THEORY OF ELASTICITY AND PLASTICITY

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GOVERNING EQUATIONS IN THE THEORY OF PLASTICITY

GOVERNING EQUATIONS OF THE THEORY OF PLASTICITY

Quantities used for the description of **elastic-plastic deformation**:

- Displacement vector : $\mathbf{u}$
- Total strain tensor: $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{el} + \boldsymbol{\varepsilon}^{pl}$
- Elastic strain tensor: $\boldsymbol{\varepsilon}^{el}$
- Plastic strain tensor: $\boldsymbol{\varepsilon}^{pl}$
- Stress tensor: $\boldsymbol{\sigma}$
- Tensor of increment of total strain: $\mathrm{d}\boldsymbol{\varepsilon} = \mathrm{d}\boldsymbol{\varepsilon}^{el} + \mathrm{d}\boldsymbol{\varepsilon}^{pl}$
- tensor of increment of elastic strain: $\mathrm{d}\boldsymbol{\varepsilon}^{el}$
- tensor of increment of plastic strain: $\mathrm{d}\boldsymbol{\varepsilon}^{pl}$
- tensor of increment of stress: $\mathrm{d}\boldsymbol{\sigma}$

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GOVERNING EQUATIONS OF THE THEORY OF PLASTICITY

Quantities used for the description of **elastic-plastic deformation**:

- Hydrostatic stress: $p = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = \frac{1}{3}\sigma_{kk}$
- Volumetric strain: $\theta = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \varepsilon_{kk}$
- Isotropic stress tensor: $p\mathbf{1} \quad \Leftrightarrow \quad \frac{1}{3}\sigma_{kk}\delta_{ij}$
- Isotropic total strain tensor: $\frac{\theta}{3}\mathbf{1} \quad \Leftrightarrow \quad \frac{1}{3}\varepsilon_{kk}\delta_{ij}$
- Stress deviator: $\mathbf{s} = \boldsymbol{\sigma} - p\mathbf{1} \quad \Leftrightarrow \quad s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$
- Total strain deviator: $\mathbf{e} = \boldsymbol{\varepsilon} - \frac{\theta}{3}\mathbf{1} \quad \Leftrightarrow \quad e_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}$

(isotropic and deviatoric parts of tensors of increment of elastic strain, plastic strain and stress are defined in an analogous way)

GOVERNING EQUATIONS OF THE THEORY OF PLASTICITY

Governing equations of the problem of **elastic-plastic deformation**:

- Equilibrium equations: $$\sigma_{ij,j} + b_i = 0$$
- Kinematic relations: $$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$
- Yield condition: $$f\left(\boldsymbol{\sigma}\right) = 0$$
- **Constitutive relations** in the region of **elastic deformation**: $$\varepsilon_{ij} = \frac{1}{E}\left[\left(1+\nu\right)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\right]$$ $$\sigma_{ij} = 2G\varepsilon_{ij} + \Lambda\varepsilon_{kk}\delta_{ij}$$
- **Constitutive relations** in the region of **elastic-plastic deformation**: $$\boldsymbol{\varepsilon} = g\left(\boldsymbol{\sigma}\right)$$

Boundary conditions:

- Static boundary conditions: $$\sigma_{ij} n_j = q_i$$
- Kinematic boundary conditions: $$u_i = \hat{u}_i$$
- Equilibrium at the boundary of elastic and plastic region: $$\begin{cases} \sigma_{\mathbf{n}}^{(el)} = \sigma_{\mathbf{n}}^{(el-pl)} \\ \tau_{\mathbf{n}}^{(el)} = \tau_{\mathbf{n}}^{(el-pl)} \end{cases}$$

GOVERNING EQUATIONS OF THE THEORY OF PLASTICITY

In the configuration of a body we distinguish:

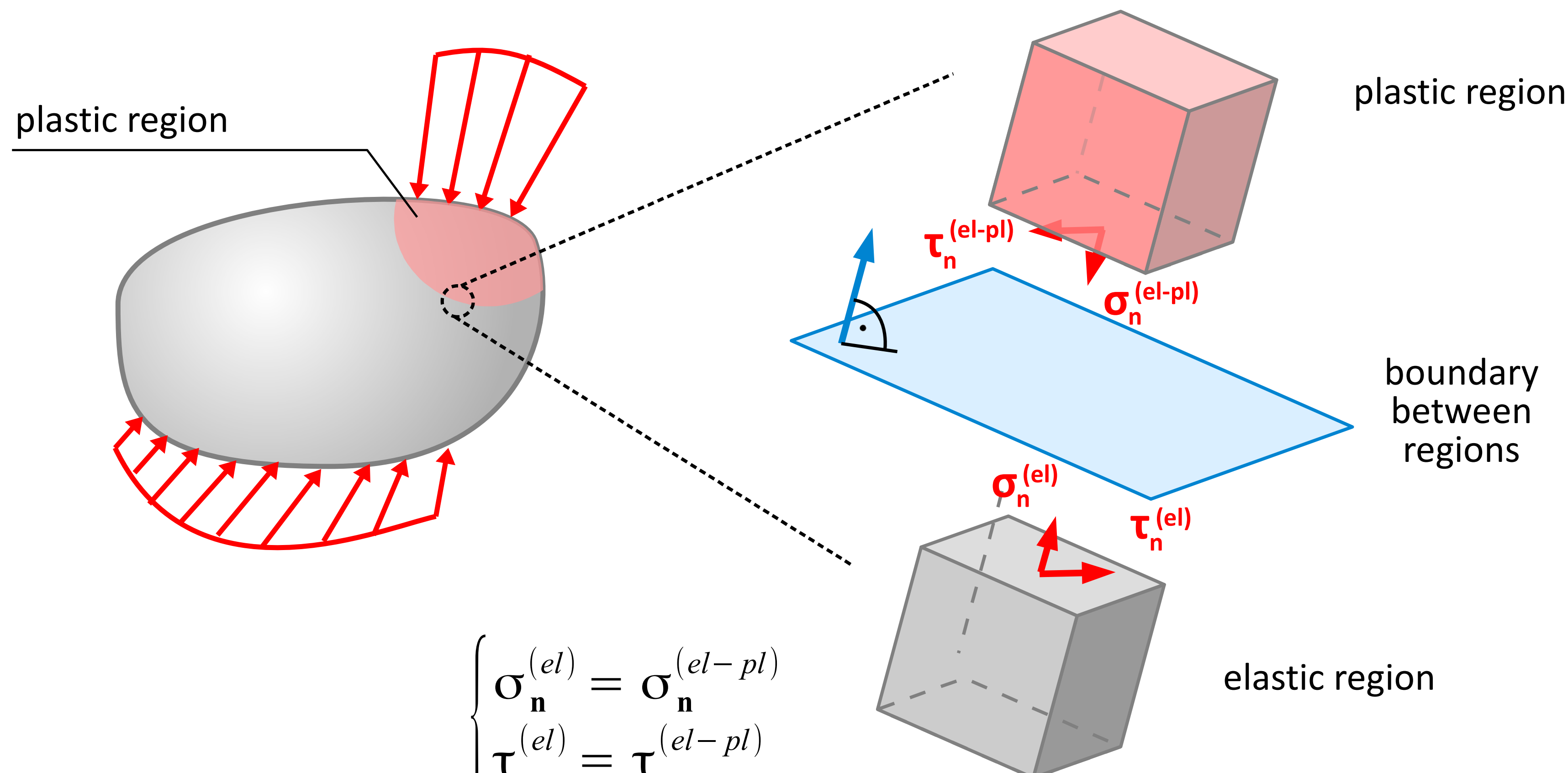
- Region of elastic deformation
- Region of elastic-plastic deformation
- Boundary between the above regions

