THEORY OF ELASTICITY AND PLASTICITY

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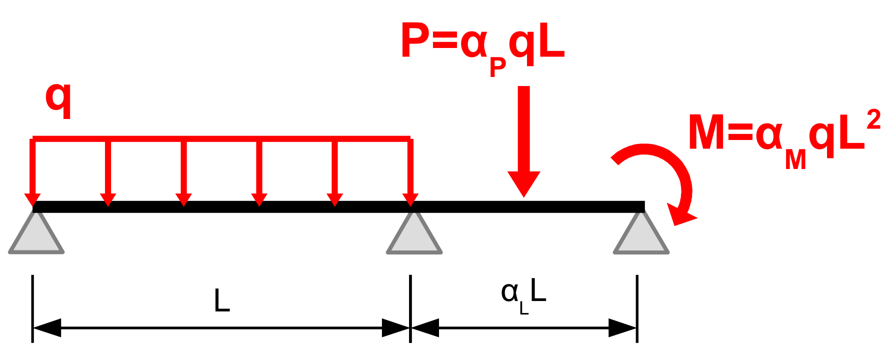
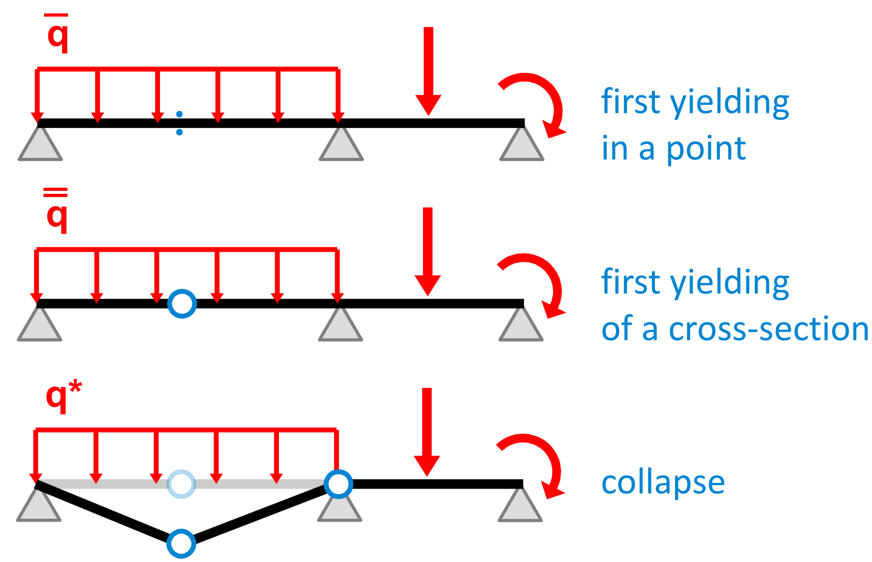
LIMIT BEARING CAPACITY

limit elastic bearing capacity $\bar{q}$ – the **value of a load parameter** q, for which the **first yielding in a point** occurs in the system.

limit plastic bearing capacity $\bar{\bar{q}}$ – the **value of a load parameter** q, for which the **first yielding of a cross-section** occurs in the system.

limit bearing capacity q^* – the **value of a load parameter** q, for which a **stable mechanical system becomes unstable** (becomes a **mechanism**)





ELASTIC-PLASTIC DEFORMATION OF SOLID BARDS

TENSION

TENSION

ASSUMPTIONS:

- **uniaxial stress state** both in elastic state and in elastic-plastic state.
- **uniform stress distribution in each cross-section** in both states.
- **yielding** occurs **simultaneously in all points.**
- within **elastic** range the **Hooke's Law** is valid.
- within **plastic** range the material exhibits **no hardening**.

Within elastic range:

stress tensor:

$$\boldsymbol{\sigma} = \frac{N}{A}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

elastic strain tensor:

$$\boldsymbol{\varepsilon}^{el} = \frac{N}{EA}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & -\nu \end{bmatrix}$$

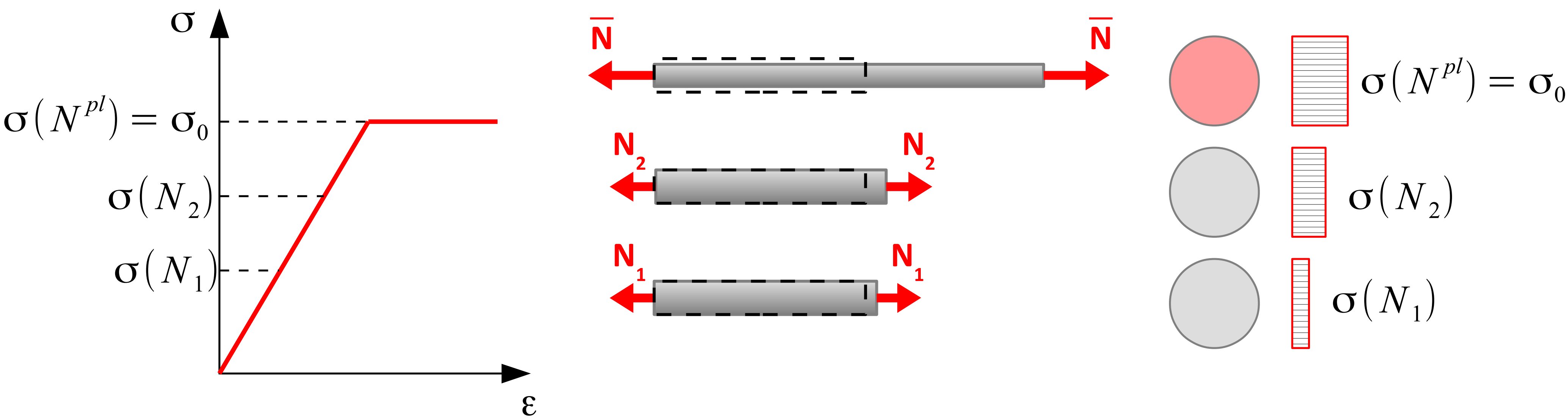
plastic strain tensor:

$$\boldsymbol{\varepsilon}^{pl} = \mathbf{0}$$

total strain tensor:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{el} + \boldsymbol{\varepsilon}^{pl} = \boldsymbol{\varepsilon}^{el}$$

TENSION



REMARKS:

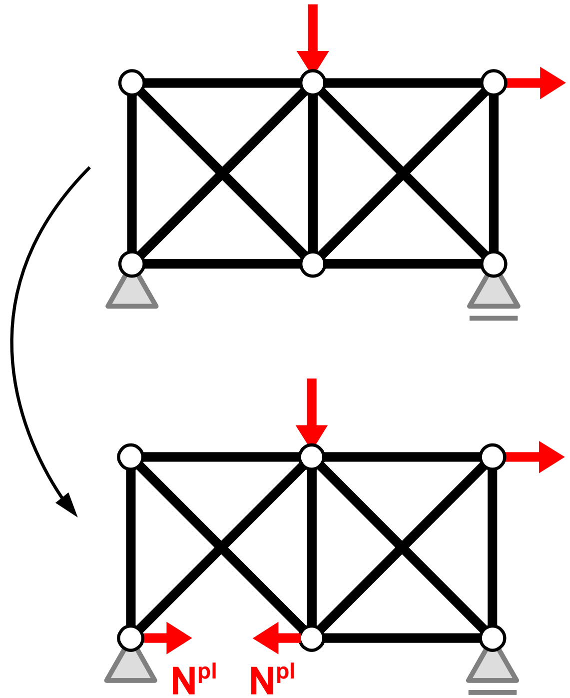
- Yielding occurs simultaneously in all point of cross-section and of a bar.
- Limit elastic bearing capacity = limit plastic bearing capacity: $\bar{N} = \bar{\bar{N}}$

TENSION

REMARKS:

- after yielding:
  - **the bar has zero longitudinal stiffness** (free elongation)
  - **the bar cannot transmit larger force than** $N^{pl} = \sigma_0 A$, it will only elongate further
  - **stress is remains constant**, equal to the yield stress: $\sigma = \sigma_0$
  - **elastic strain remains constant**, equal to: $\varepsilon^{el} = \sigma_0 / E$
  - only plastic strain increases

- In order to **model an elastic-plastic deformation** in systems, in which truss bars yield, one should:
  - **remove the bar from the system** (free deformation → zero stiffness → no contribution to the system stiffness)
  - **Introduce two opposite forces** parallel to the axis of the bar, and applied at end nodes of the bar. Magnitude of those forces is equal to the limit plastic bearing capacity of the section (the bar transmits force, and when there is no hardening the magnitude of this force remains constant after yielding).



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ELASTIC-PLASTIC DEFORMATION OF SOLID BARDS

BENDING

BENDING

ASSUMPTIONS:

- **linear distribution of strain** (**Bernoulli's hypothesis of** plane cross-sections) both in elastic state and in elastic-plastic state.
- within **elastic** range the **Hooke's Law is valid**.
- within **plastic** range the material exhibits **no hardening**.

