EXERCISE 2

Find the eigenvelues and eigenvectors of the following tensor:

$$\mathbf{T}=\begin{bmatrix}\sqrt{2} & 0 & -1\\ 0 & \sqrt{2} & -1\\ -1 & -1 & \sqrt{2}\end{bmatrix}$$

SOLUTION:

1st invariant: $I=T_{11}+T_{22}+T_{33}=\sqrt{2}+\sqrt{2}+\sqrt{2}=3\sqrt{2}$

2nd invariant:

$$II=\begin{vmatrix}T_{11} & T_{12}\\ T_{21} & T_{22}\end{vmatrix}+\begin{vmatrix}T_{22} & T_{23}\\ T_{32} & T_{33}\end{vmatrix}+\begin{vmatrix}T_{11} & T_{13}\\ T_{31} & T_{33}\end{vmatrix}=$$

$$=\begin{vmatrix}\sqrt{2} & 0\\ 0 & \sqrt{2}\end{vmatrix}+\begin{vmatrix}\sqrt{2} & -1\\ -1 & \sqrt{2}\end{vmatrix}+\begin{vmatrix}\sqrt{2} & -1\\ -1 & \sqrt{2}\end{vmatrix}=2+1+1=4$$

3rd invariant:

$$III=\begin{vmatrix}T_{11} & T_{12} & T_{13}\\ T_{21} & T_{22} & T_{23}\\ T_{31} & T_{32} & T_{33}\end{vmatrix}=\begin{vmatrix}\sqrt{2} & 0 & -1\\ 0 & \sqrt{2} & -1\\ -1 & -1 & \sqrt{2}\end{vmatrix}=0$$

Secular equation:

$$T^3-I\,T^2+II\,T-III=0 \quad\Rightarrow\quad T^3-3\sqrt{2}\,T^2+4T=0$$

$$T\left(T^2-3\sqrt{2}\,T+4\right)=0 \quad\Rightarrow\quad T=0 \;\vee\; T=\frac{3\sqrt{2}-\sqrt{(-3\sqrt{2})^2-4\cdot 4\cdot 1}}{2}=\sqrt{2} \;\vee\; T=\frac{3\sqrt{2}+\sqrt{(-3\sqrt{2})^2-4\cdot 4\cdot 1}}{2}=2\sqrt{2}$$

We have three distinct eigenvalues – three one-dimensional, mutually orthogonal eigensubspaces correspond with them, namely we may find three mutually orthogonal eigenvectors $\mathbf{u},\mathbf{v},\mathbf{w}$.

FINDING THEE EIGENVECTORS – METHOD I

Eigenvectors corresponding with $T_1=0$:

$$(\mathbf{T}-T_1\mathbf{I})\cdot\mathbf{u}=\mathbf{0} \quad\Leftrightarrow\quad \begin{bmatrix}\sqrt{2}-0 & 0 & -1\\ 0 & \sqrt{2}-0 & -1\\ -1 & -1 & \sqrt{2}-0\end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Since the corresponding eigensubspace is one-dimensional (considered eigenvalue is a single root of the secular equation), one of the components of the eigenvector may be assumed to be a parameter. Let it be u_3 . We chose now any two from the above equations, e.g. the 1st and the

2[nd] one, and we determine the remaining components of the eigenvector with the use u_3 :

$$\begin{cases}\sqrt{2}\,u_1 - u_3 = 0\\ \sqrt{2}\,u_2 - u_3 = 0\end{cases} \Rightarrow \begin{cases}u_1 = \dfrac{u_3}{\sqrt{2}}\\ u_2 = \dfrac{u_3}{\sqrt{2}}\end{cases} \Rightarrow \mathbf{u} = \left[\frac{u_3}{\sqrt{2}}\,;\,\frac{u_3}{\sqrt{2}}\,;\,u_3\right]$$

We calculate the length of this vector, and then we normalize the vector:

$$|\mathbf{u}| = \sqrt{u_1^2+u_2^2+u_3^2} = \sqrt{\frac{u_3^2}{2}+\frac{u_3^2}{2}+u_3^2} = u_3\sqrt{2} \quad\Rightarrow\quad \mathbf{t_1} = \frac{\mathbf{u}}{|\mathbf{u}|} = \left[\frac{1}{2}\,;\,\frac{1}{2}\,;\,\frac{1}{\sqrt{2}}\right]$$

Eigenvectors corresponding with $T_2 = \sqrt{2}$:

$$(\mathbf{T} - T_2\mathbf{I})\cdot\mathbf{v} = \mathbf{0} \;\Leftrightarrow\; \begin{bmatrix}\sqrt{2}-\sqrt{2} & 0 & -1\\ 0 & \sqrt{2}-\sqrt{2} & -1\\ -1 & -1 & \sqrt{2}-\sqrt{2}\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Since the corresponding eigensubspace is one-dimensional (considered eigenvalue is a single root of the secular equation), one of the components of the eigenvector may be assumed to be a parameter. Let it be v_1 . We chose now any two from the above equations, e.g. the 1[st] and the 3[rd] one (the 2[nd] one is identical with the 1[st] one), and we determine the remaining components of the eigenvector with the use v_1 :

$$\begin{cases}-v_3 = 0\\ -v_1 - v_2 = 0\end{cases} \Rightarrow \begin{cases}v_3 = 0\\ v_2 = -v_1\end{cases} \Rightarrow \mathbf{v} = [v_1\,;\,-v_1\,;\,0]$$

We calculate the length of this vector, and then we normalize the vector:

$$|\mathbf{v}| = \sqrt{v_1^2+v_2^2+v_3^2} = \sqrt{v_1^2+(-v_1)^2+0^2} = v_1\sqrt{2} \quad\Rightarrow\quad \mathbf{t_2} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left[\frac{1}{\sqrt{2}}\,;\,-\frac{1}{\sqrt{2}}\,;\,0\right]$$

Eigenvectors corresponding with $T_3 = 2\sqrt{2}$:

The 3[rd] of the eigenvectors must be perpendicular to the other two, so it can be determined as a cross product of them $\mathbf{t_3} = \mathbf{t_1}\times\mathbf{t_2}$: it will be already normalized and its sense (orientation) will be such that the sequence of vectors $\mathbf{t_1}, \mathbf{t_2}, \mathbf{t_3}$ will constitute a right-handed coordinate system:

$$\mathbf{t_3} = \mathbf{t_1}\times\mathbf{t_2} = \left[\frac{1}{2}\,;\,\frac{1}{2}\,;\,\frac{1}{\sqrt{2}}\right]\times\left[\frac{1}{\sqrt{2}}\,;\,-\frac{1}{\sqrt{2}}\,;\,0\right] = \left[\frac{1}{2}\,;\,\frac{1}{2}\,;\,-\frac{1}{\sqrt{2}}\right]$$

FINDING THEE EIGENVECTORS – METHOD II

Eigenvectors corresponding with $T_2=\sqrt{2}$ $T_1=0$:

$$(\mathbf{T}-T_1\mathbf{I})\cdot\mathbf{u}=\mathbf{0} \quad \Leftrightarrow \quad \begin{bmatrix} \sqrt{2} & 0 & -1 \\ 0 & \sqrt{2} & -1 \\ -1 & -1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The above system of equations may be interpreted as a system of three dot products which are equal to 0 – these are products of the vector $\mathbf{u}$ and three vectors, the components of which are equal the coefficients from the three rows of the matrix of coefficients. Since all three dot products are equal to 0, this means that $\mathbf{u}$ is perpendicular to those vectors – in particular, vector $\mathbf{u}$ may be found as a cross-product of any two vectors composed from the rows of the matrix of coefficients. Cross-product will be perpendicular to both of them. Since the determinant of the matrix of coefficients is equal to 0, it means that the triple product of three vectors represented by rows of that matrix is also 0. A 0 triple product of three vectors means that those vectors are coplanar, so a vector perpendicular to any two of them will be also perpendicular to the third one. Let's chose e.g. first two rows:

$$\mathbf{u}=[\sqrt{2}\ ;\ 0\ ;\ -1]\times[0\ ;\ \sqrt{2}\ ;\ -1]=[\sqrt{2}\ ;\ \sqrt{2}\ ;\ 2]$$

Vector is then normalized: $$\mathbf{t_1}=\frac{\mathbf{u}}{|\mathbf{u}|}=\frac{[\sqrt{2}\ ;\ \sqrt{2}\ ;\ 2]}{\sqrt{2+2+4}}=\left[\frac{1}{2}\ ;\ \frac{1}{2}\ ;\ \frac{1}{\sqrt{2}}\right]$$