In the **region of elastic-plastic deformation** as well as in the **points at the boundary** separating this region from the region of purely elastic deformation the yield condition **is satisfied**.

GOVERNING EQUATIONS OF THE THEORY OF PLASTICITY

Equilibrium at the boundary of elastic and plastic region:

Right-sided and left-sided values of normal stress perpendicular to the boundary as well as of shear stress tangent to the boundary must be the same.



$$\begin{cases} \sigma_{\mathbf{n}}^{(el)} = \sigma_{\mathbf{n}}^{(el-pl)} \\ \tau_{\mathbf{n}}^{(el)} = \tau_{\mathbf{n}}^{(el-pl)} \end{cases}$$

GOVERNING EQUATIONS OF THE THEORY OF PLASTICITY

REMARKS:

- **Equilibrium equations** and **kinematic relations** are **the same in both elastic and plastic region** – they are the same as in the theory of elasticity, since they are derived from fundamental principles of Newtonian dynamics and kinematics of continua.
- **Boundary conditions** at the external surface of the body are prescribed in the same way as in theory of elasticity
- It is necessary to prescribe **compatibility conditions at the boundary of elastic and plastic region**:
  - **Displacement** distribution must be **continuous** (one-sided values are the same)
  - Distribution of **stress vector components corresponding with a unit normal vector of the boundary** between elastic and plastic region must be **continuous** (one-sided values are the same)
  - Distribution of **all other components of the stress tensor** may be **discontinuous** – one-sided values corresponding with elastic region and plastic region may have different values.

GOVERNING EQUATIONS OF THE THEORY OF PLASTICITY

REMARKS:

- The most important difference concerns the **constitutive relations**.
  - Due to plastic yielding the **relation between** stress **and** total strain **changes**.
  - We assume that elastic properties of material do not change due to plastic deformation. **Relation between** stress **and** elastic strain **remains unchanged** – it is the same as in the theory of elasticity.
  - The fundamental problem in the statement of the **theory of plasticity** is a proper **choice of relation between** stress **and** plastic strain.

GOVERNING EQUATIONS OF THE THEORY OF PLASTICITY

In the theory of plasticity we distinguish **two kinds of processes of deformation**:

- active process – or the process of loading – this is an **irreversible process** – energy is dissipated due to occurrence and development of plastic strains. During an active process **both elastic strain and plastic strain tensor changes**. For active processes:
  - **Yield condition is satisfied**

$$f\left(\boldsymbol{\sigma}\right) = 0$$

  - **AND increment of the left-hand side of the yield condition** corresponding with an increment of stress is **non-negative**. (interpretation: **increment of stress results with increment of plastic strain**)

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} \cdot \mathrm{d}\boldsymbol{\sigma} \geqslant 0$$

GOVERNING EQUATIONS OF THE THEORY OF PLASTICITY

In the theory of plasticity we distinguish **two kinds of processes of deformation**:

- Passive process – process during which **energy is not dissipated**. These are processes of **elastic deformation** or processes of **unloading**. During a passive process **only elastic strain changes**. For passive processes:
  - **Yield condition is not satisfied** (material is in elastic state)

$$f(\boldsymbol{\sigma}) < 0$$

  - **OR yield condition is satisfied** but **increment of the left-hand side of the yield condition** corresponding with an increment of stress is **negative**. (interpretation: **increment of stress does not result with increment of plastic strain** – the material is in a limit state but it is the **beginning of** unloading **process**)

$$f(\boldsymbol{\sigma}) = 0 \quad \wedge \quad \frac{\partial f}{\partial \boldsymbol{\sigma}} \cdot \mathrm{d}\boldsymbol{\sigma} < 0$$

MODELS OF ASYMPTOTIC PLASTICITY

MODELS OF ASYMPTOTIC PLASTICITY

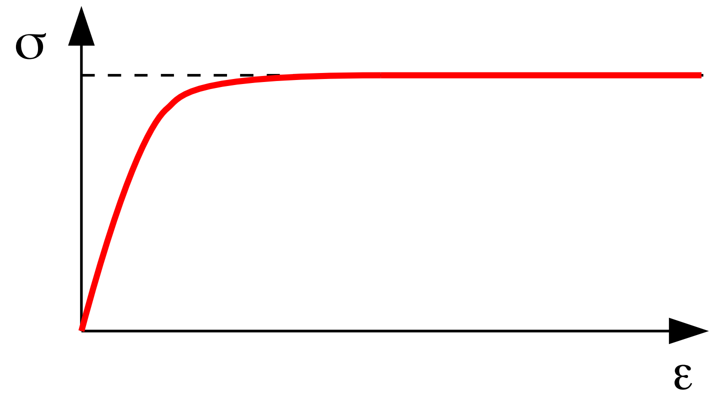
In the case of **simple mechanical states** (simple **tension**, simple **shear**) and when **loading process is monotonic**, when it is sufficient to use only a single measure of stress and a single measure of total strain, one may use so called model of asymptotic plasticity – **one-to-one relations between stress and total strain**.

These models may be considered the non-linear elastic models.

