrogradski theorem**:

$$\iint_S \sigma_{ij}^{(1)} n_j u_i^{(2)} \,\mathrm{d}S + \iiint_V \left[-\sigma_{ij}^{(1)} u_{i,j}^{(2)} + b_i^{(1)} u_i^{(2)}\right] \mathrm{d}V = \iint_S \sigma_{ij}^{(2)} n_j u_i^{(1)} \,\mathrm{d}S + \iiint_V \left[-\sigma_{ij}^{(2)} u_{i,j}^{(1)} + b_i^{(2)} u_i^{(1)}\right] \mathrm{d}V$$

Due to **symmetry of the stress tensor**:

$$\iint_S \sigma_{ij}^{(1)} n_j u_i^{(2)} \,\mathrm{d}S + \iiint_V \left[-\sigma_{ij}^{(1)} \frac{u_{j,i}^{(2)} + u_{i,j}^{(2)}}{2} + b_i^{(1)} u_i^{(2)}\right] \mathrm{d}V =$$

$$= \iint_S \sigma_{ij}^{(2)} n_j u_i^{(1)} \,\mathrm{d}S + \iiint_V \left[-\sigma_{ij}^{(2)} \frac{u_{j,i}^{(1)} + u_{i,j}^{(1)}}{2} + b_i^{(2)} u_i^{(1)}\right] \mathrm{d}V$$

Due to **additivity** of integration:

$$\iint_S \sigma_{ij}^{(1)} n_j u_i^{(2)} \,\mathrm{d}S - \iiint_V \sigma_{ij}^{(1)} \frac{u_{i,j}^{(2)} + u_{j,i}^{(2)}}{2} \,\mathrm{d}V + \iiint_V b_i^{(1)} u_i^{(2)} \,\mathrm{d}V =$$

$$\iint_S \sigma_{ij}^{(2)} n_j u_i^{(1)} \,\mathrm{d}S - \iiint_V \sigma_{ij}^{(2)} \frac{u_{i,j}^{(1)} + u_{j,i}^{(1)}}{2} \,\mathrm{d}V + \iiint_V b_i^{(2)} u_i^{(1)} \,\mathrm{d}V$$

BETTI – MAXWELL RECIPROCAL WORK THEOREM

Stress fields satisfy **static boundary conditions** on S_q: $\sigma_{ij}^{(K)} n_j = q^{(K)} \qquad (K=1,2)$

Displacement fields satisfy **kinematic relations**: $\varepsilon_{ij}^{(K)} = \frac{1}{2}\left(u_{i,j}^{(K)} + u_{j,i}^{(K)}\right) \qquad (K=1,2)$

We may write:

$$\iint_S q_i^{(1)} u_i^{(2)} \,\mathrm{d}S - \iiint_V \sigma_{ij}^{(1)} \varepsilon_{ij}^{(2)} \,\mathrm{d}V + \iiint_V b_i^{(1)} u_i^{(2)} \,\mathrm{d}V = \iint_S q_i^{(2)} u_i^{(1)} \,\mathrm{d}S - \iiint_V \sigma_{ij}^{(2)} \varepsilon_{ij}^{(1)} \,\mathrm{d}V + \iiint_V b_i^{(2)} u_i^{(1)} \,\mathrm{d}V$$

REMARK:

- Quantity $q^{(K)} = \sigma_{ij}^{(K)} n_j$ calculated on supported boundary S_u is a system of **near-surface stresses resulting from enforced displacements prescribed according to the kinematic boundary conditions** – those stresses may be to to some extent identified with **support reactions**.