Within elastic range**:**

stress tensor:

$$\boldsymbol{\sigma} = \frac{M}{I}\begin{bmatrix} x_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

elastic strain tensor:

$$\boldsymbol{\varepsilon}^{el} = \frac{M}{EI}\begin{bmatrix} x_3 & 0 & 0 \\ 0 & -\nu x_3 & 0 \\ 0 & 0 & -\nu x_3 \end{bmatrix}$$

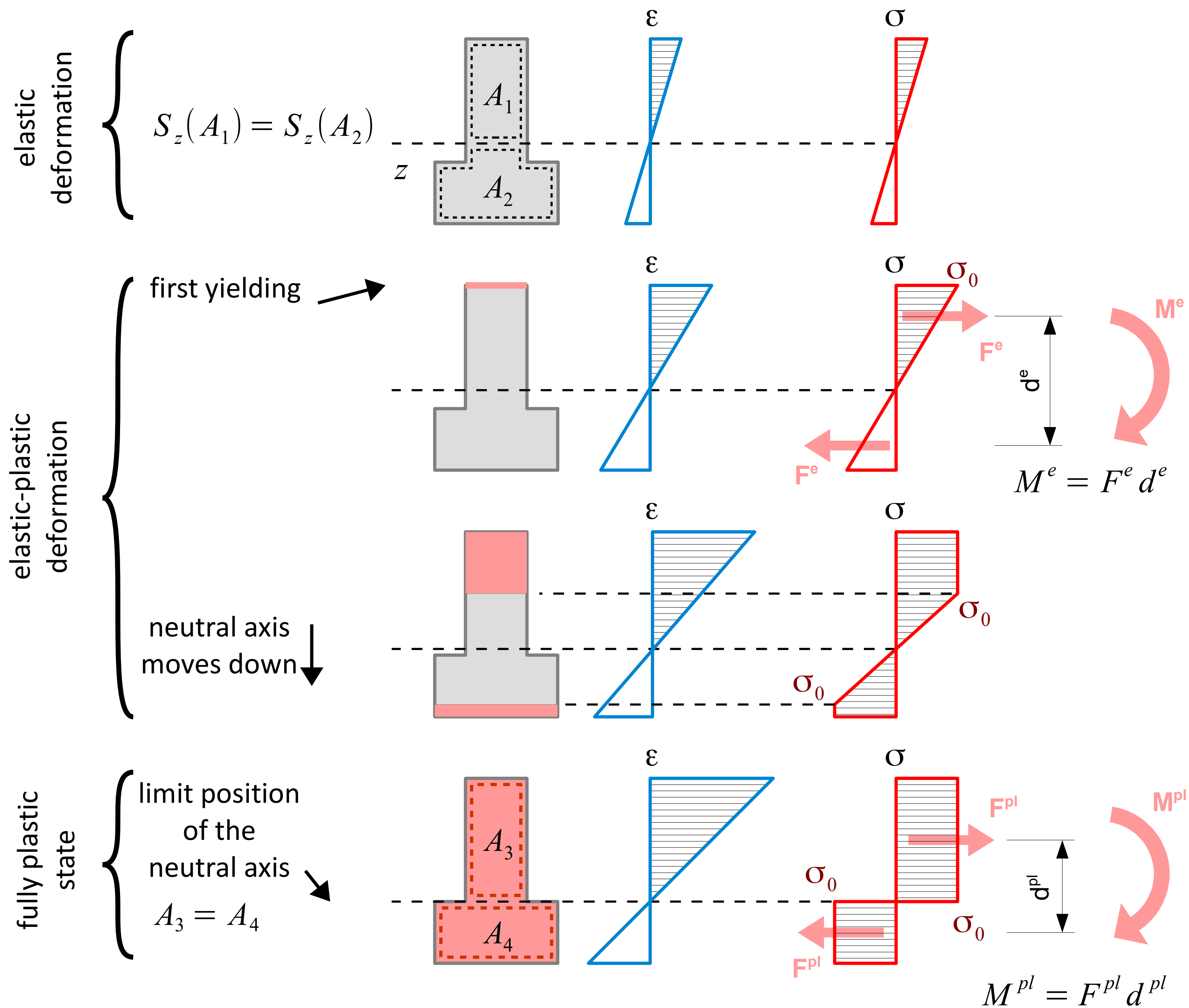
plastic strain tensor:

$$\boldsymbol{\varepsilon}^{pl} = \mathbf{0}$$

total strain tensor:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{el} + \boldsymbol{\varepsilon}^{pl} = \boldsymbol{\varepsilon}^{el}$$

BENDING


elastic deformation
$S_z(A_1) = S_z(A_2)$
z
A_1
A_2
ε
σ
elastic-plastic deformation
first yielding
σ_0
M^e
F^e
d^e
$M^e = F^e d^e$
neutral axis moves down
fully plastic state
limit position of the neutral axis
$A_3 = A_4$
A_3
A_4
F^{pl}
M^{pl}
d^{pl}
$M^{pl} = F^{pl} d^{pl}$

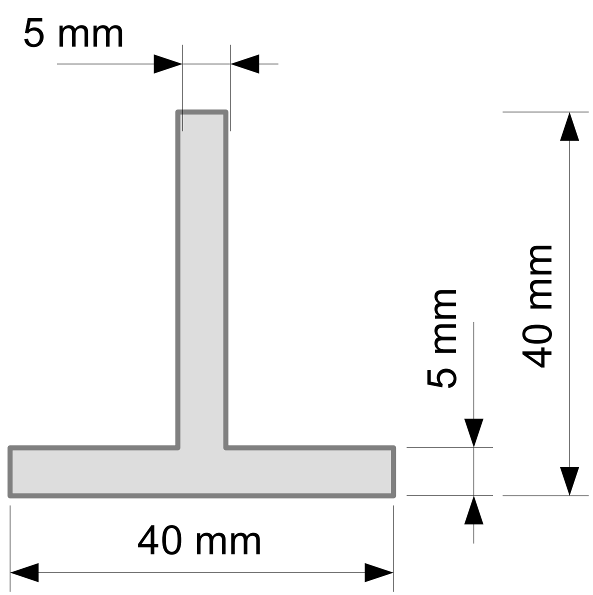
BENDING

REMARKS:

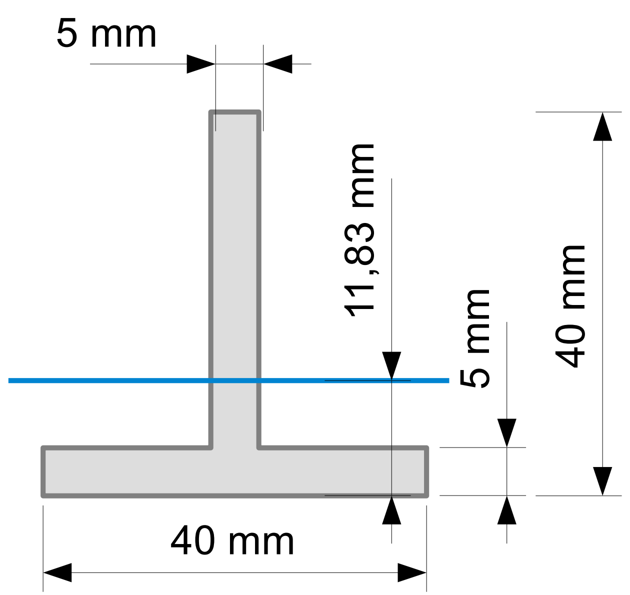
- Within thee **elastic** range the **neutral axis passes through the centroid**.
- **First yielding** occurs in a fibre which is **most distant from the neutral axis**.
- Limit elastic bearing capacity: $\bar{M} = W \sigma_0$
- Plastic zone advances **from the top and from the bottom**.
- In non-symmetric cross-sections the neutral axis changes its position.
- In case of **fully plastic state** the cross-section is divided into **two zones** (compressed and stretched) **which are loaded uniformly with a normal stress equal to the** yield stress**:** σ_0
  - **axial normal stress distribution is discontinuous** (there is a step from $-\sigma_0$ to σ_0)
  - **surface areas of both zones are equal** – according to this property the limit plastic bearing capacity of a bent section is determined.

ELASTIC AND PLASTIC LIMIT BEARING CAPACITY OF SECTION - EXAMPLE

Find the limit elastic and limit plastic bearing capacity of the T-section shown in the figure. Yield stress is equal to 200 MPa:



ELASTIC AND PLASTIC LIMIT BEARING CAPACITY OF SECTION - EXAMPLE



$$\bar{M} = W\sigma_0 = 337{,}5\ \text{Nm}$$

area of cross-section:

$$A = [40\cdot 5] + [35\cdot 5] = 375\ \text{mm}^2$$

statical moment:

$$S = \left[40\cdot 5\cdot \frac{5}{2}\right] + \left[35\cdot 5\cdot\left(5+\frac{35}{2}\right)\right] = 4437{,}5\ \text{mm}^3$$

centroid:

$$y = \frac{S}{A} \approx 11{,}83\ \text{mm}$$

2nd moment of area:

$$I = \left[\frac{40\cdot 5^3}{12} + 40\cdot 5\cdot\left(\frac{5}{2}-y\right)^2\right] + \left[\frac{5\cdot 35^3}{12} + 35\cdot 5\cdot\left(5+\frac{35}{2}-y\right)^2\right] = 47531{,}25\ \text{mm}^4$$

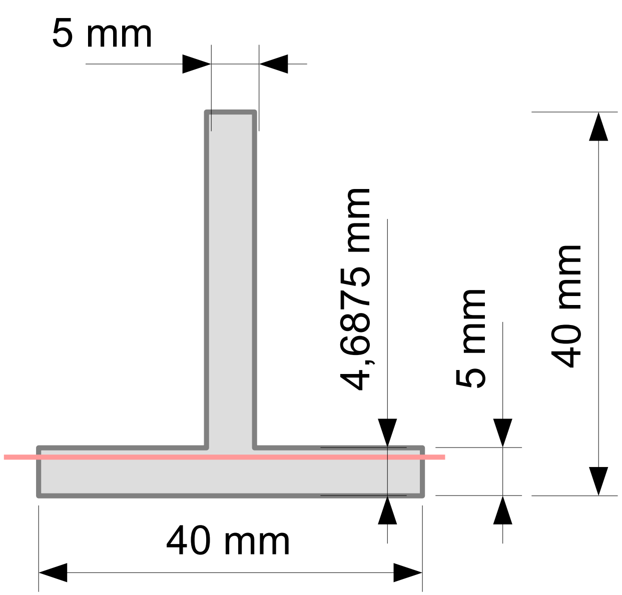
Most distant fibre:

$$y_{max} = \max(y\ ,\ 40-y) \approx 28{,}17\ \text{mm}$$

Resistance moment:

$$W = \frac{I}{y_{max}} = 1687{,}5\ \text{mm}^3$$

ELASTIC AND PLASTIC LIMIT BEARING CAPACITY OF SECTION - EXAMPLE



$$\bar{\bar{M}} = \sigma_0 A_d d_{pl} = \sigma_0 A_g d_{pl}$$

$$\bar{\bar{M}} = 711{,}72\ \text{Nm}$$

division into two equal zones:

$$40\cdot y_{pl} = 35\cdot 5 + 40\cdot(5 - y_{pl}) \quad \Rightarrow \quad y_{pl} = 4.6875\ \text{mm}$$

areas of both zones:

$$A_d = [40\cdot y_{pl}] = 187{,}5\ \text{mm}^2$$

$$A_g = [35\cdot 5 + 40\cdot(5 - y_{pl})] = 187{,}5\ \text{mm}^2$$

statical moments:

$$S_d = \left[40\cdot y_{pl}\cdot\frac{y_{pl}}{2}\right] \approx 439{,}45\ \text{mm}^3$$

$$S_g = \left[35\cdot 5\cdot\left(5 + \frac{35}{2}\right)\right] + \left[40\cdot(5 - y_{pl})\cdot\left(y_{pl} + \frac{5 - y_{pl}}{2}\right)\right] \approx$$

$$\approx 3998{,}05\ \text{mm}^3$$

centroids:

$$y_d = \frac{S_d}{A_d} \approx 2{,}34\ \text{mm} \qquad y_g = \frac{S_g}{A_g} \approx 21{,}32\ \text{mm}$$

distance between resultants:

$$d_{pl} = y_g - y_d \approx 18.98\,\text{mm}$$

PERMANENT STRAIN AND RESIDUAL STRESS - EXAMPLE

A bent **rectangular section** of dimensions b x h is given. Find:

- **limit elastic bearing capacity** of the bent cross-section:
- **limit plastic bearing capacity** of the bent cross-section:
- value of the **moment M_1**, for which **1/3 of total height of the cross-section yields.**
- **Distribution of residual stress after unloading** of the section which was formerly loaded with M_1

SOLUTION:

limit elastic bearing capacity $$\bar{M} = \frac{1}{6} b h^2 \sigma_0 \approx 0{,}167\, b h^2 \sigma_0$$

limit plastic bearing capacity $$\bar{\bar{M}} = \frac{1}{4} b h^2 \sigma_0 \approx 0{,}250\, b h^2 \sigma_0$$

moment M_1: $$M_1 = 2\left[\left(\sigma_0 \cdot b \cdot \frac{h}{6} \cdot \left(\frac{h}{3} + \frac{1}{2} \cdot \frac{h}{6}\right)\right) + \frac{1}{2} \sigma_0 \cdot b \cdot \frac{h}{3} \cdot \frac{2}{3} \cdot \frac{h}{3}\right] = \frac{23}{108} b h^2 \sigma_0 \approx 0{,}213\, b h^2 \sigma_0$$

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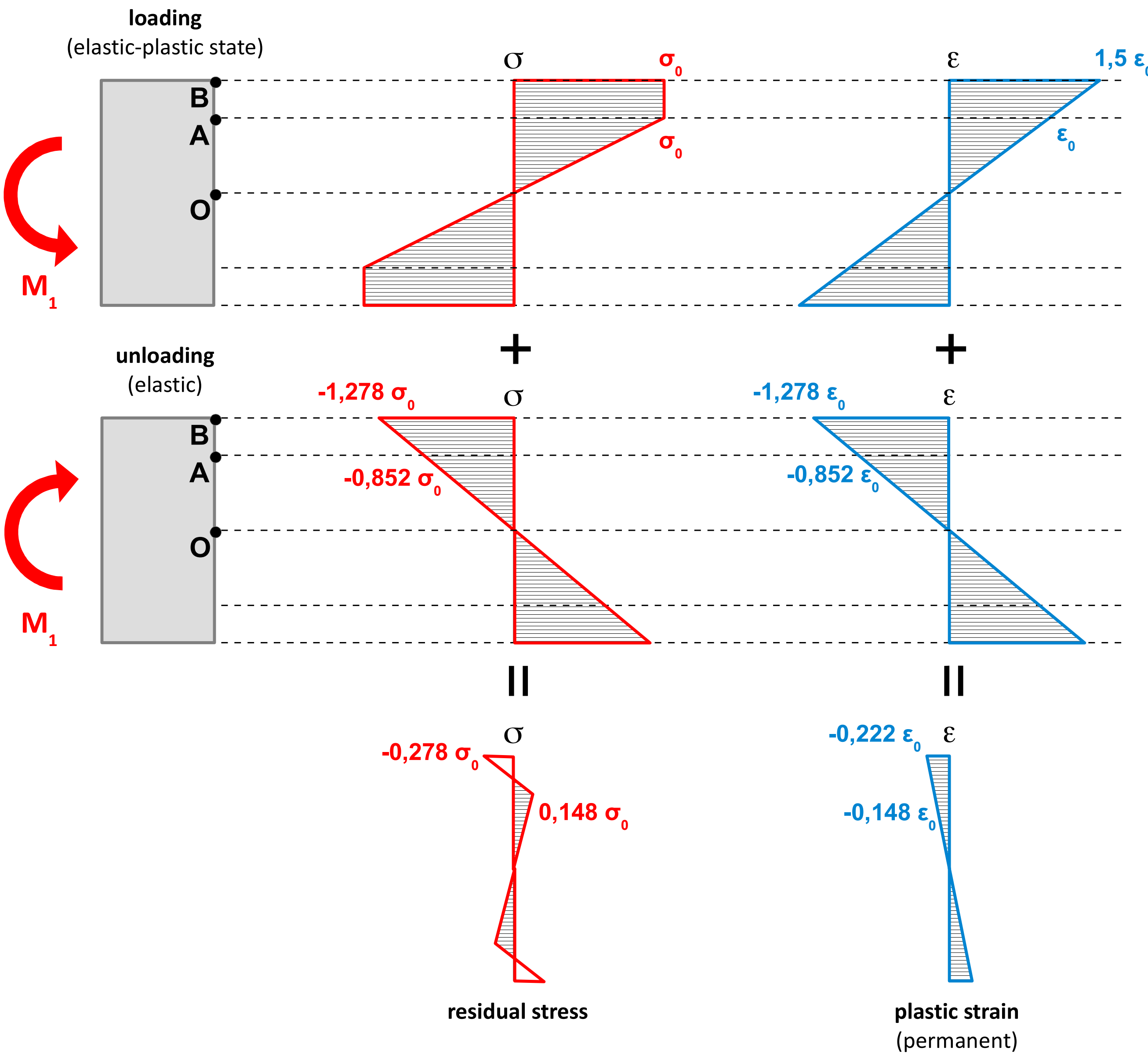
PERMANENT STRAIN AND RESIDUAL STRESS - EXAMPLE



$$\sigma_B = \sigma_0 \qquad \varepsilon_B = \frac{z_B}{z_A}\varepsilon_A = 1{,}5\frac{\sigma_0}{E}$$

$$\sigma_A = \sigma_0 \qquad \varepsilon_A = \varepsilon_0 = \frac{\sigma_0}{E}$$

$$\sigma_O = 0 \qquad \varepsilon_O = 0$$

$$\sigma_B = -\frac{M_1}{I}z_B \qquad \varepsilon_B = -\frac{M_1}{EI}z_B$$

$$\sigma_A = -\frac{M_1}{I}z_A \qquad \varepsilon_A = -\frac{M_1}{EI}z_A$$

$$\sigma_O = 0 \qquad \varepsilon_O = -\frac{M_1}{EI}z_O = 0$$

$$\sigma_B = \sigma_0 - \frac{M_1}{I}z_B \qquad \varepsilon_B = 1{,}5\,\varepsilon_0 - \frac{M_1}{EI}z_B$$

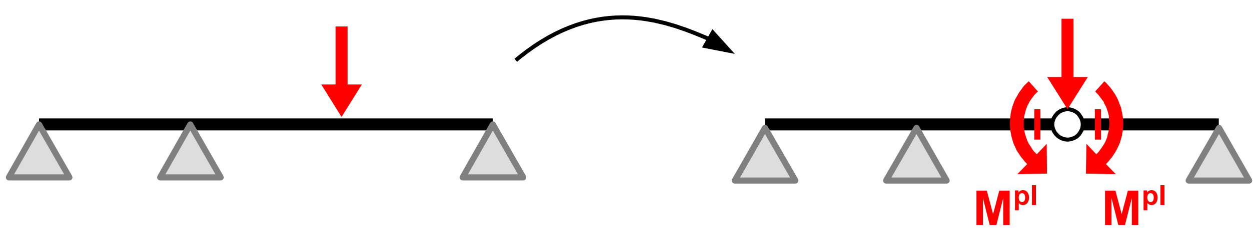
$$\sigma_A = \sigma_0 - \frac{M_1}{I}z_A \qquad \varepsilon_A = \varepsilon_0 - \frac{M_1}{EI}z_A$$

$$\sigma_O = 0 - 0 = 0 \qquad \varepsilon_O = 0 - 0 = 0$$

BENDING

REMARKS:

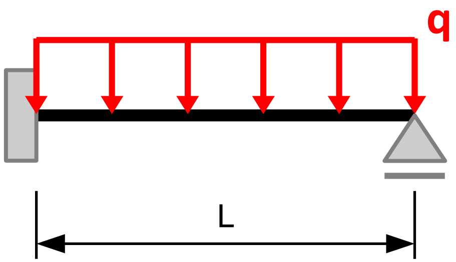
- **A cross-section in fully plastic state** (for materials without hardening) has **freedom in rotation** – plastic hinge is created there.
  - Plastic hinge may **rotate freely only in one direction** – in an opposite direction it exhibits flexural stiffness of a cross-section in elastic state (unloading has elastic character)
  - The difference between plastic hinge and regular hinge is that the plastic hinge **transmits bending moment** – its value is **constant and equal to the limit plastic bearing capacity of the cross-section**.
  - Presence of a plastic hinge in the structure is modelled in such a way, that a **hinge is introduced in the structure** and it is **loaded from both sided with moments corresponding with limit plastic bearing capacity of the bent cross-section**.



- Modelling of an **elastic-plastic deformation of a steel structure** with the use of **plastic hinges** is admissible only for **class 1 sections**.

ELASTIC-PLASTIC ANALYSIS OF A BEAM - EXAMPLE

Find the limit bearing capacity of the statically indeterminate beam with the use of elastic-plastic analysis.

