THEORY OF ELASTICITY AND PLASTICITY

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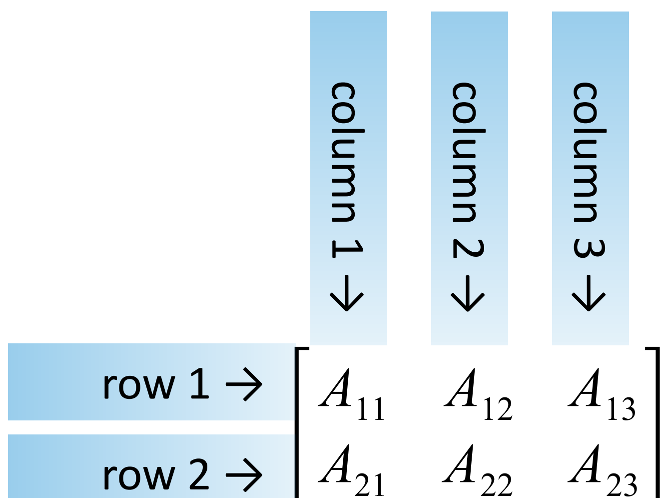
MATRIX CALCULUS

MATRIX CALCULUS

Matrix is a rectangular ("two-dimensional") **table** (**array**) **of numbers** which are referred to with the use of two indices.

- **The first index** inform us about the **row** of the array in which number is located
- **The second index** inform us about the **column** of the array in which number is located

We will refer to the numbers put in the matrix as to its elements or entries.



$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

RACHUNEK MACIERZOWY

REMARK:

- In the **countinuum mechanics** matrices that we use are almost always representations of tensors, namely their **entries** are equal the **components of a tensor in the considered coordinate system**.

- **Components of tensors** (including vectors) **change**, when the **coordinate system is changed**. As a consequence **also their matrix representations chance** (the values of the entries of that matrix are changed)

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NOTATION

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NOTATION

We will make use of the following symbols

KRONECKER'S DELTA

$$\delta_{ij} = \begin{cases} 1 & \Leftrightarrow \quad i = j \\ 0 & \Leftrightarrow \quad i \neq j \end{cases}$$

LEVI-CIVITA PERMUTATION SYMBOL

$$\epsilon_{ijk} = \begin{cases} 1 & \Leftrightarrow \quad (i,j,k) \in \{(1,2,3);(2,3,1);(3,1,2)\} & \leftarrow \text{even permutations of } (1,2,3) \\ -1 & \Leftrightarrow \quad (i,j,k) \in \{(2,1,3);(1,3,2);(3,2,1)\} & \leftarrow \text{odd permutations of } (1,2,3) \\ 0 & \Leftrightarrow \quad \text{any two indices have the same value} \end{cases}$$

SUMMATION CONVENTION

We will make us of the so called Einstein's summation convention:

> *If in an index notation in a single expression being a product of entries numbered with indices* ***certain index is repeated****, once in the superscript one in the subscript,* ***it means summation with respect to that index*** *for all values that index may take.*

REMARKS:

- In our considerations (Cartesian coordinates, orthogonal transformations) it is not necessary to distinguish between subscripts and superscripts – **all indices will be written in the subscript**.
- **An index with respect to which summation is performed** is called a dummy index.
- An index which is not a dummy index is called a free index.
- Dummy indices may be changed within a single expression.
- Differentiation is denoted in index notation with an additional index (indicating the variable with respect to which differentiation is performed) written after a comma. Such an index obey the above rules.

SUMMATION CONVENTION

EXAMPLES:

- (single inner) dot product of tensors (matrix multiplication):

$$C_{ij} = A_{ik} B_{kj} \quad \Rightarrow \quad C_{ij} = \sum_{k=1}^{N} A_{ik} B_{kj} = A_{i1} B_{1j} + A_{i2} B_{2j} + \ldots + A_{iN} B_{Nj}$$

i, j – free indices, k – dummy index $\qquad \left(C_{ij} = A_{ik} B_{kj} = A_{im} B_{mj} \text{ etc.}\right)$

- Contraction of a tensor:

$$\theta = \varepsilon_{ii} \quad \Rightarrow \quad \theta = \sum_{i=1}^{N} \varepsilon_{ii} = \varepsilon_{11} + \varepsilon_{22} + \ldots + \varepsilon_{NN}$$

- Length of a vector:

$$|\mathbf{v}| = \sqrt{v_k v_k} \quad \Rightarrow \quad |\mathbf{v}| = \sqrt{\sum_{k=1}^{3} v_k v_k} = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

SUMMATION CONVENTION

EXAMPLES:

- Divergence of a vector field:

$$\operatorname{div}\mathbf{v} = v_{k,k} \quad \Rightarrow \quad \operatorname{div}\mathbf{v} = \sum_{k=1}^{3} v_{k,k} = v_{1,1} + v_{2,2} + v_{3,3} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$$