Eigenvectors corresponding with $T_2=\sqrt{2}$:

$$(\mathbf{T}-T_2\mathbf{I})\cdot\mathbf{v}=\mathbf{0} \quad \Leftrightarrow \quad \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We chose the 1st and the 3rd row (the 1st and the 2nd row give us parallel vectors, and a cross-product of parallel vectors is a zero vector which cannot be considered an eigenvector):

$$\mathbf{v}=[0\ ;\ 0\ ;\ -1]\times[-1\ ;\ -1\ ;\ 0]=[-1\ ;\ 1\ ;\ 0]$$

Vector is then normalized: $$\mathbf{t_2}=\frac{\mathbf{v}}{|\mathbf{v}|}=\frac{[-1\ ;\ 1\ ;\ 0]}{\sqrt{1+1+0}}=\left[-\frac{1}{\sqrt{2}}\ ;\ \frac{1}{\sqrt{2}}\ ;\ 0\right]$$

Eigenvectors corresponding with $T_3=2\sqrt{2}$:

The 3rd of the eigenvectors must be perpendicular to the other two, so it can be determined as a cross product of them $\mathbf{t_3}=\mathbf{t_1}\times\mathbf{t_2}$: it will be already normalized and its sense (orientation) will be such that the sequence of vectors $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$ will constitute a right-handed coordinate system:

$$\mathbf{t}_3 = \mathbf{t}_1 \times \mathbf{t}_2 = \left[\frac{1}{2}\ ;\ \frac{1}{2}\ ;\ \frac{1}{\sqrt{2}}\right] \times \left[-\frac{1}{\sqrt{2}}\ ;\ \frac{1}{\sqrt{2}}\ ;\ 0\right] = \left[-\frac{1}{2}\ ;\ -\frac{1}{2}\ ;\ \frac{1}{\sqrt{2}}\right]$$

REMARK:
We can observe that eigenvectors obtained with the use of those two methods have different orientation – it does not matter, since each vector which is parallel to an eigenvector (also an opposite vector) is also an eigenvector and the choice of orientation is a matter of agreement.

TRANSITION MATRIX

Transition matrix is defined as follows: $\mathbf{A}:\ A_{ij} = \mathbf{t}_i \cdot \mathbf{e}_j$

Component *ij* of the transition matrix is the *j*-th component of the *i*-th eigenvector in the original coordinate system. Let's use the results obtained with the use of the method I:

$$\mathbf{A} = \begin{bmatrix} 1/2 & 1/2 & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/2 & 1/2 & -1/\sqrt{2} \end{bmatrix}$$

Transition matrix has the following properties:

- *i*-th row of a transition matrix is an *i*-th eigenvector represented in the basis of the original coordinate system.
- *J*-th column of a transition matrix is an *j*-th basis vector of the original coordinate system represented in the basis of eigenvectors
- sum of squares of entries in each column is equal to 1 (vectors of the original basis are normalized)
- sum of squares of entries in each row is equal to 1 (eigenvectors are normalized)
- dot product of any two distinct columns is equal to 0 (vectors of the original basis are mutually orthogonal)
- dot product of any two distinct rows is equal to 0 (eigenvectors are mutually orthogonal)
- determinant of the transition matrix is equal to 1 (it is an orthogonal matrix)

The transition matrix may be used in order to transform the representation matrix of a tensor from the original coordinate system to the system of eigenaxes of the tensor:

$$\mathbf{T}' = \mathbf{A} \cdot \mathbf{T} \cdot \mathbf{A}^{\mathrm{T}}$$

$$\mathbf{A} \cdot \mathbf{T} \cdot \mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 1/2 & 1/2 & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/2 & 1/2 & -1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} \sqrt{2} & 0 & -1 \\ 0 & \sqrt{2} & -1 \\ -1 & -1 & \sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1/2 & 1/\sqrt{2} & 1/2 \\ 1/2 & -1/\sqrt{2} & 1/2 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ \sqrt{2} & \sqrt{2} & -2 \end{bmatrix} \cdot \begin{bmatrix} 1/2 & 1/\sqrt{2} & 1/2 \\ 1/2 & -1/\sqrt{2} & 1/2 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 2\sqrt{2} \end{bmatrix}$$

EXERCISE 3

Find the eigenvalues and eigenvectors of the following tensor: $\mathbf{T}=\begin{bmatrix}102 & 36 & 0\\ 36 & 123 & 0\\ 0 & 0 & 75\end{bmatrix}$

SOLUTION:

1st invariant: $I=102+123+75=300$

2nd invariant: $II=\begin{vmatrix}102 & 36\\ 36 & 123\end{vmatrix}+\begin{vmatrix}102 & 0\\ 0 & 75\end{vmatrix}+\begin{vmatrix}123 & 0\\ 0 & 75\end{vmatrix}=28125$

3rd invariant: $III=\begin{vmatrix}102 & 36 & 0\\ 36 & 123 & 0\\ 0 & 0 & 75\end{vmatrix}=843750$

Secular equations: $T^3-I\,T^2+II\,T-III=0 \quad\Rightarrow\quad T^3-300\,T^2+28125\,T-843750=0$

Roots of the secular equations may be found numerically or with the use of the Cardano's formulae:

$$T_1=150,\quad T_2=75,\quad T_3=75$$

FINDING THEE EIGENVECTORS – METHOD I

Eigenvectors corresponding with $T_1=150$:

$$(\mathbf{T}-T_1\mathbf{I})\cdot\mathbf{u}=\mathbf{0} \quad\Leftrightarrow\quad \begin{bmatrix}102-150 & 36 & 0\\ 36 & 123-150 & 0\\ 0 & 0 & 75-150\end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Since the corresponding eigensubspace is one-dimensional (considered eigenvalue is a single root of the secular equation), one of the components of the eigenvector may be assumed to be a parameter. We cannot chose u_3 , since the 3rd equation gives us $u_3=0$. Let's chose u_1 to be a parameter. We chose any two from the above equations and we solve them trying to express the remaining components with the use of u_1 . Equations 1 and 2 are the same. We will chose the 1st and the 3rd equation:

$$\begin{cases}-48\,u_1+36\,u_2=0\\ -75\,u_3=0\end{cases} \quad\Rightarrow\quad \begin{cases}u_2=\dfrac{4}{3}u_1\\ u_3=0\end{cases} \quad\Rightarrow\quad \mathbf{u}=\left[u_1\ ;\ \frac{4}{3}u_1\ ;\ 0\right]$$

We calculate the length of this vector, and then we normalize the vector:

$$|\mathbf{u}|=\sqrt{u_1^2+u_2^2+u_3^2}=\sqrt{u_1^2+\frac{16}{9}u_1^2+0^2}=\frac{5}{3}u_1 \quad\Rightarrow\quad \mathbf{t}_1=\frac{\mathbf{u}}{|\mathbf{u}|}=\left[\frac{3}{5}\ ;\ \frac{4}{5}\ ;\ 0\right]$$

Eigenvectors corresponding with $T_2=T_3=75$:

$$(\mathbf{T}-T_2\mathbf{I})\cdot\mathbf{v}=\mathbf{0} \quad\Leftrightarrow\quad \begin{bmatrix}102-75 & 36 & 0\\ 36 & 123-75 & 0\\ 0 & 0 & 75-75\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Since for a double eigenvalue the corresponding eigensubspace is two-dimensional, so two unknown components of the eigenvector are assumed to be independent parameters. We will obtain a two-parameter family of eigenvectors, corresponding with $T_2 = T_3$ - all of them will lie in a plane which is perpendicular to $\mathbf{t_1}$. Among them we will chose any two, which are mutually orthogonal and which constitute (together with $\mathbf{t_1}$) a right-handed coordinate system. We will do it is such a way, that one of those vectors will be determined by an arbitrary choice of the value of parameters (they cannot be all equal to 0), the remaining one will be determined with the use of th cross-product of the two already known.

The third equation suggests that one of the parameters should be v_3 , since this equation is satisfied always, for any value of v_3 . Let's choose v_1 as the second parameter. We shall choose any equation (except the third one). In fact equations 1 and 2 are equivalent one to another. We have then:

$$\begin{cases} 27 v_1 + 36 v_2 = 0 \\ v_3 - \text{dowolne} \end{cases} \Rightarrow \begin{cases} v_2 = -\dfrac{3}{4} v_1 \\ v_3 - \text{dowolne} \end{cases} \Rightarrow \mathbf{v} = \left[v_1 \ ; \ -\frac{3}{4} v_1 \ ; \ v_3 \right]$$

We may choose any eigenvector, corresponding to e.g. $v_1 = 0, v_3 = 1$. Such an eigenvector is already normalized.

$$\mathbf{t_1} = [0 ; 0 ; 1]$$

The 3rd of the eigenvectors must be perpendicular to the other two, so it can be determined as a cross product of them $\mathbf{t_3} = \mathbf{t_1} \times \mathbf{t_2}$: it will be already normalized and its sense (orientation) will be such that the sequence of vectors $\mathbf{t_1}, \mathbf{t_2}, \mathbf{t_3}$ will constitute a right-handed coordinate system:

$$\mathbf{t_3} = \mathbf{t_1} \times \mathbf{t_2} = \left[\frac{3}{5} \ ; \ \frac{4}{5} \ ; \ 0 \right] \times [0 \ ; \ 0 \ ; \ 1] = \left[\frac{4}{5} \ ; \ -\frac{3}{5} \ ; \ 0 \right]$$