The equation takes the following form:

$$\iint_S q_i^{(1)} u_i^{(2)} \,\mathrm{d}S + \iiint_V b_i^{(1)} u_i^{(2)} \,\mathrm{d}V + \Theta = \iint_S q_i^{(2)} u_i^{(1)} \,\mathrm{d}S + \iiint_V b_i^{(2)} u_i^{(1)} \,\mathrm{d}V$$

where:

$$\Theta = \iiint_V \left(\sigma_{ij}^{(2)} \varepsilon_{ij}^{(1)} - \sigma_{ij}^{(1)} \varepsilon_{ij}^{(2)}\right) \mathrm{d}V$$

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BETTI – MAXWELL RECIPROCAL WORK THEOREM

According to the **generalized Hooke's Law** and **internal symmetries of elasticity tensors**:

$$\Theta = \iiint_V \left(\sigma_{ij}^{(2)}\varepsilon_{ij}^{(1)} - \sigma_{ij}^{(1)}\varepsilon_{ij}^{(2)}\right) \mathrm{d}V = \iiint_V \left(\mathsf{S}_{ijkl}\,\varepsilon_{kl}^{(2)}\varepsilon_{ij}^{(1)} - \mathsf{S}_{ijkl}\,\varepsilon_{kl}^{(1)}\varepsilon_{ij}^{(2)}\right) \mathrm{d}V =$$

$$= \iiint_V \left(\mathsf{S}_{klij}\,\varepsilon_{kl}^{(2)}\varepsilon_{ij}^{(1)} - \mathsf{S}_{ijkl}\,\varepsilon_{kl}^{(1)}\varepsilon_{ij}^{(2)}\right) \mathrm{d}V = \iiint_V \left(\mathsf{S}_{ijkl}\,\varepsilon_{ij}^{(2)}\varepsilon_{kl}^{(1)} - \mathsf{S}_{ijkl}\,\varepsilon_{kl}^{(1)}\varepsilon_{ij}^{(2)}\right) \mathrm{d}V = 0$$

We may formulate:

BETTI – MAXWEL RECIPROCAL WORK THEOREM

When a **linear-elastic** body is **independently subjected to two systems of external loads**, then work of forces of the 1st system along displacements caused by the 2nd system is the same as work of forces of the 2nd system along displacements caused by the 1st system:

$$\iint_S q_i^{(1)} u_i^{(2)} \,\mathrm{d}S + \iiint_V b_i^{(1)} u_i^{(2)} \,\mathrm{d}V = \iint_S q_i^{(2)} u_i^{(1)} \,\mathrm{d}S + \iiint_V b_i^{(2)} u_i^{(1)} \,\mathrm{d}V$$

BETTI – MAXWELL RECIPROCAL WORK THEOREM

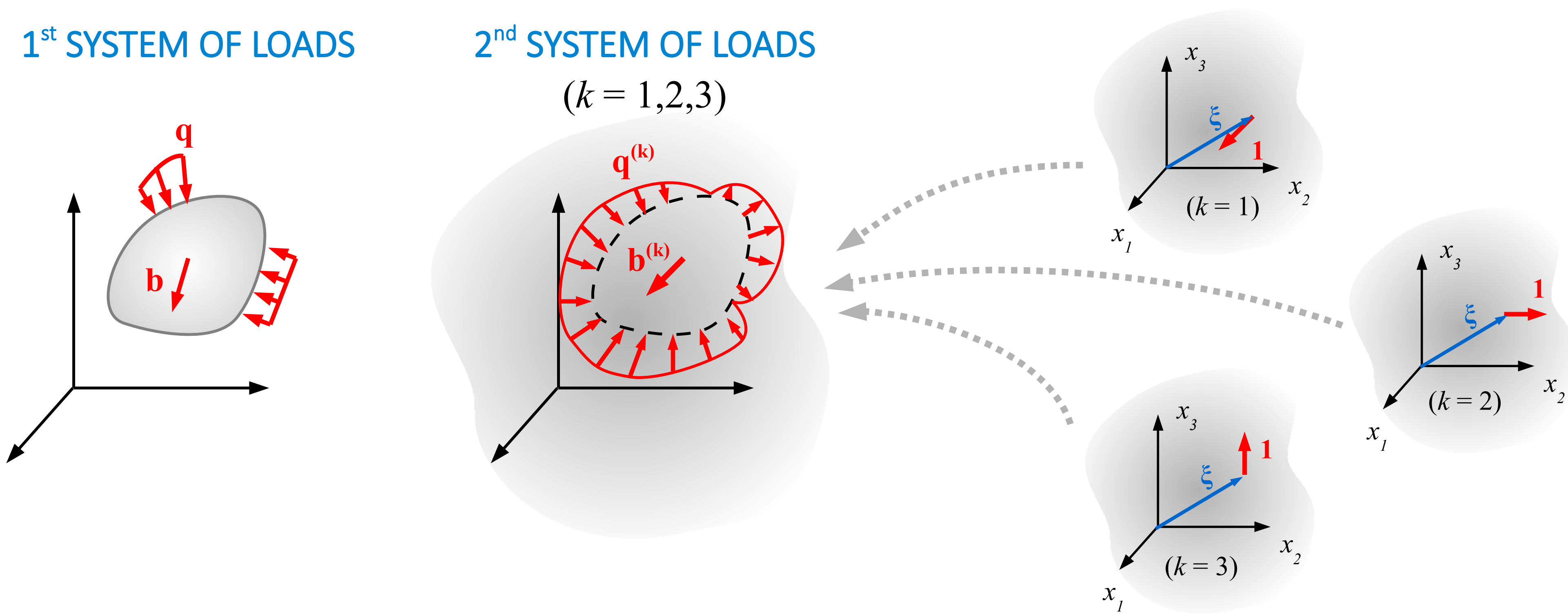
REMARKS:

- The Betti-Maxwell reciprocal theorem is **true only for** linear elastic solids (**Hooke's materials**).
- This theorem is fundamental for the set of reciprocal theorems and computational methods which are used in the **structural mechanics**:
  - **reciprocal work theorem**
  - **reciprocal displacement theorem**
  - **reciprocal reaction theorem**
  - **reciprocal displacement and reaction theorem**
  - **Maxwell – Mohr formula** (with the use of PVD)
  - **Force method** (flexibility method, method of consistent deformations)

SOMIGLIANA'S FORMULA

A specific example of **application of the Betti - Maxwell theorem** is the application for **three pairs of systems of loads**. In each on those pairs:

- The 1st system is the system of **true external loads**,
- The 2nd system is a **point force parallel to the k-th axis of the coordinate system, applied in point ξ** and the **system of stresses resulting form the Kelvin solution and corresponding with the boundary** of the body.



SOMIGLIANA'S FORMULA

According to the **reciprocal work theorem**:

$$\iint\limits_S q_i^{(k)} u_i \,\mathrm{d}S + \iiint\limits_V b_i^{(k)} u_i \,\mathrm{d}V = \iint\limits_S q_i u_i^{(k)} \,\mathrm{d}S + \iiint\limits_V b_i u_i^{(k)} \,\mathrm{d}V$$

After transformation:

$$\iiint\limits_V b_i^{(k)} u_i \,\mathrm{d}V = \iint\limits_S \left(q_i u_i^{(k)} - q_i^{(k)} u_i\right) \mathrm{d}S + \iiint\limits_V b_i u_i^{(k)} \,\mathrm{d}V$$

Body forces in the component **Kelvin solutions** are **prescribed with the use of the** Dirac delta distribution, which has the property, that when it is integrated with any other function, the result is the value of that other function in the point in which the point force is applied:

$$\iiint\limits_V \delta_k(\mathbf{x}-\boldsymbol{\xi}) u_k(\boldsymbol{\xi}) \,\mathrm{d}V = u_k(\mathbf{x})$$

We obtain:

$$u_k = \iint\limits_S \left(q_i u_i^{(k)} - q_i^{(k)} u_i\right) \mathrm{d}S + \iiint\limits_V b_i u_i^{(k)} \,\mathrm{d}V$$