- Equations of motion of a continuum in spatial description:

$$\sigma_{ji,j} + b_i = \rho \ddot{u}_i \quad \Rightarrow \quad \begin{cases} \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{21}}{\partial x_2} + \dfrac{\partial \sigma_{31}}{\partial x_3} + b_1 = \rho \dfrac{\partial^2 u_1}{\partial t^2} \\ \dfrac{\partial \sigma_{12}}{\partial x_1} + \dfrac{\partial \sigma_{22}}{\partial x_2} + \dfrac{\partial \sigma_{32}}{\partial x_3} + b_2 = \rho \dfrac{\partial^2 u_2}{\partial t^2} \\ \dfrac{\partial \sigma_{13}}{\partial x_1} + \dfrac{\partial \sigma_{23}}{\partial x_2} + \dfrac{\partial \sigma_{33}}{\partial x_3} + b_3 = \rho \dfrac{\partial^2 u_3}{\partial t^2} \end{cases}$$

SUMMATION CONVENTION

EXAMPLES:

- Geometric relations of non-linear theory of elasticity

$$E_{ij}=\frac{1}{2}\left(u_{i,j}+u_{j,i}+u_{k,i}u_{k,j}\right) \quad\Rightarrow\quad \begin{cases} E_{11}=\frac{1}{2}\left[2\frac{\partial u_1}{\partial x_1}+\left(\frac{\partial u_1}{\partial x_1}\right)^2+\left(\frac{\partial u_2}{\partial x_1}\right)^2+\left(\frac{\partial u_3}{\partial x_1}\right)^2\right] \\ E_{22}=\frac{1}{2}\left[2\frac{\partial u_2}{\partial x_2}+\left(\frac{\partial u_1}{\partial x_2}\right)^2+\left(\frac{\partial u_2}{\partial x_2}\right)^2+\left(\frac{\partial u_3}{\partial x_2}\right)^2\right] \\ E_{33}=\frac{1}{2}\left[2\frac{\partial u_3}{\partial x_3}+\left(\frac{\partial u_1}{\partial x_3}\right)^2+\left(\frac{\partial u_2}{\partial x_3}\right)^2+\left(\frac{\partial u_3}{\partial x_3}\right)^2\right] \\ E_{23}=\frac{1}{2}\left[\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}+\frac{\partial u_1}{\partial x_2}\frac{\partial u_1}{\partial x_3}+\frac{\partial u_2}{\partial x_2}\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\frac{\partial u_3}{\partial x_3}\right] \\ E_{31}=\frac{1}{2}\left[\frac{\partial u_3}{\partial x_1}+\frac{\partial u_1}{\partial x_3}+\frac{\partial u_1}{\partial x_3}\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_3}\frac{\partial u_2}{\partial x_1}+\frac{\partial u_3}{\partial x_3}\frac{\partial u_3}{\partial x_1}\right] \\ E_{12}=\frac{1}{2}\left[\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}+\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_1}\frac{\partial u_3}{\partial x_2}\right] \end{cases}$$

NOTATION

When referring to the **tensors** or their **matrix representations**, we will use one of **three types of notation**:

$$A_{ij}$$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{ lub } \left[A_{ij} \right]$$

$$\mathbf{A}$$

INDEX NOTATION

- We refer to the components of tensors (entries of matrices) in a certain coordinate system

MATRIX NOTATION

- We recall the whole matrix. It is most useful in the case of practical calculations.
- Sometimes we will use a simplified matrix notation. It concerns the situations when certain operations on tensors written in index notation may be more easily interpreted within the matrix calculus.

ABSOLUTE NOTATION

- We refer to a tensor (matrix) without reference to its components (entries) in any coordinate system

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NOTATION

	ABSOLUTE NOTATION	INDEX NOTATION
• scalar	α	α
• vector	$\mathbf{a}$	a_i
• tensor	$\mathbf{A}$	A_{ij}
• dot product of vectors	$\alpha = \mathbf{a} \cdot \mathbf{b}$	$\alpha = a_i b_i$
• length of a vector	$\lvert\mathbf{v}\rvert = \sqrt{\mathbf{v} \cdot \mathbf{v}}$	$\lvert\mathbf{v}\rvert = \sqrt{v_i v_i}$
• cross product of vectors	$\mathbf{c} = \mathbf{a} \times \mathbf{b}$	$c_i = \epsilon_{ijk} a_j b_k$
• dot product of tensor and vector	$\mathbf{y} = \mathbf{A}\,\mathbf{x}$	$y_i = A_{ij} x_j$
• single dot product of tensors	$\mathbf{C} = \mathbf{A}\,\mathbf{B}$	$C_{ij} = A_{ik} B_{kj}$
• (double) dot product of tensors	$\alpha = \mathbf{A} \cdot \mathbf{B}$	$\alpha = A_{ij} B_{ij}$
• unit tensor	$\mathbf{1}$	δ_{ij}

OPERATIONS ON TENSORS

OPERATIONS ON TENSORS

REMARKS:

- Since we are dealing only with matrix representation of 3-dimensional tensors and vectors, we shall use only
  - **Single-row** and **single-column matrices –** representations of **vectors**
  - **3 x 3 square matrices –** representations of **2^{nd} rank tensors**
- **Vectors** will be represented by **single-column** (three-row) **matrices**. In certain circumstances we will use also transposes of those matrices..