FINDING THEE EIGENVECTORS – METHOD II

Eigenvectors corresponding with $T_1 = 150$:

$$(\mathbf{T} - T_1 \mathbf{I}) \cdot \mathbf{u} = \mathbf{0} \quad \Leftrightarrow \quad \begin{bmatrix} -48 & 36 & 0 \\ 36 & -27 & 0 \\ 0 & 0 & -75 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We choose the vectors corresponding with the 1st and the 3rd row of matrix (the 1st and the 2nd would give us parallel vectors). Their cross-product:

$$\mathbf{u} = [-48 \ ; \ 36 \ ; \ 0] \times [0 \ ; \ 0 \ ; \ -75] = [-2700 \ ; \ -3600 \ ; \ 0]$$

Vector is then normalized:

$$\mathbf{t}_1 = \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{[-2700\ ;\ -3600\ ;\ 2]}{\sqrt{2700^2+3600^2+0}} = \left[-\frac{2700}{4500}\ ;-\frac{3600}{4500}\ ;\ 0\right] = \left[-\frac{3}{5}\ ;-\frac{4}{5}\ ;\ 0\right]$$

Eigenvectors corresponding with $T_2 = T_3 = 75$:

$$(\mathbf{T}-T_2\mathbf{I})\cdot\mathbf{v} = \mathbf{0} \quad \Leftrightarrow \quad \begin{bmatrix} 27 & 36 & 0 \\ 36 & 48 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Vectors corresponding with the 1st and the 2nd row are parallel, and the 3rd row corresponds with a zero vector. We will look for the eigenvector in another way – we can notice, that the vectors corresponding with the first two rows have their third component equal to 0, so they both lie in plane (x_1, x_2) . A vector which is perpendicular to them (and which is satisfying the above system of equations) is the vector:

$$\mathbf{t}_2 = [0\,;0\,;1]$$

The 3rd of the eigenvectors must be perpendicular to the other two, so it can be determined as a cross product of them $\mathbf{t}_3 = \mathbf{t}_1 \times \mathbf{t}_2$: it will be already normalized and its sense (orientation) will be such that the sequence of vectors $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$ will constitute a right-handed coordinate system:

$$\mathbf{t}_3 = \mathbf{t}_1 \times \mathbf{t}_2 = \left[-\frac{3}{5}\ ;-\frac{4}{5}\ ;\ 0\right] \times [0\ ;\ 0\ ;\ 1] = \left[-\frac{4}{5}\ ;\ \frac{3}{5}\ ;\ 0\right]$$

TRANSITION MATRIX

Using the results obtained with the use of the method II:

$$\mathbf{A} = \begin{bmatrix} -3/5 & -4/5 & 0 \\ 0 & 0 & 1 \\ -4/5 & 3/5 & 0 \end{bmatrix}$$

Check:

$$\mathbf{A}\cdot\mathbf{T}\cdot\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} -3/5 & -4/5 & 0 \\ 0 & 0 & 1 \\ -4/5 & 3/5 & 0 \end{bmatrix} \cdot \begin{bmatrix} 102 & 36 & 0 \\ 36 & 123 & 0 \\ 0 & 0 & 75 \end{bmatrix} \cdot \begin{bmatrix} -3/5 & 0 & -4/5 \\ -4/5 & 0 & 3/5 \\ 0 & 1 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} -90 & -120 & 0 \\ 0 & 0 & 75 \\ -60 & 45 & 0 \end{bmatrix} \cdot \begin{bmatrix} -3/5 & 0 & -4/5 \\ -4/5 & 0 & 3/5 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 150 & 0 & 0 \\ 0 & 75 & 0 \\ 0 & 0 & 75 \end{bmatrix}$$

EXERCISE 4

Find the eigenvalues and eigenvectors of the following tensor: $\mathbf{T} = \begin{bmatrix} 34 & -12 \\ -12 & 41 \end{bmatrix}$

SOLUTION:

Eigenvalues:

$$T_1 = \frac{T_{11}+T_{12}}{2} + \sqrt{\left(\frac{T_{11}-T_{22}}{2}\right)^2 + T_{12}^2} = 50$$

$$T_2 = \frac{T_{11}+T_{12}}{2} - \sqrt{\left(\frac{T_{11}-T_{22}}{2}\right)^2 + T_{12}^2} = 25$$

An angle between axis x_1 and the axis corresponding to the first (larger) eigenvalue:

$$\phi = \mathrm{arctg}\frac{T_{12}}{T_1 - T_{22}} = \mathrm{arctg}\left(-\frac{4}{3}\right) \approx -53{,}13^{\circ}$$

Eigenvectors:

$$\mathbf{t_1} = [\cos\phi \ ; \ \sin\phi] = \left[\frac{3}{5} \ ; -\frac{4}{5}\right]$$

$$\mathbf{t_2} = [-\sin\phi \ ; \ \cos\phi] = \left[\frac{4}{5} \ ; \ \frac{3}{5}\right]$$

Transition matrix:

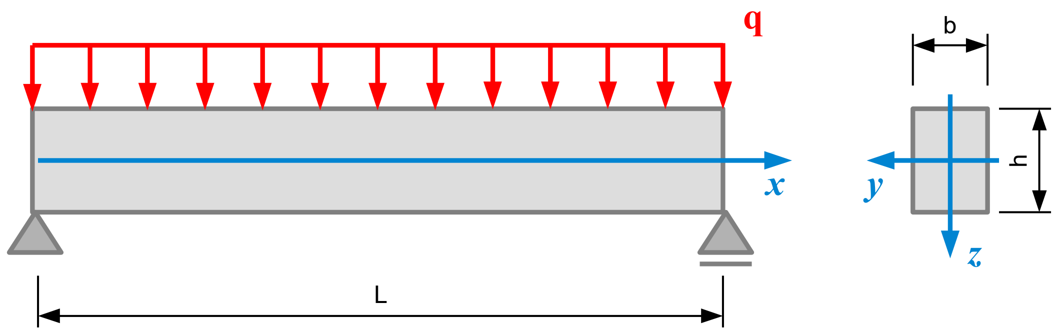
$$\mathbf{A} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}$$

Check:

$$\mathbf{A}\cdot\mathbf{T}\cdot\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix} \cdot \begin{bmatrix} 34 & -12 \\ -12 & 41 \end{bmatrix} \cdot \begin{bmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{bmatrix} = \begin{bmatrix} 30 & -40 \\ 20 & 15 \end{bmatrix} \cdot \begin{bmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{bmatrix} = \begin{bmatrix} 50 & 0 \\ 0 & 25 \end{bmatrix}$$

EXERCISE 5

A simply-supported beam is given. Its length is L , it has a rectangular cross-section of dimensions $b\times h$ and it is loaded with a uniformly distributed load of density q . Find the pricipal stresses and their orientation for any point of the beam.



- What is the direction of maximum principal stress near the edge of the cross-section in the middle od the span of the beam?
- What is the direction of maximum principal stress in the center of the cross-section above supports?

SOLUTION:

Stress state in any cross-section is of the following form: $\boldsymbol{\sigma}(x,y,z)=\begin{bmatrix}\sigma(x,z) & \tau(x,z)\\ \tau(x,z) & 0\end{bmatrix}$

Normal stress: $\sigma(x,z)=\frac{M_y(x)}{I_y}z$

Shear stress: $\tau(x,z)=\frac{Q_z(x)S_y(z)}{bI_y}$

Bending moment distribution: $M_y(x)=\frac{qL}{2}x-\frac{q}{2}x^2=\frac{qx}{2}(L-x)$

Shear force distribution: $Q_z(x)=\frac{qL}{2}-qx=q\left(\frac{L}{2}-x\right)$

2nd moment of area of the cross-section: $I=\frac{bh^3}{12}$

Statical moment of a part of cross-section: $S_y(z)=b\left(\frac{h}{2}-z\right)\left(z+\frac{\frac{h}{2}-z}{2}\right)=\frac{b}{2}\left[\frac{h^2}{4}-z^2\right]$

Stress state components distribution: $\sigma(x,z)=\frac{M_y(x)}{I_y}z=\frac{6q}{bh^3}zx(L-x)$

$$\tau(x,z)=\frac{6q}{bh^3}\left(\frac{L}{2}-x\right)\left(\frac{h^2}{4}-z^2\right)$$

