THEORY OF ELASTICITY AND PLASTICITY

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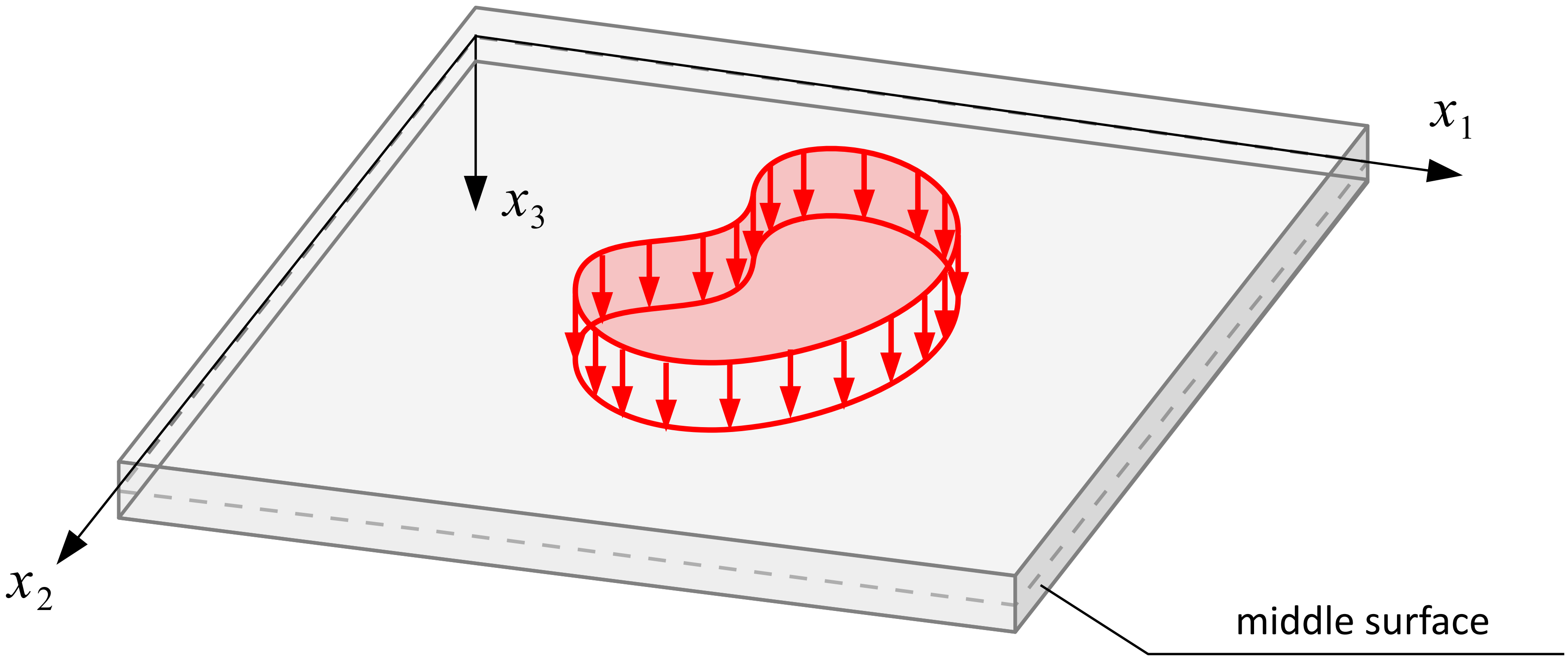
THE KIRCHHOFF – LOVE THEORY OF THIN ELASTIC PLATES

THE KIRCHHOFF – LOVE THEORY OF THIN PLATES

Elastic plate – it is an **elastic surface system**, the **transverse dimension of which is much smaller than its dimensions in plan**, it is **loaded in a way perpendicular to the middle surface** (in a way parallel to the smallest dimensions).



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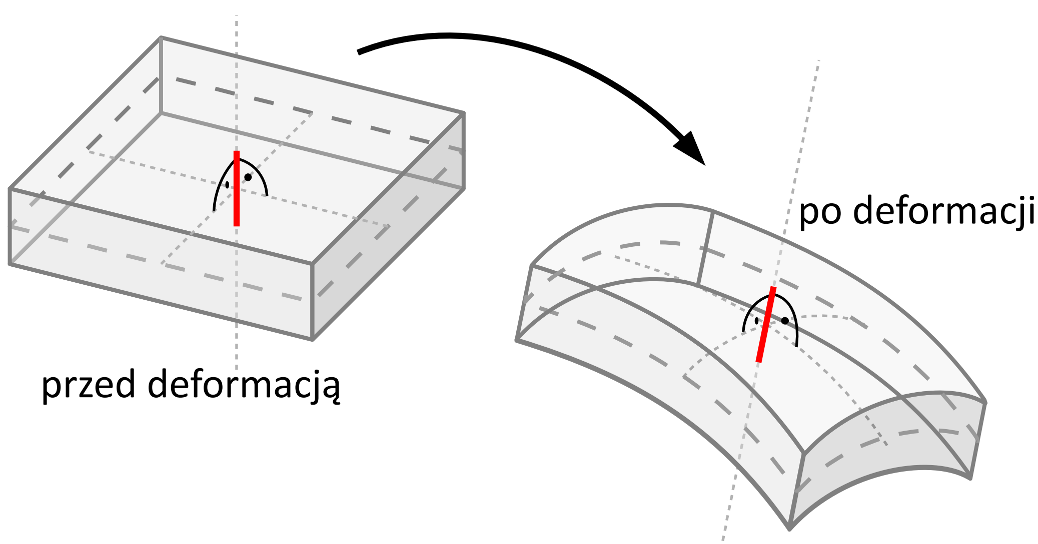
THE KIRCHHOFF – LOVE THEORY OF THIN PLATES

One of the models describing elastic plates is the theory of thin plates by Kirchhoff and Love. The fundamental assumption of this theory concerns the deformation of the plate and it is a **two-dimensional generalization of the Bernoulli's hypothesis of plane cross-sections**.