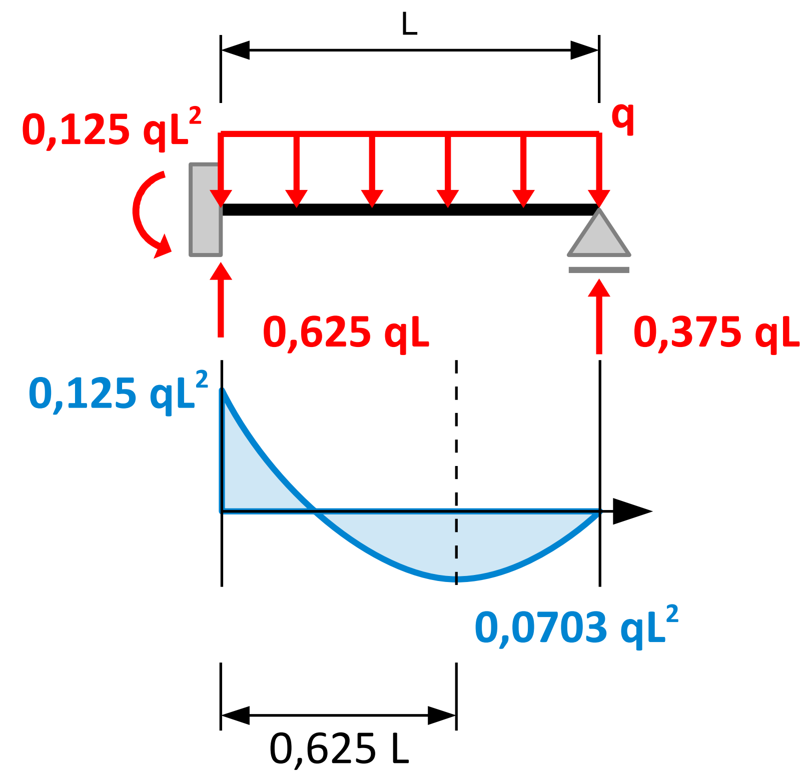


ELASTIC-PLASTIC ANALYSIS OF A BEAM - EXAMPLE

STEP 1 – Elastic analysis

There is only one hyperstatic reaction.

Maximal moment occurs in the fixed support – it is the place, where first yielding occurs.



limit elastic bearing capacity – value of the load parameter, for which **first yielding in a point occurs**:

$$0{,}125\, q\, L^2 = M^{el} = W\, \sigma_0 \quad \Rightarrow \quad \bar{q} = 8\frac{W^{el}\sigma_0}{L^2}$$

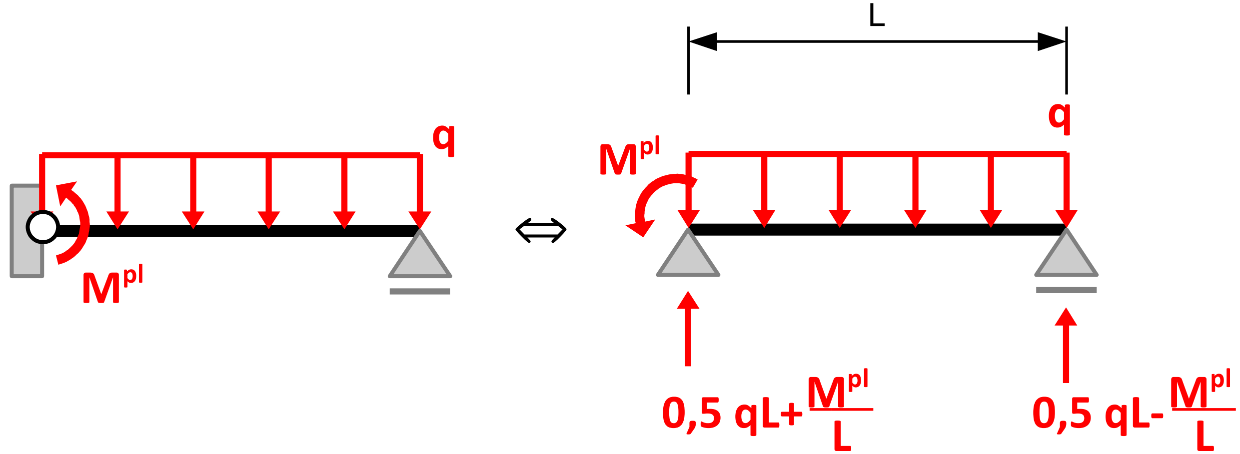
limit plastic bearing capacity – value of the load parameter, for which **first yielding of a cross-section occurs**:

$$0{,}125\, q\, L^2 = M^{pl} \quad \Rightarrow \quad \bar{\bar{q}} = 8\frac{M^{pl}}{L^2}$$

ELASTIC-PLASTIC ANALYSIS OF A BEAM - EXAMPLE

STEP 2 – Elastic-plastic analysis

We introduce a plastic hinge. The structure becomes statically determinate



Distribution of bending moment:

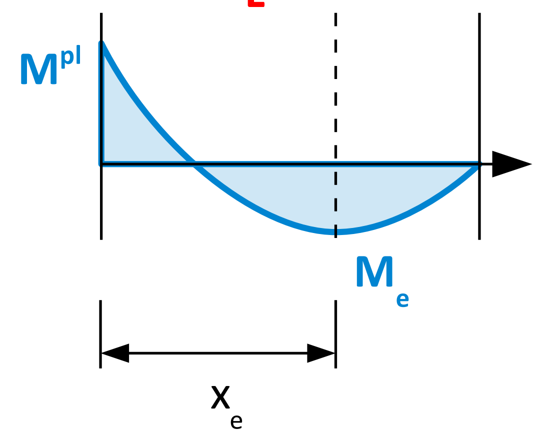
$$M(x) = -M^{pl} + \left(\frac{qL}{2} + \frac{M^{pl}}{L}\right)x - q\frac{x^2}{2}$$

Position of the maximal mid-span moment:

$$\frac{\mathrm{d}M}{\mathrm{d}x} = 0 \quad \Rightarrow \quad x_e = \frac{L}{2} + \frac{M^{pl}}{qL}$$

Maximal mid-span bending moment:

$$M(x_e) = \frac{(M^{pl})^2}{2qL^2} - \frac{M^{pl}}{2} + \frac{qL^2}{2}$$

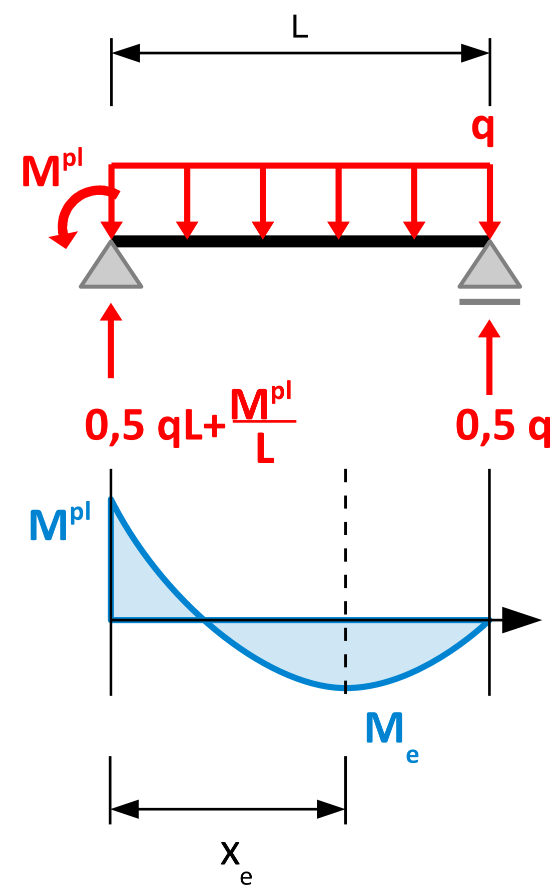


ELASTIC-PLASTIC ANALYSIS OF A BEAM - EXAMPLE

STEP 2 – Elastic-plastic analysis

When yielding occurs: $M(x_e) = +M^{pl}$

$$4(M^{pl})^2 - 12\,q\,L^2 M^{pl} + q^2 L^4 = 0 \quad \Rightarrow \quad \vee \quad \begin{array}{l} q_1 \approx 0{,}34315 \dfrac{M^{pl}}{L^2} \\ q_2 \approx 11{,}657 \dfrac{M^{pl}}{L^2} \end{array}$$



position of the second plastic hinge:

$$x_{e,1} = \frac{L}{2} + \frac{M^{pl}}{q_1 L} \approx 3{,}4142\,L$$ ← solution without physical meaning

$$x_{e,2} = \frac{L}{2} + \frac{M^{pl}}{q_1 L} \approx 0{,}58579\,L$$

After yielding in the second cross-section the system collapses.

Limit bearing capacity: $$q^* = \frac{(M^{pl})^2}{2 q_2 L^2} - \frac{M^{pl}}{2} + \frac{q_2 L^2}{2} \approx 11{,}657 \frac{M^{pl}}{L^2}$$

ELASTIC-PLASTIC DEFORMATION OF SOLID BARDS

TORSION

TORSION

ASSUMPTIONS:

- within **elastic** range the **cross-section rotates and warps**.
- **unconstrained torsion** (of de Saint-Venant) – the **cross-section may warp freely**
- **the only non-zero components** of the stress tensor are **shear stresses in the plane of cross-section**
- within **elastic** range the **Hooke's Law** is valid.
- within **plastic** range the material exhibits **no hardening**.

Within elastic range:

displacement vector:

$$\mathbf{u} = \Theta \begin{bmatrix} \psi(x_1, x_2) \\ -x_1 x_3 \\ x_1 x_2 \end{bmatrix}$$

elastic strain tensor:

$$\boldsymbol{\varepsilon}^e = \frac{\Theta}{2} \begin{bmatrix} 0 & \left(\frac{\partial \psi}{\partial x_2} - x_3\right) & \left(\frac{\partial \psi}{\partial x_3} + x_2\right) \\ & 0 & 0 \\ \text{sym} & & 0 \end{bmatrix}$$

stress tensor:

$$\boldsymbol{\sigma} = \Theta G \begin{bmatrix} 0 & \left(\frac{\partial \psi}{\partial x_2} - x_3\right) & \left(\frac{\partial \psi}{\partial x_3} + x_2\right) \\ & 0 & 0 \\ \text{sym} & & 0 \end{bmatrix}$$

TORSION

We'll make use of the Prandtl's stress function ϕ :

$$\sigma_{12} = \frac{\partial \phi}{\partial x_3}, \quad \sigma_{31} = -\frac{\partial \phi}{\partial x_2}$$

Stresses expressed in such a way satisfy the equilibrium equations identically.