SOMIGLIANA'S FORMULA

It can be rewritten in the following form:

$$\mathbf{u} = \int_V \underbrace{\begin{bmatrix} u_1^{(1)} & u_2^{(1)} & u_3^{(1)} \\ u_1^{(2)} & u_2^{(2)} & u_3^{(2)} \\ u_1^{(3)} & u_2^{(3)} & u_3^{(3)} \end{bmatrix}}_{\boldsymbol{\Gamma}} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \mathrm{d}V + \int_S \underbrace{\begin{bmatrix} u_1^{(1)} & u_2^{(1)} & u_3^{(1)} \\ u_1^{(2)} & u_2^{(2)} & u_3^{(2)} \\ u_1^{(3)} & u_2^{(3)} & u_3^{(3)} \end{bmatrix}}_{\boldsymbol{\Gamma}_u} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \mathrm{d}S - \int_S \underbrace{\begin{bmatrix} q_1^{(1)} & q_2^{(1)} & q_3^{(1)} \\ q_1^{(2)} & q_2^{(2)} & q_3^{(2)} \\ q_1^{(3)} & q_2^{(3)} & q_3^{(3)} \end{bmatrix}}_{\boldsymbol{\Gamma}_q} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \mathrm{d}S$$

or equivalently:

$$\mathbf{u}(\mathbf{x}) = \iiint_V \boldsymbol{\Gamma}(\mathbf{x}-\boldsymbol{\xi})\cdot\mathbf{b}(\boldsymbol{\xi})\,\mathrm{d}V + \iint_S \boldsymbol{\Gamma}_u(\mathbf{x}-\boldsymbol{\xi})\cdot\mathbf{q}(\boldsymbol{\xi})\,\mathrm{d}S - \iint_S \boldsymbol{\Gamma}_q(\mathbf{x}-\boldsymbol{\xi})\cdot\mathbf{u}(\boldsymbol{\xi})\,\mathrm{d}S$$

The above formula is termed the Somigliana's formula.

SOMIGLIANA'S FORMULA

$$\mathbf{u}(\mathbf{x}) = \iiint_V \mathbf{\Gamma}(\mathbf{x}-\boldsymbol{\xi})\cdot\mathbf{b}(\boldsymbol{\xi})\,\mathrm{d}V + \iint_S \mathbf{\Gamma}_u(\mathbf{x}-\boldsymbol{\xi})\cdot\mathbf{q}(\boldsymbol{\xi})\,\mathrm{d}S - \iint_S \mathbf{\Gamma}_q(\mathbf{x}-\boldsymbol{\xi})\cdot\mathbf{u}(\boldsymbol{\xi})\,\mathrm{d}S$$

REMARKS:

- This is a **system of integral equations** for components of the **true displacement field**.
- If only we know the fundamental (Kelvin) solution, then the solution of **any other problem of linear theory of elasticity will depend solely on static and kinematic boundary conditions and body forces**.
- Matrices $\mathbf{\Gamma}$, $\mathbf{\Gamma}_u$, $\mathbf{\Gamma}_q$ are **known** – they are **determined** by the **Kelvin solution**.
- Matrix $\mathbf{\Gamma}$ is prescribed for V. Matrices $\mathbf{\Gamma}_u$ and $\mathbf{\Gamma}_q$ are prescribed for S.
- Somigliana's formula is fundamental for a numerical method of solving the problems of linear theory of elasticity, the Boundary Element Method. In this method **only the boundary is discretized** and **only boundary values are determined numerically** – **internal values are determined with the use of fundamental solutions**.

RITZ METHODS

RITZ METHODS

We will use the term of Ritz methods in the sense of methods in which a certain **field, which is a solution** of linear theory of elasticity, is **approximated** with the use of **assumed functions depending on a finite number of parameters**.

Regarding the energy principles we may speak of two methods:

- Lagrange – Ritz method
  - we approximate a **kinematically admissible displacement field**:

$$\mathbf{u}(\mathbf{x}) \approx \sum_{i=1}^{N} \alpha_i \mathbf{u}_i(\mathbf{x})$$

  - coefficients of this approximation are found according to the Lagrange theorem **on the minimum of the total potential energy**:

$$\Pi[\mathbf{u}] = \iiint_V \frac{1}{8} S_{ijkl}(u_{k,l}+u_{l,k})(u_{i,j}+u_{j,i})\,\mathrm{d}V - \left[\iiint_V b_i u_i\,\mathrm{d}V + \iint_{S_q} q_i u_i\,\mathrm{d}S\right] \quad \to \quad \min$$

$$\frac{\partial \Pi}{\partial \alpha_i} = 0 \quad i=1,2,\dots,N \quad \to \quad \alpha_1, \alpha_2, \dots, \alpha_N$$

RITZ METHODS

We will use the term of Ritz methods in the sense of methods in which a certain **field, which is a solution** of linear theory of elasticity, is **approximated** with the use of **assumed functions depending on a finite number of parameters**.