Principal stresses:

$$\sigma_{max}=\frac{\sigma_{11}+\sigma_{22}}{2}+\sqrt{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2+\sigma_{12}^2}=\frac{1}{2}\left[\sigma+\sqrt{\sigma^2+4\tau^2}\right]$$
$$=\frac{3q}{bh^3}\left[zx(L-x)+\sqrt{z^2x^2(L-x)^2+4\left(\frac{L}{2}-x\right)^2\left(\frac{h^2}{4}-z^2\right)^2}\right]$$

$$\sigma_{min}=\frac{\sigma_{11}+\sigma_{22}}{2}-\sqrt{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2+\sigma_{12}^2}=\frac{1}{2}\left[\sigma-\sqrt{\sigma^2+4\tau^2}\right]$$
$$=\frac{3q}{bh^3}\left[zx(L-x)-\sqrt{z^2x^2(L-x)^2+4\left(\frac{L}{2}-x\right)^2\left(\frac{h^2}{4}-z^2\right)^2}\right]$$

An angle between axis of maximum principal stress and the axis of the beam:

$$\mathrm{tg}\,\phi=\frac{\sigma_{12}}{\sigma_{max}-\sigma_{22}}=\frac{2\tau}{\sigma+\sqrt{\sigma^2+4\tau^2}}=\frac{4\left(\frac{L}{2}-x\right)\left(\frac{h^2}{4}-z^2\right)}{zx(L-x)+\sqrt{z^2x^2(L-x)^2+4\left(\frac{L}{2}-x\right)^2\left(\frac{h^2}{4}-z^2\right)^2}}$$

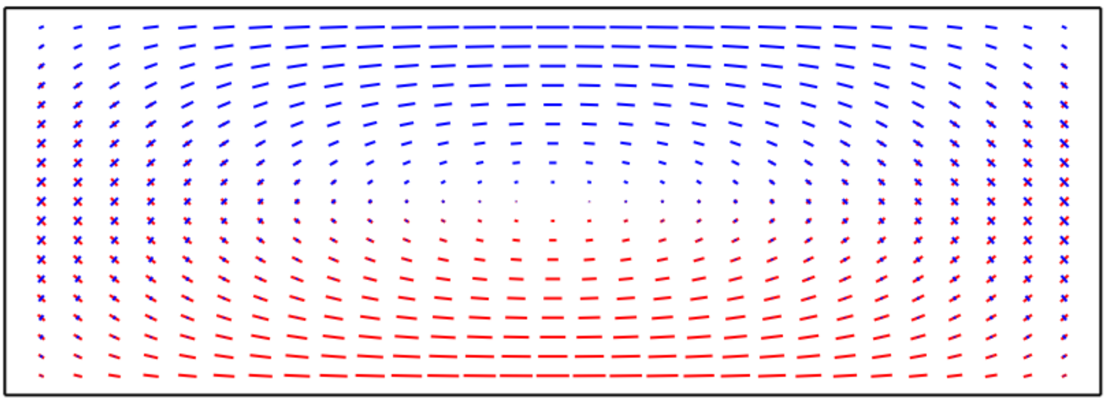
Direction of maximum principal stress near the edge of the cross-section in the middle od the span of the beam:

$$\mathrm{tg}\,\phi\left(x=\frac{L}{2}\ ;\ z=\frac{h}{2}\right)=0\quad\Rightarrow\quad\phi=0^{\circ}$$

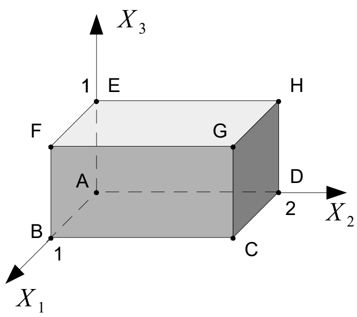
Direction of maximum principal stress in the center of the cross-section above supports

$$\mathrm{tg}\,\phi(x=0\ ;\ z=0)=1\quad\Rightarrow\quad\phi=45^{\circ}$$

In the picture below, the orientation of maximum (red) and minimum (blue) principal stress is marked. Length of lines of that markings is proportional to the magnitude of corresponding stress. Trajectories of the principal stress can be seen.



EXERCISE 8



Deformation is given in material description.

$$\begin{cases} x_1 = X_1 - X_3 \\ x_2 = 2X_2 \\ x_3 = X_3 \end{cases}$$

a) Check if these relation are invertible. If so, find deformation relations in spatial description.
b) Find displacement field in both spatial and material description.
c) Make a sketch of an actual configuration.
d) What would be the shape of material fiber AG after deformation? What is its length?
e) What is an equation of BCGF surface after deformation?
f) What is the surface area of BCGF face before and after deformation?

SOLUTION:
Invertibility is checked by calculating the Jacobian determinant:

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2$$

Determinant $J > 0$, so the relations are locally invertible. These are linear equations with respect to material coordinates, so it can be solved, to obtain following result:

$$\begin{cases} X_1 = x_1 + x_3 \\ X_2 = \frac{1}{2} x_2 \\ X_3 = x_3 \end{cases}$$

DISPLACEMENT FIELD

- **material description:**

$$\begin{cases} u_1(\mathbf{X}) = x_1((\mathbf{X})) - X_1 = -X_3 \\ u_2(\mathbf{X}) = x_2((\mathbf{X})) - X_2 = X_2 \\ u_3(\mathbf{X}) = x_3((\mathbf{X})) - X_3 = 0 \end{cases}$$

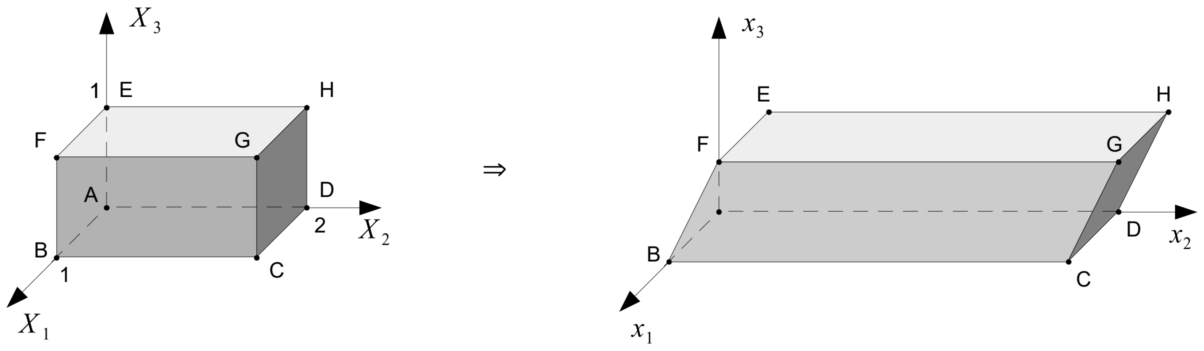
- **spatial description:**

$$\begin{cases} u_1(\mathbf{x}) = x_1 - X_1(\mathbf{x}) = -x_3 \\ u_2(\mathbf{x}) = x_2 - X_2(\mathbf{x}) = \frac{1}{2} x_2 \\ u_3(\mathbf{x}) = x_3 - X_3(\mathbf{x}) = 0 \end{cases}$$

DEFORMATION

Point A:	$\mathbf{X}_A=[0;0;0]\rightarrow\mathbf{x}_A=[0;0;0]$	Point B:	$\mathbf{X}_B=[1;0;0]\rightarrow\mathbf{x}_B=[1;0;0]$
Point C:	$\mathbf{X}_C=[1;2;0]\rightarrow\mathbf{x}_C=[1;4;0]$	Point D:	$\mathbf{X}_D=[0;2;0]\rightarrow\mathbf{x}_D=[0;4;0]$
Point E:	$\mathbf{X}_E=[0;0;1]\rightarrow\mathbf{x}_E=[-1;0;1]$	Point F:	$\mathbf{X}_F=[1;0;1]\rightarrow\mathbf{x}_E=[0;0;1]$
Point G:	$\mathbf{X}_G=[1;2;1]\rightarrow\mathbf{x}_G=[0;4;1]$	Point H:	$\mathbf{X}_H=[0;2;1]\rightarrow\mathbf{x}_H=[-1;4;1]$

Reference configuration　　　　　　Actual configuration



DEFORMATION OF LINE AG

Line containing A and G before deformation is given by parametric equations:

$$AG:\quad \mathbf{X}_{AG}(\lambda)=\mathbf{X}_A+\lambda(\mathbf{X}_G-\mathbf{X}_A)=\begin{cases}X_1=\lambda\\X_2=2\lambda\\X_3=\lambda\end{cases}\qquad \lambda\in\langle 0\,;\,1\rangle$$

Fibre length before deformation:

$$|AG|=\int_{AG}\mathrm{d}S=\int_{AG}\sqrt{\mathrm{d}X_1^2+\mathrm{d}X_2^2+\mathrm{d}X_3^2}=\int_0^1\sqrt{\left(\frac{\mathrm{d}X_1}{\mathrm{d}\lambda}\right)^2+\left(\frac{\mathrm{d}X_2}{\mathrm{d}\lambda}\right)^2+\left(\frac{\mathrm{d}X_3}{\mathrm{d}\lambda}\right)^2}\,\mathrm{d}\lambda=$$
$$\int_0^1\sqrt{1^2+2^2+1^2}\,\mathrm{d}\lambda=\sqrt{6}\int_0^1\mathrm{d}\lambda=\sqrt{6}$$