OPERATIONS ON TENSORS

ADDITION OF VECTORS

- **Addition of vectors** is associated with the **addition of their matrix representations**
- Addition of vectors (matrices) is **associative** $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$
- Addition of vectors (matrices) is **commutative** $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$

$$\mathbf{w}=\mathbf{u}+\mathbf{v} \qquad \Leftrightarrow \qquad w_i=u_i+v_i$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix}$$

OPERATIONS ON TENSORS

ADDITION OF TENSORS

- **Addition of tensors** is associated with the **addition of their matrix representations**
- We can add only the tensors of the same type (matrices of the same sizes)
- Addition of tensors (matrices) is **associative** $(\mathbf{A}+\mathbf{B})+\mathbf{C} = \mathbf{A}+(\mathbf{B}+\mathbf{C})$
- Addition of tensors (matrices) is **commutative** $\mathbf{A}+\mathbf{B} = \mathbf{B}+\mathbf{A}$

$$\mathbf{C}=\mathbf{A}+\mathbf{B} \quad \Leftrightarrow \quad C_{ij}=A_{ij}+B_{ij}$$

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} = \begin{bmatrix} A_{11}+B_{11} & A_{12}+B_{12} & A_{13}+B_{13} \\ A_{21}+B_{21} & A_{22}+B_{22} & A_{23}+B_{23} \\ A_{31}+B_{31} & A_{32}+B_{32} & A_{33}+B_{33} \end{bmatrix}$$

OPERATIONS ON TENSORS

TRANSPOSITION OF A TENSOR

- **Transposition of a tensor** is associated with **transposition of its matrix representation**, namely:
  - Columns become rows, rows become columns
  - Indices are switched
- **Transpose of a transpose** results in the original matrix $(\mathbf{A}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{A}$

$$\mathbf{B} = \mathbf{A}^{\mathrm{T}} \quad \Leftrightarrow \quad B_{ij} = A_{ji}$$

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$\mathbf{w} = \mathbf{v}^{\mathrm{T}} \quad \Leftrightarrow \quad w_i = v_i$$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$$

OPERATIONS ON TENSORS

SINGLE INNER DOT PRODUCT OF TENSORS

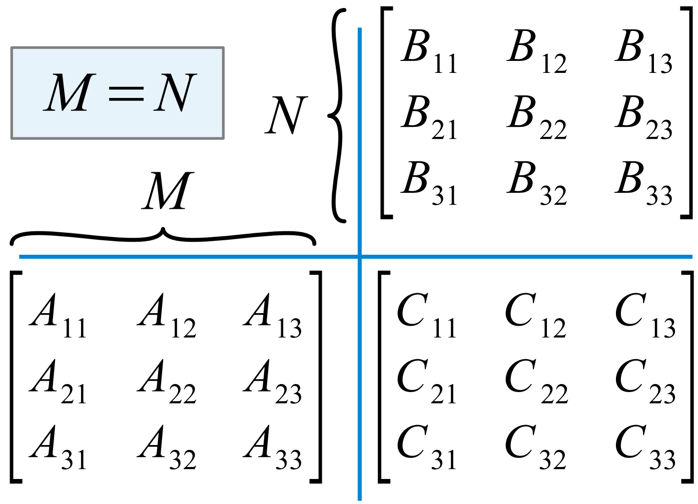
- **Single dot product of tensors** is associated with the **multiplication of their matrix representations**
- Product of tensors (matrices) is **associative** $(\mathbf{A}\mathbf{B})\mathbf{C} = \mathbf{A}(\mathbf{B}\mathbf{C})$
- Product of tensors (matrices) is **not commutative** $\mathbf{A}\mathbf{B} \neq \mathbf{B}\mathbf{A} = (\mathbf{A}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}})^{\mathrm{T}}$

$$\mathbf{C} = \mathbf{A}\mathbf{B} \quad \Leftrightarrow \quad C_{ij} = A_{ik} B_{kj}$$