Regarding the energy principles we may speak of two methods:

- Castigliano – Ritz method
  - we approximate a **statically admissible stress field**:

$$\boldsymbol{\sigma}(\mathbf{x}) \approx \sum_{i=1}^{N} \alpha_i \boldsymbol{\sigma}_i(\mathbf{x})$$

  - coefficients of this approximation are found according to the Castigliano's theorem **on the minimum of the total complementary energy**:

$$\Psi[\boldsymbol{\sigma}] = \iiint_V \frac{1}{2} \mathsf{C}_{ijkl} \sigma_{ij} \sigma_{kl} \, \mathrm{d}V - \iint_S \sigma_{ij} n_j u_i \, \mathrm{d}S \quad \rightarrow \quad \min$$

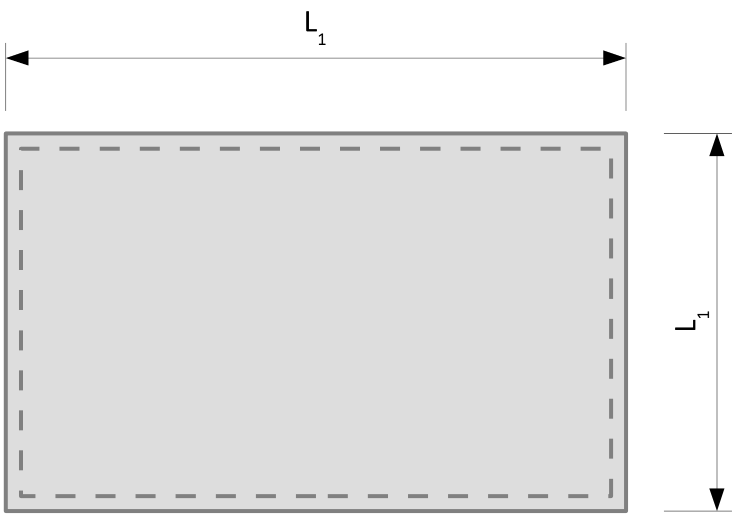
$$\frac{\partial \Psi}{\partial \alpha_i} = 0 \quad i = 1, 2, \ldots, N \quad \rightarrow \quad \alpha_1, \alpha_2, \ldots, \alpha_N$$

RITZ METHODS - EXAMPLE

With the use of the **Lagrange – Ritz method** find an **approximate distribution of deflection of a thin elastic** rectangular **plate,** which is simply supported along the boundary and uniformly loaded.

Parameters:

- plate's length $L_1 = 8\,\mathrm{m}$
- plate's width $L_2 = 6\,\mathrm{m}$
- plate's thickness $h = 30\,\mathrm{cm}$
- Young's modulus $E = 32\,\mathrm{GPa}$
- Poisson's ratio: $\nu = 0{,}2$
- load: $q = 10\,\mathrm{kN/m}$



RITZ METHODS - EXAMPLE

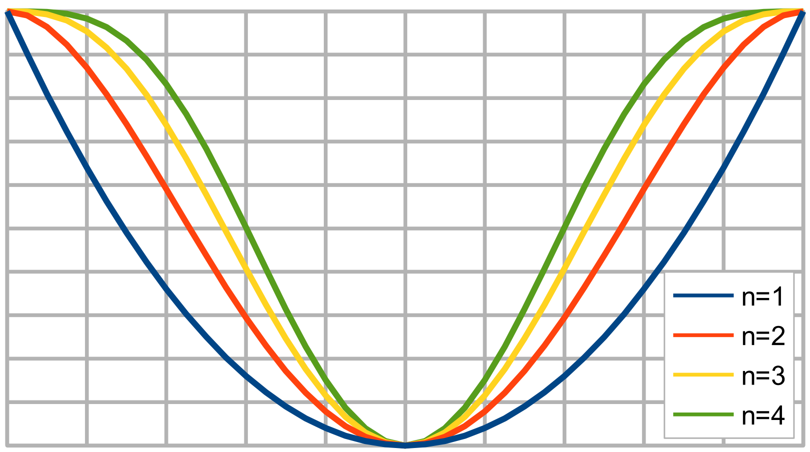
Approximation of the displacement:

$$u_{11} = \left[\left(\frac{L_1}{2} - x_1\right)\left(\frac{L_1}{2} + x_1\right)\right]^1 \qquad u_{21} = \left[\left(\frac{L_2}{2} - x_2\right)\left(\frac{L_2}{2} + x_2\right)\right]^1$$

$$u_{12} = \left[\left(\frac{L_1}{2} - x_1\right)\left(\frac{L_1}{2} + x_1\right)\right]^2 \qquad u_{22} = \left[\left(\frac{L_2}{2} - x_2\right)\left(\frac{L_2}{2} + x_2\right)\right]^2$$

$$u_{13} = \left[\left(\frac{L_1}{2} - x_1\right)\left(\frac{L_1}{2} + x_1\right)\right]^3 \qquad u_{23} = \left[\left(\frac{L_2}{2} - x_2\right)\left(\frac{L_2}{2} + x_2\right)\right]^3$$

$$u_{14} = \left[\left(\frac{L_1}{2} - x_1\right)\left(\frac{L_1}{2} + x_1\right)\right]^4 \qquad u_{24} = \left[\left(\frac{L_2}{2} - x_2\right)\left(\frac{L_2}{2} + x_2\right)\right]^4$$



$$\begin{aligned} u = {} & \alpha_{11} u_{11} u_{21} + \alpha_{12} u_{11} u_{22} + \alpha_{13} u_{11} u_{23} + \alpha_{14} u_{11} u_{24} + \\ & + \alpha_{21} u_{12} u_{21} + \alpha_{22} u_{12} u_{22} + \alpha_{23} u_{12} u_{23} + \alpha_{24} u_{12} u_{24} + \\ & + \alpha_{31} u_{13} u_{21} + \alpha_{32} u_{13} u_{22} + \alpha_{33} u_{13} u_{23} + \alpha_{34} u_{13} u_{24} + \\ & + \alpha_{41} u_{14} u_{21} + \alpha_{42} u_{14} u_{22} + \alpha_{43} u_{14} u_{23} + \alpha_{44} u_{14} u_{24} \end{aligned}$$

RITZ METHODS - EXAMPLE

Displacement (deflection):

$$w = \alpha_{11} u_{11} u_{21} + \alpha_{12} u_{11} u_{22} + \alpha_{13} u_{11} u_{23} + \alpha_{14} u_{11} u_{24} + \alpha_{21} u_{12} u_{21} + \alpha_{22} u_{12} u_{22} + \alpha_{23} u_{12} u_{23} + \alpha_{24} u_{12} u_{24} + \\ + \alpha_{31} u_{13} u_{21} + \alpha_{32} u_{13} u_{22} + \alpha_{33} u_{13} u_{23} + \alpha_{34} u_{13} u_{24} + \alpha_{41} u_{14} u_{21} + \alpha_{42} u_{14} u_{22} + \alpha_{43} u_{14} u_{23} + \alpha_{44} u_{14} u_{24}$$

Strain:

$$\varepsilon_{11} = -\frac{\partial^2 w}{\partial x_1^2} \cdot x_3, \quad \varepsilon_{22} = -\frac{\partial^2 w}{\partial x_2^2} \cdot x_3, \quad \varepsilon_{33} = 0, \quad \varepsilon_{23} = 0, \quad \varepsilon_{31} = 0, \quad \varepsilon_{12} = -\frac{\partial^2 w}{\partial x_1 \partial x_2} \cdot x_3,$$