Example 1:

Prager's model

$$\varepsilon = \frac{\sigma_0}{E} \operatorname{arctgh}\left(\frac{\sigma}{\sigma_0}\right)$$



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MODELS OF ASYMPTOTIC PLASTICITY

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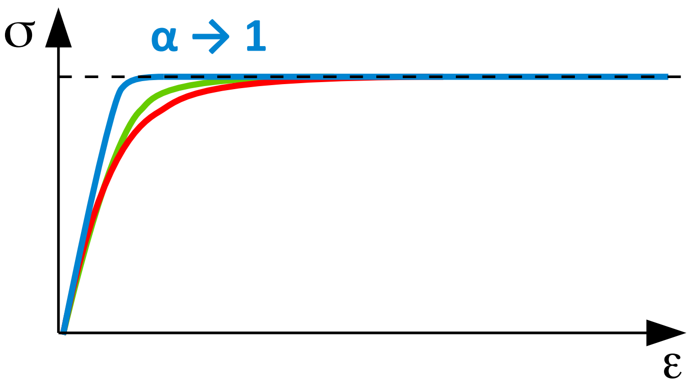
Example 2:

Ylinen's model

$$\varepsilon = \frac{1}{E}\left[\alpha\sigma - (1-\alpha)\sigma_0 \ln\left(1-\frac{\sigma}{\sigma_0}\right)\right]$$

$$\alpha \in \langle 0\,;\,1\rangle$$



MODELS OF ASYMPTOTIC PLASTICITY

In the case of **simple mechanical states** (simple **tension**, simple **shear**) and when **loading process is monotonic**, when it is sufficient to use only a single measure of stress and a single measure of total strain, one may use so called model of asymptotic plasticity **– one-to-one relations between stress and total strain**.

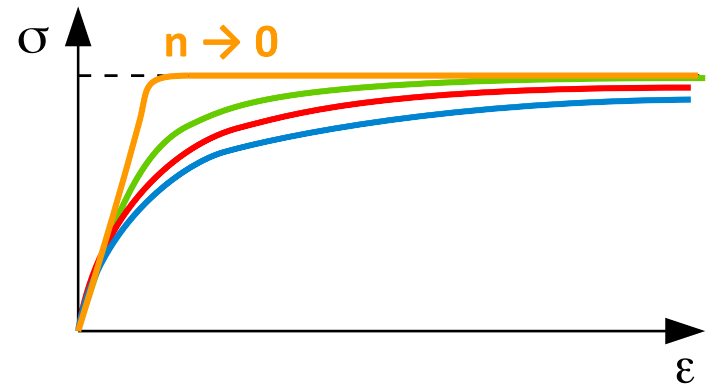
These models may be considered the non-linear elastic models.

Example 3:

Życzkowski's model

$$\varepsilon = \frac{\sigma}{E}\left(1-\frac{\sigma}{\sigma_0}\right)^{-n}$$

$$n \geqslant 0$$



DEFORMATION THEORIES OF PLASTICITY

DEFORMATION THEORIES OF PLASTICITY

Models of plasticity in which constitutive **relations are unique relations between stress and strain tensor** are termed the **deformation theories of plasticity.**

Perhaps the most popular and most commonly used theory of that kind is the **Nádai - Hencky – Ilyushin deformation theory of plasticity**.

DEFORMATION THEORIES OF PLASTICITY

Assumptions of the Nádai - Hencky – Ilyushin theory

- Volumetric strain is proportional to the hydrostatic stress and bulk modulus is the same in both elastic and plastic state

$$p = K\theta$$

$p = \frac{1}{3}(\sigma_{11}+\sigma_{22}+\sigma_{33})$ – hydrostatic stress

$\theta = \varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}$ – volumetric strain

$K = \frac{E}{3(1-2\nu)}$ – bulk modulus (Helmholtz modulus)

- **Hydrostatic stress** is **proportional** to the norm of **isotropic part of stress tensor**
- **Volumetric strain** is **proportional** to the norm of **isotropic part of strain tensor**

- The above relation is a relation between isotropic tensors (Law of the volume change)

DEFORMATION THEORIES OF PLASTICITY

Assumptions of the Nádai - Hencky – Ilyushin theory

- Stress intensity σ_i is a function solely of strain intensity ε_i ,

$$\varepsilon_i = h(\sigma_i)$$

namely, stress intensity do not depend of volumetric strain, and strain intensity do not depend on hydrostatic stress.

$$\sigma_i = \frac{1}{\sqrt{2}}\sqrt{(\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2 + (\sigma_{11}-\sigma_{22})^2 + 6(\sigma_{23}^2+\sigma_{31}^2+\sigma_{12}^2)}$$ – stress intensity

$$\varepsilon_i = \frac{1}{\sqrt{2}}\sqrt{(\varepsilon_{22}-\varepsilon_{33})^2 + (\varepsilon_{33}-\varepsilon_{11})^2 + (\varepsilon_{11}-\varepsilon_{22})^2 + 6(\varepsilon_{23}^2+\varepsilon_{31}^2+\varepsilon_{12}^2)}$$ – strain intensity

- **Stress intensity** is **proportional** to the norm of the **stress deviator**.
- **Strain intensity** is **proportional** to the norm of the **strain deviator.**

DEFORMATION THEORIES OF PLASTICITY

Assumptions of the **Nádai - Hencky – Ilyushin theory**

- **Eigenvectors of the stress tensor and strain tensor are the same** (those tensors are coaxial)
  - Each tensor may be **uniquely decompose into its isotropic and deviatoric part**.
  - **Any axis is an eigenaxis of an isotropic tensor**
  - Coaxiality of stress tensors and strain tensors is thus equivalennt to the **coaxiality of their deviatoric parts**.

CONCLUSIONS:

- Since the stress and strain deviators are coaxial and their norms are related with a certain function, then this function constitutes a relation between the deviators themselves
- Function $h(\sigma_i)$ determines a **constitutive relation between deviators** (Law of the change of shape). This function may be decomposed into a part describing elastic deformation and a part describing plasit deformation.