KIRCHHOFF'S HYPOTHESIS

Straight segment which is perpendicular to the middle surface before deformation remains straight and perpendicular to the deformed middle surface



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THE KIRCHHOFF – LOVE THEORY OF THIN PLATES

KIRCHHOFF'S HYPOTHESIS

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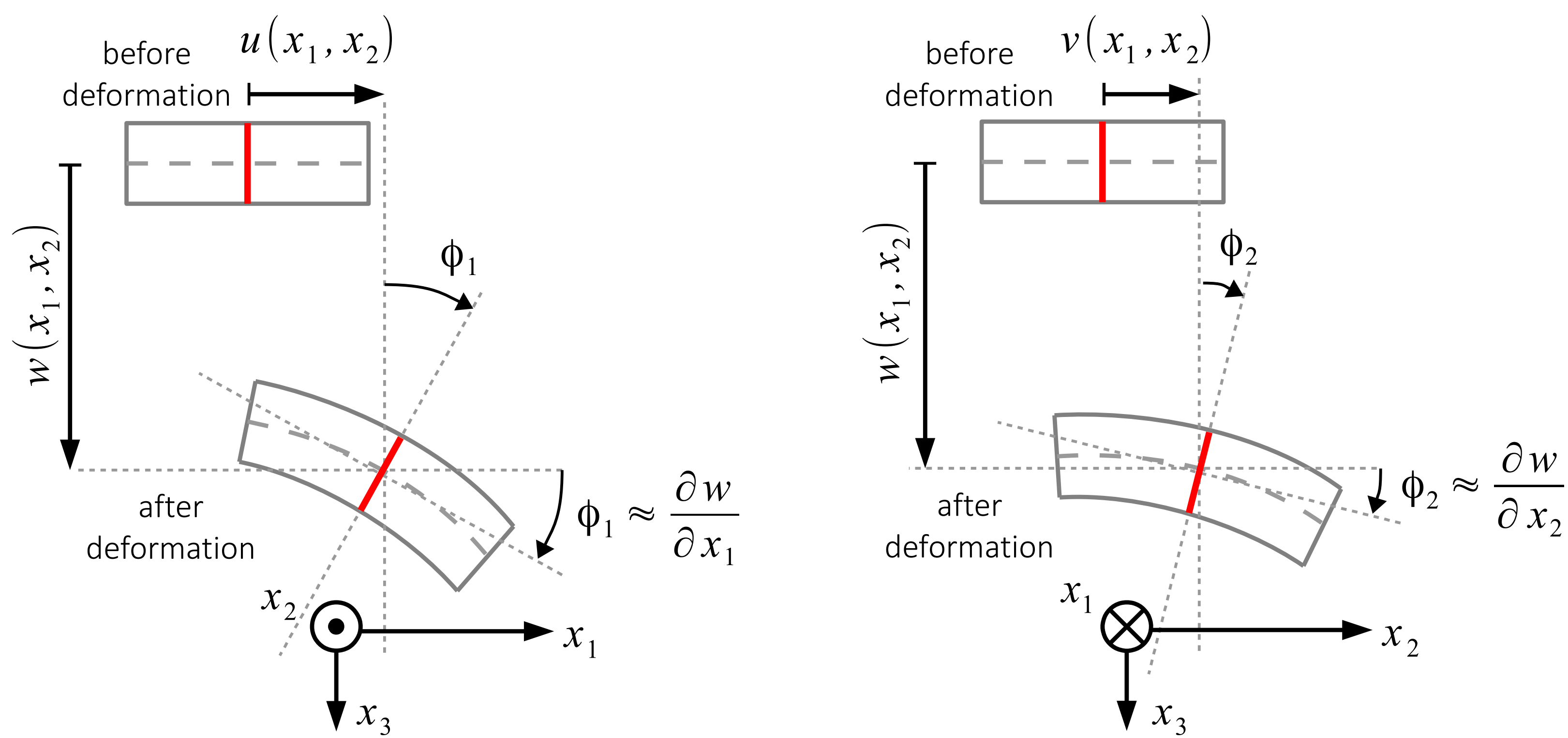


THE KIRCHHOFF – LOVE THEORY OF THIN PLATES

ASSUMPTIONS:

- **Linear theory of elasticity** is valid**:**
  - small displacements
  - small strains – linear kinematic relations
  - linear constitutive relations (linear elastic Hooke's material)
- **Kirchhoff's hypothesis** is valid.
- The transverse dimension of the plate is much smaller than the dimensions in plan.
- **normal stress perpendicular to the middle surface is negligibly small:** $\sigma_{33} \approx 0$

KINEMATIC RELATIONS IN THIN PLATES

Displacement vector of the points of the middle surface $(z \neq 0)$:

$$\mathbf{u}(x_1, x_2, 0) = \begin{bmatrix} u(x_1, x_2) \\ v(x_1, x_2) \\ w(x_1, x_2) \end{bmatrix}$$

Displacement vector of other point of coordinate $z \neq 0$ is determined according to the **Kirchhoff's hypothesis**:

$$\mathbf{u}(x_1, x_2, x_3) = \begin{bmatrix} u(x_1, x_2) - \dfrac{\partial w(x_1, x_2)}{\partial x_1} \cdot x_3 \\ v(x_1, x_2) - \dfrac{\partial w(x_1, x_2)}{\partial x_2} \cdot x_3 \\ w(x_1, x_2) \end{bmatrix}$$

KINEMATIC RELATIONS IN THIN PLATES

General **kinematic relation**:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Enable us to determine the components of the **strain tensor** :

$$\varepsilon_{11} = \frac{\partial u}{\partial x_1} - \frac{\partial^2 w}{\partial x_1^2}\cdot x_3, \quad \varepsilon_{22} = \frac{\partial v}{\partial x_2} - \frac{\partial^2 w}{\partial x_2^2}\cdot x_3, \quad \varepsilon_{33} = \frac{\partial w}{\partial x_3} = 0,$$

$$\varepsilon_{12} = \frac{1}{2}\left[\frac{\partial}{\partial x_1}\left(v - \frac{\partial w}{\partial x_2}x_3\right) + \frac{\partial}{\partial x_2}\left(u - \frac{\partial w}{\partial x_1}x_3\right)\right] = \frac{1}{2}\left[\frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1} - 2\frac{\partial w}{\partial x_1 \partial x_2}x_3\right],$$

$$\varepsilon_{23} = \frac{1}{2}\left[\frac{\partial}{\partial x_3}\left(v - \frac{\partial w}{\partial x_2}x_3\right) + \frac{\partial w}{\partial x_2}\right] = 0, \quad \varepsilon_{31} = \frac{1}{2}\left[\frac{\partial}{\partial x_3}\left(u - \frac{\partial w}{\partial x_1}x_3\right) + \frac{\partial w}{\partial x_1}\right] = 0$$

KINEMATIC RELATIONS IN THIN PLATES

REMARKS:

- Vanishing of the distortional strains ε_{31} and ε_{23} is due to **Kirchhoff's hypothesis**.
- Due to linearity of constitutive relations also corresponding stresses are vanishing: $\sigma_{31}=\sigma_{23}=0$
- We suspect, however, that in a transversally loaded plate (along x_3 variable) also shear stress σ_{31} and σ_{23} should be present.