Strain compatibility conditions:

$$\frac{\partial}{\partial x_3}\left[\frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{31}}{\partial x_2} - \frac{\partial \varepsilon_{12}}{\partial x_3}\right] = \frac{\partial^2 \varepsilon_{33}}{\partial x_1 \partial x_2}$$

All other conditions are either dependent or satisfied identically.

Since $\varepsilon_{23} = \varepsilon_{33} = 0$:

$$\frac{\partial \varepsilon_{31}}{\partial x_2} - \frac{\partial \varepsilon_{12}}{\partial x_3} = const. \quad \Leftrightarrow \quad \frac{\partial \sigma_{31}}{\partial x_2} - \frac{\partial \sigma_{12}}{\partial x_3} = const.$$

Indeed:

$$\frac{\partial \sigma_{31}}{\partial x_2} - \frac{\partial \sigma_{12}}{\partial x_3} = \Theta G\left[\left(\frac{\partial^2 \psi}{\partial x_2 \partial x_3} + 1\right) - \left(\frac{\partial^2 \psi}{\partial x_2 \partial x_3} - 1\right)\right] = 2\Theta G$$

If we express the stresses in terms of the **Prandtl's function**, we will obtain:

$$\nabla^2 \phi = \frac{\partial \phi^2}{\partial x_2^2} + \frac{\partial \phi^2}{\partial x_3^2} = -2G\Theta$$

TORSION

$$\nabla^2 \phi = \frac{\partial \phi^2}{\partial x_2^2} + \frac{\partial \phi^2}{\partial x_3^2} = -2G\Theta$$

REMARKS:

- Function ϕ which satisfies the above **Poisson's equation** determines a stress state that satisfies the **strain compatibility conditions** and **equilibrium equations**. This state will be the solution of the problem if it also satisfy the static boundary conditions (constitutive relations are assumed to be satisfied)
- **Directional derivative** of function ϕ determines a **shear stress of direction which is perpendicular to the direction of differentiation**:

$$\partial_{\mathbf{n}} \phi = \nabla \phi \cdot \mathbf{n} = \tau_{\perp \mathbf{n}}$$

- **Static boundary condition** require that the **shear stress normal to the boundary of the cross-section is equal to 0.** This requirement is equivalent to that the **directional derivative of ϕ along direction which is tangent to the boundary is 0, so the distribution of ϕ along boundary must be constant.** This values may be chosen arbitrary – let's assume that it is 0.
- Solution of the Poisson's equation with the above boundary determines such a function that determines the stress state, which is the **solution of the problem**, since no kinematic boundary conditions are prescribed.

TORSION

$$\nabla^2 \phi = \frac{\partial \phi^2}{\partial x_2^2} + \frac{\partial \phi^2}{\partial x_3^2} = -2 G \Theta$$

REMARKS:

- The above equation with homogeneous (zero) boundary condition describes also the **deformation of an elastic membrane fixed at the boundary, the shape of which is the same as the contour of the cross-section**. Membrane is loaded uniformly and function ϕ describes the **deflection of the membrane**.

- **Directional derivative** of function ϕ along direction $\mathbf{n}$ is the slope of the membrane in a plane perpendicular to the membrane and parallel to $\mathbf{n}$. We know also that:

$$\partial_{\mathbf{n}} \phi = \nabla \phi \cdot \mathbf{n} = \tau_{\perp \mathbf{n}}$$

this means that **slope of a membrane is a measure of magnitude of the shear stress**.

- This observation is referred to as the Prandtl's membrane analogy.

TORSION

PRANDTL'S ANALOGY

Total twisting moment corresponding with a continuous system of shear stresses is equal to:

$$M = \iint_A \left(\sigma_{12} x_3 - \sigma_{31} x_2\right) \mathrm{d}A = \iint_A \left(\frac{\partial \phi}{\partial x_3} x_3 + \frac{\partial \phi}{\partial x_2} x_2\right) \mathrm{d}A$$

Let's integrate it by parts:

$$M = \iint_A \left(\frac{\partial \phi}{\partial x_3} x_3 + \frac{\partial \phi}{\partial x_2} x_2\right) \mathrm{d}A = \oint_{\partial A} \left[\phi\left(x_2 n_2 + x_3 n_3\right)\right] \mathrm{d}A - \iint_A \phi \left[\frac{\partial}{\partial x_2}(x_2) + \frac{\partial}{\partial x_3}(x_3)\right] \mathrm{d}A$$

Since ϕ **is equal to 0 at the boundary**, the line integral over the boundary is equal to 0. We obtain:

$$M = -2 \iint_A \phi \, \mathrm{d}A$$

If ϕ denotes deflection of the membrane, then the above integral is equal twice the **volume** (with “-” sign) **between deformed membrane and plane** (x_2, x_3) – this **volume** is then the **measure of total twisting moment within the elastic range**.

$$M = 2V$$

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TORSION

When the **material begins to yield**, for materials exhibiting **no hardening**, **shear stress cannot exceed the value corresponding with the yield stress** in torsion – in all those points in which the **yield condition** is satisfied it must be:

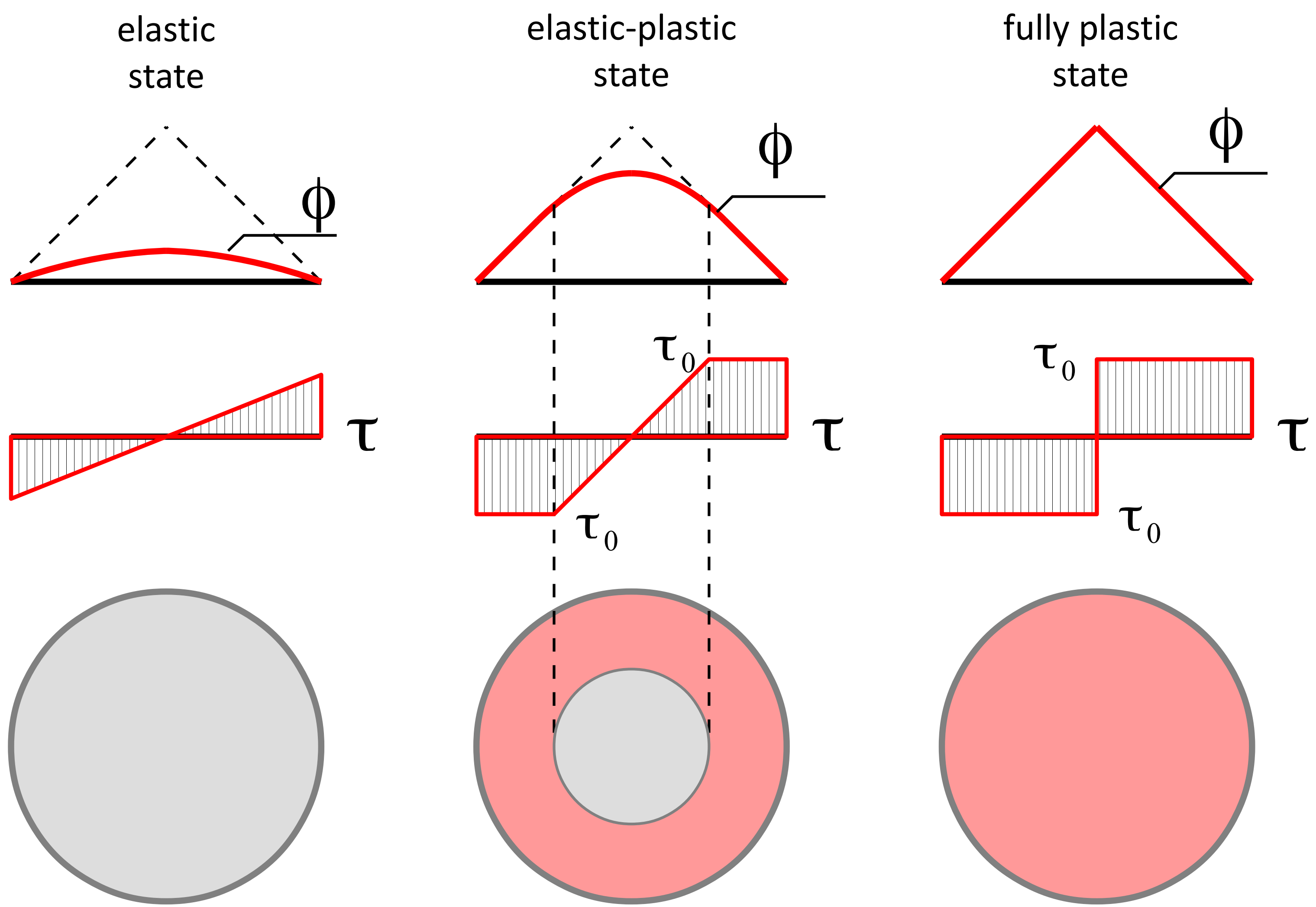
$$\sqrt{\sigma_{12}^2+\sigma_{31}^2}=\tau_0 \qquad \Rightarrow \qquad \left(\frac{\partial\phi}{\partial x_2}\right)^2+\left(\frac{\partial\phi}{\partial x_3}\right)^2=\left|\mathrm{grad}\,\phi\right|^2=\tau_0^2=const.$$

If we make use to the membrane analogy, then the above relation means that **in the plastic region the slope of a membrane is constant** – the measure of that limit value of slope is the yield stress:

$$\tau_0^2=\mathrm{tg}^2\alpha=const.$$

It may be illustrated in such a way, that deformed membrane encounters a rigid roof of given slope. **In the plastic region the membrane remains in contact with the roof**. This observation is referred to as the Nádai roof analogy.