DEFORMATION THEORIES OF PLASTICITY

Constitutive relation in the Nádai - Hencky – Ilyushin theory:

$$\begin{cases} e_{ij} = \left(\phi(s_i) + \frac{1}{2G}\right) s_{ij} \\ p = K\theta \end{cases}$$

elastic deformation: 3[rd] form of the Hooke's Law

$$\begin{cases} e_{ij}^{e} = \frac{s_{ij}}{2G} & \text{(Law of the change of shape)} \\ p = K\theta & \text{(Law of the volume change)} \end{cases}$$

plastic deformation:

$$\varepsilon_{ij}^{pl} = e_{ij}^{pl} = \phi(s_i)\, s_{ij}$$

REMARKS:

- **Total strain** is decomposed into **elastic strain** and **plastic strain**: $\varepsilon_{ij} = \varepsilon_{ij}^{el} + \varepsilon_{ij}^{pl}$
- **Constitutive relation** for **elastic** deformation is the **generalized Hooke's Law**.
- **Constitutive relation** for **plastic** deformation is the **relation between deviators**.
- Plastic volumetric strain is zero: $\mathrm{tr}(\varepsilon_{ij}^{pl}) = 0 \quad \Rightarrow \quad \varepsilon_{ij}^{pl} = e_{ij}^{pl}$
- Function ϕ is equal:

$$\begin{cases} \phi(s_i) > 0 & \text{for active processes} \\ \phi(s_i) = 0 & \text{for passive processes} \end{cases}$$

DEFORMATION THEORIES OF PLASTICITY

REMARKS:

- In the case of **unloading after yielding** we determine the increment of stress and strain with respect to the state in which the unloading began. Those increments are calculated with the use of derived constitutive relation after substituting $\phi = 0$ and then they are added to the state in which unloading began.

- **It is impossible to describe multiple loading-unloading processes with the use of deformation theories:**
  - Let's assume that the body was deformed plastically.
  - Then the material is unloaded in such a way that the yield condition is not satisfied ina ny point. Plastic strain remain present.
  - Then the material is loaded again in such a way that the stress deviator in the limit state is different than when the material was yielding previously. According to the constitutive relation this new deviator determines new plastic strains – but they are different than those, which were obtained previously. Such a solution has no physical sense.

DEFORMATION THEORIES OF PLASTICITY

REMARKS:

- **Deformation theories may be used in the cases of monotonic processes of elastic-plastic deformation** – these are processes in which all ratios of components of stress and strain tensors remain unchanged and only norm of those tensors varies.

- **Deformation theories are simpler in analysis and less computationally complex than incremental models.**

INCREMENTAL THEORIES OF PLASTICITY (FLOW THEORIES)

INCREMENTAL THEORIES OF PLASTICITY

An alternative is to use incremental models. Any **constitutive relation** which has a general form:

$$\varepsilon = F(\sigma)$$

may be rewritten in the **incremental form**:

$$\mathrm{d}\varepsilon = \underbrace{\frac{\mathrm{d}F(\sigma)}{\mathrm{d}\sigma}}_{G}\mathrm{d}\sigma = G(\sigma, \mathrm{d}\sigma)$$

In incremental models it is assumed that:

Increment of plastic strain depend only on actual stress state and it is independent of the increment of stress

- Magnitude of increment of plastic strain for a given load depend on the magnitude of stress and orientation of stress with respect to possible slip or twinning planes.
- It does not depend on the rate of stress (magnitude of stress increment in given instant of time)
- Plastic strains in next instant of time will depend again on new stress state (nad not on increments)

INCREMENTAL THEORIES OF PLASTICITY

General form of the **constitutive relation** in incremental theories of plasticity (flow theories) is as follows:

$$\mathrm{d}\,\varepsilon_{ij}^{pl} = \mathrm{d}\lambda \frac{\partial \Psi}{\partial \sigma_{ij}}$$

where:

$\Psi(\boldsymbol{\sigma})$ – plastic potential

$\mathrm{d}\lambda$ – **scalar parameter**, which depends on mechanical properties of the material as well as on the history of the process of deformation

$$\begin{cases} \mathrm{d}\lambda > 0 & \text{for active processes} \\ \mathrm{d}\lambda = 0 & \text{for passive processes} \end{cases}$$

- This is a relation between **increment of plastic strain** and **stress**.
- Plastic potential plays a similar role in constitutive relations in plasticity as elastic potential in elasticity.

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INCREMENTAL THEORIES OF PLASTICITY

General form of the **constitutive relation** in incremental theories of plasticity (flow theories) is as follows:

$$\mathrm{d}\varepsilon_{ij}^{pl} = \mathrm{d}\lambda \frac{\partial \Psi}{\partial \sigma_{ij}}$$

- If the **plastic potential has the same mathematical form as the yield condition**

$$\Psi(\boldsymbol{\sigma}) = f(\boldsymbol{\sigma})$$

  then we're speaking of the associated flow rule.

- For the case of an **associated flow rule** the following uniqueness theorem is true:

> If the strain is sufficiently small, then for an incremental model with an **associated flow rule** for **given static boundary conditions** the **distribution of stress in an elastic-plastic solid** is unique.

INCREMENTAL THEORIES OF PLASTICITY

Most commonly used incremental model of plasticity is the Prandtl – Reuss flow theory.

- we assume an **associated flow rule:**
- we assume the **Huber – Mises yield condition:**

$$\Psi(\boldsymbol{\sigma}) = f(\boldsymbol{\sigma})$$

$$f(\boldsymbol{\sigma}) = \sigma_{eq} - \sigma_0 = \sqrt{\frac{3}{2} s_{ij} s_{ij}} - \sigma_0$$

Increment of plastic strain:

$$\mathrm{d}\varepsilon_{ij}^{pl} = \mathrm{d}\lambda \frac{\partial \Psi}{\partial \sigma_{ij}} = \mathrm{d}\lambda\, s_{ij}$$

Increment of elastic strain:

$$\mathrm{d}\varepsilon_{kk}^{el} = \frac{1}{3K}\mathrm{d}\sigma_{kk}$$

$$\mathrm{d}e_{ij}^{el} = \frac{1}{2G}\mathrm{d}s_{ij}$$

increment of plastic strain is proportional to the stress deviator

Constitutive relation in the Prandtl – Reuss theory:

$$\begin{cases} \mathrm{d}\varepsilon_{ij} = \mathrm{d}\lambda\, s_{ij} + \dfrac{1}{2G}\mathrm{d}s_{ij} \\ \mathrm{d}\varepsilon_{kk} = \dfrac{1}{3K}\mathrm{d}\sigma_{kk} \end{cases}$$

INCREMENTAL THEORIES OF PLASTICITY

A special case of the Prandtl – Reuss theory is an earlier **Lévy – Mises flow theory**. Constitutive relation for plastic strain is the same, however it concerns the **rigid-plastic solid**, in which elastic strain are zero.