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} = \begin{bmatrix} (A_{11}B_{11}+A_{12}B_{21}+A_{13}B_{31}) & (A_{11}B_{12}+A_{12}B_{22}+A_{13}B_{32}) & (A_{11}B_{13}+A_{12}B_{23}+A_{13}B_{33}) \\ (A_{21}B_{11}+A_{22}B_{21}+A_{23}B_{31}) & (A_{21}B_{12}+A_{22}B_{22}+A_{23}B_{32}) & (A_{21}B_{13}+A_{22}B_{23}+A_{23}B_{33}) \\ (A_{31}B_{11}+A_{32}B_{21}+A_{33}B_{31}) & (A_{31}B_{12}+A_{32}B_{22}+A_{33}B_{32}) & (A_{31}B_{13}+A_{32}B_{23}+A_{33}B_{33}) \end{bmatrix}$$

OPERATIONS ON TENSORS

MATRIX MULTIPLICATION



$$
\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}
\begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}
\;\rightarrow\;
\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}
$$

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OPERATIONS ON TENSORS

MATRIX MULTIPLICATION

column j

$j=1$

$$\begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}$$

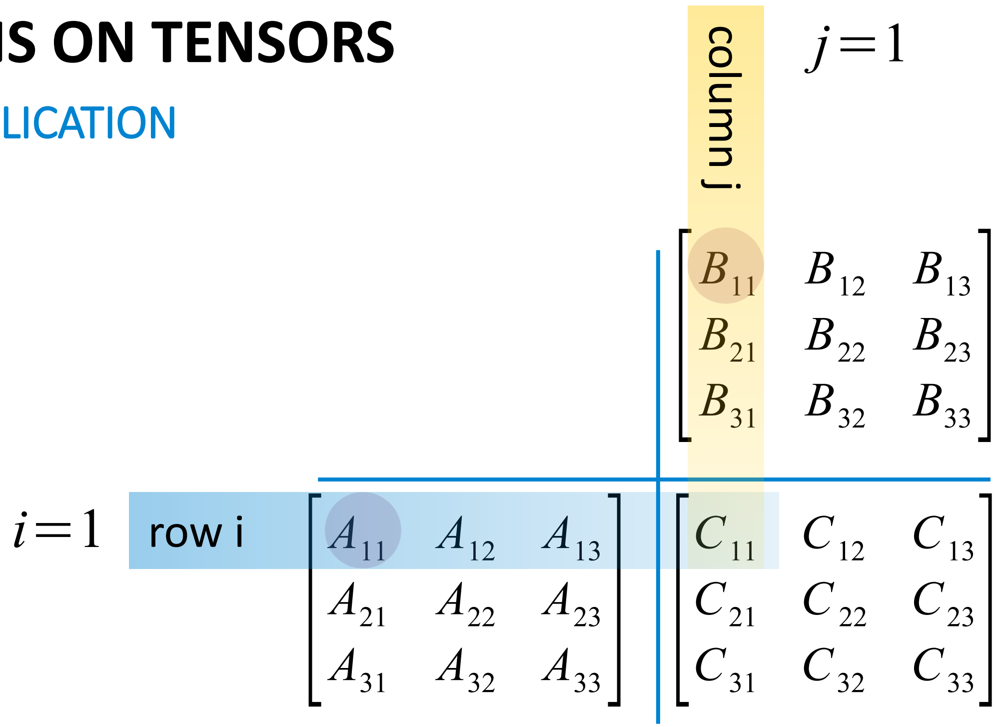
$i=1$ row i

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \quad \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$C_{ij} = C_{11} =$$

OPERATIONS ON TENSORS

MATRIX MULTIPLICATION



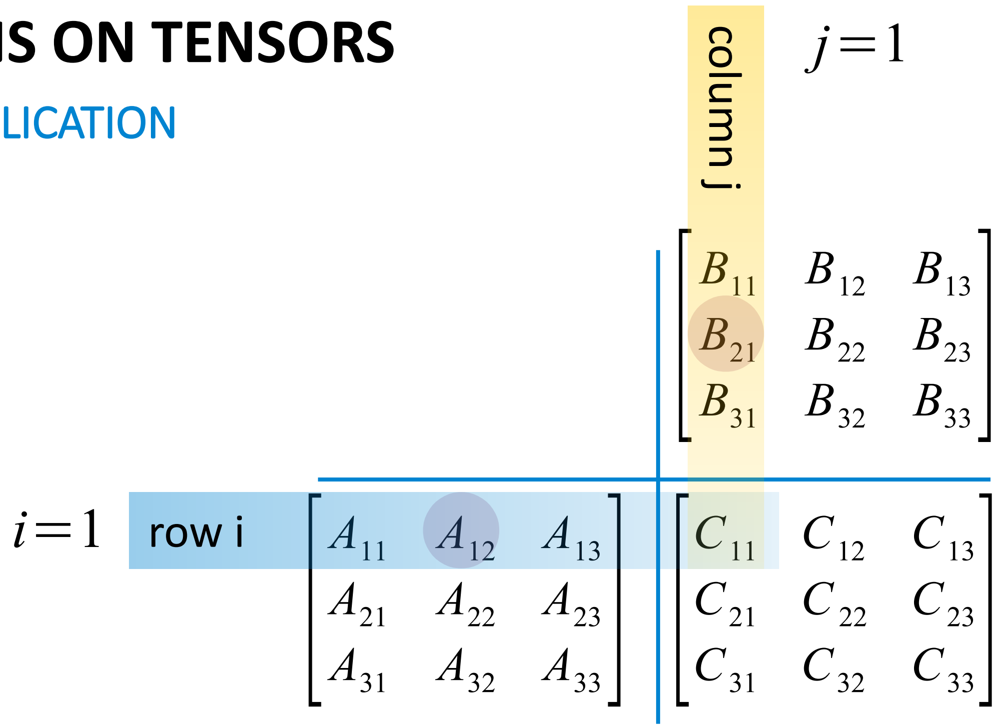
$$\begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}$$

