THEORY OF ELASTICITY AND PLASTICITY

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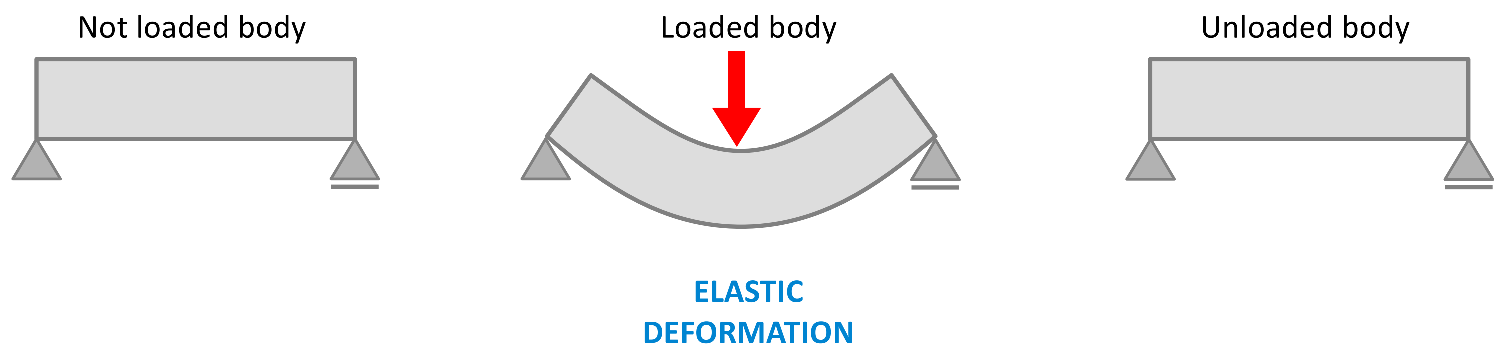
PLASTIC DEFORMATION

PLASTIC DEFORMATION

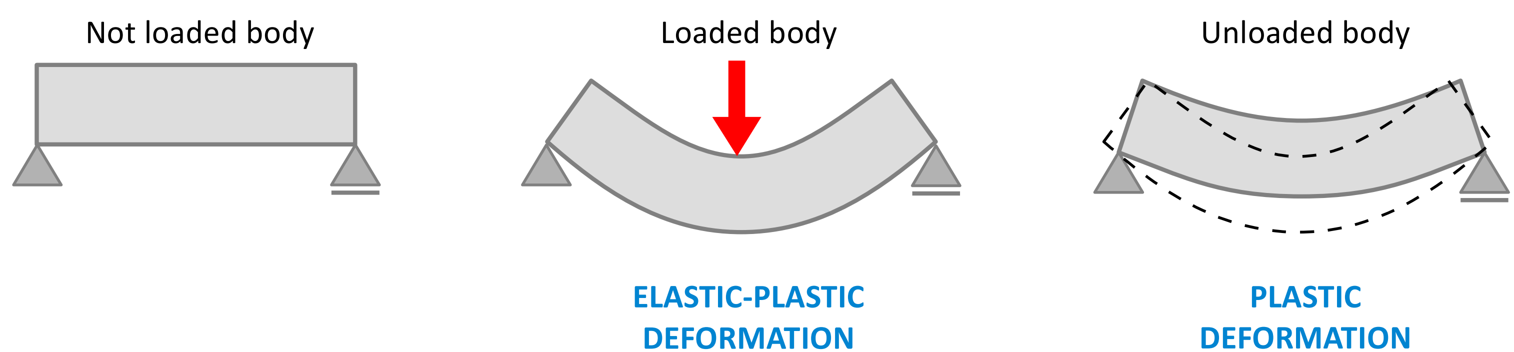
elastic strain = reversible strain

ELASTIC MATERIAL



plastic strain = permanent strain

ELASTIC-PLASTIC MATERIAL

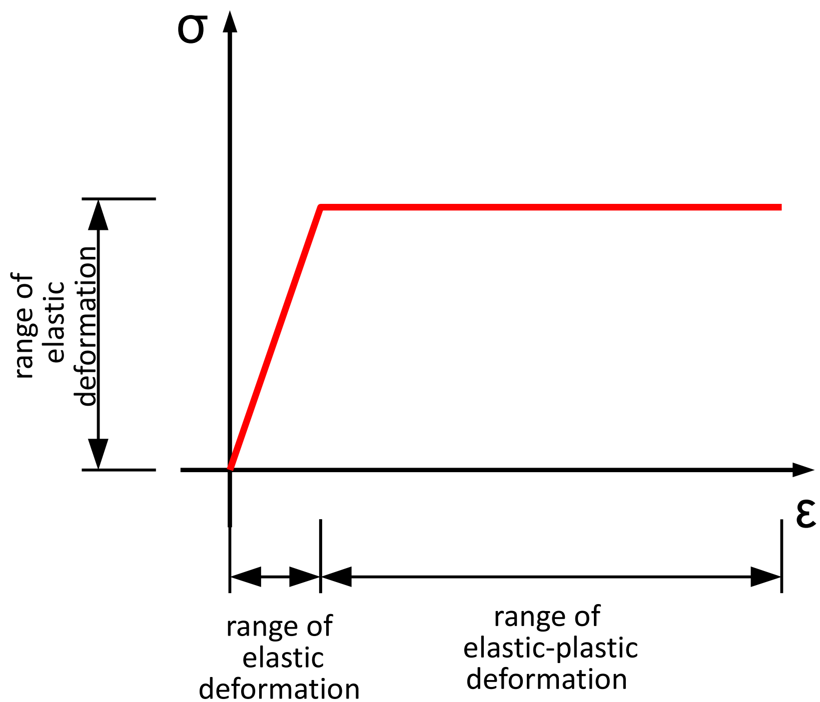


PLASTIC DEFORMATION

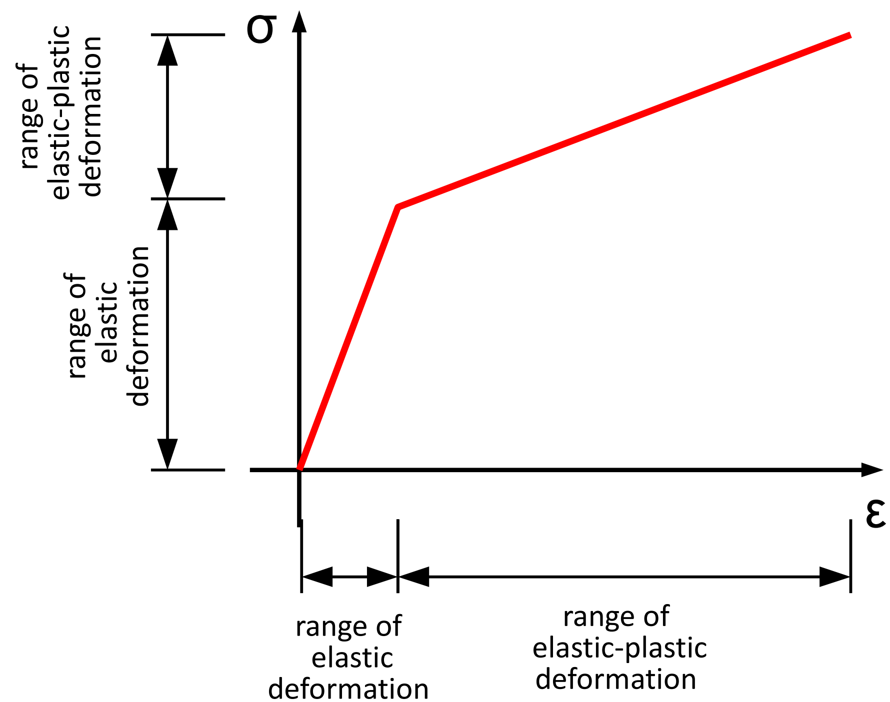
Theory of plasticity – part of the **continuum mechanics** dealing with the description of deformation of **elastic-plastic** solids, **rigid-plastic** solids and **ideally plastic** solids.