TORSION


elastic state
elastic-plastic state
fully plastic state
ϕ
ϕ
ϕ
τ
τ
τ
τ0
τ0
τ0
τ0

TORSION

In the **fully plastic state** the **membrane has the shape of the roof** of given slope. This roof has a form of a **surface of constant slope**, which is the solution of a non-linear differential equation:

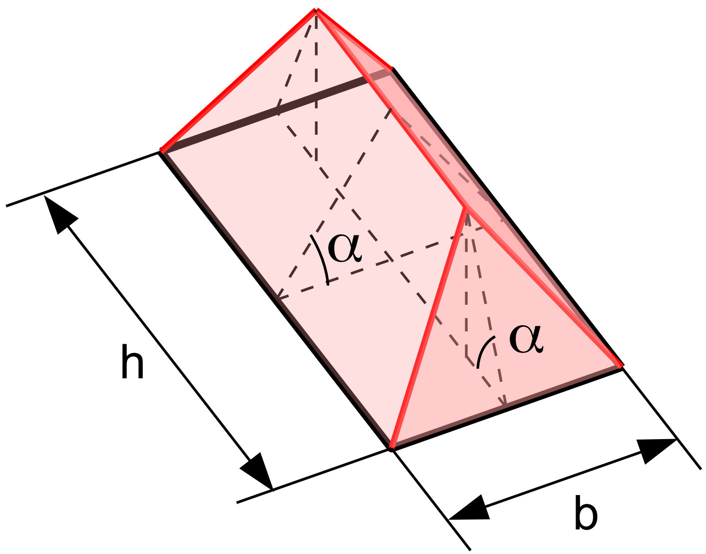
$$\left(\frac{\partial \phi}{\partial x_2}\right)^2 + \left(\frac{\partial \phi}{\partial x_3}\right)^2 = const.$$

This is also the **shape of a hill which is created when piling up a dry, ideally powdery sand on an area of the shape of the considered cross-section**. The **volume of this sand hill** is then a **measure of plastic bearing capacity of the cross-section**. This observation is referred to as the Nádai sand hill analogy.

A generalization of the sand hill analogy for the case of a cross-section with openings is referred to as the Sadovsky analogy:

- Infinitely long tube is placed where the openings are
- we pile up the sand – the interior of tubes remains empty
- we lower each tubes until the sand pours into the tube along whole circumference of that tube
- the volume of a sand hill created in such a way is a measure of limit plastic bearing capacity of a twisted cross-section.

TORSION

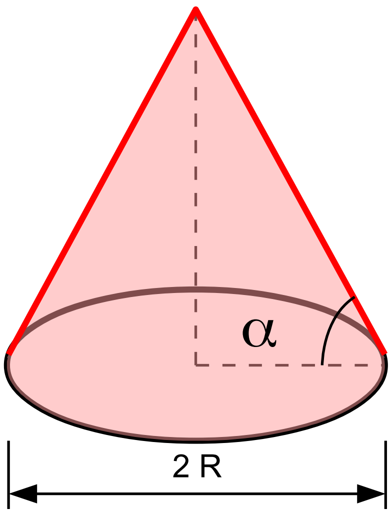


RECTANGULAR CROSS-SECTION

volume of the sand hill: $$V = \frac{1}{12} b^2 (3h - b)\ \mathrm{tg}\,\alpha$$

limit bearing capacity: $$M^{pl} = \frac{1}{6} b^2 (3h - b) \tau_0$$

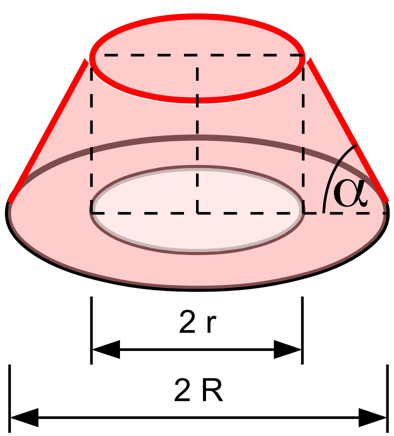
TORSION



CIRCULAR CROSS-SECTION

volume of the sand hill: $$V = \frac{1}{3}\pi R^3 \operatorname{tg}\alpha$$

limit bearing capacity: $$M^{pl} = \frac{2}{3}\pi R^3 \tau_0$$



HOLLOW TUBE CROSS-SECTION

volume of the sand hill: $$V = \frac{1}{3}\pi\left(R^3 - r^3\right)\operatorname{tg}\alpha$$

limit bearing capacity: $$M^{pl} = \frac{2}{3}\pi\left(R^3 - r^3\right)\tau_0$$

TORSION

TRUCK-SHAPED CROSS-SECTION

PLANE-SHAPED CROSS-SECTION

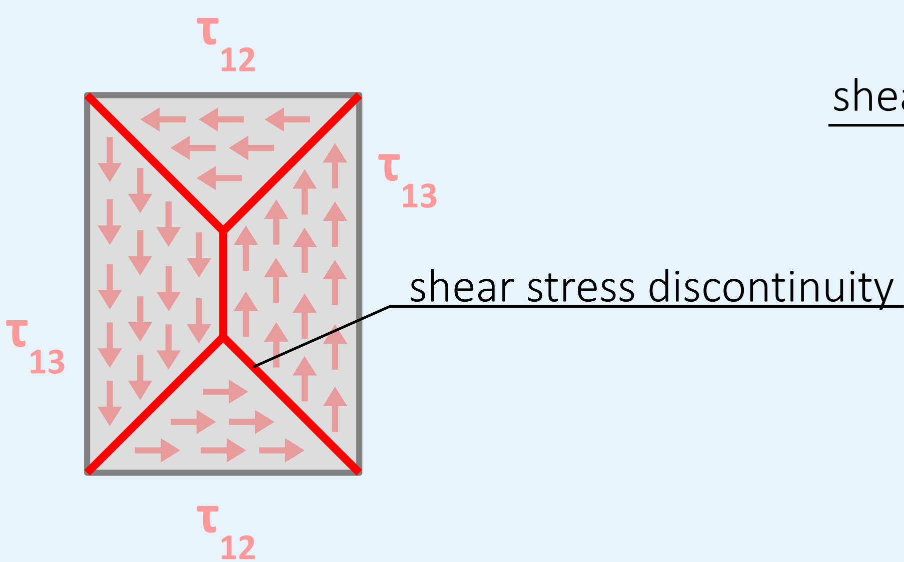
models performed by: Paweł, Klara, Józef, Barnaba Szeptyński

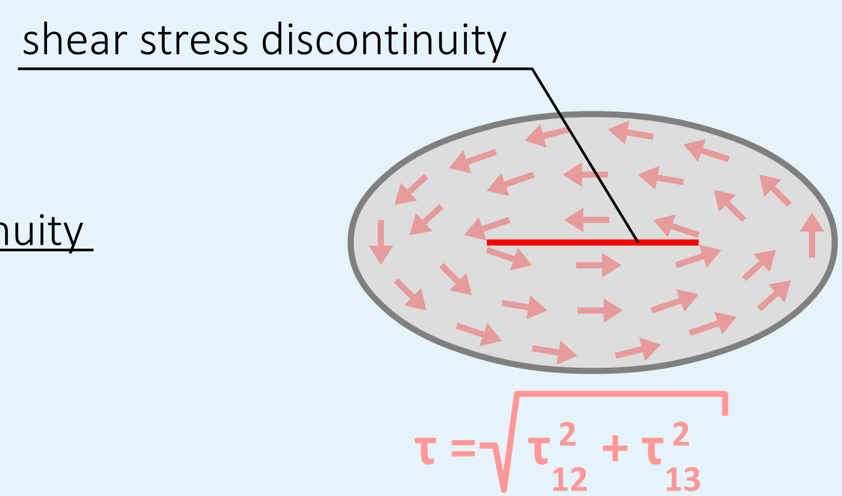
TORSION

$$\partial_{\mathbf{n}}\phi = \nabla\phi\cdot\mathbf{n} = \tau_{\perp\mathbf{n}}$$

- **Slope of the roof** is a measure of **magnitude of the shear stress.**
- **Direction of slope** of the roof is **perpendicular to the direction of the shear stress.**
- In those places where the **direction of slope of the roof changes rapidly** (in a step-wise manner), also the **shear stress changes its direction in a discontinuous manner**.

In a cross-section in a fully plastic state **shear stresses** are distributed **discontinuously**.





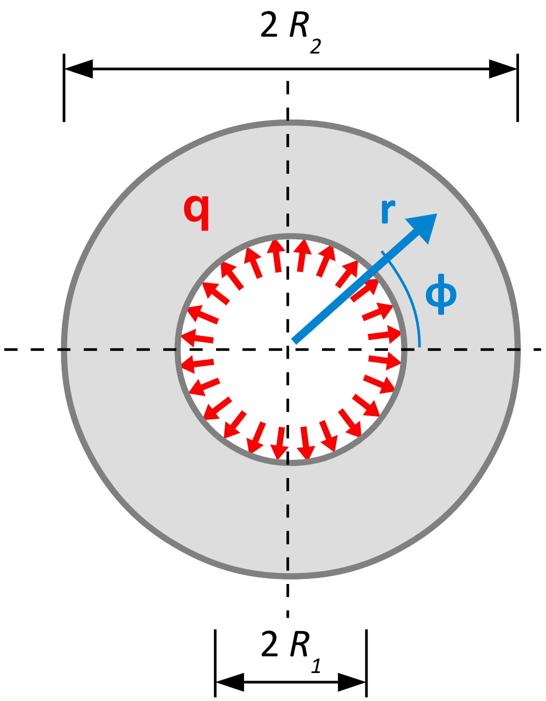
PLANE ELASTIC-PLASTIC DEFORMATION

PLASTIC DEFORMATION OF A THICK-WALLED TUBE

Axis-symmetric thick-walled tube is loaded with a **uniform internal pressure**.