Stress:

$$\sigma_{11} = \frac{E}{1-\nu^2}(\varepsilon_{11} + \nu \varepsilon_{22}) = -\frac{E}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x_1^2} + \nu \frac{\partial^2 w}{\partial x_2^2}\right) \cdot x_3 \qquad \sigma_{33} = 0$$

$$\sigma_{22} = \frac{E}{1-\nu^2}(\varepsilon_{22} + \nu \varepsilon_{11}) = -\frac{E}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x_2^2} + \nu \frac{\partial^2 w}{\partial x_1^2}\right) \cdot x_3 \qquad \sigma_{31} = 0$$

$$\sigma_{12} = \frac{E}{1+\nu} \varepsilon_{12} = -\frac{E}{(1+\nu)}\left(\frac{\partial w}{\partial x_1 \partial x_2}\right) \cdot x_3 \qquad \sigma_{23} = 0$$

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RITZ METHODS - EXAMPLE

Total potential energy:

$$\Pi[\mathbf{u}] = \frac{1}{2}\iiint\limits_V \sigma_{ij}\varepsilon_{ij}\,\mathrm{d}V - \left[\iiint\limits_V b_i u_i\,\mathrm{d}V + \iint\limits_{S_q} q_i u_i\,\mathrm{d}S\right] =$$

$$= \frac{1}{2}\int\limits_{x_1=-\frac{L_1}{2}}^{\frac{L_1}{2}}\int\limits_{x_2=-\frac{L_2}{2}}^{\frac{L_2}{2}}\int\limits_{x_3=-\frac{h}{2}}^{\frac{h}{2}}\left[\sigma_{11}\varepsilon_{11}+\sigma_{22}\varepsilon_{22}+\sigma_{33}\varepsilon_{33}+2\left(\sigma_{23}\varepsilon_{23}+\sigma_{31}\varepsilon_{31}+\sigma_{12}\varepsilon_{23}\right)\right]\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 -$$

$$-\int\limits_{x_1=-\frac{L_1}{2}}^{\frac{L_1}{2}}\int\limits_{x_2=-\frac{L_2}{2}}^{\frac{L_2}{2}}\int\limits_{x_3=-\frac{h}{2}}^{\frac{h}{2}}[0\cdot w]\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 -$$

$$-\int\limits_{x_1=-\frac{L_1}{2}}^{\frac{L_1}{2}}\int\limits_{x_2=-\frac{L_2}{2}}^{\frac{L_2}{2}}[q\,w]\,\mathrm{d}x_1\,\mathrm{d}x_2$$

RITZ METHODS - EXAMPLE

According to the Lagrange theorem:

$$\Pi \;\to\; \min \qquad \Rightarrow \qquad \frac{\partial \Pi}{\partial \alpha_i} = 0 \quad i=1,2,\ldots,N$$

We obtain a **system of linear algebraic equations for the coefficients** $\alpha_{ij}(i\,,\,j=1,2,3,4)$

$a11 = 5.801754686024868\ 10^{-6}$
$a12 = -1.505170024636658\ 10^{-7}$
$a13 = -4.951668552290187\ 10^{-9}$
$a14 = -1.499627766133912\ 10^{-10}$
$a21 = -8.021862334214811\ 10^{-8}$
$a22 = 5.066493675879956\ 10^{-9}$
$a23 = 2.598920163357196\ 10^{-10}$
$a24 = 3.618870844231519\ 10^{-12}$
$a31 = -3.103672759106117\ 10^{-9}$
$a32 = 3.269527853006765\ 10^{-10}$
$a33 = 6.422700908763642\ 10^{-11}$
$a34 = 2.457188083111769\ 10^{-12}$
$a41 = -4.453036506355453\ 10^{-11}$
$a42 = 7.815979494356788\ 10^{-12}$
$a43 = 2.198636144944349\ 10^{-12}$
$a44 = 1.021412676869167\ 10^{-13}$

Maximum deflection in the middle of the plate

$$w(0,0) = 1{,}145\ \mathrm{mm}$$

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THANK YOU FOR YOUR ATTENTION