PLASTIC DEFORMATION

Elastic-plastic solid (**Prandtl model**) – **within certain range of deformation**, which is characteristic for the considered material, it exhibits **elastic properties**. **Beyond this range** it exhibits **both elastic and plastic** properties.



model of elastic-plastic material without hardening



model of elastic-plastic material with hardening

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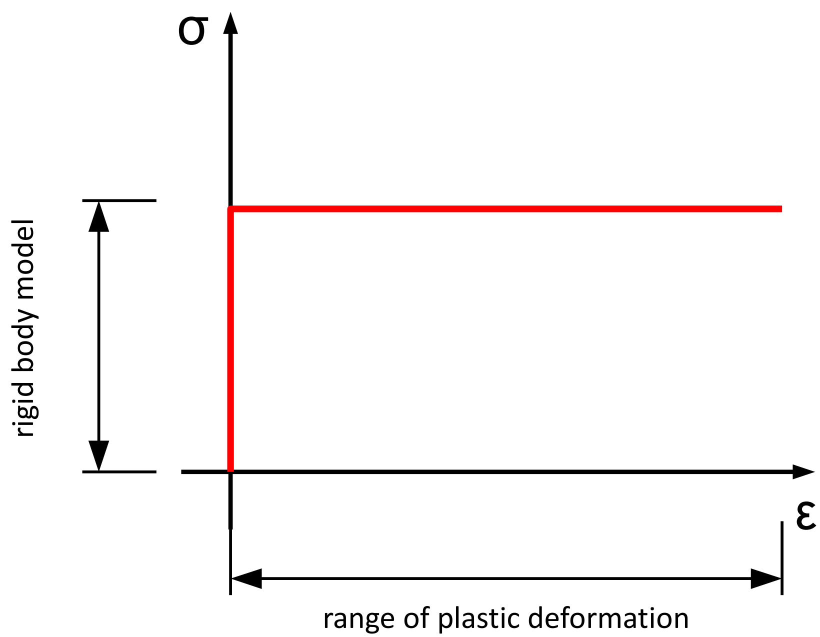
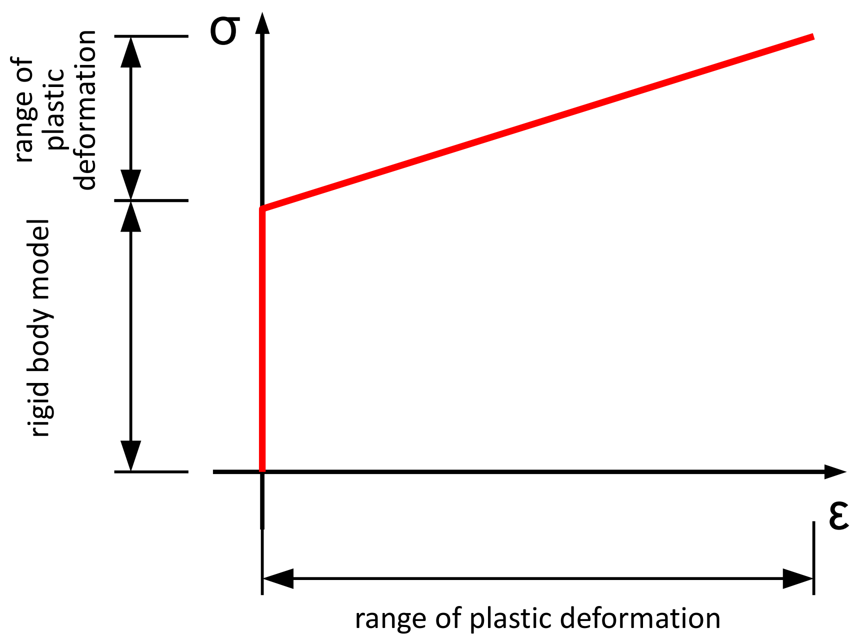
PLASTIC DEFORMATION

Rigid-plastic solid (**Lévy – Mises model**) – **within certain range of deformation**, which is characteristic for the material, it **doesn't undergo any deformation** and behaves as a **rigid body**. **Beyond that range** it exhibits **solely plastic properties**.

It is a **limit case of elastic-plastic solid** and it may be used as an **approximation** of the latter if only **elastic strain is considerably smaller than plastic strain**.



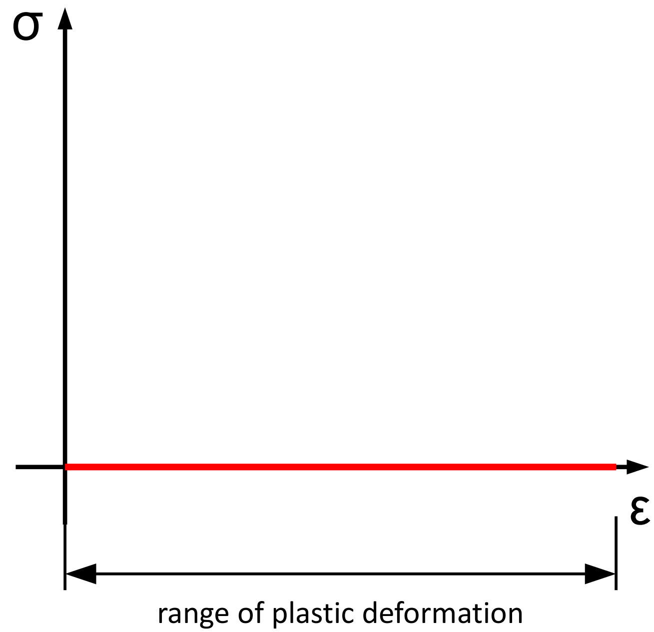
model of rigid-plastic material
without hardening



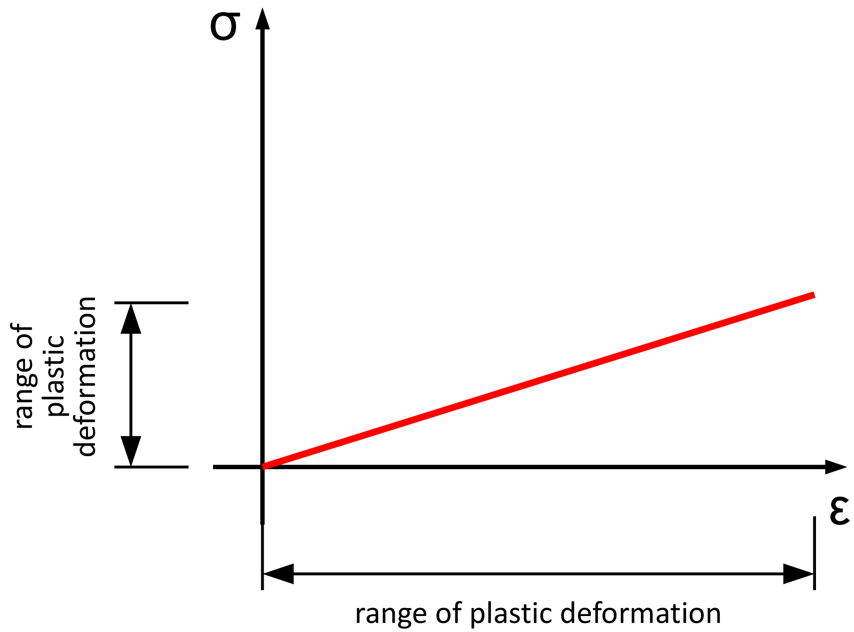
model of rigid-plastic material
with hardening

PLASTIC DEFORMATION

Ideally plastic model – **within the whole range of deformation** it exhibits **solely plastic properties**.



model of ideally plastic material
without hardening



model of ideally plastic material
with hardening

PLASTIC DEFORMATION

The range of elastic deformation depends on:

- Mechanical properties of the material

  limit value of stress (yield stress) or **limit value of elastic strain**, beyond which plastic deformation is observed, is a mechanical property which is **characteristic for the considered material**.

- Type of load / deformation

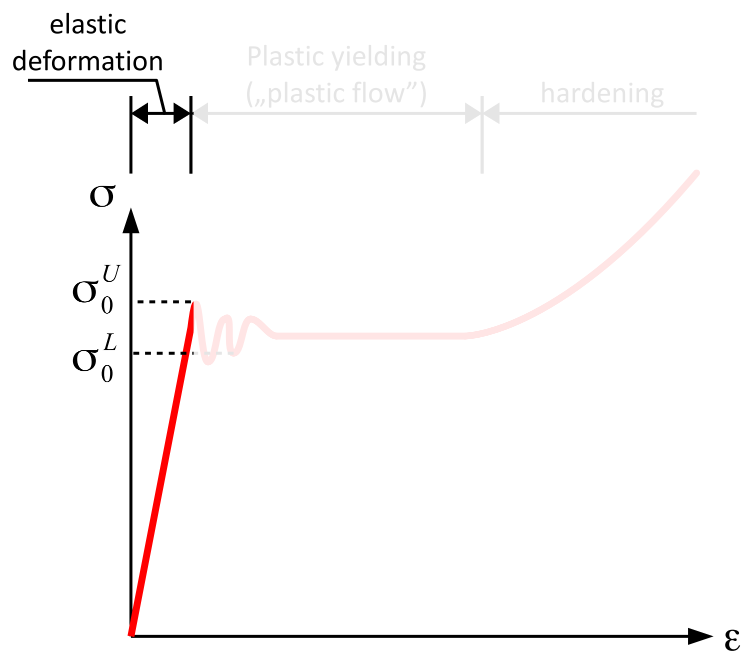
  For **different types of mechanical states plastic deformation is initialized for different magnitude of load or deformation**, e.g. plastic yielding occurs for lower values of shear stress in shearing than the limit values of normal stress in tension / compression. The condition which must be satisfied by components of the stress tensor or strain tensor for the plastic deformation to be initiated is termed the yield condition.