After deformation the line may be described as:

$$AG:\quad \mathbf{x}_{AG}(\lambda)=\begin{cases}x_1=X_1-X_3\\x_2=2X_2\\x_3=X_3\end{cases}=\begin{cases}x_1=0\\x_2=4\lambda\\x_3=\lambda\end{cases}\qquad \lambda\in\langle 0\,;\,1\rangle$$

Fibre length after deformation

$$|AG|=\int_{AG}\mathrm{d}s=\int_{AG}\sqrt{\mathrm{d}x_1^2+\mathrm{d}x_2^2+\mathrm{d}x_3^2}=\int_0^1\sqrt{\left(\frac{\mathrm{d}x_1}{\mathrm{d}\lambda}\right)^2+\left(\frac{\mathrm{d}x_2}{\mathrm{d}\lambda}\right)^2+\left(\frac{\mathrm{d}x_3}{\mathrm{d}\lambda}\right)^2}\,\mathrm{d}\lambda=$$
$$\int_0^1\sqrt{0+4^2+1^2}\,\mathrm{d}\lambda=\sqrt{17}\int_0^1\mathrm{d}\lambda=\sqrt{17}$$

The length of the AG fibre may be also calculated with the use of an integral over a reference configuration. In order to do that, we need to determine the material deformation gradient $\mathbf{F}$ and material deformation tensor $\mathbf{C}$:

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 4 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

The relation between an infinitely small linear element before deformation $\mathrm{d}S = \sqrt{\mathrm{d}X_i \mathrm{d}X_i}$ and after deformation $\mathrm{d}s$ is as follows:

$$\mathrm{d}s = \sqrt{C_{ij}\,\mathrm{d}X_i \mathrm{d}X_j}$$

The length of the fibre is calculated with the use of an integral over a reference configuration (we are using the parametrization of a curve before deformation):

$$|AG| = \int_{AG} \mathrm{d}s = \int_{AG} \sqrt{C_{ij}\,\mathrm{d}X_i \mathrm{d}X_j} = \int_0^1 \sqrt{C_{ij}\frac{\mathrm{d}X_i}{\mathrm{d}\lambda}\frac{\mathrm{d}X_j}{\mathrm{d}\lambda}}\,\mathrm{d}\lambda =$$

$$= \int_0^1 \sqrt{C_{11}\left(\frac{\mathrm{d}X_1}{\mathrm{d}\lambda}\right)^2 + C_{22}\left(\frac{\mathrm{d}X_2}{\mathrm{d}\lambda}\right)^2 + C_{33}\left(\frac{\mathrm{d}X_3}{\mathrm{d}\lambda}\right)^2 2C_{23}\frac{\mathrm{d}X_2}{\mathrm{d}\lambda}\frac{\mathrm{d}X_3}{\mathrm{d}\lambda} + 2C_{31}\frac{\mathrm{d}X_3}{\mathrm{d}\lambda}\frac{\mathrm{d}X_1}{\mathrm{d}\lambda} + 2C_{12}\frac{\mathrm{d}X_1}{\mathrm{d}\lambda}\frac{\mathrm{d}X_2}{\mathrm{d}\lambda}}\,\mathrm{d}\lambda =$$

$$= \int_0^1 \sqrt{1\cdot(1)^2 + 4\cdot(2)^2 + 2\cdot(1)^2 + 2\cdot 0\cdot 2\cdot 1 + 2\cdot(-1)\cdot 1\cdot 1 + 0\cdot 1\cdot 2}\,\mathrm{d}\lambda = \int_0^1 \sqrt{17}\,\mathrm{d}\lambda = \sqrt{17}\int_0^1 \mathrm{d}\lambda = \sqrt{17}$$

DEFORMATION OF BCGF FACE

Face BCGF before deformation is a face perpendicular to X_1 axis and containing point for which $X_1 = 1$. An equation of surface containing BCGF face is:

$$BCGF: \quad X_1 - 1 = 0$$

Surface area of face BCGF before deformation is calculated by a double definite integral:

$$A_R = \iint\limits_{BCGF} \mathrm{d}A_R = \int\limits_{X_2=0}^{2} \int\limits_{X_3=0}^{1} \mathrm{d}X_2 \mathrm{d}X_3 = 2$$

Surface containing BCGF face may be found by expressing material coordinates in the above equation in terms of spatial ones, according to inverted relations $\mathbf{X}(\mathbf{x})$:

$$BCGF: \quad X_1 - 1 = 0$$
$$x_1 + x_3 - 1 = 0$$

Integral in actual configuration may be calculated by expressing the area of infinitely small surface element after deformation by the area of infinitely small surface element before deformation:

$$\mathrm{d}A = \mathrm{d}A_R \cdot J \cdot \sqrt{(\mathbf{N}^{\mathbf{T}} \cdot \mathbf{F}^{-1})^{\mathrm{T}} \cdot (\mathbf{N}^{\mathbf{T}} \cdot \mathbf{F}^{-1})}$$

Inverse of material deformation gradient (spatial deformation gradient):

$$\mathbf{F}^{-1} = \mathbf{f} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0{,}5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Unit normal vector for BCGF face: $\mathbf{N} = [1 \; ; \; 0 \; ; \; 0]^{\mathrm{T}}$

$$\mathbf{N}^{\mathbf{T}} \cdot \mathbf{F}^{-1} = [1 \; ; \; 0 \; ; \; 0] \cdot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0{,}5 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [1 \; ; \; 0 \; ; \; 1]$$

$$(\mathbf{N}^{\mathbf{T}} \cdot \mathbf{F}^{-1})^{\mathrm{T}} \cdot (\mathbf{N}^{\mathbf{T}} \cdot \mathbf{F}^{-1}) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot [1 \; ; \; 0 \; ; \; 1] = 2 \quad \Rightarrow \quad \sqrt{(\mathbf{N}^{\mathbf{T}} \cdot \mathbf{F}^{-1})^{\mathrm{T}} \cdot (\mathbf{N}^{\mathbf{T}} \cdot \mathbf{F}^{-1})} = \sqrt{2}$$

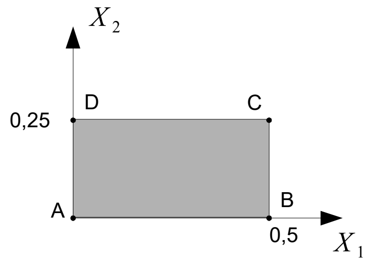
$$J = \det \mathbf{F} = 2$$

Surface area of face BCGF after deformation is calculated by a double definite integral:

$$A = \iint\limits_{BCGF} \mathrm{d}A = \iint\limits_{BCGF} J \sqrt{(\mathbf{N}^{\mathbf{T}} \cdot \mathbf{F}^{-1})^{\mathrm{T}} \cdot (\mathbf{N}^{\mathbf{T}} \cdot \mathbf{F}^{-1})} \, \mathrm{d}A_R = \int\limits_{X_2=0}^{2} \int\limits_{X_3=0}^{1} 2\sqrt{2} \, \mathrm{d}X_2 \mathrm{d}X_3 = 4\sqrt{2}$$

EXERCISE 9

Deformation is given in material description.



$$\begin{cases} x_1 = X_1 \\ x_2 = X_2 + 2X_1^3 \end{cases}$$

a) Check if these relation are invertible. If so, find deformation relations in spatial description.
b) Find displacement field in both spatial and material description.
c) Make a sketch of an actual configuration.
d) What would be the shape of material fiber AC after deformation? What is its length?

SOLUTION:

Invertibility is checked by calculating the Jacobian determinant:

$$J = \begin{vmatrix} \dfrac{\partial x_1}{\partial X_1} & \dfrac{\partial x_1}{\partial X_2} \\ \dfrac{\partial x_2}{\partial X_1} & \dfrac{\partial x_2}{\partial X_2} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 6X_1^2 & 1 \end{vmatrix} = 1$$

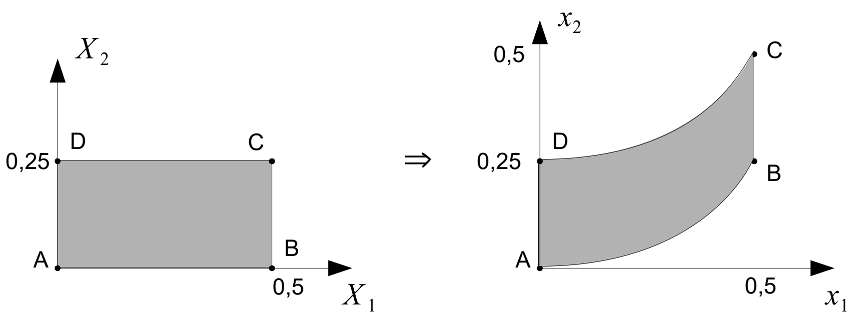
Determinant $J>0$, so the relations are locally invertible. It is easy to find inverse relations:

$$\begin{cases} X_1 = x_1 \\ X_2 = x_2 - 2x_1^3 \end{cases}$$

DEFORMATION

Point A: $\mathbf{X}_A = [0;0] \rightarrow \mathbf{x}_A = [0;0]$ Point B: $\mathbf{X}_B = [0{,}5;0] \rightarrow \mathbf{x}_B = [0{,}5;0{,}25]$
Point C: $\mathbf{X}_C = [0{,}5;0{,}25] \rightarrow \mathbf{x}_C = [0{,}5;0{,}5]$ Point D: $\mathbf{X}_D = [0;0{,}25] \rightarrow \mathbf{x}_D = [0;0{,}25]$