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \quad \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$C_{ij} = C_{11} = A_{11} B_{11}$$

OPERATIONS ON TENSORS

MATRIX MULTIPLICATION



$$C_{ij} = C_{11} = A_{11} B_{11} + A_{12} B_{21}$$

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OPERATIONS ON TENSORS

MATRIX MULTIPLICATION

column j $\quad j=1$

$$\begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}$$

$i=1$ row i

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \quad \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$C_{ij} = C_{11} = A_{11} B_{11} + A_{12} B_{21} + A_{13} B_{31} = \sum_{k=1}^{3} A_{1k} B_{k1} \qquad \Rightarrow \qquad C_{ij} = A_{ik} B_{kj}$$

OPERATIONS ON TENSORS

MATRIX MULTIPLICATION

column j $\quad j=3$

$$\begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}$$

$i=2 \quad$ row i

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \quad \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$C_{ij} = C_{23} =$$

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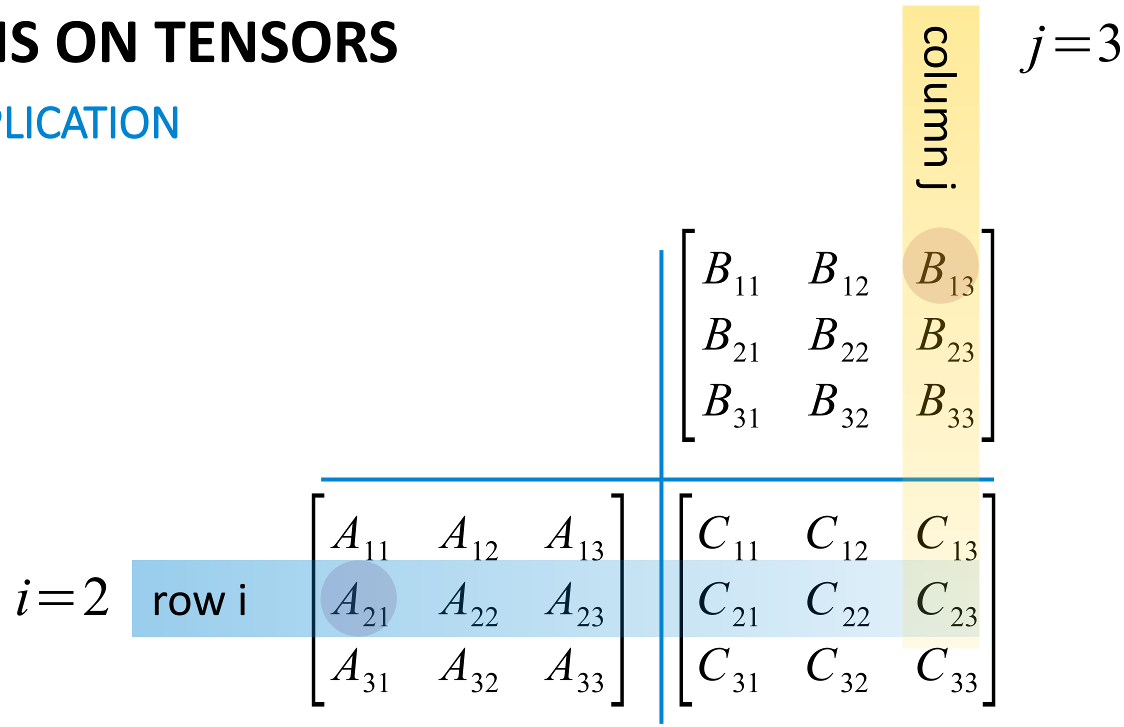
OPERATIONS ON TENSORS