- History of deformation

  Most of materials exhibit the effect of strain hardening. Past plastic deformation results in the **increase of range of elastic deformation**. Some materials do not exhibit any measurable hardening and some exhibit **softening**.

PLASTIC DEFORMATION

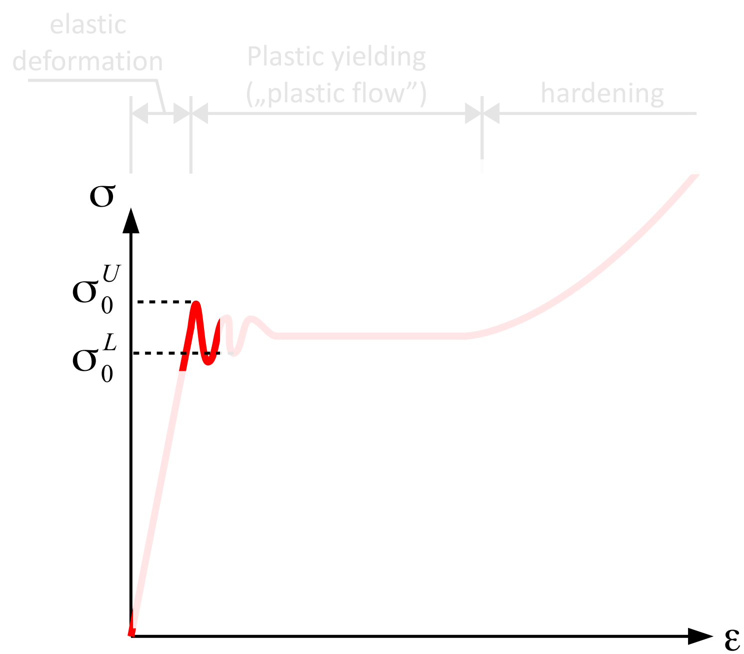
Characteristics of materials exhibiting sharp yield stress



- Range of elastic deformation – plastic deformation does not occur or it is negligibly small. In fact small plastic strain can be observed even for very small load / deformation, however, within certain range they are very small, and only beyond that range their increment becomes important.

PLASTIC DEFORMATION

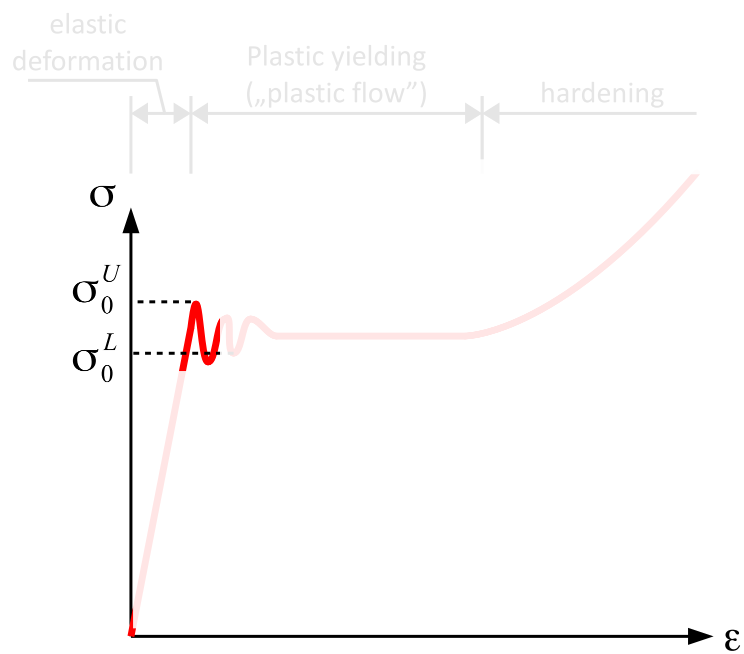
Characteristics of materials exhibiting sharp yield stress



- **Limit state** – stress or strain state satisfy the yield condition. Mechanisms of plastic deformation are initiated.

PLASTIC DEFORMATION

Characteristics of materials exhibiting sharp yield stress

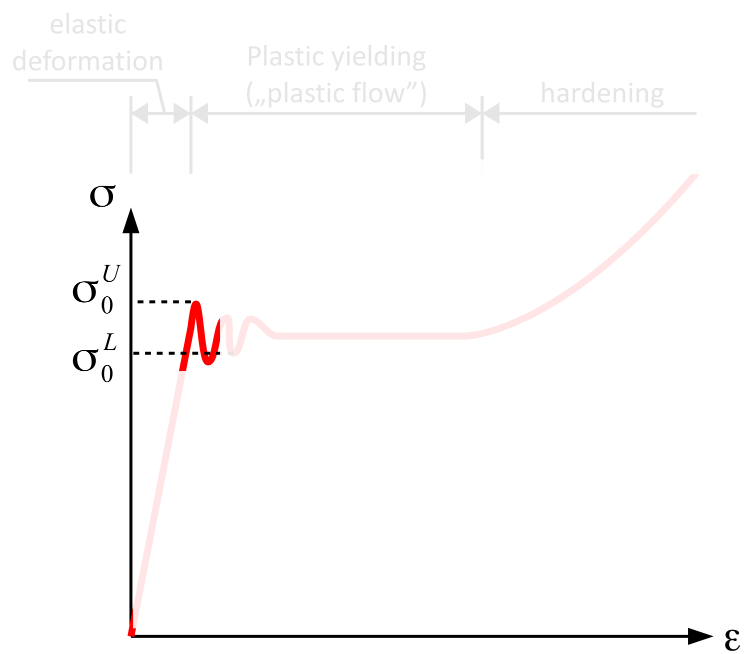


Lower yield stress σ_0^L :

- **Minimum value** of stress which is **necessary for further increment of plastic strain** after initiation of plastic deformation (once the yield conditions has been satisfied),
- **Minimum value** of stress **within the range of plastic deformation**, **excluding the first local minimum** after initiation of plastic deformation,
- **Last local minimum** of value stress within the range of plastic deformation, **before hardening** occurs.

PLASTIC DEFORMATION

Characteristics of materials exhibiting sharp yield stress

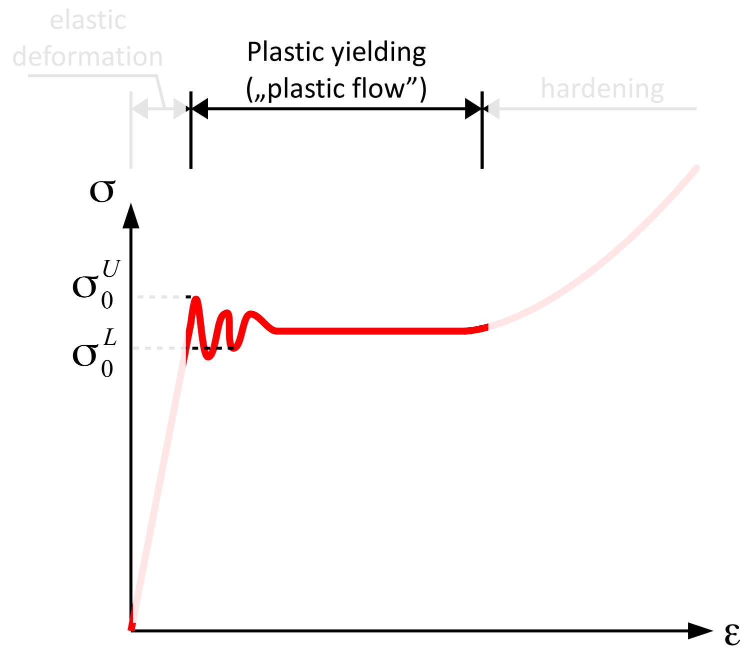


Upper yield stress σ_0^U:

- **First local maximum** of the value of stress which is after initiation of plastic deformation (once the yield conditions has been satisfied),
- **Maximum value** of stress **within the range of plastic deformation**

PLASTIC DEFORMATION

Characteristics of materials exhibiting sharp yield stress

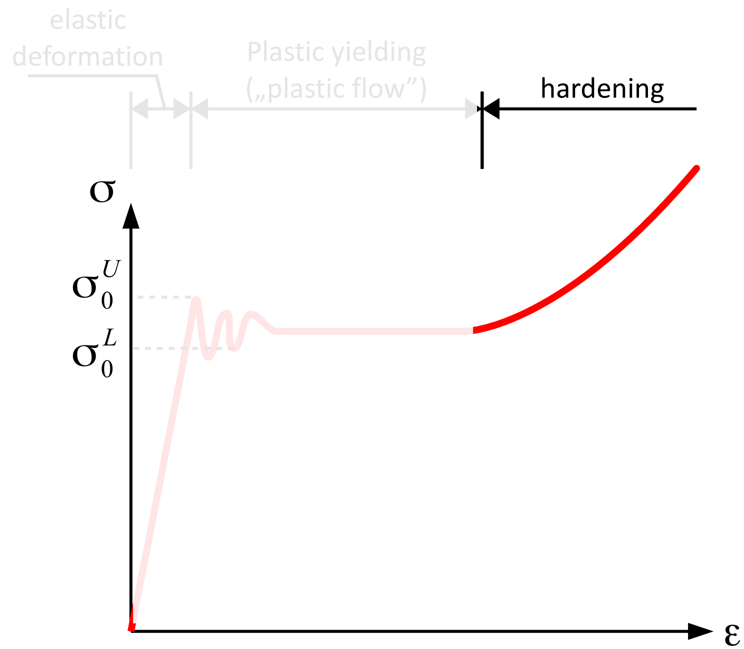


Plastic yielding ("plastic flow"):