KINEMATIC RELATIONS IN THIN PLATES

Kinematic relations may be formally rewritten in a different way

$$\varepsilon_{11} = \bar{\varepsilon}_{11} + \varkappa_{11} x_3 , \qquad \varepsilon_{22} = \bar{\varepsilon}_{22} + \varkappa_{22} x_3 , \qquad \varepsilon_{33} = 0,$$

$$\varepsilon_{12} = \bar{\varepsilon}_{12} + \varkappa_{12} x_3 \qquad \varepsilon_{31} = \frac{1}{2}\left(\phi_1 + \frac{\partial w}{\partial x_1}\right) \qquad \varepsilon_{23} = \frac{1}{2}\left(\phi_2 + \frac{\partial w}{\partial x_2}\right)$$

where:

$$\bar{\varepsilon}_{11} = \frac{\partial u}{\partial x_1}, \qquad \bar{\varepsilon}_{22} = \frac{\partial v}{\partial x_2}, \qquad \bar{\varepsilon}_{12} = \frac{1}{2}\left(\frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1}\right), \qquad \phi_1 = -\frac{\partial w}{\partial x_1}, \qquad \phi_2 = -\frac{\partial w}{\partial x_2},$$

$$\varkappa_{11} = \frac{\partial \phi_1}{\partial x_1} = -\frac{\partial^2 w}{\partial x_1^2}, \qquad \varkappa_{22} = \frac{\partial \phi_2}{\partial x_2} = -\frac{\partial^2 w}{\partial x_2^2}, \qquad \varkappa_{12} = \left(\frac{\partial \phi_1}{\partial x_2} + \frac{\partial \phi_2}{\partial x_1}\right) = -2\frac{\partial^2 w}{\partial x_1 \partial x_2}$$

The „trick” is that we are considering ϕ_i and $\dfrac{\partial w}{\partial x_i}$ as independent quantities.

CONSTITUTIVE RELATIONS IN THIN PLATES

- Mechanical state, in which the particles of the plate are, is a **plane strain state**.
- Due to **negligibly small magnitude of the transverse normal stress** $\sigma_{33} \approx 0$ we may consider it to be also approximately **plane stress state**.
- Generalized elastic constants in the constitutive relations for plane states may be considered identical with true elastic constant describing three-dimensional problems.
- Strictly speaking, this corresponds with a problem, in which the **Poisson ratio is zero**.

CONSTITUTIVE RELATIONS IN THIN PLATES

Constitutive relations for **isotropic** plates:

$$\sigma_{11}=\frac{E}{1-\nu^2}(\varepsilon_{11}+\nu\varepsilon_{22})=\frac{E}{1-\nu^2}\left[\left(\frac{\partial u}{\partial x_1}-\frac{\partial^2 w}{\partial x_1^2}\cdot x_3\right)+\nu\left(\frac{\partial v}{\partial x_2}-\frac{\partial^2 w}{\partial x_2^2}\cdot x_3\right)\right]$$

$$\sigma_{22}=\frac{E}{1-\nu^2}(\varepsilon_{22}+\nu\varepsilon_{11})=\frac{E}{1-\nu^2}\left[\left(\frac{\partial v}{\partial x_2}-\frac{\partial^2 w}{\partial x_2^2}\cdot x_3\right)+\nu\left(\frac{\partial u}{\partial x_1}-\frac{\partial^2 w}{\partial x_1^2}\cdot x_3\right)\right]$$

$$\sigma_{12}=\frac{E}{1+\nu}\varepsilon_{12}=\frac{E}{2(1+\nu)}\left[\frac{\partial u}{\partial x_2}+\frac{\partial v}{\partial x_1}-2\frac{\partial w}{\partial x_1\partial x_2}x_3\right]$$

In **matrix** notation:

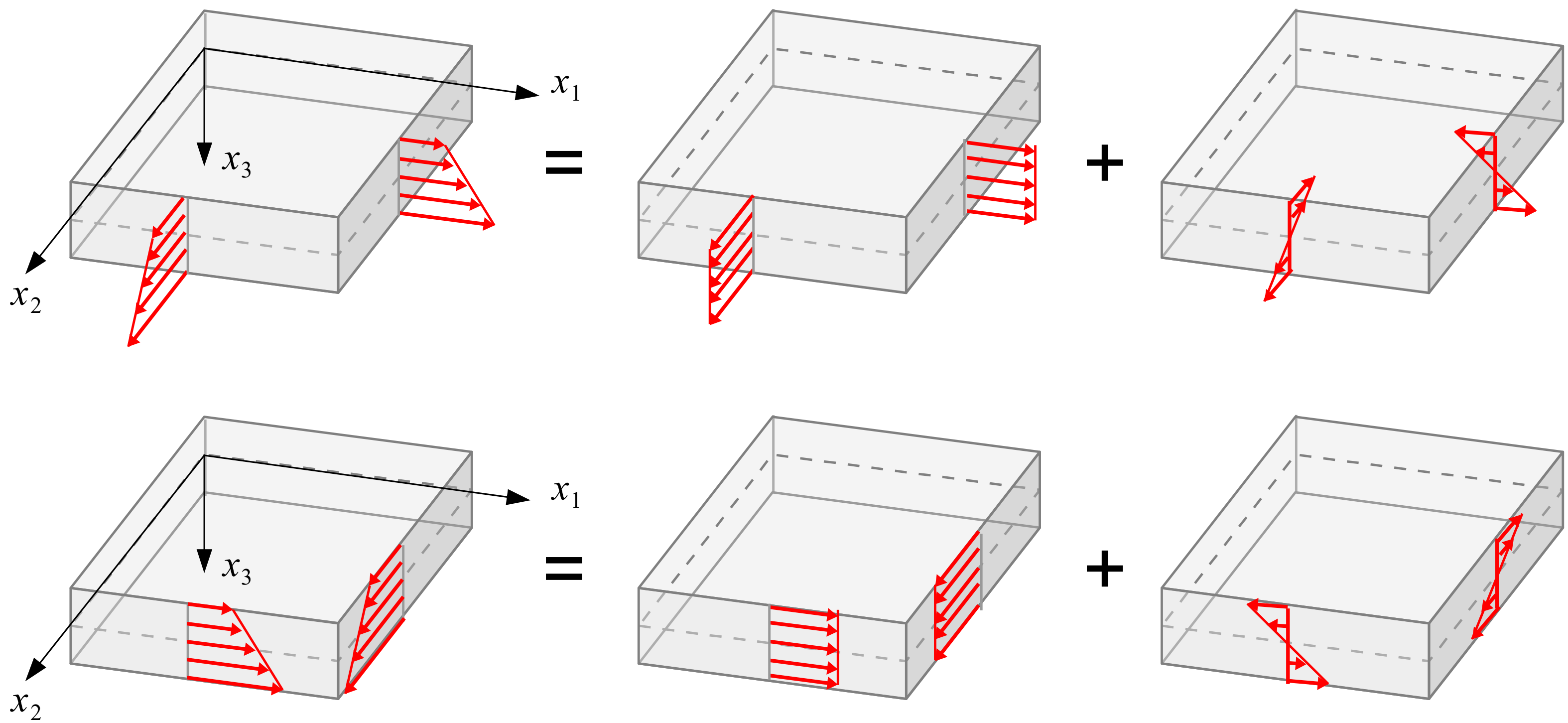
$$\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{12}\end{bmatrix}=\frac{1}{E}\begin{bmatrix}1 & -\nu & 0\\ & 1 & 0\\ \text{sym} & & 1+\nu\end{bmatrix}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix}\quad\Leftrightarrow\quad\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix}=\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ & 1 & 0\\ \text{sym} & & 1-\nu\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{12}\end{bmatrix}$$