Elastic – plastic material:

- within **elastic** range: Hooke's material
- **yield condition:** Coulomb – Tresca – Guest
- within **plastic** range: no hardening



Solution for purely elastic state (Lamé's solution):

displacement field:

$$u_r(r) = C_1 r + \frac{C_2}{r}$$

strain tensor field:

$$\varepsilon^e_{rr}(r) = C_1 - \frac{C_2}{r^2}$$

$$\varepsilon^e_{\phi\phi}(r) = C_1 + \frac{C_2}{r^2}$$

stress tensor field:

$$\sigma_{rr}(r) = 2C_1(G+\lambda) - \frac{2C_2G}{r^2}$$

$$\sigma_{\phi\phi}(r) = 2C_1(G+\lambda) + \frac{2C_2G}{r^2}$$

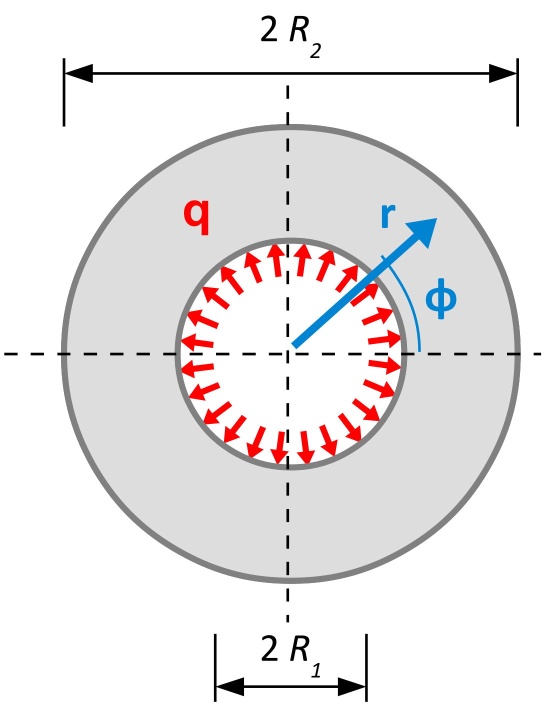
$$C_1 = \frac{q R_1^2}{(2G+\lambda)(R_2^2 - R_1^2)}$$

$$C_2 = \frac{q R_1^2 R_2^2}{2G(R_2^2 - R_1^2)}$$

PLASTIC DEFORMATION OF A THICK-WALLED TUBE

We assume that the **character of distribution of stress in the elastic zone** (I) is **the same as in the Lamé's solution**, however the constants of integration may be different.

$$\sigma^{I}_{rr}(r) = D_1 - \frac{D_2}{r^2} \qquad \sigma^{I}_{\phi\phi}(r) = D_1 + \frac{D_2}{r^2}$$



Boundary conditions:

- outer surface: $\sigma^{I}_{rr}(R_2) = 0$
- **boundary between elastic** (I) and **plastic** (II) **zone**: $\sigma^{II}_{rr}(R_{pl}) = \sigma^{I}_{rr}(R_{pl})$

  $\sigma^{II}_{r\phi}(R_{pl}) = \sigma^{I}_{r\phi}(R_{pl})$ (satisfied, since $\sigma^{II}_{r\phi} = \sigma^{I}_{r\phi} = 0$)
- inner surface: $\sigma^{II}_{rr}(R_q) = -q$

PLASTIC DEFORMATION OF A THICK-WALLED TUBE

CTG yield condition in the case of axial symmetry for the elastic solution:

$$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{\sigma_{\phi\phi} - \sigma_{rr}}{2} = \frac{2 C_2 G}{r^2}$$

FIRST YIELDING

The above expression gets maximal value for minimal r. **Yielding will begin at inner surface**. The magnitude of internal pressure that leads to first yielding is equal to:

$$2\tau_{max} = \sigma_0 \quad \Rightarrow \quad \frac{4 G C_2}{R_1^2} = \frac{2 q_0 R_2^2}{(R_2^2 - R_1^2)} = \sigma_0 \quad \Rightarrow \quad q_0 = \frac{\sigma_0}{2}\left(1 - \frac{R_1^2}{R_2^2}\right)$$

BOUNDARY OF THE PLASTIC ZONE

Yield condition written down for the stress distribution in elastic zone:

$$2\tau_{max}^{I}(R_{pl}) = \sigma_{\phi\phi}^{I}(R_{pl}) - \sigma_{rr}^{I}(R_{pl}) = \frac{2 D_2}{R_{pl}^2} = \sigma_0 \quad \Rightarrow \quad \boxed{R_{pl} = \sqrt{\frac{2 D_2}{\sigma_0}} \quad \Leftrightarrow \quad D_2 = \frac{1}{2}\sigma_0 R_{pl}^2}$$

PLASTIC DEFORMATION OF A THICK-WALLED TUBE

SOLUTION IN PLASTIC ZONE

In each point of the plastic zone the yield condition must be satisfied:

$$2\tau_{max}^{II} = \sigma_0 \qquad \Rightarrow \qquad \sigma_{\phi\phi}^{II} - \sigma_{rr}^{II} = \sigma_0$$

Equilibrium equations for axis-symmetric problem (the same in both zones):

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\phi\phi}}{r} = 0$$

After substituting the yield condition we obtain an equilibrium equation in plastic zone:

$$\frac{\partial \sigma_{rr}^{II}}{\partial r} = \frac{\sigma_0}{r}$$

The **solution** of the above equation is:

$$\sigma_{rr}^{II} = \sigma_0 \ln r + D_3$$

PLASTIC DEFORMATION OF A THICK-WALLED TUBE

SOLUTION IN PLASTIC ZONE

Having determined the radial stress, we are able now to determine the **circumferential stress** according to the yield condition:

$$\sigma_{\phi\phi}^{II}-\sigma_{rr}^{II}=\sigma_0 \qquad \Rightarrow \qquad \sigma_{\phi\phi}^{II}=\sigma_0(1+\ln r)+D_3$$

Constants of integration are determined according to the yield condition written for the boundary between zones and according to boundary conditions (4 equations for 4 unknowns $D_1,\ D_2,\ D_3,\ R_{pl}$):

$$D_2=\frac{1}{2}\sigma_0 R_{pl}^2 \qquad \leftarrow \text{yield condition at the boundary of zones}$$

$$\begin{cases} \sigma_{rr}^{I}(R_2)=0 & \Rightarrow \quad D_1-\dfrac{D_2}{R_2^2}=0 & \leftarrow \text{outer surface} \\ \sigma_{rr}^{II}(R_1)=-q & \Rightarrow \quad D_3+\sigma_0\ln R_1=-q & \leftarrow \text{inner surface} \\ \sigma_{rr}^{I}(R_{pl})-\sigma_{rr}^{II}(R_{pl})=0 & \Rightarrow \quad D_1-\dfrac{\sigma_0}{2}-D_3-\sigma_0\ln R_{pl}=0 & \leftarrow \text{at the boundary of zones} \end{cases}$$

$$D_1=\frac{\sigma_0 R_{pl}^2}{2R_2^2} \qquad D_2=\frac{1}{2}\sigma_0 R_{pl}^2 \qquad D_3=\frac{\sigma_0 R_{pl}^2}{2R_2^2}\left(1-\frac{R_2^2}{R_{pl}^2}\right)-\sigma_0\ln R_{pl}$$

PLASTIC DEFORMATION OF A THICK-WALLED TUBE

The **boundary of the plastic zone** is determined according to the **boundary condition for inner surface**:

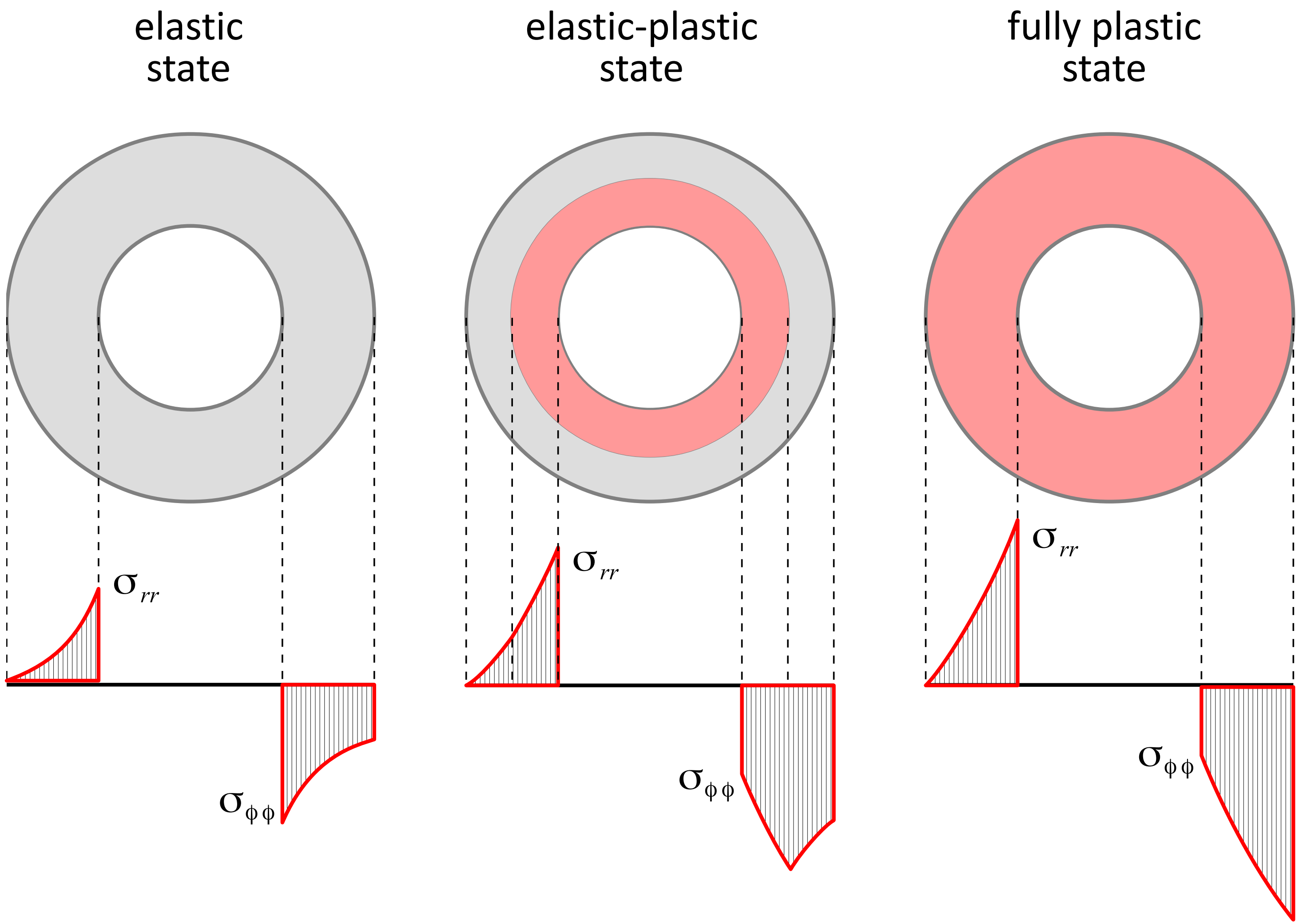
$$\sigma_{rr}^{II}(R_1) = -q \qquad \Rightarrow \qquad \ln\frac{R_{pl}}{R_1} + \frac{1}{2}\left(1 - \frac{R_{pl}^2}{R_2^2}\right) = \frac{q}{\sigma_0}$$

This is a non-linear algebraic equation with respect to R_{pl} and it must be solved numerically.