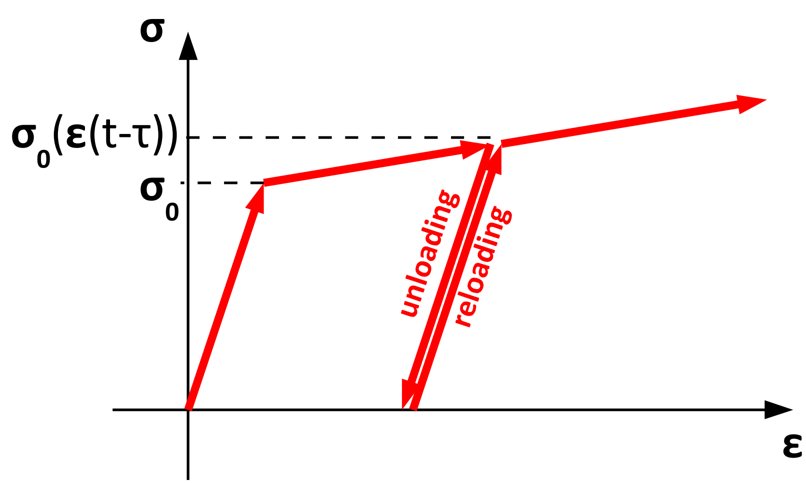
$$\mathrm{d}\,\varepsilon_{ij}^{pl} = \mathrm{d}\,\lambda\, s_{ij}$$

This theory is simpler in analysis and it is often used in numerical simulation, especially in those cases in which **elastic deformation is much smaller than plastic deformation** (e.g. industrial **plastic forming processes of metals**)

MODELS OF HARDENING

MODELS OF HARDENING

The phenomenon of hardening is a situation in which **it is necessary to apply larger stress in order to continue plastic deformation**. The reason is that dislocations of similar type block their motion when they are concentrated in a small region.



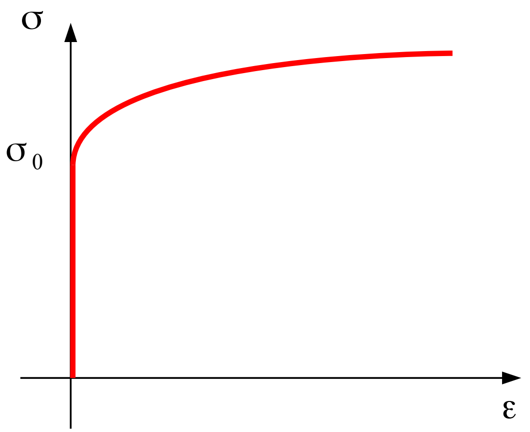
- After unloading of a material which has already undergone plastic deformation with hardening, when it is loaded again, **plastic yielding occurs for the higher value of stress**.
- In this sense, modelling of hardening may be done by the assumption that the **yield condition is not constant**, but that it **varies depending on what was the history of deformation**.
- Modelling of such kind may be graphically illustrated by the fact, that the **yield surface moves and changes its shape** in the space of principal stresses.

MODELS OF HARDENING

Phenomenon of hardening may be modelled in simple load cases and for monotonic load with the use of **empirical formulae**:

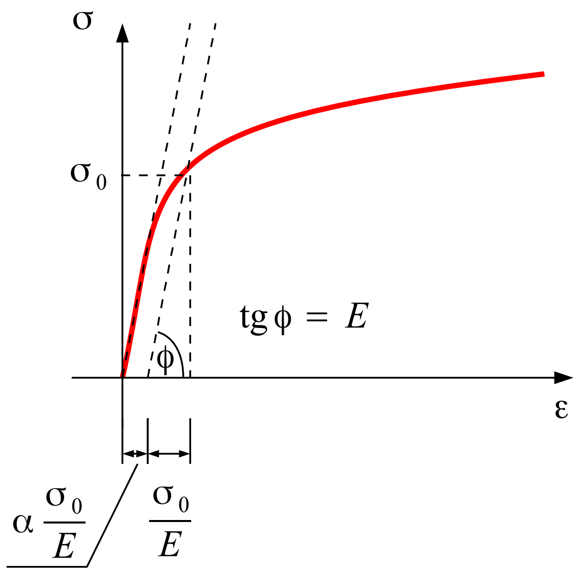
- Ludwik model

$$\sigma = \sigma_0 + K\left(\varepsilon^{pl}\right)^n$$



- Ramberg – Osgood model:

$$\varepsilon = \frac{\sigma}{E} + \alpha\frac{\sigma}{E}\left(\frac{\sigma}{\sigma_0}\right)^{n-1}$$



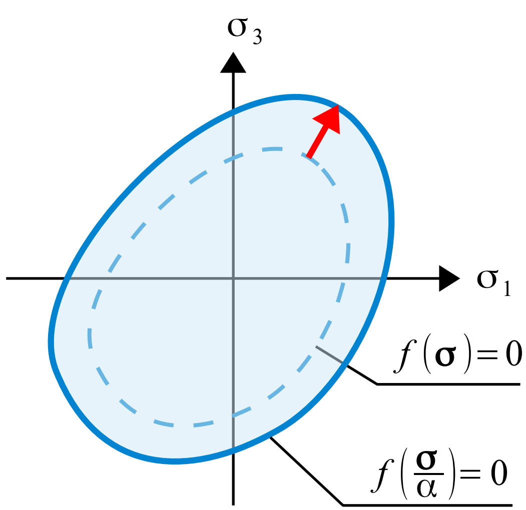
MODELS OF HARDENING

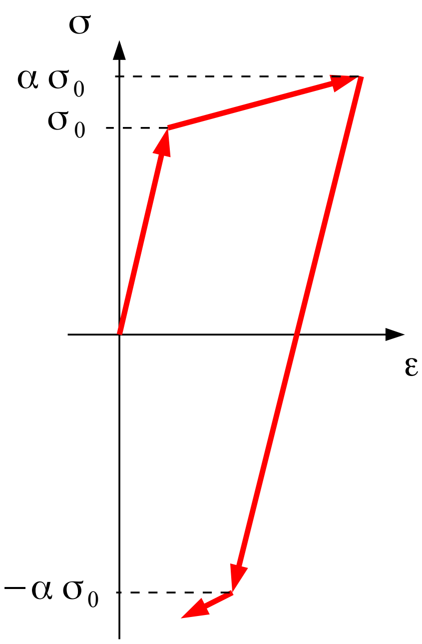
ISOTROPIC HARDENING:

- **Yield surface changes its dimensions uniformly in all directions**, but it **preseves its shape**.

$$f(\boldsymbol{\sigma}) = 0 \qquad \rightarrow \qquad f\left(\frac{\boldsymbol{\sigma}}{\alpha}\right) = 0 \; , \quad \alpha \geqslant 1$$

- **Absolute values of a limit stress in opposite stress states increase equally**.