CROSS-SECTIONAL FORCES IN THIN PLATES

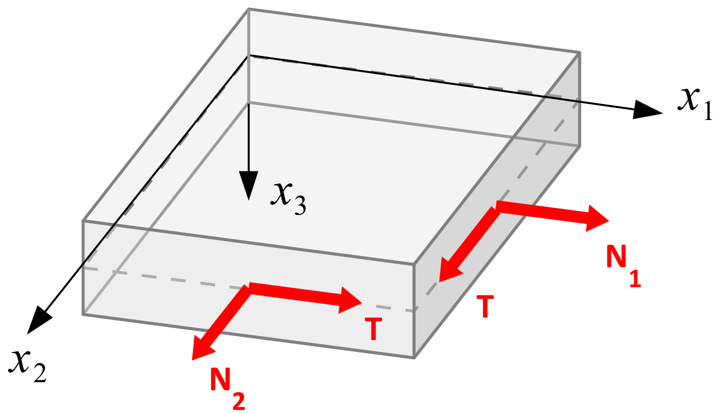
According to the constitutive relations it can be noticed that for a fixed material fibre which is perpendicular to the middle surface, stresses varies linearly along x_3 .

Normal and shear stresses can be decomposed then into:

- symmetric **constant** component
- skew-symmetric component varying **linearly**



CROSS-SECTIONAL FORCES IN THIN PLATES



Normal force – sum of the **symmetric** part of **normal** stresses

$$N_1 = \int_{x_3=-h/2}^{h/2} \sigma_{11}\, \mathrm{d}x_3 = \frac{Eh}{1-\nu^2}\left[\frac{\partial u}{\partial x_1} + \nu \frac{\partial v}{\partial x_2}\right]$$

$$N_2 = \int_{x_3=-h/2}^{h/2} \sigma_{22}\, \mathrm{d}x_3 = \frac{Eh}{1-\nu^2}\left[\frac{\partial v}{\partial x_2} + \nu \frac{\partial u}{\partial x_1}\right]$$

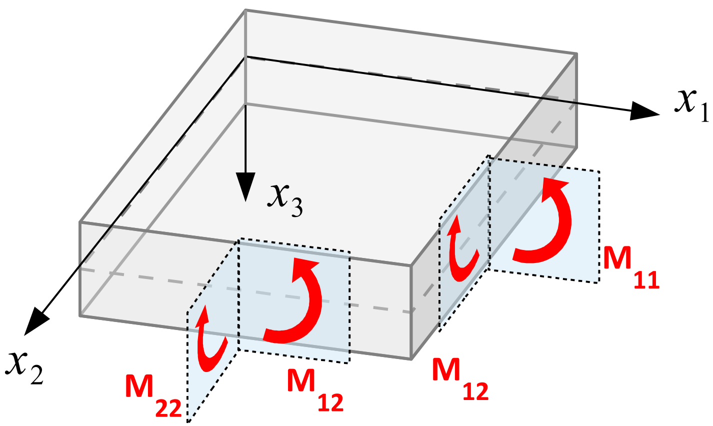
Shear force – sum of the **symmetric** part of **shear** stresses

$$T = \int_{x_3=-h/2}^{h/2} \sigma_{12}\, \mathrm{d}x_3 = \frac{Eh}{1-\nu^2}\frac{1-\nu}{2}\left[\frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1}\right]$$

REMARK:

- **physical dimension** of **normal** and **shear forces** is $\mathrm{N/m}$ – it is linear density of forces related to a cross-section of unit width.

CROSS-SECTIONAL FORCES IN THIN PLATES



Bending moment – moment of **skew-symmetric** part of **normal** stresses about a point in the middle surface.

$$M_{11} = \int_{x_3=-h/2}^{h/2} x_3 \sigma_{11} \, d x_3 = -\frac{E h^3}{12(1-\nu^2)} \left[\frac{\partial^2 w}{\partial x_1^2} + \nu \frac{\partial^2 w}{\partial x_2^2} \right]$$

$$M_{22} = \int_{x_3=-h/2}^{h/2} x_3 \sigma_{22} \, d x_3 = -\frac{E h^3}{12(1-\nu^2)} \left[\frac{\partial^2 w}{\partial x_2^2} + \nu \frac{\partial^2 w}{\partial x_1^2} \right]$$

Twisting moment – **moment of skew-symmetric** part of **shear** stresses about a point in the middle surface.

$$M_{12} = \int_{x_3=-h/2}^{h/2} x_3 \sigma_{12} \, d x_3 = -(1-\nu) \frac{E h^3}{12(1-\nu^2)} \left[\frac{\partial^2 w}{\partial x_1 \partial x_2} \right]$$

REMARK:

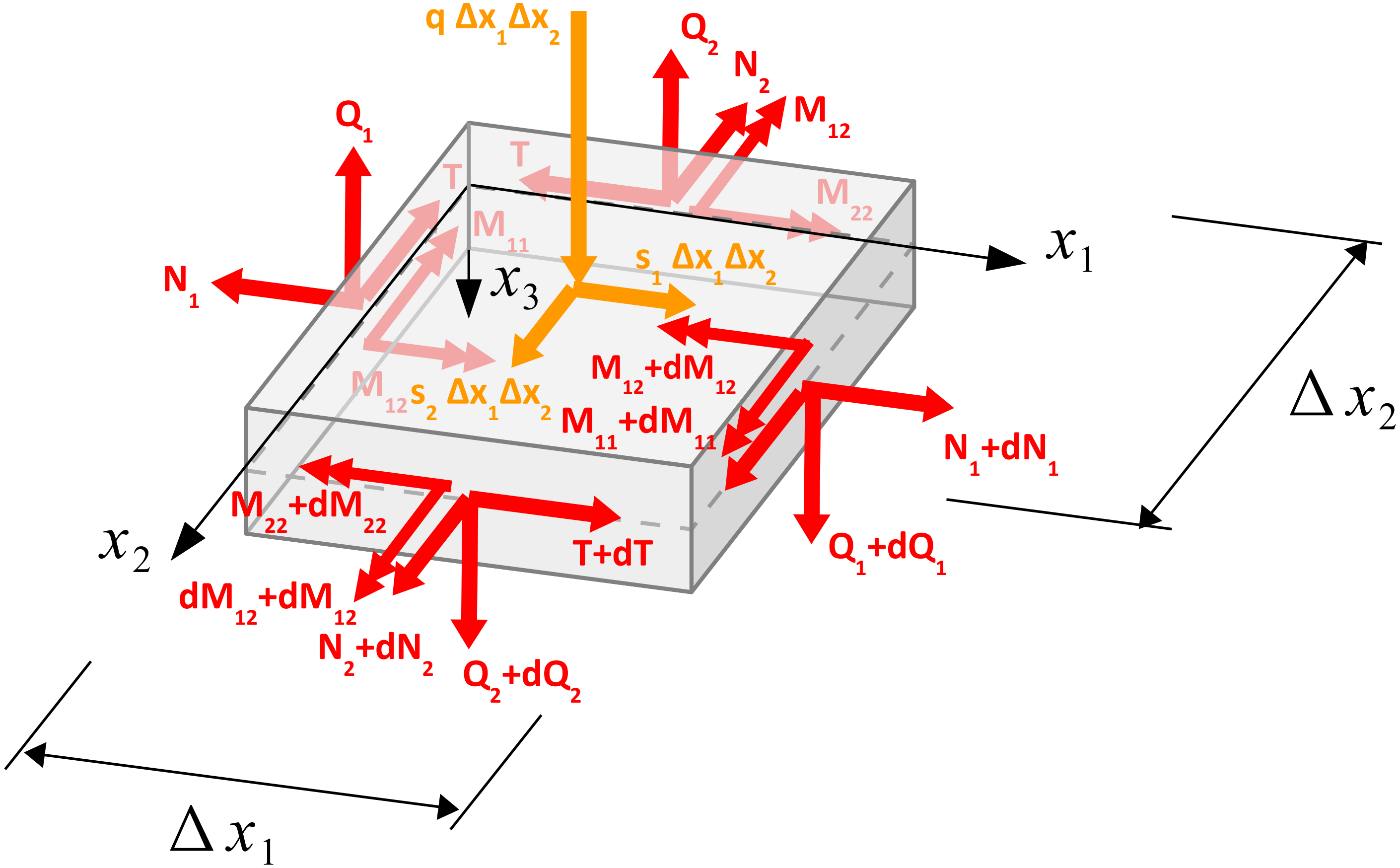
- **physical dimension** of **bending** and **twisting moments** is $\mathrm{N/m}$ – it is linear density of moments related to a cross-section of unit width.

CROSS-SECTIONAL FORCES IN THIN PLATES

REAMRK:

- According to the assumptions of the Kirchhoff – Love theory, **shear stresses** $\sigma_{31}, \sigma_{23} = 0$
- **Transverse shear force in thin plates** is formally equal to **zero**.
- We can observe such forces, however – it is important in verification of the bearing-capacity of plates. In case of plates of moderate thickness it can be also observed that shear stresses influence the magnitude of deflection.
- In order to derive the **equilibrium equations** of an infinitely small plate element it is assumed that there exist **non-zero transverse shear forces**.

EQUILIBRIUM EQUATIONS FOR THIN PLATES



We assume that an increment in the values of each cross-sectional force related to the increment of coordinate x_i with approximately equal:

$$dF = \frac{\partial F}{\partial x_i} \Delta x_i$$

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EQUILIBRIUM EQUATIONS FOR THIN PLATES

Equilibrium equations for a considered plate element for $\Delta x_1, \Delta x_2 \to 0$

$$\Sigma F_1 = 0: \quad \frac{\partial N_1}{\partial x_1} + \frac{\partial T}{\partial x_2} + s_1 = 0$$

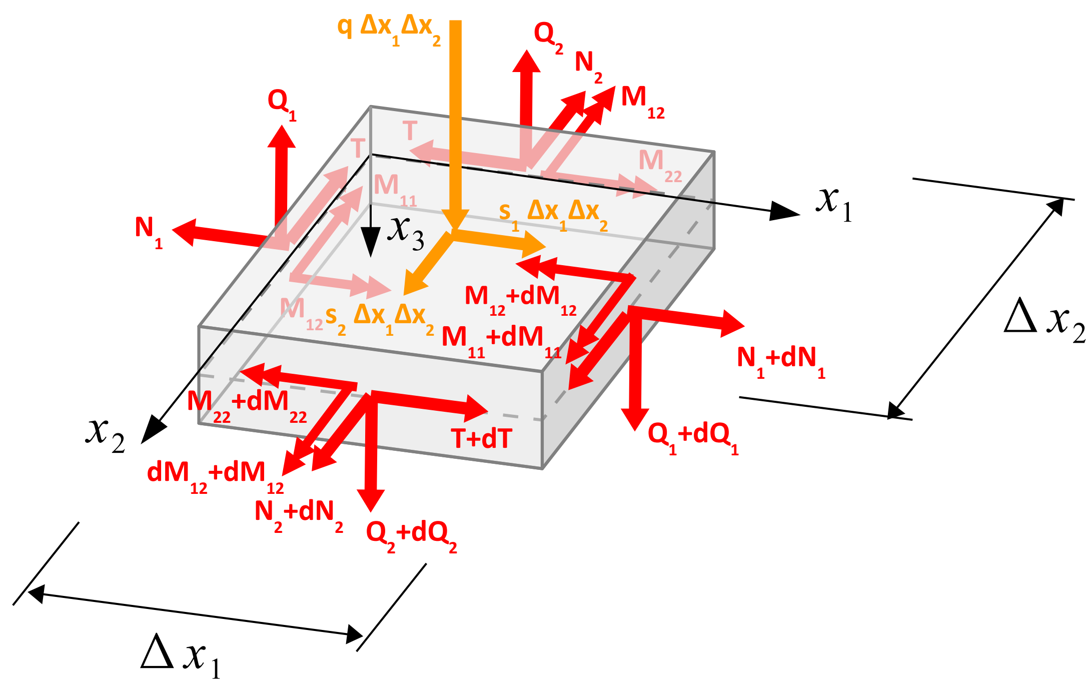
$$\Sigma F_2 = 0: \quad \frac{\partial N_2}{\partial x_2} + \frac{\partial T}{\partial x_1} + s_2 = 0$$