MATRIX MULTIPLICATION



$$C_{ij} = C_{23} = A_{21} B_{13}$$

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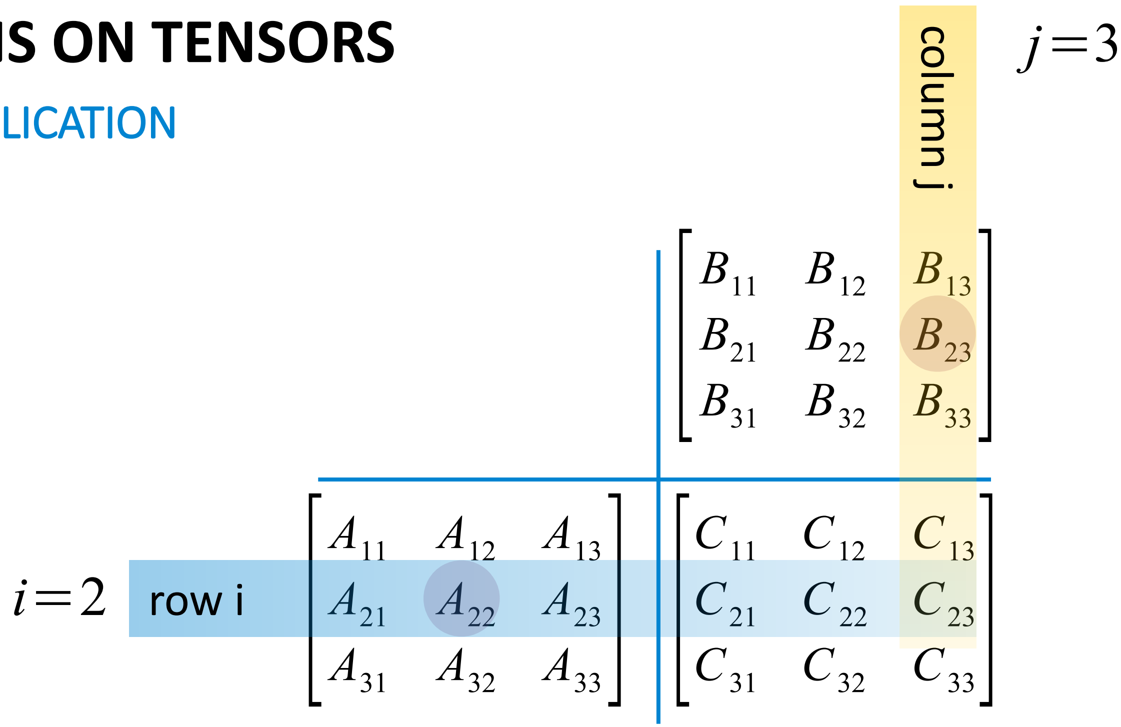
OPERATIONS ON TENSORS

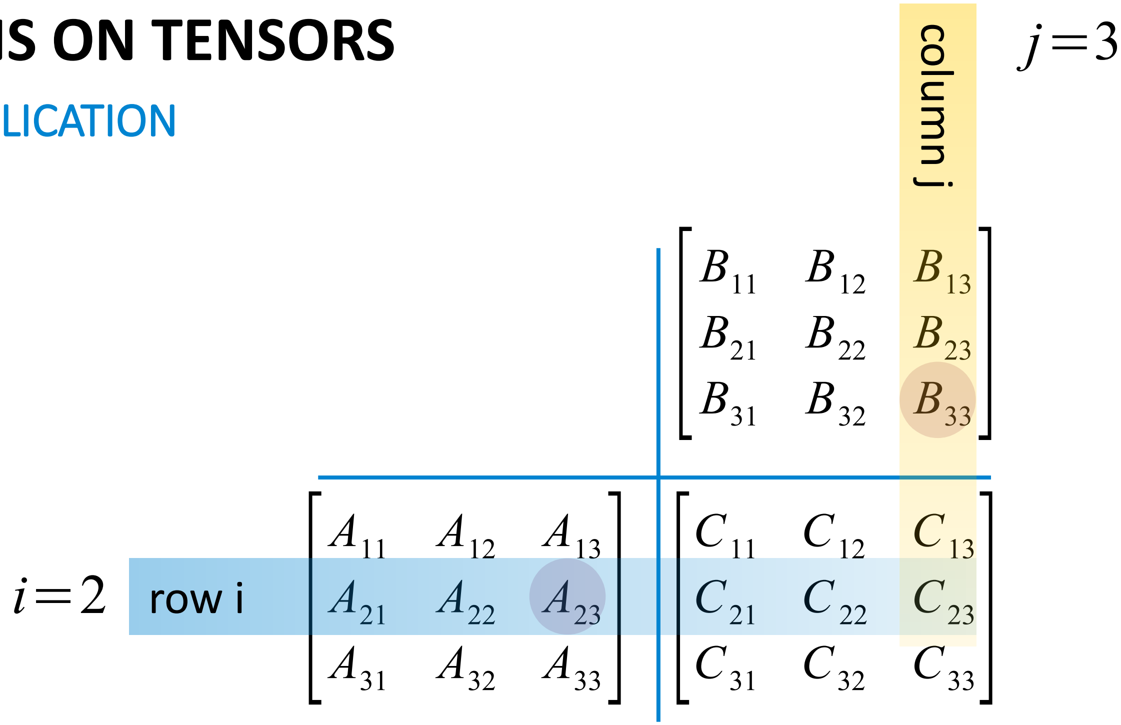
MATRIX MULTIPLICATION



$$C_{ij} = C_{23} = A_{21} B_{13} + A_{22} B_{23}$$

OPERATIONS ON TENSORS

MATRIX MULTIPLICATION



$$C_{ij} = C_{23} = A_{21}B_{13} + A_{22}B_{23} + A_{23}B_{33} = \sum_{1}^{3} A_{2k}B_{k3} \quad \Rightarrow \quad C_{ij} = A_{ik}B_{kj}$$