MODELS OF HARDENING

ISOTROPIC HARDENING:

Yield condition for a material exhibiting hardening may be expressed in the following form:

$$f(\boldsymbol{\sigma}) = c$$

Parameter c takes into account the **history of deformation**. Following models are commonly discussed:

- Taylor – Quinney **isotropic hardening model**
  - Hardening rate depends of the **work of stresses along plastic strains**

$$c = c(W_{pl}) \qquad \text{where} \qquad W_{pl} = \int \sigma_{ij} \, \mathrm{d}\,\varepsilon_{ij}^{pl}$$

- Odquist – Hill **isotropic hardening model**
  - Hardening rate depends of the **total length of the plastic strain-path**

$$c = c(d_{pl}) \qquad \text{where} \qquad d_{pl} = \int \sqrt{\mathrm{d}\,\varepsilon_{ij}^{pl} \, \mathrm{d}\,\varepsilon_{ij}^{pl}}$$

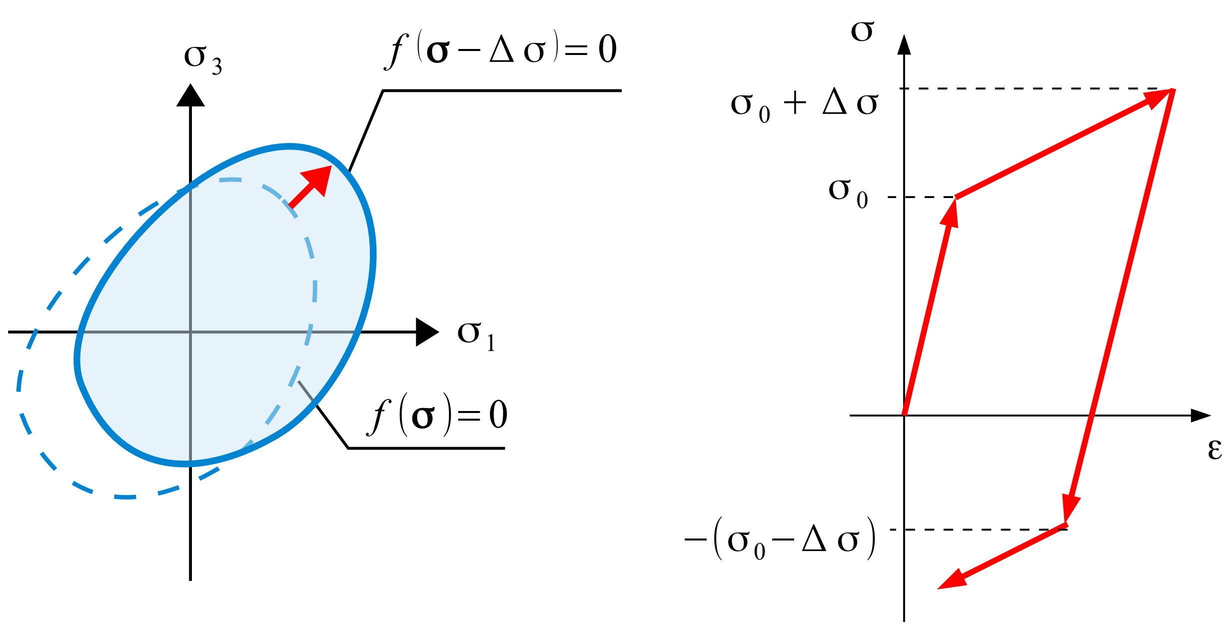
MODELS OF HARDENING

KINEMATIC HARDENING:

- **Yield surface moves in the space of principal stresses** but it **preserves its shape and size**.

$$f(\boldsymbol{\sigma}) = 0 \qquad \rightarrow \qquad f(\boldsymbol{\sigma} - \Delta\boldsymbol{\sigma}) = 0$$

- If the **absolute value of a limit stress in a certain state increases**, then the **absolute value of the limit stress in an opposite state decreases by the same magnitude**.



MODELS OF HARDENING

KINEMATIC HARDENING:

Yield condition for a material exhibiting hardening may be expressed in the following form:

$$f(\boldsymbol{\sigma}-\boldsymbol{\alpha}) = 0$$

Quantity $\boldsymbol{\alpha}$ **may take into account the history of deformation** and it may be determined according to various models, e.g.:

- **Prager model of kinematic hardening**

$$\mathrm{d}\boldsymbol{\alpha} = c\,\mathrm{d}\mathbf{e}^{pl} \quad \text{where} \quad c = \text{const.}$$

c is a constant, which is characteristic for the considered material

- **Ziegler model of kinematic hardening**

$$\mathrm{d}\boldsymbol{\alpha} = (\boldsymbol{\sigma}-\boldsymbol{\alpha})\,\mathrm{d}\mu \quad \text{where} \quad \mathrm{d}\mu = \mathrm{d}\mu(\mathrm{d}\mathbf{e}^{pl})$$

$\mathrm{d}\mu$ is a function of plastic strain increment – it is characteristic for considered material.