$$\Sigma F_3 = 0: \quad \frac{\partial Q_1}{\partial x_1} + \frac{\partial Q_2}{\partial x_2} + q = 0$$

$$\Sigma M_1 = 0: \quad Q_2 - \frac{\partial M_{12}}{\partial x_1} - \frac{\partial M_{22}}{\partial x_2} = 0$$

$$\Sigma M_2 = 0: \quad Q_1 - \frac{\partial M_{11}}{\partial x_1} - \frac{\partial M_{12}}{\partial x_2} = 0$$

$$\Sigma M_3 = 0: \quad T_{12} - T_{21} = 0 \quad \Rightarrow \quad T_{12} = T_{21} = T$$



EQUILIBRIUM EQUATIONS FOR THIN PLATES

Equations of equilibrium of moments become the **definition of** transverse shear forces:

$$Q_1 = \frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2} = -\frac{E h^3}{12(1-\nu^2)}\left[\frac{\partial^3 w}{\partial x_1^3} + \frac{\partial^3 w}{\partial x_1 \partial x_2^2}\right] = -\frac{E h^3}{12(1-\nu^2)}\frac{\partial}{\partial x_1}\nabla^2 w$$

$$Q_2 = \frac{\partial M_{12}}{\partial x_1} + \frac{\partial M_{22}}{\partial x_2} = -\frac{E h^3}{12(1-\nu^2)}\left[\frac{\partial^3 w}{\partial x_2^3} + \frac{\partial^3 w}{\partial x_2 \partial x_1^2}\right] = -\frac{E h^3}{12(1-\nu^2)}\frac{\partial}{\partial x_2}\nabla^2 w$$

EQUILIBRIUM EQUATIONS FOR THIN PLATES – MEMBRANE STATE

Let's consider **equilibrium equations in the middle surface**: $\Sigma F_1 = 0, \quad \Sigma F_2 = 0.$

$$\begin{cases} \dfrac{\partial N_1}{\partial x_1} + \dfrac{\partial T}{\partial x_2} + s_1 = 0 \\ \dfrac{\partial N_2}{\partial x_2} + \dfrac{\partial T}{\partial x_1} + s_2 = 0 \end{cases}$$

Cross-sectional forces may be expressed in terms of **displacements**

$$\begin{cases} \dfrac{Eh}{1-\nu^2}\dfrac{\partial}{\partial x_1}\left[\dfrac{\partial u}{\partial x_1} + \nu\dfrac{\partial v}{\partial x_2}\right] + \dfrac{Eh}{1-\nu^2}\dfrac{1-\nu}{2}\dfrac{\partial}{\partial x_2}\left[\dfrac{\partial u}{\partial x_2} + \dfrac{\partial v}{\partial x_1}\right] + s_1 = 0 \\ \dfrac{Eh}{1-\nu^2}\dfrac{\partial}{\partial x_2}\left[\dfrac{\partial v}{\partial x_2} + \nu\dfrac{\partial u}{\partial x_1}\right] + \dfrac{Eh}{1-\nu^2}\dfrac{1-\nu}{2}\dfrac{\partial}{\partial x_1}\left[\dfrac{\partial u}{\partial x_2} + \dfrac{\partial v}{\partial x_1}\right] + s_2 = 0 \end{cases}$$

EQUILIBRIUM EQUATIONS FOR THIN PLATES – MEMBRANE STATE

Let's denote: $\bar{\sigma}_{11} = \frac{E}{1-\nu^2}[\bar{\varepsilon}_{11} - \nu\bar{\varepsilon}_{22}], \quad \bar{\sigma}_{22} = \frac{E}{1-\nu^2}[\bar{\varepsilon}_{22} - \nu\bar{\varepsilon}_{11}], \quad \bar{\sigma}_{12} = \frac{E}{(1+\nu)}\bar{\varepsilon}_{12}$

where: $\bar{\varepsilon}_{11} = \frac{\partial u}{\partial x_1}, \quad \bar{\varepsilon}_{22} = \frac{\partial v}{\partial x_2}, \quad \bar{\varepsilon}_{12} = \frac{1}{2}\left(\frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1}\right),$

Then the **equilibrium equations in the middle surface** are as follows:

$$\begin{cases} \frac{\partial \bar{\sigma}_{11}}{\partial x_1} + \frac{\partial \bar{\sigma}_{12}}{\partial x_2} + \frac{s_1}{h} = 0 \\ \frac{\partial \bar{\sigma}_{12}}{\partial x_1} + \frac{\partial \bar{\sigma}_{22}}{\partial x_2} + \frac{s_2}{h} = 0 \end{cases}$$

These are **equilibrium equations** for a plane problem in the middle surface of the plate.