Reference configuration Actual configuration



DISPLACEMENT FIELD

- **material description:**

$$\begin{cases} u_1(\mathbf{X}) = x_1((\mathbf{X})) - X_1 = 0 \\ u_2(\mathbf{X}) = x_2((\mathbf{X})) - X_2 = 2X_1^3 \end{cases}$$

- **spatial description:**

$$\begin{cases} u_1(\mathbf{x}) = x_1 - X_1(\mathbf{x}) = 0 \\ u_2(\mathbf{x}) = x_2 - X_2(\mathbf{x}) = 2x_1^3 \end{cases}$$

DEFORMATION OF FIBER AC

Line containing A and C before deformation:

$$AC: \quad \mathbf{X}_{AC} = \mathbf{X}_A + \lambda(\mathbf{X}_C - \mathbf{X}_A) \quad \Leftrightarrow \quad \begin{cases} X_1 = 0{,}5\lambda \\ X_2 = 0{,}25\lambda \end{cases} \qquad \lambda \in \langle 0\ ;\ 1\rangle$$

Length of AC before deformation:

$$|AC| = \int_{AC} \mathrm{d}S = \int_{AC} \sqrt{\mathrm{d}X_1^2 + \mathrm{d}X_2^2} = \int_{\lambda=0}^{1} \sqrt{\left(\frac{\mathrm{d}X_1}{\mathrm{d}\lambda}\right)^2 + \left(\frac{\mathrm{d}X_2}{\mathrm{d}\lambda}\right)^2}\,\mathrm{d}X_1 = \int_0^1 \sqrt{(0{,}5)^2 + (0{,}25)^2}\,\mathrm{d}\lambda =$$
$$= \frac{\sqrt{5}}{4}\int_0^1 \mathrm{d}\lambda = \frac{\sqrt{5}}{4} \approx 0{,}560$$

Accounting for deformation relations:

$$AC: \quad \mathbf{x} = \begin{cases} x_1 = X_1 \\ x_2 = X_2 + 2X_1^3 \end{cases} \quad \Leftrightarrow \quad \begin{cases} x_1 = 0{,}5\lambda \\ x_2 = 0{,}25\lambda + 2(0{,}5\lambda)^3 = 0{,}25\lambda + 0{,}25\lambda^3 \end{cases}$$

Length of AC after deformation:

$$|AC| = \int_{AC} \mathrm{d}s = \int_{AC} \sqrt{\mathrm{d}x_1^2 + \mathrm{d}x_2^2} = \int_{\lambda=0}^{1} \sqrt{\left(\frac{\mathrm{d}x_2}{\mathrm{d}\lambda}\right)^2 + \left(\frac{\mathrm{d}x_2}{\mathrm{d}\lambda}\right)^2}\,\mathrm{d}\lambda = \int_0^1 \sqrt{(0{,}5)^2 + (0{,}25 + 0{,}75\lambda^2)^2}\,\mathrm{d}\lambda \approx 0{,}723$$

The length of the AC fibre may be also calculated with the use of an integral over a reference configuration. In order to do that, we need to determine the material deformation gradient $\mathbf{F}$ and material deformation tensor $\mathbf{C}$:

$$\mathbf{F}=\begin{bmatrix}1 & 0\\ 6X_1^2 & 1\end{bmatrix}\qquad \mathbf{C}=\mathbf{F}^{\mathrm{T}}\mathbf{F}=\begin{bmatrix}1 & 6X_1^2\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 6X_1^2 & 1\end{bmatrix}=\begin{bmatrix}36X_1^4+1 & 6X_1^2\\ 6X_1^2 & 1\end{bmatrix}$$

The relation between an infinitely small linear element before deformation $\mathrm{d}S=\sqrt{\mathrm{d}X_i\,\mathrm{d}X_i}$ and after deformation $\mathrm{d}s$ is as follows:

$$\mathrm{d}s=\sqrt{C_{ij}\,\mathrm{d}X_i\,\mathrm{d}X_j}$$

The length of the fibre is calculated with the use of an integral over a reference configuration (we are using the parametrization of a curve before deformation):

$$|AC|=\int_{AC}\mathrm{d}s=\int_{AC}\sqrt{C_{ij}\,\mathrm{d}X_i\,\mathrm{d}X_j}=\int_0^1\sqrt{C_{ij}\frac{\mathrm{d}X_i}{\mathrm{d}\lambda}\frac{\mathrm{d}X_j}{\mathrm{d}\lambda}}\,\mathrm{d}\lambda=$$

$$=\int_0^1\sqrt{C_{11}\left(\frac{\mathrm{d}X_1}{\mathrm{d}\lambda}\right)^2+C_{22}\left(\frac{\mathrm{d}X_2}{\mathrm{d}\lambda}\right)^2+2C_{12}\frac{\mathrm{d}X_1}{\mathrm{d}\lambda}\frac{\mathrm{d}X_2}{\mathrm{d}\lambda}}\,\mathrm{d}\lambda=$$

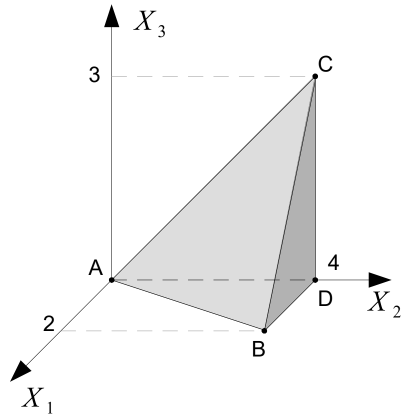
$$=\int_0^1\sqrt{(36X_1^4+1)\cdot(0{,}5)^2+1\cdot(0{,}25)^2+2\cdot(6X_1^2)\cdot 0{,}5\cdot 0{,}25}=\int_0^1\sqrt{9X_1^4+1{,}5X_1^2+0{,}3125}\,\mathrm{d}\lambda=$$

$$=\int_0^1\sqrt{9\cdot(0{,}5\lambda)^4+1{,}5(0{,}5\lambda)^2+0{,}3125}\,\mathrm{d}\lambda=$$

$$=\int_0^1\sqrt{0{,}5625\lambda^4+0{,}375\lambda^2+0{,}3125}\,\mathrm{d}\lambda=0{,}723$$

EXERCISE 11

Deformation is given in material description:



$$\begin{cases} x_1 = X_1 - X_3 + 2 \\ x_2 = -2X_1 + X_2 - X_3 + 4 \\ x_3 = X_1 + X_3 \end{cases}$$

a) Check if these relation are invertible. If so, find deformation relations in spatial description.
b) Find displacement field in both spatial and material description.
c) Make a sketch of an actual configuration.
d) What would be the shape of material fiber BC after deformation? What is its length?
e) What is an equation of ABC surface after deformation?
f) What is the volume of the body before and after deformation?

SOLUTION:

Invertibility is checked by calculating the Jacobian determinant:

$$J = \begin{vmatrix} \dfrac{\partial x_1}{\partial X_1} & \dfrac{\partial x_1}{\partial X_2} & \dfrac{\partial x_1}{\partial X_3} \\ \dfrac{\partial x_2}{\partial X_1} & \dfrac{\partial x_2}{\partial X_2} & \dfrac{\partial x_2}{\partial X_3} \\ \dfrac{\partial x_3}{\partial X_1} & \dfrac{\partial x_3}{\partial X_2} & \dfrac{\partial x_3}{\partial X_3} \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 \\ -2 & 1 & -1 \\ 1 & 0 & 1 \end{vmatrix} = 2$$

Determinant $J > 0$, so the relations are locally invertible. These are linear equations with respect to material coordinates, so it can be solved, to obtain following result:

$$\begin{cases} X_1 = \dfrac{1}{2}x_1 + \dfrac{1}{2}x_3 - 1 \\ X_2 = \dfrac{1}{2}x_1 + x_2 + \dfrac{3}{2}x_3 - 5 \\ X_3 = -\dfrac{1}{2}x_1 + \dfrac{1}{2}x_3 + 1 \end{cases}$$

DISPLACEMENT FIELD

- **material description:**

$$\begin{cases} u_1(\mathbf{X}) = x_1((\mathbf{X})) - X_1 = -X_3 + 2 \\ u_2(\mathbf{X}) = x_2((\mathbf{X})) - X_2 = -2X_1 - X_3 + 4 \\ u_3(\mathbf{X}) = x_3((\mathbf{X})) - X_3 = X_1 \end{cases}$$