OPERATIONS ON TENSORS

INNER DOT PRODUCT OF A TENSOR AND OF A VECTOR

- An inner dot product of a tensor and of a vector gives a vector as a result. Tensor may be then identified with a certain map (and also with a certain matrix) which associates a vector to a vector.

$$\mathbf{y} = \mathbf{A}\,\mathbf{x} \quad \Leftrightarrow \quad y_i = A_{ij} x_j$$

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} A_{11} x_1 + A_{12} x_2 + A_{13} x_3 \\ A_{21} x_1 + A_{22} x_2 + A_{23} x_3 \\ A_{31} x_1 + A_{32} x_2 + A_{33} x_3 \end{bmatrix}$$

- If the **vectors** are represented by **single-column matrices** then such a map is realized by **left-side inner dot product** (**left-side multiplication by a square matrix**)

- In an index notation such a product is denoted by **repeated neighbouring indices**.

OPERATIONS ON TENSORS

INNER DOT PRODUCT OF A TENSOR AND OF A VECTOR

- Right-side inner dot product is calculated in the same way:

$$\mathbf{y} = \mathbf{x}\,\mathbf{A} \qquad \Leftrightarrow \qquad y_i = x_j A_{ji}$$

but in the matrix notation the vector should be then represented by a single-row matrix

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} (A_{11}x_1 + A_{21}x_2 + A_{31}x_3) & (A_{12}x_1 + A_{22}x_2 + A_{32}x_3) & (A_{31}x_1 + A_{23}x_2 + A_{33}x_3) \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} A_{11}x_1 + A_{21}x_2 + A_{31}x_3 \\ A_{12}x_1 + A_{22}x_2 + A_{32}x_3 \\ A_{31}x_1 + A_{23}x_2 + A_{33}x_3 \end{bmatrix}^{\mathrm{T}}$$

$$[x_j]^{\mathrm{T}}[A_{ji}] = [y_i]^{\mathrm{T}}$$

- Operations on tensors are not always associated in an easy and natural way with operations of matrix calculus.

OPERATIONS ON TENSORS

SINGLE INNER DOT PRODUCT OF TENSORS

$$\mathbf{A}\mathbf{1} = \mathbf{A} \quad \Leftrightarrow \quad k=1,2,3: \quad A_{ij}\delta_{jk} = A_{i1}\delta_{1k} + A_{i2}\delta_{2k} + A_{i3}\delta_{3k} = A_{ik}$$

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$\mathbf{1}\mathbf{v} = \mathbf{v} \quad \Leftrightarrow \quad j=1,2,3: \quad v_i\delta_{ij} = v_1\delta_{1j} + v_2\delta_{2j} + v_3\delta_{3j} = v_j$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

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OPERATIONS ON TENSORS

SINGLE INNER DOT PRODUCT OF TENSORS

- Transpose of a product is a product of transposes with switched sequence of multiplication

$$\mathbf{C} = \mathbf{A}\,\mathbf{B} \quad \Rightarrow \quad C_{ij} = A_{ik}\,B_{kj}$$

$$\mathbf{C}^{\mathrm{T}} = (\mathbf{A}\,\mathbf{B})^{\mathrm{T}} \quad \Rightarrow \quad (C^{\mathrm{T}})_{ij} = C_{ji} = A_{jk}\,B_{ki} = (A^{\mathrm{T}})_{kj}(B^{\mathrm{T}})_{ik} = (B^{\mathrm{T}})_{ik}(A^{\mathrm{T}})_{kj} \quad \Rightarrow \quad \boxed{(\mathbf{A}\,\mathbf{B})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\,\mathbf{A}^{\mathrm{T}}}$$

- Inverse matrix:

$$\boxed{\mathbf{B} = \mathbf{A}^{-1} \quad \Leftrightarrow \quad \mathbf{A}\,\mathbf{B} = \mathbf{B}\,\mathbf{A} = \mathbf{1} \quad \Leftrightarrow \quad A_{ik}\,B_{kj} = B_{ik}\,A_{kj} = \delta_{ij}}$$