EQUILIBRIUM EQUATIONS FOR THIN PLATES – PLATE STATE

Equilibrium equation $\Sigma F_3 = 0$ after accounting for the moment equilibrium equations has the form:

$$\frac{\partial^2 M_{11}}{\partial x_1^2} + 2\frac{\partial^2 M_{12}}{\partial x_1 \partial x_2} + \frac{\partial^2 M_{22}}{\partial x_2^2} + q$$

Cross-sectional forces may be expressed in terms of **displacements**:

$$\frac{\partial^2}{\partial x_1^2}\left[\frac{\partial^2 w}{\partial x_1^2} + \nu\frac{\partial^2 w}{\partial x_2^2}\right] + 2\frac{\partial^2}{\partial x_1 \partial x_2}\left[(1-\nu)\frac{\partial^2 w}{\partial x_1 \partial x_2}\right] + \frac{\partial^2}{\partial x_2^2}\left[\frac{\partial^2 w}{\partial x_2^2} + \nu\frac{\partial^2 w}{\partial x_1^2}\right] = \frac{q}{D_b}$$

After transformations we obtain displacement equation governing the **plate** (**flexural**) **state**:

$$\frac{\partial^4 w}{\partial x_1^4} + 2\frac{\partial^4 w}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 w}{\partial x_2^4} = \frac{q}{D_b} \quad \Leftrightarrow \quad \nabla^4 w = \frac{q}{D_b}$$

where:

$$D_b = \frac{E h^3}{12(1-\nu^2)}$$ - **flexural rigidity**

EQUILIBRIUM EQUATIONS FOR THIN PLATES

REMARKS:

- If only the boundary conditions are formulated in such a way that
  - in-plane displacements and membrane forces are not determined by deflection and moments
  - deflection and moments are not determined by in-plane displacements and membrane forces

- Then the **membrane** state and **plate** state are **independent problems**:
  - membrane state is determined as a solution of a **plane problem**, e.g. with the use of the **Airy stress function**, which is a solution of a **homogeneous biharmonic equation**:

$$\nabla^4 F = 0$$

  - plate state is determined by the distribution of **deflection**, which is a solution of an **inhomogeneous biharmonic equation**:

$$\nabla^4 w = \frac{q}{D_b}$$

BOUNDARY CONDITIONS

- Kinematic boundary conditions for membrane **state**:

in-plane displacements at the boundary:	$\mathbf{u}(\mathbf{x}) = \mathbf{u_0}\ , \qquad \mathbf{x} \in \partial\Omega$

- Static boundary conditions for membrane **state**:

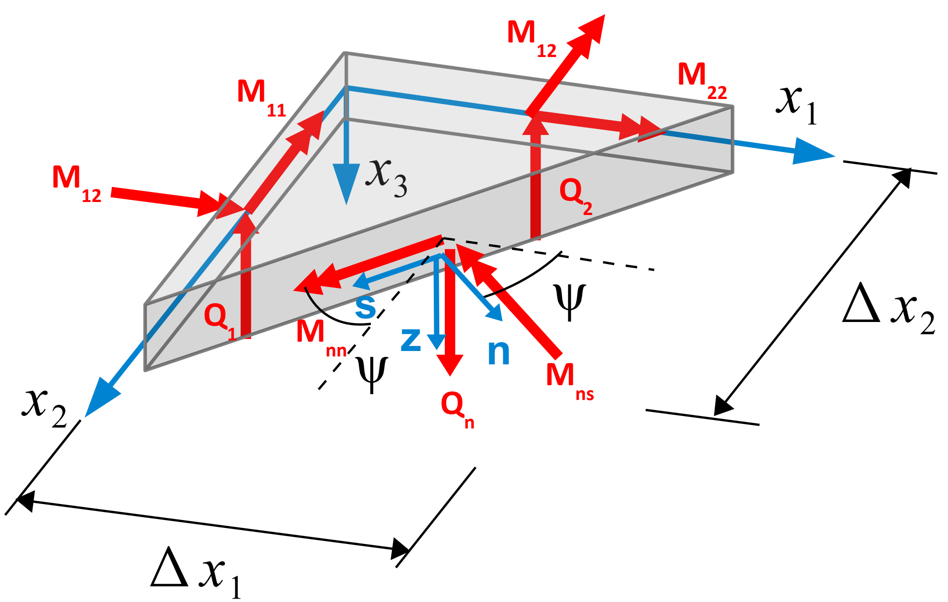
tractions at the boundary:	$(\bar{\boldsymbol{\sigma}} \cdot \mathbf{n})\big\|_{\mathbf{x}} = \mathbf{q}\ , \qquad \mathbf{x} \in \partial\Omega$

- Kinematic boundary conditions for plate **state**:

deflection of the boundary:	$w(\mathbf{x}) = w_0\ , \qquad \mathbf{x} \in \partial\Omega$
rotation angle at the boundary:	$\phi_n(\mathbf{x}) = \nabla w \cdot \mathbf{n} = \frac{\partial w}{\partial x_1} n_1 + \frac{\partial w}{\partial x_2} n_2 = \phi_0\ , \quad \mathbf{x} \in \partial\Omega$

BOUNDARY CONDITIONS

- static boundary conditions for plate **state**:
  - we need formulas for **cross-sectional forces at boundary** for any orientation of the boundary with respect to the axes of the considered coordinate system. We can derive them from the **equilibrium conditions** of the plate element below:



membrane state:

$$\begin{cases} N_n = N_{11}\cos^2\psi + N_{22}\sin^2\psi + T\sin 2\psi \\ T_{ns} = \dfrac{1}{2}(N_{22} - N_{11})\sin 2\psi + T\cos 2\psi \end{cases}$$

plate state:

$$\begin{cases} M_{ns} = \dfrac{1}{2}(M_{22} - M_{11})\sin 2\psi + M_{12}\cos 2\psi \\ M_{nn} = M_{11}\cos^2\psi + M_{22}\sin^2\psi + M_{12}\sin 2\psi \\ Q_n = Q_1\cos\psi + Q_2\sin\psi \end{cases}$$