- **spatial description:**

$$\begin{cases} u_1(\mathbf{x}) = x_1 - X_1(\mathbf{x}) = \frac{1}{2}x_1 - \frac{1}{2}x_3 + 1 \\ u_2(\mathbf{x}) = x_2 - X_2(\mathbf{x}) = -\frac{1}{2}x_1 - \frac{3}{2}x_3 + 5 \\ u_3(\mathbf{x}) = x_3 - X_3(\mathbf{x}) = \frac{1}{2}x_1 + \frac{1}{2}x_3 - 1 \end{cases}$$

DEFORMATION

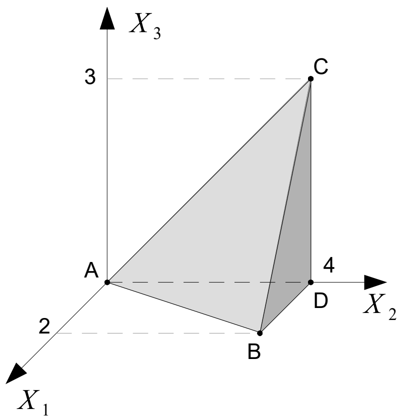
Point A: $\mathbf{X}_A = [0;0;0] \rightarrow \mathbf{x}_A = [2;4;0]$

Point B: $\mathbf{X}_B = [2;4;0] \rightarrow \mathbf{x}_B = [4;4;2]$

Point C: $\mathbf{X}_C = [0;4;4] \rightarrow \mathbf{x}_C = [-2;4;4]$

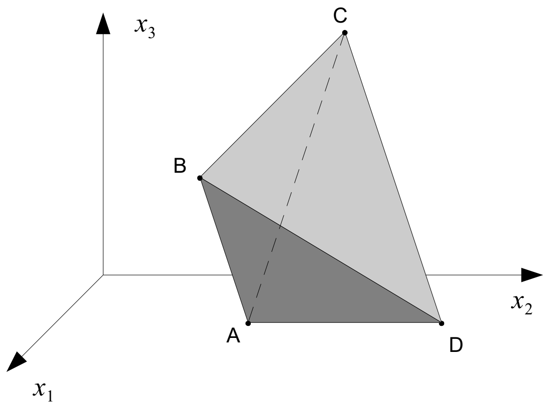
Point D: $\mathbf{X}_D = [0;4;0] \rightarrow \mathbf{x}_D = [2;8;0]$

Reference configuration



⇒

Actual configuration



DEFORMATION OF FIBER BC

Line containing B and C before deformation:

$$\mathbf{X}_{BC}(\lambda) = \mathbf{X}_B + \lambda(\mathbf{X}_C - \mathbf{X}_B) \quad \Leftrightarrow \quad \begin{cases} X_1 = 2 + \lambda(0-2) = 2 - 2\lambda \\ X_2 = 4 + \lambda(4-4) = 4 \\ X_3 = 0 + \lambda(4-0) = 4\lambda \end{cases} \quad \lambda \in \langle 0,1 \rangle$$

Length of BC before deformation:

$$|BC| = \int_K \mathrm{d}S = \int_K \sqrt{\mathrm{d}X_1^2 + \mathrm{d}X_2^2 + \mathrm{d}X_3^2} = \int_{\lambda=0}^{1} \sqrt{\left(\frac{\mathrm{d}X_1}{\mathrm{d}\lambda}\right)^2 + \left(\frac{\mathrm{d}X_2}{\mathrm{d}\lambda}\right)^2 + \left(\frac{\mathrm{d}X_3}{\mathrm{d}\lambda}\right)^2}\,\mathrm{d}\lambda =$$

$$= \int_0^1 \sqrt{(-2)^2 + 0 + (4)^2}\,\mathrm{d}\lambda = 2\sqrt{5}\int_0^1 \lambda = 2\sqrt{5}$$

Curve containing B and C after deformation:

$$\mathbf{x}(\lambda)=\begin{cases} x_1 = X_1 - X_3 + 2 \\ x_2 = -2X_1 + X_2 - X_3 + 4 \\ x_3 = X_1 + X_3 \end{cases} \quad \Leftrightarrow \quad \begin{cases} x_1 = 4 - 6\lambda \\ x_2 = 4 \\ x_3 = 2 + 2\lambda \end{cases} \qquad \lambda \in \langle 0,1 \rangle$$

Length of fiber BC after deformation may be calculated in two ways:

- integral along deformed BC curve in actual configuration
- integral along undeformed BC line in reference configuration with the use of deformation tensor

Length of BC after deformation:

The 1st approach:

$$|BC| = \int_K \mathrm{d}s = \int_K \sqrt{\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2} = \int_{\lambda=0}^{1} \sqrt{\left(\frac{\mathrm{d}x_1}{\mathrm{d}\lambda}\right)^2 + \left(\frac{\mathrm{d}x_2}{\mathrm{d}\lambda}\right)^2 + \left(\frac{\mathrm{d}x_3}{\mathrm{d}\lambda}\right)^2}\,\mathrm{d}\lambda =$$
$$= \int_0^1 \sqrt{(-6)^2 + 0 + (2)^2}\,\mathrm{d}\lambda = 2\sqrt{10}\int_0^1 \lambda = 2\sqrt{10}$$

The 2nd approach:

Deformation gradient:

$$\mathbf{F} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

Deformation tensor:

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}} \cdot \mathbf{F} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

$$|BC| = \int_K \mathrm{d}s = \int_K \sqrt{C_{ij}\,\mathrm{d}X_i\,\mathrm{d}X_j} = \int_{\lambda=0}^{1} \sqrt{C_{ij}\frac{\mathrm{d}X_i}{\mathrm{d}\lambda}\frac{\mathrm{d}X_j}{\mathrm{d}\lambda}}\,\mathrm{d}\lambda =$$
$$= \int_0^1 \sqrt{C_{11}\frac{\mathrm{d}X_1}{\mathrm{d}\lambda}\frac{\mathrm{d}X_1}{\mathrm{d}\lambda} + C_{12}\frac{\mathrm{d}X_1}{\mathrm{d}\lambda}\frac{\mathrm{d}X_2}{\mathrm{d}\lambda} + \ldots + C_{33}\frac{\mathrm{d}X_3}{\mathrm{d}\lambda}\frac{\mathrm{d}X_3}{\mathrm{d}\lambda}}\,\mathrm{d}\lambda =$$
$$= \int_0^1 \sqrt{[6\cdot(-2)\cdot(-2)] + 2[(-2)\cdot(-2)\cdot 0] + 2[2\cdot(-2)\cdot 4] + [1\cdot 0\cdot 0] + 2[(-1)\cdot 0\cdot 4] + [3\cdot 4\cdot 4]}\,\mathrm{d}\lambda =$$
$$\int_0^1 \sqrt{24 - 32 + 48}\,\mathrm{d}\lambda = 2\sqrt{10}$$

In the sum with respect to *ij* indices (inside the square root) we've made us of symmetry of $\mathbf{C}$:

$$C_{ij}\frac{\mathrm{d}X_i}{\mathrm{d}\lambda}\frac{\mathrm{d}X_j}{\mathrm{d}\lambda} = C_{ji}\frac{\mathrm{d}X_j}{\mathrm{d}\lambda}\frac{\mathrm{d}X_i}{\mathrm{d}\lambda}$$

DEFORMATION OF FACE ABC

Equation of a surface perpendicualr to vector $\mathbf{n}=[a,b,c]$ is: $a X_1+b X_2+c X_3+d=0$

Normal vector of a surface may be found as a cross product of two vectors connecting three non-collinear point on that surface:

$$\vec{AB}=[2;4;0] \quad \vec{AC}=[0;4;4] \quad \Rightarrow \quad \mathbf{n}=\vec{AB}\times\vec{AC}=[16;-8;8]$$

Any parallel vector may be chose, e.g.: $\mathbf{n}=[2,-1,1]$. **Equation of plane containing ABC face before deformation** is:

$$2X_1-X_2+X_3+d=0$$

Parameter *d* is found with the use of condition that point A, B and C belong to that plane. Writing down this condition for A gives us: $2\cdot 0-0+0+d=0 \quad \Rightarrow \quad d=0$. It can be checked that this equations is satisfied also by coordinates of B and C. Equation of plane ABC before deoformation:

$$ABC: \quad 2X_1-X_2+X_3=0$$

Equation of plane containing ABC face before deformation is obtained by substitution of $\mathbf{X}=\mathbf{X}(\mathbf{x})$:

$$ABC: \quad 2\cdot\left(\frac{1}{2}x_1+\frac{1}{2}x_3-1\right)-\left(\frac{1}{2}x_1+x_2+\frac{3}{2}x_3-5\right)+\left(-\frac{1}{2}x_1+\frac{1}{2}x_3+1\right)=0$$

$$-x_2+4=0$$

VOLUME CHANGE

Reference volume is equal the volume of a pyramid:

$$V=\frac{1}{3}\cdot P_p\cdot H=\frac{1}{3}\cdot\left(\frac{1}{2}\cdot 2\cdot 4\right)\cdot 4=\frac{16}{3}$$