- A transpose of inverse matrix is an **inverse of transposed matrix**:

$$\left.\begin{aligned} \mathbf{1}^{\mathrm{T}} = \mathbf{1} \quad &\Rightarrow \quad (\mathbf{A}\,\mathbf{A}^{-1})^{\mathrm{T}} = \mathbf{1} \quad \Rightarrow \quad (\mathbf{A}^{-1})^{\mathrm{T}}\,\mathbf{A}^{\mathrm{T}} = \mathbf{1} \\ \mathbf{1}^{\mathrm{T}} = \mathbf{1} \quad &\Rightarrow \quad (\mathbf{A}^{-1}\,\mathbf{A})^{\mathrm{T}} = \mathbf{1} \quad \Rightarrow \quad \mathbf{A}^{\mathrm{T}}(\mathbf{A}^{-1})^{\mathrm{T}} = \mathbf{1} \end{aligned}\right\} \quad \Rightarrow \quad (\mathbf{A}^{-1})^{\mathrm{T}}\,\mathbf{A}^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}}(\mathbf{A}^{-1})^{\mathrm{T}} = \mathbf{1} \quad \Rightarrow$$

$$\boxed{(\mathbf{A}^{-1})^{\mathrm{T}} = (\mathbf{A}^{\mathrm{T}})^{-1} = \mathbf{A}^{-\mathrm{T}}}$$

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OPERATIONS ON TENSORS

SINGLE INNER DOT PRODUCT

- **Dot product of two vectors** – scalar product

$$\mathbf{a}\cdot\mathbf{b} = a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$$

- Dot product of two vectors may be realized by a matrix multiplication of a transpose of the matrix representation of the first vector and of the matrix representation of the second vector:

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{a}^{\mathrm{T}}\mathbf{b} \quad \Leftrightarrow \quad [a_i]^{\mathrm{T}}[b_i]$$

	$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$
$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$	$a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$

We will use the following rules in notation:

- **Dot** will be used to denote **scalar product**.
- **Single inner dot product** associated with **matrix multiplication** will be denoted w**ithout any sign**.

- Length of a vector:

$$|\mathbf{v}| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{\mathbf{v}^{\mathrm{T}}\mathbf{v}} = \sqrt{v_i v_i} = \sqrt{v_1^2+v_2^2+v_3^2}$$

OPERATIONS ON TENSORS

DOUBLE INNER DOT PRODUCT OF TENSORS – SCALAR PRODUCT OF TENSORS

- Scalar product of two tensors will be calculated in a similar way as in case of vectors as a sum of products of respective components. This may be referred to as a double inner dot product, since summation is performed with respect to two indices.

$$\mathbf{A}\cdot\mathbf{B} = A_{ij}B_{ij}$$

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \cdot \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} = \begin{matrix} A_{11}B_{11} + A_{12}B_{12} + A_{13}B_{13} + \\ + A_{21}B_{21} + A_{22}B_{22} + A_{23}B_{23} + \\ + A_{31}B_{31} + A_{32}B_{32} + A_{33}B_{33} \end{matrix}$$

OPERATIONS ON TENSORS

CONTRACTION OF A TENSOR

- Contraction of a tensor means **summation with respect to chosen pair of indices**:

$$A_{ij}\delta_{ij} = A_{11}+A_{22}+A_{33} = A_{ii} = A_{jj} = A_{kk}$$

- Trace of a matrix $\quad \mathrm{tr}(\mathbf{A}) = \mathbf{A}\cdot\mathbf{1} \quad \Leftrightarrow \quad \mathrm{tr}(\mathbf{A}) = A_{ij}\delta_{ij} = A_{ii} = A_{11}+A_{22}+A_{33}$
- Scalar product of tensors $\quad \mathrm{tr}(\mathbf{A}\,\mathbf{B}^{\mathrm{T}}) = \mathbf{A}\cdot\mathbf{B} \quad \Leftrightarrow \quad \delta_{ik}A_{kj}(B^{\mathrm{T}})_{ji} = \delta_{ik}A_{kj}B_{ij} = A_{ij}B_{ij}$
- Divergence of a vector field $\quad \mathrm{tr}(\mathbf{v}\otimes\nabla) = \nabla\cdot\mathbf{v} \quad \Leftrightarrow \quad v_{i,j}\delta_{ij} = v_{i,i} = \dfrac{\partial v_1}{\partial x_1}+\dfrac{\partial v_2}{\partial x_2}+\dfrac{\partial v_3}{\partial x_3}$
- Laplacian of a scalar field $\quad \mathrm{tr}(\phi\otimes\nabla\otimes\nabla) = \Delta\phi \quad \Leftrightarrow \quad \phi_{,ij}\delta_{ij} = \phi_{,ii} = \dfrac{\partial^2\phi}{\partial x_1^2}+\dfrac{\partial^2\phi}{\partial x_2^2}+\dfrac{\partial^2\phi}{\partial x_3^2}$

THANK YOU FOR YOUR ATTENTION