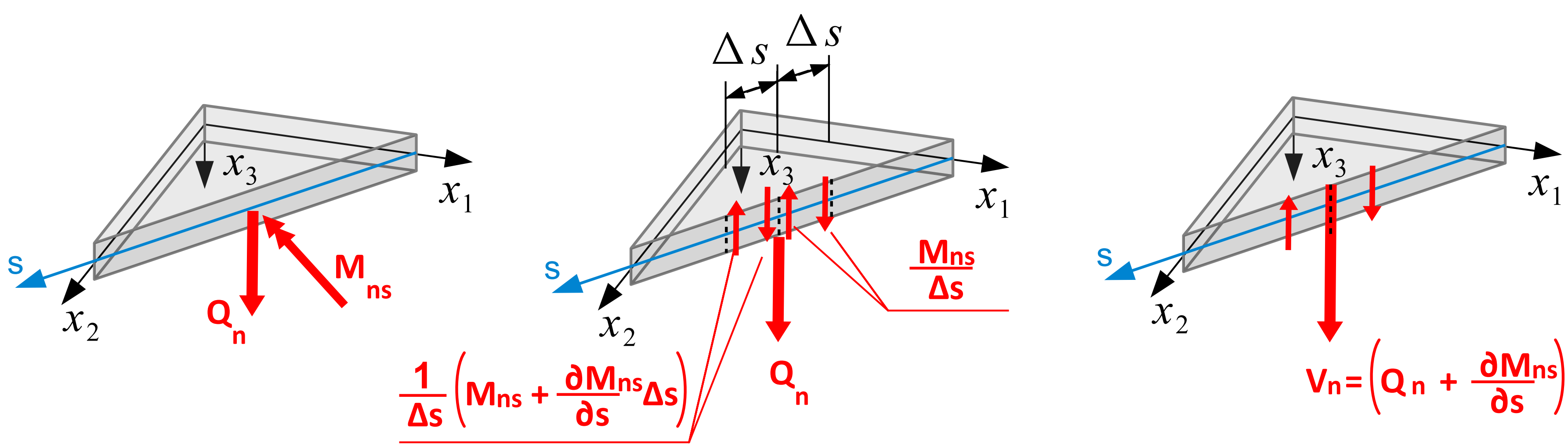
BOUNDARY CONDITIONS

- static boundary conditions for plate state:

bending moment at the boundary: $M_{nn} = \hat{M}_{nn}\,, \qquad \mathbf{x} \in \partial\Omega$

twisting moment at the boundary: $M_{ns} = \hat{M}_{ns}\,, \qquad \mathbf{x} \in \partial\Omega$

shear force at the boundary: $V_n \overset{\text{df.}}{=} Q_n + \dfrac{\partial M_{ns}}{\partial s} = \hat{V}_s\,, \qquad \mathbf{x} \in \partial\Omega$



V_n - effective shear force (**Kirchhoff transverse force**) $\quad [V_n] = [Q_n] = \mathrm{N/m}$

BOUNDARY CONDITIONS

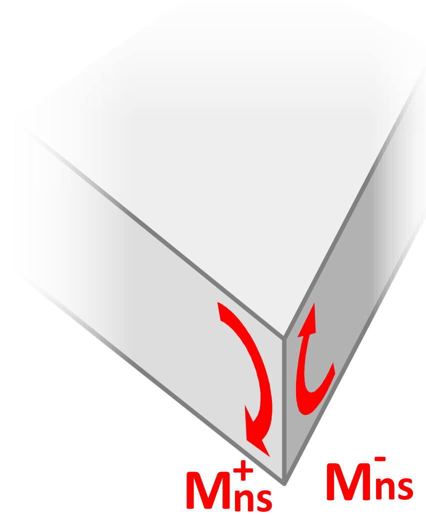
- static boundary conditions for plate state in corner points:

sum of effective shear force in a corner point of the boundary:

$$P = \lim_{e\to 0}\int_{-e}^{e} V_n \, \mathrm{d}s = \lim_{e\to 0}\int_{-e}^{e}\left(Q_n + \frac{\partial M_{ns}}{\partial s}\right)\mathrm{d}s = \lim_{e\to 0}\left[\int_{-e}^{e} Q_n \, \mathrm{d}s + \left[M_{ns}\right]_{-e}^{e}\right] =$$

$$= \lim_{e\to 0} M_{ns}(e) - \lim_{e\to 0} M_{ns}(-e) = M_{ns}^{+} - M_{ns}^{-}$$

$$[P] = [M_{ns}] = \mathrm{Nm/m} = \mathrm{N}$$



CONCLUSION:

- even if the **plate is not loaded with transverse shear forces at the boundary, equilibrium of a corner point requires presence of a point force**.
- example – for a corner with **right angle**: $P = 2M_{ns}$

CIRCULAR AND RING PLATES

CIRCULAR AND RING PLATES

For **circular and ring plates** it is more convenient to use the **polar coordinates** – in such a system of coordinates the **biharmonic** displacement equation has the following form:

$$w_{,rrrr} + \frac{2}{r^2} w_{,rr\phi\phi} + \frac{1}{r^4} w_{,\phi\phi\phi\phi} + \frac{2}{r} w_{,rrr} - \frac{2}{r^3} w_{,r\phi\phi} - \frac{1}{r^2} w_{,rr} + \frac{4}{r^4} w_{,\phi\phi} + \frac{1}{r^3} w_{,r} = \frac{q(r,\phi)}{D_b}$$

Cross-sectional forces in **polar coordinates**:

$$M_{rr} = -D_b \left[\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r} \left(\frac{\partial w}{\partial r} + \frac{1}{r} \frac{\partial^2 w}{\partial \phi^2} \right) \right]$$

$$M_{\phi\phi} = -D_b \left[\frac{1}{r} \left(\frac{\partial w}{\partial r} + \frac{1}{r} \frac{\partial^2 w}{\partial \phi^2} \right) + \nu \frac{\partial^2 w}{\partial r^2} \right]$$

$$M_{r\phi} = -D_b (1-\nu) \left[\frac{1}{r} \frac{\partial^2 w}{\partial r \, \partial \phi} - \frac{1}{r} \frac{\partial w}{\partial \phi} \right]$$

$$Q_r = -D_b \frac{\partial}{\partial r} \nabla^2 w$$

$$Q_\phi = -D_b \frac{1}{r} \frac{\partial}{\partial \phi} \nabla^2 w$$

CIRCULAR AND RING PLATES

For **circular and ring plates** the **biharmonic** displacement equation has the following form:

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial w}{\partial r}\right]\right]\right]=\frac{q(r)}{D_b}$$

Solution – the distribution of deflection:

$$w(r)=\left[A_{00}+A_{01}r^2+A_{02}r^2\ln r+A_{03}\ln r\right]+$$
$$+\frac{1}{D_b}\int\frac{1}{r}\left[\int r\left[\int\frac{1}{r}\left[\int r\,q(r)\,\mathrm{d}r+B_1\right]\mathrm{d}r+B_2\right]\mathrm{d}r+B_3\right]\mathrm{d}r+B_4$$

Constants of integration are determined according to the **boundary conditions**.

Non-zero cross-sectional forces in **polar coordinates:**

$$M_{rr}=-D_b\left[\frac{\mathrm{d}^2w}{\mathrm{d}r^2}+\frac{\nu}{r}\frac{\mathrm{d}w}{\mathrm{d}r}\right]\qquad M_{\phi\phi}=-D_b\left[\frac{1}{r}\frac{\mathrm{d}w}{\mathrm{d}r}+\nu\frac{\mathrm{d}^2w}{\mathrm{d}r^2}\right]\qquad Q_r=-D_b\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}w}{\mathrm{d}r}\right)\right]$$

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THANK YOU FOR YOUR ATTENTION