- **Extensive increment of strain** corresponding with **negligible variation of stress**
- Relation between stress and strain is no longer one-to-one
- Deformation process is strain-controlled

PLASTIC DEFORMATION

Characteristics of materials exhibiting sharp yield stress

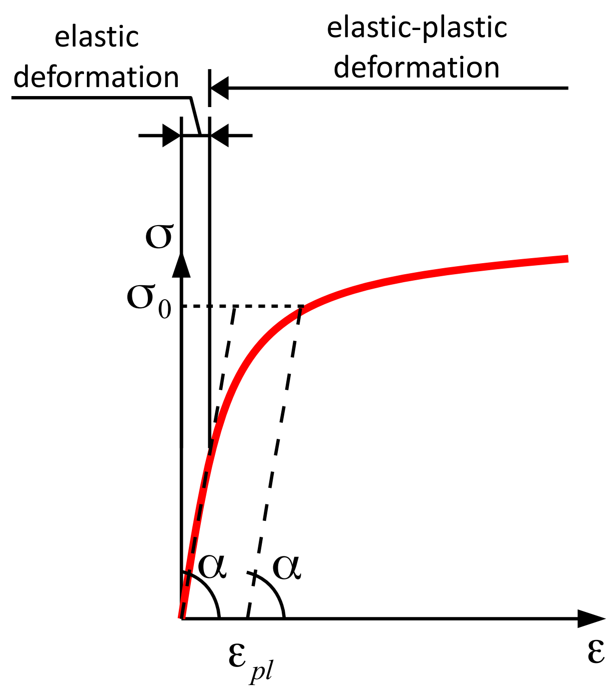


Hardening:

- **Further elastic-plastic deformation requires greater load**
- In case of materials which do not exhibit hardening the values of stress above the value corresponding with "yield plateau" is unreachable – stress is constant and strain increases until fracture occurs

PLASTIC DEFORMATION

Characteristics of materials which do not exhibit sharp yield stress



Offset yield stress:

- **The value of stress which corresponds with a certain value of permanent strain**, which is determined with an assumption that unloading process is fully elastic and that elastic properties of the material do not change due to plastic deformation. The value which is commonly used for ductile elastic-plastic materials is $\varepsilon_{pl} = 0{,}2\,\%$

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MECHANISMS OF PLASTIC DEFORMATION

MECHANISMS OF PLASTIC DEFORMATION

The presence of permanent (plastic) strains corresponds with a **permanent and irreversible change in the internal structure of material**:

> In the **transformed structure** of material a **new state of internal equilibrium is constituted**

Theory of plasticity is most widely applied in the description of ductile metals and alloys. These materials exhibit a **polycrystalline internal structure**:

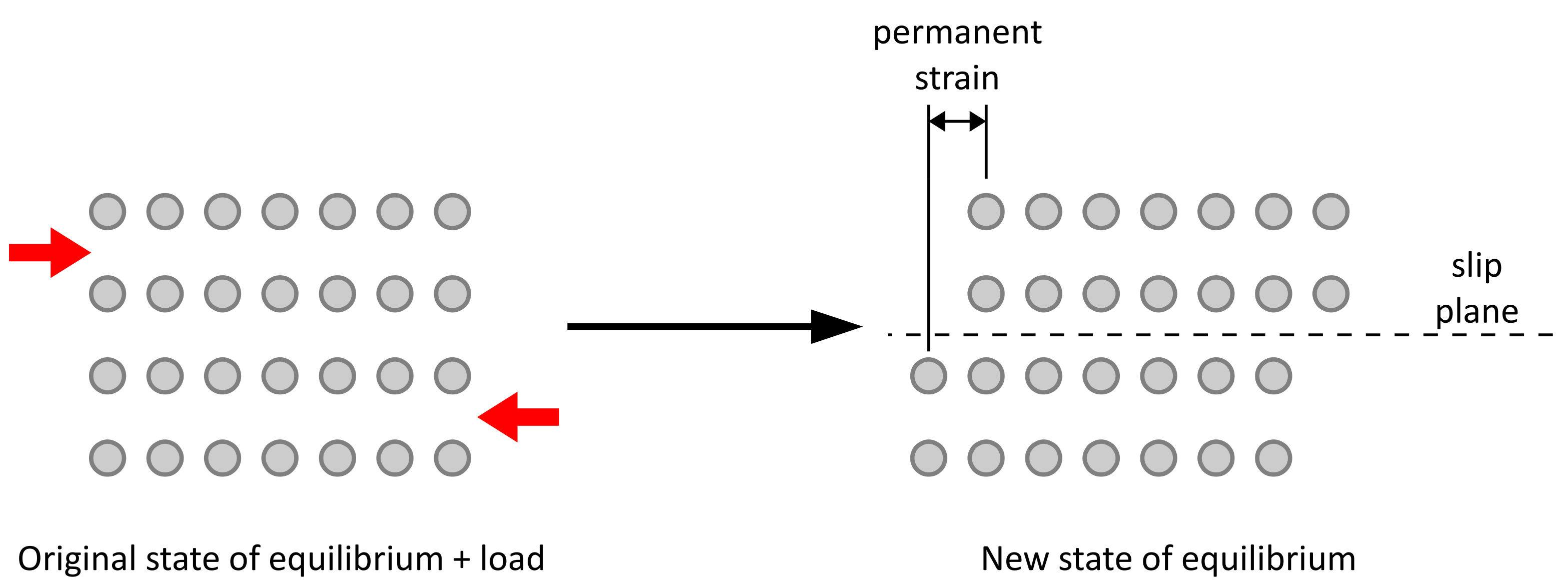
- they're made of **monocrystalline grains** - within grains the material has regular crystalline structure,
- the grains have **various sizes and shapes**,
- **Orientation of the crystalline structure of different grains** may be **different**.

> **Permanent deformation of internal structure** of a polycrystalline material may occur due to phenomena occurring:
>
> - **within a single monocrystalline grain** → slip (including: motion of dislocations) i twinning
> - **at the boundary of monocrystalline grains** → slip at the grain boundary
> - **in a region containing many grains** → e.g. shear bands

MECHANISMS OF PLASTIC DEFORMATION

A slip parallel to a plane which is most densely packed with atoms (close-packed plane):

- It occurs within a single monocrystal
- **Part of the structure is shifted (it slips) with respect to the second part of the structure** in such a way, that atoms in a part of the structure move to the places which were occupied in the original structure by their neighbouring atoms.



REMARK: Energy and load which is required for a slip to occur is very large. In fact such phenomena are observed for much smaller loads. It is due to **existence of certain defects in the crystalline structure** – they **lower the amount of energy which is required to initiate the slip**.

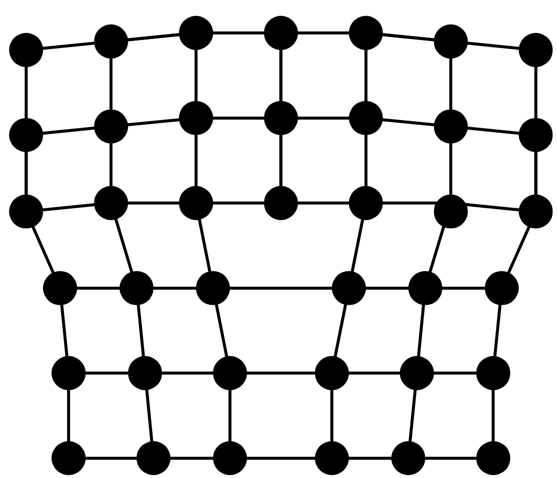
MECHANISMS OF PLASTIC DEFORMATION

We distinguish point defect (concerning a **single atom**), line defects (so called dislocations concerning a set of **atoms lying on a single line**) and plane defects (concerning a set of **atoms lying within a single plane**)

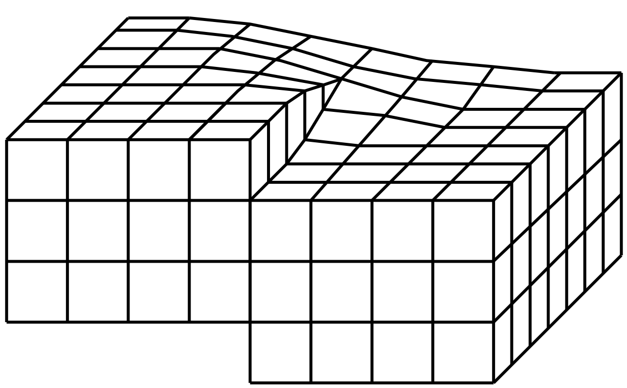
Regarding the mechanisms of plastic deformation, the **dislocations** are of greatest importance. We distinguish **3 types of dislocations**:

- edge dislocations
- screw dislocations
- mixed dislocations

crystalline structure
with edge dislocation



crystalline structure
with screw dislocation



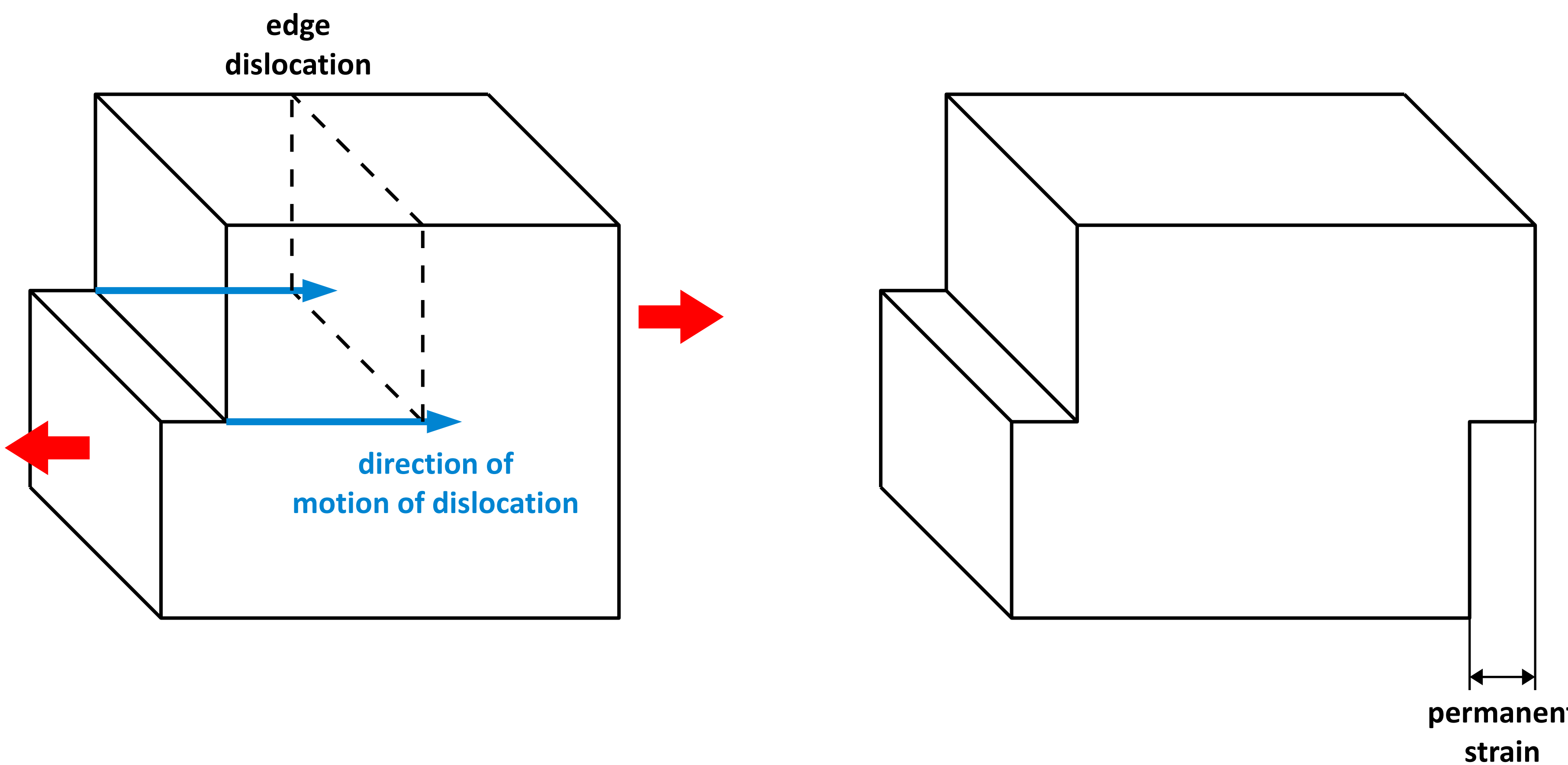
MECHANISMS OF PLASTIC DEFORMATION

Within the area close to dislocations the energy required for slip of atoms is much lower than in ideal structure

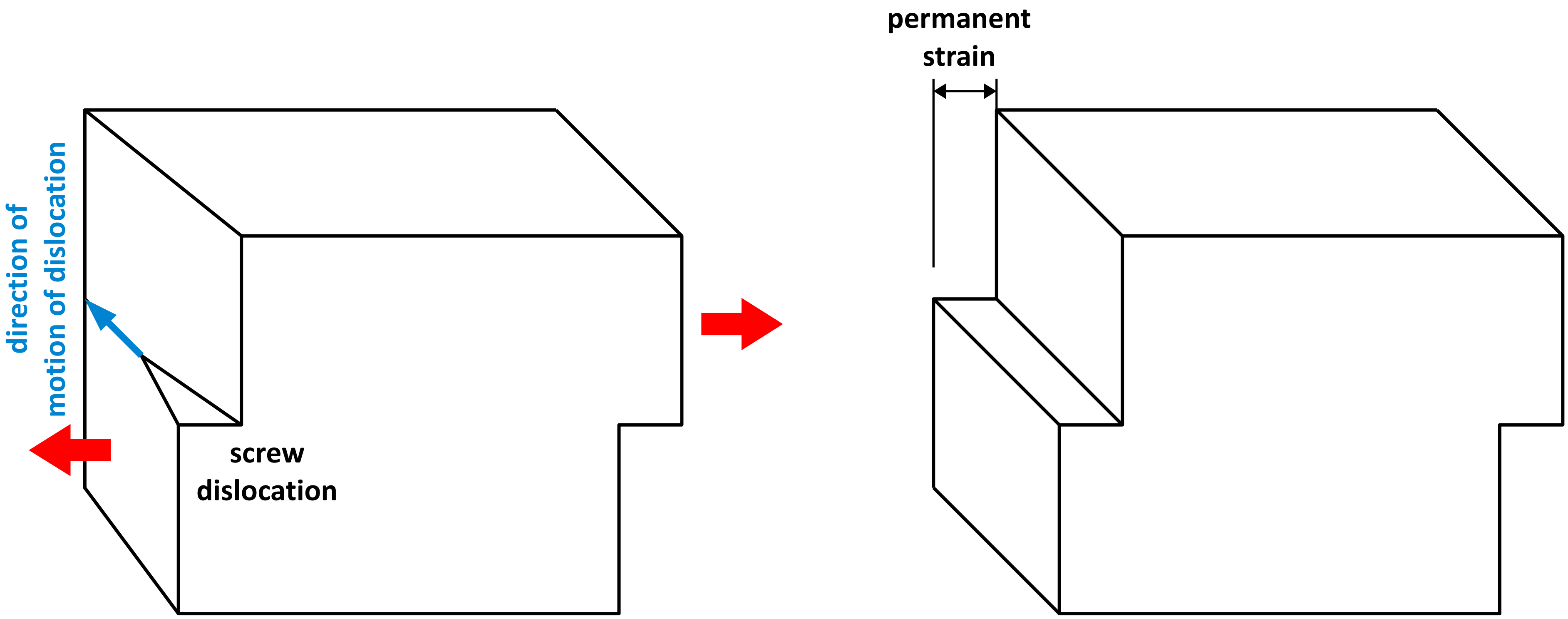
Simple model of deformation assumes that **atoms which are deflected from the original equilibrium position** corresponding with ideal structure (namely, those within the dislocation) r**eturn to the original equilibrium position**, while **neighbouring atoms are deflected from their original equilibrium position**.

As a result dislocation (considered a geometric system or a disturbance of ideal structure) "**moves**" - we are speaking then of motion of dislocations.