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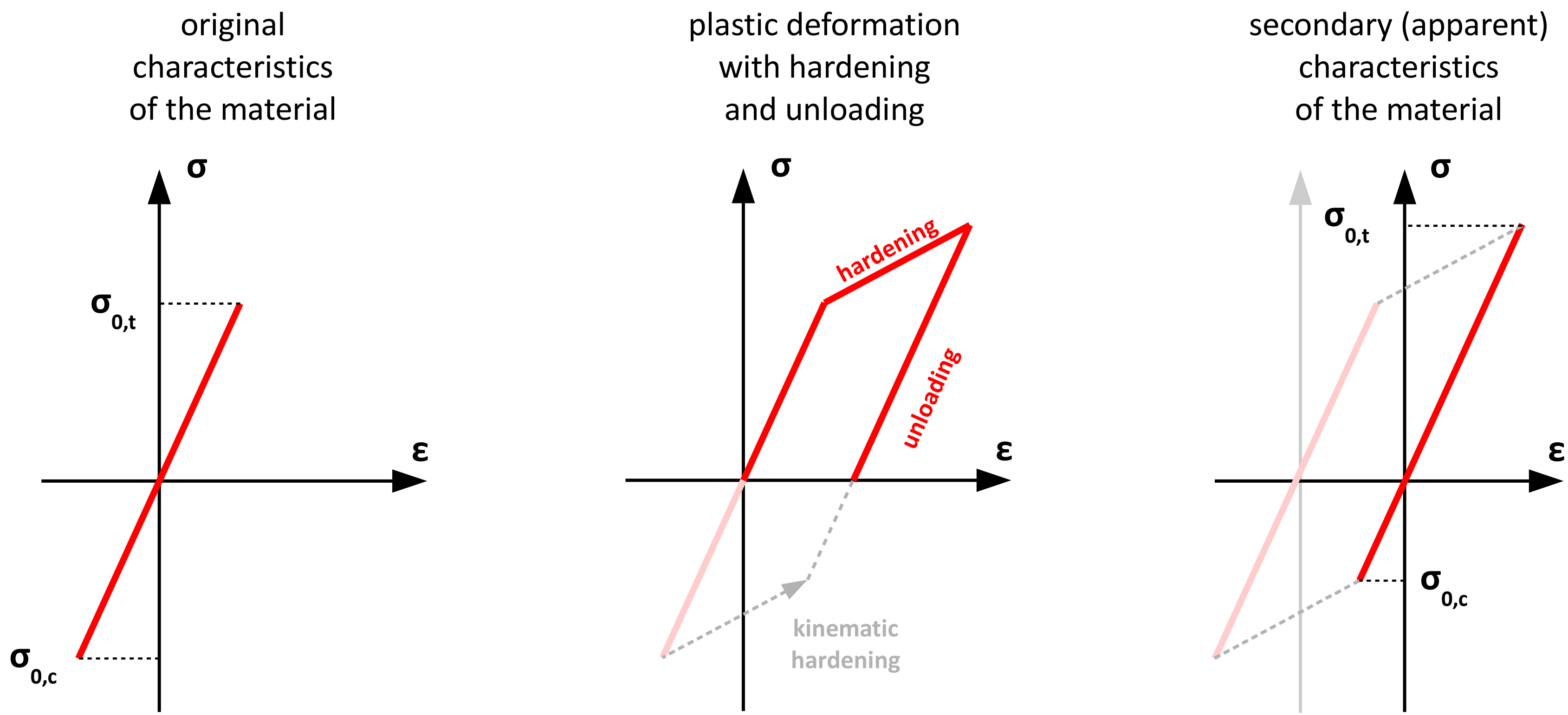
MODELS OF HARDENING

Bauschinger effect – due to plastic deformation with unloading **residual stresses** occur in the material – they correspond with a new state of equilibrium in the inhomogeneous transformed internal structure of material. Presence of residual stresses make the **mechanisms of plastic deformation be initiated for different values of limit stress** than originally, before plastic deformation.



MODELS OF HARDENING

MIXED HARDENING – it is a **composition of isotropic and kinematic hardening** model

$$f(\boldsymbol{\sigma}) = 0 \qquad \rightarrow \qquad f\left(\frac{\boldsymbol{\sigma}}{\alpha} - \Delta\boldsymbol{\sigma}\right) = 0$$

ANISOTROPIC HARDENING – **yield surface changes its shape and position** in the space of principal stresses.

MATERIAL STABILITY

MATERIAL STABILITY

Drucker's stability postulate:

In a stable material the **work performed by the increment of surface tractions** $\Delta\mathbf{q}$ and **increment of body forces** $\Delta\mathbf{b}$ a**long corresponding increment of displacements** $\Delta\mathbf{u}$ is non-negative.

$$W = \int_{t_0}^{t_k} \left[\iint_S (\Delta\mathbf{q}\,\Delta\dot{\mathbf{u}})\,\mathrm{d}S + \iiint_V (\Delta\mathbf{b}\,\Delta\dot{\mathbf{u}})\,\mathrm{d}V \right] \mathrm{d}t \geqslant 0$$

- If the **considered interval of time is arbitrary** (**finitely**) **large**, then we speak of “stability in large”
- If the **considered interval of time is infinitely small**, tj. $(t_k - t_0) \to 0$, then we speak of “stability in small”

MATERIAL STABILITY

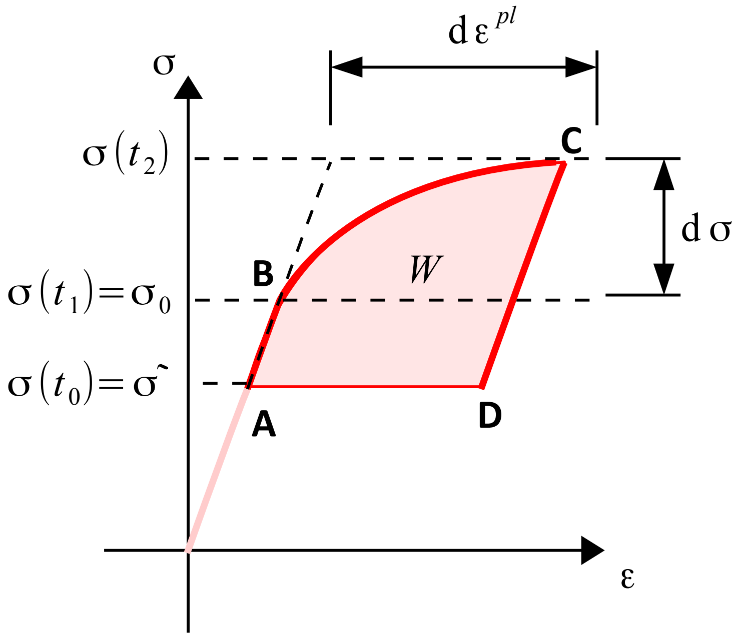
Let's consider a **cycle of loading with unloading**:

1) In instant t_0 a stress state $\boldsymbol{\sigma}^{\sim}$ is given. It corresponds with initial external loads. The process of **loading with an additional load begins**.

2) In instant t_1 the yield condition is satisfied and new plastic strain occurs.

3) In instant t_2 **additional loading is stopped and unloading process begins**.