Finally, **distribution of stress** in the cross-section of the tube is as follows:

	elastic zone:	**plastic** zone:
radial stress:	$\sigma_{rr}^{I}(r) = \frac{\sigma_0 R_{pl}^2}{2R_2^2}\left(1 - \frac{R_2^2}{r^2}\right)$	$\sigma_{rr}^{II} = \sigma_0\left[\frac{R_{pl}^2}{2R_2^2} - \frac{1}{2} - \ln\frac{R_{pl}}{r}\right]$
circumferential stress:	$\sigma_{\phi\phi}^{I}(r) = \frac{\sigma_0 R_{pl}^2}{2R_2^2}\left(1 + \frac{R_2^2}{r^2}\right)$	$\sigma_{\phi\phi}^{II} = \sigma_0\left[\frac{R_{pl}^2}{2R_2^2} + \frac{1}{2} - \ln\frac{R_{pl}}{r}\right]$

PLASTIC DEFORMATION OF A THICK-WALLED TUBE



REMARKS:

- This solution was obtained with the assumption of plane stress state and CTG yield condition.
- In the case of plane strain state or when another yield condition is chosen, the solution is different.

UPPER AND LOWER BOUND ESTIMATE THEOREMS

LIMIT BEARING CAPACITY THEORY

In practical problems of design of elastic-plastic systems, one of the key issue is to determine the **limit bearing capacity of the system**, namely such a value of the load parameter, for which the **system collapses**.

For the model of a rigid- ideally plastic solid (with no hardening), namely for the Lévy – Mises model, **two theorems** may be proved, which enable us to **estimate the lower bound and upper bound of the limit bearing capacity**. If **both bound estimates are the same**, then it is **exactly the value of the limit bearing capacity**.

LOWER BOUND ESTIMATE THEOREM

LOWER BOUND ESTIMATE THEOREM

The **system won't collapse** or at most it will be in a limit equilibrium state due to given external load, if it is possible to **find a statically admissible stress field** corresponding with that load. In such situation the limit bearing capacity of the system is equal to that external load but it may be even higher.

Static admissibility of a **stress field** requires that:

- the stress field is in **equilibrium with external load**
- the stress field satisfies the **internal equilibrium conditions**
- the stress field satisfies the **static boundary conditions**
- stress does **not exceed the value of the yield stress** σ_0

UPPER BOUND ESTIMATE THEOREM

UPPER BOUND ESTIMATE THEOREM

The **system will collapse**, if it is **possible to find a kinematically admissible velocity field** such, that total **power of external loads is greater or equal to the power of internal forces**. In such situation the limit bearing capacity is equal to the value of corresponding load, however it may be lower.

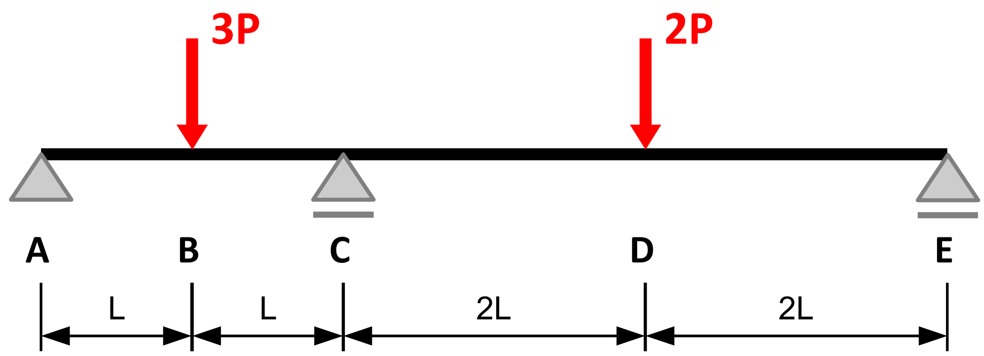
Kinematic admissibility of the **velocity field** requires that:

- velocity field **satisfies the kinematic boundary conditions** (it is consistent with given constraints, i.e. supports)
- velocity field is such that corresponding **displacement field is continuous**
- **total power of external loads is positive.**

REMARK: Velocity field is determined according to the rules of kinematics of system of rigid bodies (in a similar way as virtual displacements)

EXAMPLE

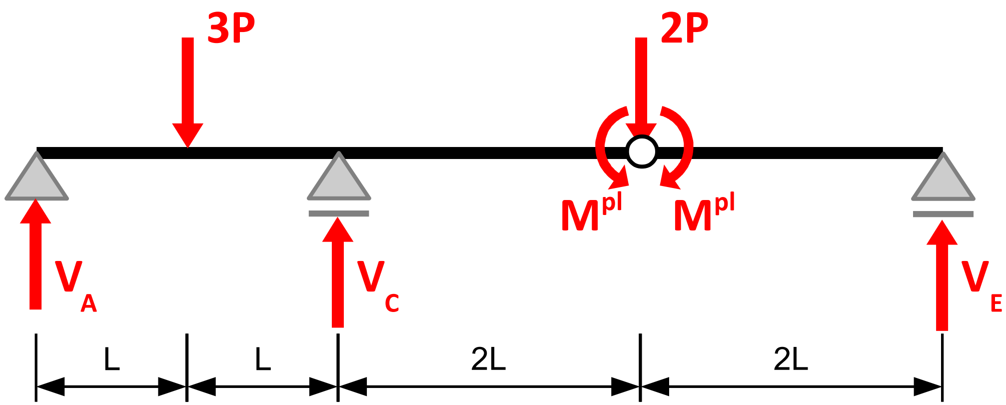
Estimate the limit bearing capacity of the beam:



EXAMPLE

Lower bound estimate – static approach:

Let's assume that first yielding will take place in cross-section D. Then:



$$\begin{cases} V_A = -\frac{1}{2}P + \frac{M^{pl}}{L} \\ V_B = \frac{11}{2}P - \frac{3}{2}\frac{M^{pl}}{L} \\ V_E = \frac{M^{pl}}{2L} \end{cases}$$

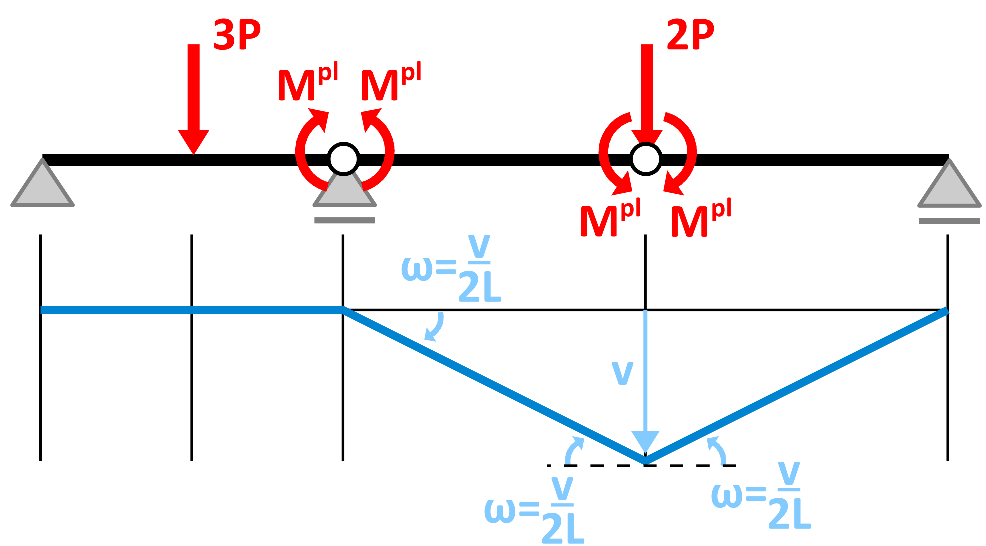
The beam is loaded with point forces. The distribution of moments is piece-wise linear. Maximal moments will occur in cross-section B or C – these are the places where next yielding may occur:

Next yielding in B: $M_B = M^{pl} \quad \Rightarrow \quad P = 0$ then $M_C = 2M^{pl} > M^{pl}$ **statically inadmissible**

Next yielding in C: $M_C = -M^{pl} \quad \Rightarrow \quad P = \frac{3}{4}\frac{M^{pl}}{L}$ then $M_B = \frac{5}{8}M^{pl}$ **statically admissible**

EXAMPLE

Upper bound estimate – kinematic approach:



Power of external forces: $\dot{\Phi}_z = 2P \cdot v \quad \Rightarrow \quad P = \frac{3}{4}\frac{M^{pl}}{L}$

Power of internal forces: $\dot{\Phi}_w = M^{pl} \cdot \frac{v}{2L} + M^{pl} \cdot \frac{v}{2L} + M^{pl} \cdot \frac{v}{2L}$

$$\Rightarrow \quad \dot{\Phi}_w = \dot{\Phi}_z \quad \Rightarrow \quad P = \frac{3}{4}\frac{M^{pl}}{L}$$

EXAMPLE

According to the **lower bound estimate theorem**: $P^* \geqslant \frac{3}{4}\frac{M^{pl}}{L}$

According to the **upper bound estimate theorem**: $P^* \leqslant \frac{3}{4}\frac{M^{pl}}{L}$

Limit bearing capacity is then equal to: $P^* = \frac{3}{4}\frac{M^{pl}}{L}$

REMARKS:

- If the mechanism of collapse is no to obvious to be predicted, then usually it is required to consider few possible stress distributions (lower bound estimates) and few possible collapse mechanisms (upper bound estimates).
- It is not always easy to find equal lower bound and upper bound. Then the true limit bearing capacity is between those two values.
- **Lower bound estimate** (**static** approach) is a **safe estimate**.

THANK YOU FOR YOUR ATTENTION