MECHANISMS OF PLASTIC DEFORMATION



MECHANISMS OF PLASTIC DEFORMATION



MECHANISMS OF PLASTIC DEFORMATION

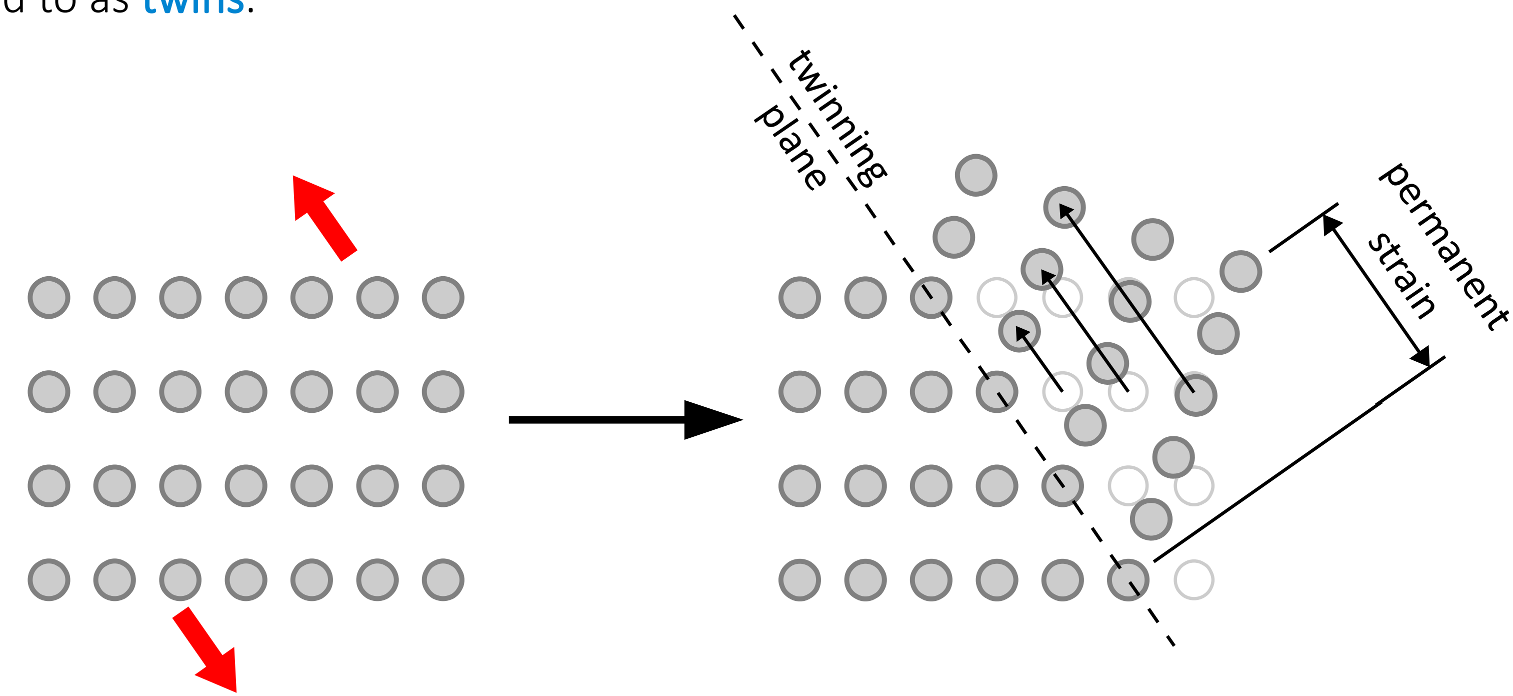
REMARKS:

- **Motion of dislocation terminates at the boundary of a monocrystalline grain**.
- **Dislocations may “attract” or “repel” each other** – energy which is needed to make those disturbances come closer is greater than energy needed to make them move away from each other (and vice versa)
- **Concentration of a great number of dislocations of similar type** in a small region make them **“repel” one from another** – as a result their motion is constrained. In order to continue the process of plastic deformation greater load is required – the material exhibits **hardening**.
- An opposite phenomenon may also occur – if in a small region **many dislocations of different type concentrate**, **they “attract” each other** and their **motion requires smaller external load to be applied** – the material exhibits **softening**.

MECHANISMS OF PLASTIC DEFORMATION

Twinning – change in the crystalline structure in which **one part of the structure is transformed with regards to the second part** with **symmetry transformation.**

- It may be **inversion** (**point reflection**), **reflection through a line**, or **reflection through a plane**.
- In the case of reflection through a plane, two crystals are obtained – they have the same composition, the same structure but the **orientation of those structures are different** – **they are symmetric with respect to the plane of reflection** which is termed the twinning plane. Those crystals are referred to as twins.



Original state of equilibrium

New state of equilibrium

YIELD CONDITION

YIELD CONDITION

Mechanisms of the plastic deformation may occur only if specific conditions are satisfied.

Locally those conditions concern:

- **Orientation of load with respect to possible slip planes or twinning planes**
- **Energy which is required in order to initiate this mechanism** (it depend on the type of atoms and on the distance between them)

In a macroscopic description those conditions are formulated in the form of a **relation which must be satisfied by the components of the stress tensor** (or strain tensor). Such a relation is termed the yield conditions, and it is usually written down in the following form:

$$f\left(\sigma_{11},\sigma_{22},\sigma_{33},\sigma_{23},\sigma_{31},\sigma_{12}\right)=0$$

For isotropic materials the yield condition must be an **isotropic scalar-valued function of tensorial argument.** It must be then possible to express it in the form of a **scalar-valued functions of invariants of the stress tensor**, in particular: in terms of principal stresses:

$$f\left(\sigma_1,\sigma_2,\sigma_3\right)=0$$

YIELD CONDITION

Alternative formulations:

- coefficients of the secular equation $f = f(I_1, I_2, I_3)$

$$I_1(\boldsymbol{\sigma}) = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

$$I_2(\boldsymbol{\sigma}) = \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} + \sigma_{11}\sigma_{22} - \sigma_{23}^2 - \sigma_{31}^2 - \sigma_{12}^2$$

$$I_3(\boldsymbol{\sigma}) = \sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{23}\sigma_{31}\sigma_{12} - \sigma_{11}\sigma_{23}^2 - \sigma_{22}\sigma_{31}^2 - \sigma_{33}\sigma_{12}^2$$

- Invariants of isotropic and deviatoric parts of the stress tensor $f = f(I_1, J_2, J_3)$

$$I_1(\boldsymbol{\sigma}) = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

$$J_2(\boldsymbol{\sigma}) = \frac{1}{6}\left[(\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2 + (\sigma_{11}-\sigma_{22})^2\right] + \left(\sigma_{23}^2+\sigma_{31}^2+\sigma_{12}^2\right)$$

$$J_3(\boldsymbol{\sigma}) = \left(\sigma_{11}-\frac{1}{3}I_1\right)\left(\sigma_{22}-\frac{1}{3}I_1\right)\left(\sigma_{33}-\frac{1}{3}I_1\right) + 2\sigma_{23}\sigma_{31}\sigma_{12} - \left(\sigma_{11}-\frac{1}{3}I_1\right)\sigma_{23}^2 - \left(\sigma_{22}-\frac{1}{3}I_1\right)\sigma_{31}^2 - \left(\sigma_{33}-\frac{1}{3}I_1\right)\sigma_{12}^2$$

- Cylindrical coordinates in the space of principal stresses $f = f(p, q, \theta)$
  (e.g. **Lode**, **Haigh-Westergaard coordinates**)

$p = \frac{1}{3}I_1$ - hydrostatic stress

$q = \sqrt{2J_2}$ - deviatoric stress

$\theta = \frac{1}{3}\arccos\left[\frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}}\right]$ - Lode angle

YIELD CONDITION

Yield condition for isotropic **materials**:

$$f(\sigma_1, \sigma_2, \sigma_3) = 0$$

may be interpreted geometrically as an equation of a **surface in a three-dimensional space of principal stresses**. Such a surface is termed the yield surface.

It is common to write down the **yield condition** in the following form:

$$\sigma_{eq} = \sigma_0$$

where:

- σ_0 – certain limit stress (e.g. offset yield stress in tension)
- σ_{eq} – equivalent stress – it is a **function of components of the stres state** which is **related to the yield condition**

YIELD CONDITION

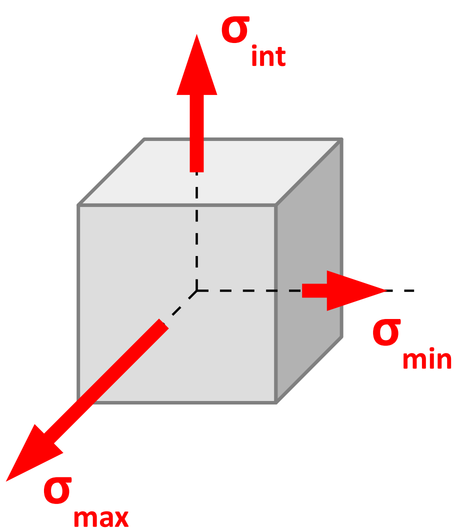
COULOMB – TRESCA – GUEST YIELD CONDITION

Plastic yielding occurs when **maximum shear stress** reaches certain limit value