4) **Unloading process is continued until actual stress state is the same as the initial one** $\boldsymbol{\sigma}^{\sim}$. An instant in which this state is reached is denoted with t_k



MATERIAL STABILITY

According to the **Principle of Virtual Works** it may be proven that:

- condition of ”stability in large” **is satisfied when**:

$$W_{pl} = \int_{t_1}^{t_2} \left[(\sigma_{ij} - \tilde{\sigma}_{ij}) \dot{\varepsilon}_{ij}^{pl} \right] \mathrm{d}t \geqslant 0$$

- condition of ”stability in small” **is satisfied when**:

$$(\sigma_{ij} - \tilde{\sigma}_{ij}) \, \mathrm{d}\varepsilon_{ij}^{pl} \geqslant 0$$

if initially the material is in an **elastic state**

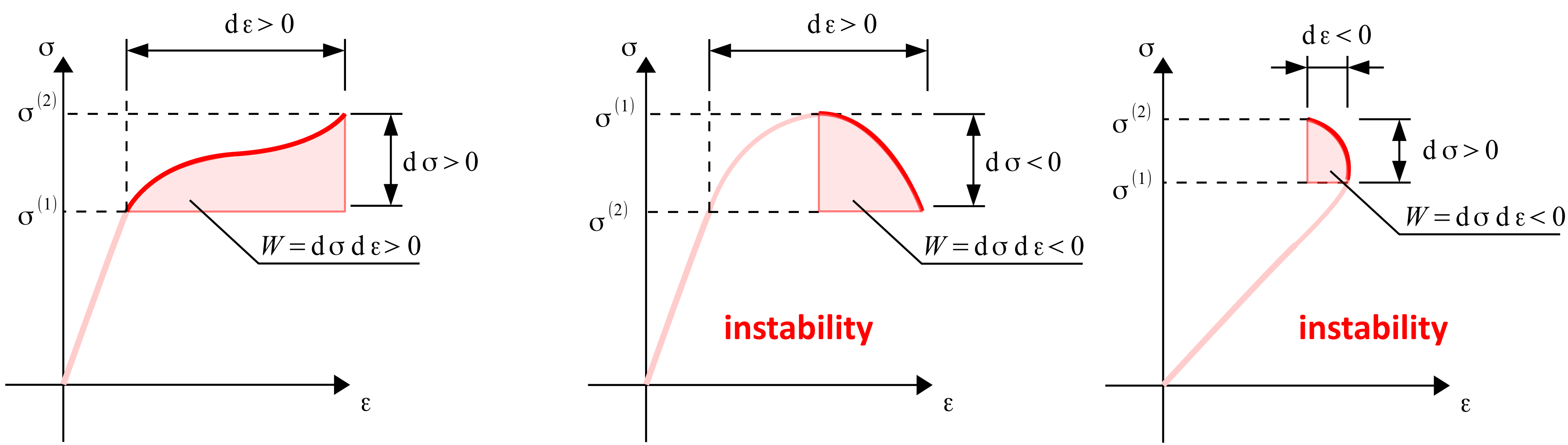
$$\mathrm{d}\sigma_{ij} \, \mathrm{d}\varepsilon_{ij}^{pl} \geqslant 0$$

if initially the material is in an **elastic - plastic state**

MATERIAL STABILITY

Condition of "stability in small" when the material is **initially in the elastic – plastic** state

$$d\sigma_{ij} d\varepsilon_{ij}^{pl} \geqslant 0$$



REMARK: Occurrence of the **lower yield stress** is a manifestation of a **local instability** in the sense of Drucker.

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Assuming that **Drucker's stability postulate is satisfied**, the **inequality of the condition for the "stability in small"** may be interpreted in a following way:

$$(\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}})\cdot \mathrm{d}\,\boldsymbol{\varepsilon}^{pl} = |\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}}||\mathrm{d}\,\boldsymbol{\varepsilon}^{pl}|\cos\sphericalangle\left[(\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}})\,;\mathrm{d}\,\boldsymbol{\varepsilon}^{pl}\right] \geqslant 0$$

Norms of tensors are positive, so it must be:

$$\sphericalangle\left[(\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}})\,;\mathrm{d}\,\boldsymbol{\varepsilon}^{pl}\right] \in \left\langle -\frac{\pi}{2}\;;\;\frac{\pi}{2}\right\rangle$$

namely, an angle in the space of stresses* between the **increment of the stress tensor** (from the **initial elastic state** $\tilde{\boldsymbol{\sigma}}$ to the **limit state** $\boldsymbol{\sigma}$ **on the yield surface**) and **tensor of increment of plastic strain** $\mathrm{d}\,\boldsymbol{\varepsilon}^{pl}$ must be less or equal to the right angle.

* in order to represent the strain tensor as a vector in the space of stresses it must be multiplied by a certain constant reference magnitude of stress (e.g. 1 Pa) to provide it with an appropriate physical dimension.

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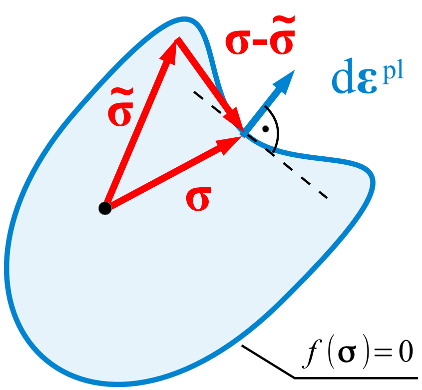
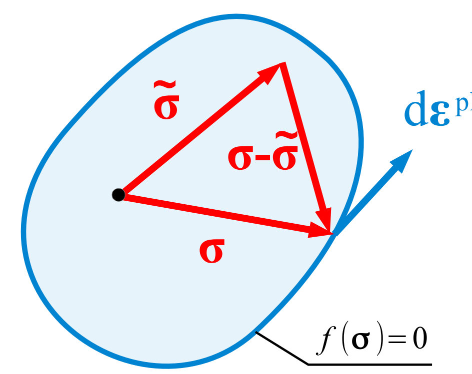
Such a condition will be satisfied if **following two conditions are satisfied simultaneously**:

- **yield surface must be convex**
- **plastic strain increment tensor** must be **perpendicular** (orthogonal) **to the yield surface**, namely it must be **parallel to the gradient of the yield condition** (normality rule)

$$d\,\varepsilon_{ij}^{pl} = d\,\lambda \frac{\partial f}{\partial \sigma_{ij}}$$



material's instability (in the sense of Drucker)
for a concave yield surface



material's instability (in the sense of Drucker)
for the tensor of increment of plastic strain which is non-orthogonal to the yield surface

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REMARKS:

- If a **flow theory of plasticity with an associated flow rule** is used, then the **normality rule is always satisfied**

$$\mathrm{d}\,\varepsilon_{ij}^{pl} = \mathrm{d}\,\lambda \frac{\partial f}{\partial \sigma_{ij}}$$

  If **additionally** the **yield surface is convex**, then the **material is stable in the sense of Drucker**.

- If we **assume the material to be stable in the sense of Drucker** and we want to describe it with the use of a **flow theory of plasticity**, then **it is necessary to use an associated flow rule** and the **yield surface must be convex**.

THANK YOU FOR YOUR ATTENTION