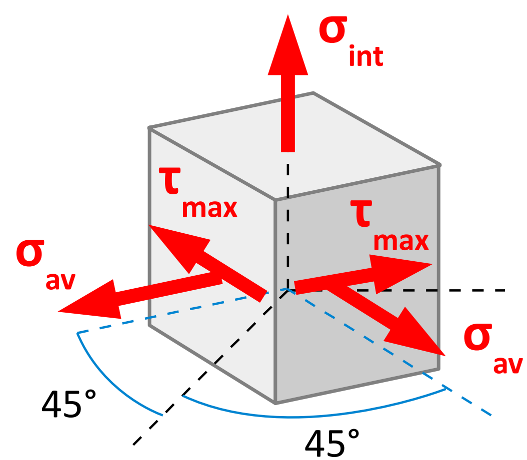
$$\tau_{max}=\tau_0 \quad \text{where} \quad \tau_{max}=\max\left(\frac{|\sigma_2-\sigma_3|}{2}\ ;\ \frac{|\sigma_3-\sigma_1|}{2}\ ;\ \frac{|\sigma_1-\sigma_2|}{2}\right)$$

Maximum shear stress corresponds with shear stress in planes which are parallel to the intermediate principal stress and which are inclined at angle 45° to the directions of maximum and minimum principal stress.

$$\boldsymbol{\sigma}=\begin{bmatrix}\sigma_{\max} & 0 & 0\\ 0 & \sigma_{\text{int}} & 0\\ 0 & 0 & \sigma_{\min}\end{bmatrix}$$



$\Leftrightarrow$



$$\boldsymbol{\sigma}=\begin{bmatrix}\sigma_{\text{av}} & 0 & \tau_{\max}\\ 0 & \sigma_{\text{int}} & 0\\ \tau_{\max} & 0 & \sigma_{\text{av}}\end{bmatrix}$$

$$\tau_{\max}=\frac{\sigma_{\max}-\sigma_{\min}}{2}$$

$$\sigma_{\text{av}}=\frac{\sigma_{\max}-\sigma_{\min}}{2}$$

YIELD CONDITION

The yield surface is a **infinitely long prism** the **base of which is a regular hexagon** and its **axis is equally inclined to all axes of the coordinate system of principal stresses**.

Yield condition expressed in terms of the **equivalent stress**: $\sigma_{eq}=\sigma_0$

$$\boxed{\sigma_{eq} = \max\left(|\sigma_2-\sigma_3| \ ; \ |\sigma_3-\sigma_1| \ ; \ |\sigma_1-\sigma_2|\right)}$$

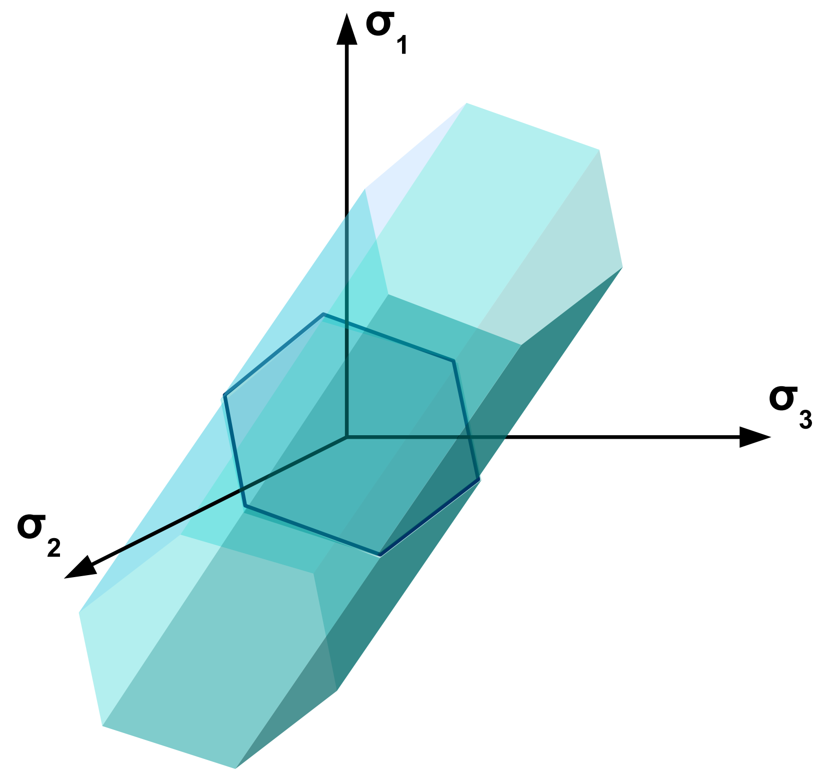
REMARKS:

- **Hydrostatic stress does not influence the velue of the equivalent stress**
- **Material exhibits the same yield stress in tension** $\sigma_{0,t}$ **and in compression** $\sigma_{0,c}$:

$$\sigma_{0,t} = \sigma_{0,c}$$

- The relation between the **yield stress in tension** and **yield stress in shear**:

$$\sigma_{0,s} = \frac{1}{2}\sigma_{0,t}$$



YIELD CONDITION

MAXWELL – HUBER – MISES – HENCKY YIELD CONDITION

Plastic yielding occurs when the **density of energy of distortional strain** reaches certain limit value:

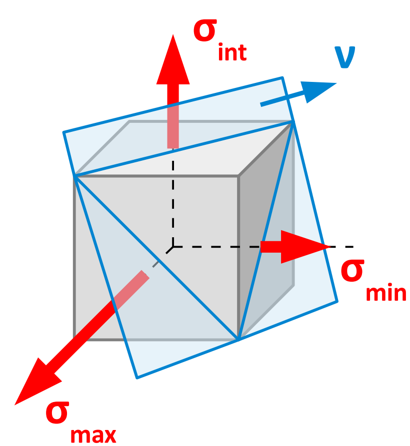
$$\phi_f = h \quad \text{where} \quad \phi_f = \frac{1}{12G}\left[(\sigma_{22}-\sigma_{33})^2 + (\sigma_{11}-\sigma_{22})^2 + (\sigma_{33}-\sigma_{11})^2 + 6(\sigma_{23}^2+\sigma_{31}^2+\sigma_{12}^2)\right]$$

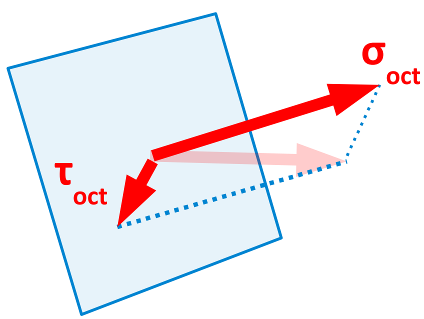
Yield condition expressed in terms of **equivalent stress**: $\sigma_{eq}=\sigma_0$

$$\sigma_{eq} = \sqrt{\frac{1}{2}\left[(\sigma_{22}-\sigma_{33})^2 + (\sigma_{11}-\sigma_{22})^2 + (\sigma_{33}-\sigma_{11})^2 + 6(\sigma_{23}^2+\sigma_{31}^2+\sigma_{12}^2)\right]} = \sqrt{3J_2} = \sqrt{\frac{3}{2}q}$$

Equivalent stress is proportional to the **tangent** (**shear**) **component of the octahedral stress** which is a stress vector corresponding with a plane which is equally inclined to all directions of principal stresses:

$$\sigma_{eq} = \frac{3}{\sqrt{2}}\tau_{oct}$$





YIELD CONDITION

The yield surface is a **infinitely long cylinder** the **axis of which is equally inclined to all axes of the coordinate system of principal stresses**.

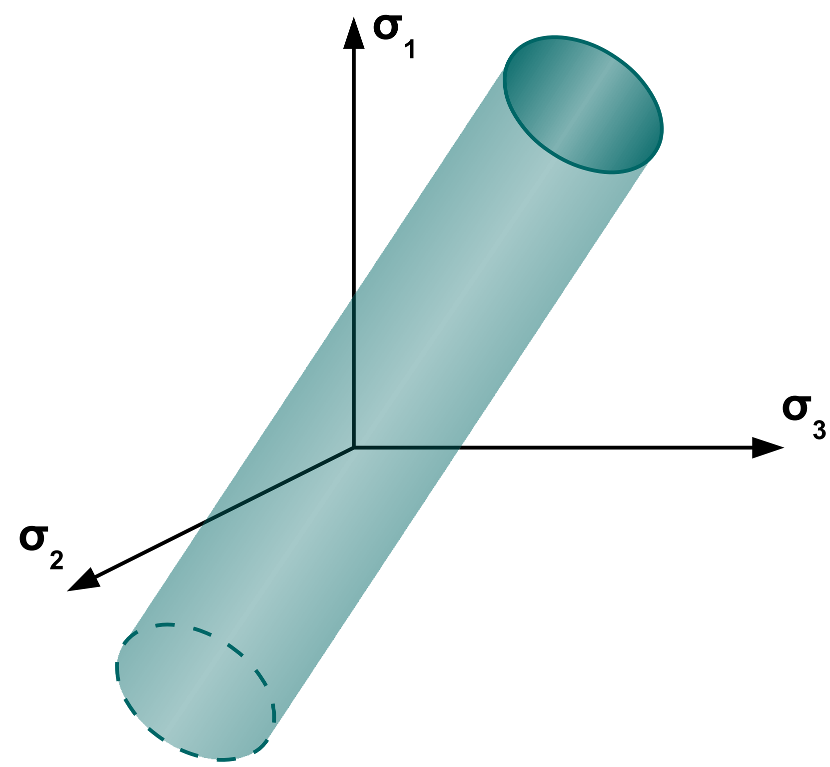
REMARKS:

- **Hydrostatic stress does not influence the velue of the equivalent stress**
- **Material exhibits the same yield stress in tension** $\sigma_{0,t}$ **and in compression** $\sigma_{0,c}$ **:**

$$\sigma_{0,t} = \sigma_{0,c}$$

- The relation between the **yield stress in tension** and **yield stress in shear**:

$$\sigma_{0,s} = \frac{1}{\sqrt{3}}\sigma_{0,t}$$



YIELD CONDITION

BURZYŃSKI YIELD CONDITION

Plastic yielding occurs when the **density of energy of distortional strain** reaches certain **limit value which depends on the magnitude of hydrostatic stress** (volumetric strain):

$$\phi_f + \left(A + \frac{B}{p} \right) \phi_v = h \qquad \Leftrightarrow \qquad \phi_f = h(p)$$

where:

density of energy of distortional strain:

$$\phi_f = \frac{1}{12G} \left[(\sigma_{22} - \sigma_{33})^2 + (\sigma_{11} - \sigma_{22})^2 + (\sigma_{33} - \sigma_{11})^2 + 6(\sigma_{23}^2 + \sigma_{31}^2 + \sigma_{12}^2) \right]$$

density of energy of volumetric strain:

$$\phi_v = \frac{1}{18K} (\sigma_{11} + \sigma_{22} + \sigma_{33})^2$$

hydrostatic stress:

$$p = \frac{1}{3} (\sigma_{11} + \sigma_{22} + \sigma_{33})$$

YIELD CONDITION

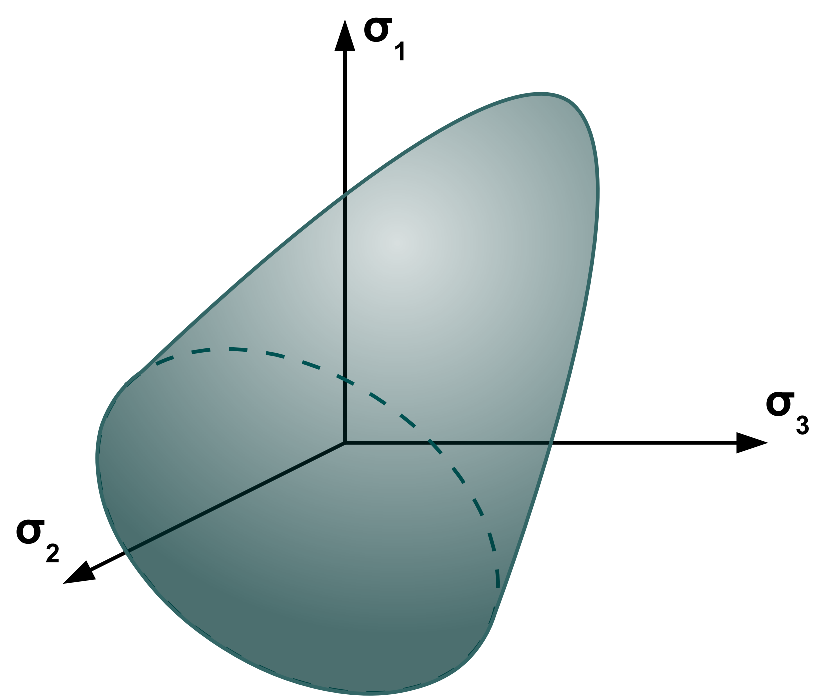
Burzyński yield condition may be written down in the following form:

$$(\sigma_{11}^2+\sigma_{22}^2+\sigma_{33}^2)-2\left(\frac{\sigma_{0,c}\sigma_{0,t}}{2\sigma_{0,s}^2}-1\right)(\sigma_{22}\sigma_{33}+\sigma_{33}\sigma_{11}+\sigma_{11}\sigma_{22})+\frac{\sigma_{0,c}\sigma_{0,t}}{\sigma_{0,s}^2}(\sigma_{23}^2+\sigma_{31}^2+\sigma_{12}^2)+(\sigma_{0,c}-\sigma_{0,t})(\sigma_{11}+\sigma_{22}+\sigma_{33})-\sigma_{0,c}\sigma_{0,t}=0$$

The yield surface is a 2nd **degree surface** (**quadric**) with a symmetry axis which is equally inclined to the directions of principal stresses – it may be a **sphere**, an **ellipsoid of revolution**, a **cylinder**, a **paraboloid of revolution**, a **cone** or a **single sheet of two-sheet hyperboloid of revolution**.

REMARKS:

- **Hydrostatic stress may influence the equivalent stress.**
- Material may exhibit **different values of the yield stress** in **tension** $\sigma_{0,t}$ and in **compression** $\sigma_{0,c}$.
- The special case of the Burzyński yield condition is the commonly used Drucker – Prager yield condition which corresponds with the yield surface in the form of a **cone**.



CHARACTERISTICS OF PLASTIC DEFORMATION

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Plastic deformation exhibits certain **specific properties**:

ELASTIC UNLOADING

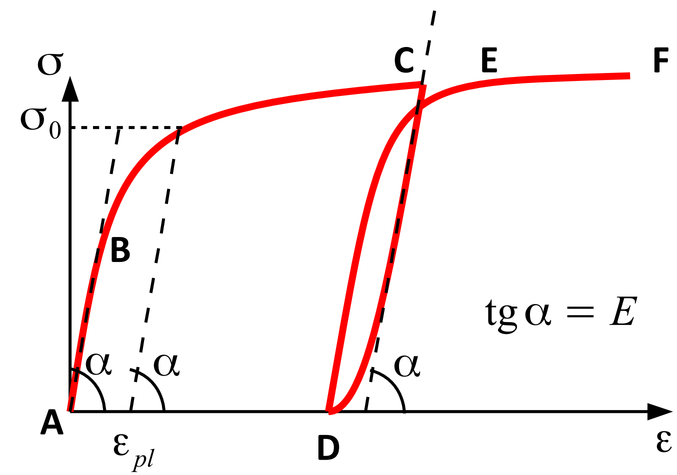
INCOMPRESSIBILITY, PRESSURE INDEPENDENCE

INDUCED ANISOTROPY

CHARACTERISTICS OF PLASTIC DEFORMATION

Plastic deformation exhibits certain **specific properties**:

- ELASTIC UNLOADING
- INCOMPRESSIBILITY, PRESSURE INDEPENDENCE
- INDUCED ANISOTROPY



- The process of **unloading** is **elastic** – only elastic strain are reversible. Reversing of the permanent strain would require initiation of new slips (additional energy is required).

- **Elastic properties** (e.g. elastic constants) do not change due to plastic deformation. Dislocations occur only in small part of the total volume of the body – outside these regions the structure of monocrystalline grains remains unchanged.

CHARACTERISTICS OF PLASTIC DEFORMATION

Plastic deformation exhibits certain **specific properties**:

ELASTIC UNLOADING

INCOMPRESSIBILITY, PRESSURE INDEPENDENCE

INDUCED ANISOTROPY

- It is observed that **during plastic deformation volume of the body remains unchanged**. The material behaves as if it was **incompressible**.
- It is also observed that the **influence of pressure on satisfying the yield condition** (value of equivalent stress) **is negligibly small**.
- In the commonly used **incremental model of plasticity with an associated flow rule** both those phenomena are accounted for by the fact, that **yield condition does not depend on hydrostatic stress**. Yield condition determines the plastic potential according to which the plastic strain is calculated. When the potential is independent of pressure, corresponding volumetric plastic strain is equal to 0.

CHARACTERISTICS OF PLASTIC DEFORMATION

Plastic deformation exhibits certain **specific properties**:

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INCOMPRESSIBILITY, PRESSURE INDEPENDENCE

INDUCED ANISOTROPY

- In fact, **there exists certain influence of the pressure** and **small volumetric deformation** corresponding with it.
- Influence of the hydrostatic stress on the yield condition results with the fact, that **plastic yielding in tension occurs for a slightly different value of the stress than in case of compression**. Such a phenomenon is termed the **SD** (**strength-differential**) **effect**.

CHARACTERISTICS OF PLASTIC DEFORMATION

Plastic deformation exhibits certain **specific properties**:

ELASTIC UNLOADING

INCOMPRESSIBILITY, PRESSURE INDEPENDENCE

INDUCED ANISOTROPY

- In a polycrystalline body **severe deformation may lead to the change of mutual orientation of monocrystalline grains**.
- If the load oriented along certain direction (e.g. uniaxial tension) then **the grains tend to rotate and elongate along the direction of maximum tensile strain.**
- Whole **internal structure of the material changes** in such a way, that **macroscopic mechanical properties of the material** (which was originally an isotropic one) **start to exhibit properties of the anisotropy induced by the load**.

THANK YOU FOR YOUR ATTENTION