It can be calculated also by a triple integral:

$$V_R=\iiint_{V_R} \mathrm{d}V_R=\int_{X_1=0}^{2}\int_{X_2=2X_1}^{4}\int_{X_3=0}^{X_2-2X_1}\mathrm{d}X_1\,\mathrm{d}X_2\,\mathrm{d}X_3=\int_{X_1=0}^{2}\int_{X_2=2X_1}^{4}[X_2-2X_1]\,\mathrm{d}X_1\,\mathrm{d}X_2=$$

$$=\int_{X_1=0}^{2}\left[\frac{X_2^2}{2}-2X_1X_2\right]_{X_2=2X_1}^{4}\mathrm{d}X_1=\int_0^2\left[8-8X_1+2X_1^2\right]\mathrm{d}X_1=\left[8X_1-4X_1^2+\frac{2}{3}X_1^3\right]_0^2=\frac{16}{3}$$

Actual volume: $$V=\iiint_V \mathrm{d}V=\iiint_{V_R} J\,\mathrm{d}V_R$$

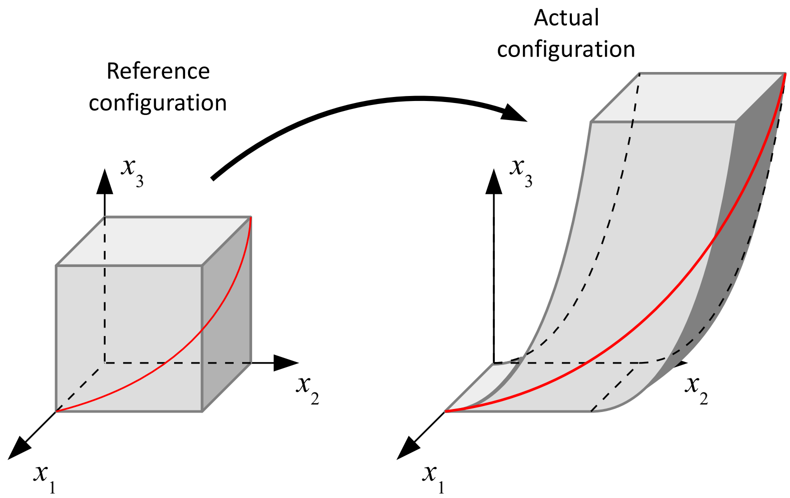
Since the Jacobian determinant is constant (equal in each point) it can be put outside integral:

$$V=J\iiint_{V_R}\mathrm{d}V_R=J\cdot V_R=\frac{2\cdot 16}{3}=\frac{32}{3}$$

EXERCISE 12

Elastic cube of side length 2 undergoes deformation according to equations:

$$\begin{cases} x_1 = X_1 \\ x_2 = X_2 + X_3 \\ x_3 = X_3^2 \end{cases}$$



What will the change of length of material curve given by equations:

$$\begin{cases} X_1 = 2(1-\lambda) \\ X_2 = 2\lambda \\ X_3 = 2\lambda^2 \end{cases} \quad , \quad \lambda \in (0,1)$$

SOLUTION:

Length of the curve before deformation is calculated as a line integral along reference configuration of that curve:

$$L_r = \int \mathrm{d}S = \int_0^1 \sqrt{\left(\frac{\mathrm{d}X_1}{\mathrm{d}\lambda}\right)^2 + \left(\frac{\mathrm{d}X_2}{\mathrm{d}\lambda}\right)^2 + \left(\frac{\mathrm{d}X_3}{\mathrm{d}\lambda}\right)^2}\,\mathrm{d}\lambda = \int_0^1 \sqrt{(-2)^2 + (2)^2 + (4\lambda)^2}\,\mathrm{d}\lambda =$$

$$= \int_0^1 \sqrt{8+16\lambda^2}\,\mathrm{d}\lambda = \operatorname{arsinh}\sqrt{2} + \sqrt{6} \approx 3{,}5957$$

In order to find its length after deformation we will need deformation gradient and deformation tensor:

$$\mathbf{F} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2X_3 \end{bmatrix}$$

Deformation tensor:

$$\mathbf{C}=\mathbf{F}^{\mathbf{T}}\cdot\mathbf{F}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 1 & 4X_3^2+1\end{bmatrix}$$

Length of the curve after deformation is calculated as a line integral of deformed line elements along reference configuration of that curve:

$$L=\int \mathrm{d}s=\int\sqrt{C_{ij}\,\mathrm{d}X_i\,\mathrm{d}X_j}=\int_0^1\sqrt{C_{ij}\frac{\mathrm{d}X_i}{\mathrm{d}\lambda}\frac{\mathrm{d}X_j}{\mathrm{d}\lambda}}\,\mathrm{d}\lambda=$$

$$=\int_0^1\sqrt{C_{11}\frac{\mathrm{d}X_1}{\mathrm{d}\lambda}\frac{\mathrm{d}X_1}{\mathrm{d}\lambda}+C_{12}\frac{\mathrm{d}X_1}{\mathrm{d}\lambda}\frac{\mathrm{d}X_2}{\mathrm{d}\lambda}+\ldots+C_{33}\frac{\mathrm{d}X_3}{\mathrm{d}\lambda}\frac{\mathrm{d}X_3}{\mathrm{d}\lambda}}\,\mathrm{d}\lambda=$$

$$=\int_0^1\sqrt{1\cdot(-2)^2+1\cdot(2)^2+1\cdot(2)\cdot(4\lambda)+1\cdot(4\lambda)\cdot(2)+(4X_3^2+1)\cdot(4\lambda)^2}\,\mathrm{d}\lambda=$$

$$=\int_0^1\sqrt{1\cdot(-2)^2+1\cdot(2)^2+1\cdot(2)\cdot(4\lambda)+1\cdot(4\lambda)\cdot(2)+(4(2\lambda^2)^2+1)\cdot(4\lambda)^2}\,\mathrm{d}\lambda=$$

$$=\int_0^1\sqrt{256\lambda^6+16\lambda^2+16\lambda+8}\,\mathrm{d}\lambda\approx 6{,}5666$$

In the above integral parametric equations of the curve were substituted in place of material coordinates – values of these coordinates for points on the curve along which we are integrating are given be those equations.

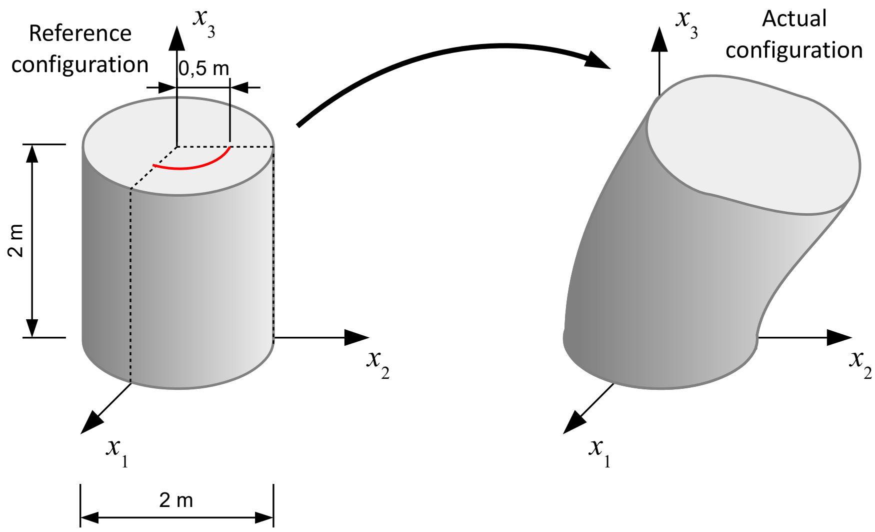
Curve length before deformation: $L_r\approx 3{,}5957$

Curve length after deformation: $L\approx 6{,}5666$

EXERCISE 13

Cylinder of heigth 2m and diameter of base 2m undergoes deformation accordint to relations:

$$\begin{cases} x_1 = X_1 \\ x_2 = X_2 + \dfrac{1}{4} X_3^2 \\ x_3 = X_3\left(1 - \dfrac{1}{4} X_2^3\right) \end{cases}$$



Find the surface area of the top face of the cylinder before and after deformation and length of a material fiber which has a shape of circular arc lying in the top face in distance 0,5m from the axis of the cylinder and contained in the 1st octant of the assumed coordinate system.

SOLUTION:

CHANGE OF LENGTH OF A CURVE:

Length of a fiber before deformation is equal quater of circumference of a circle of radius 0,5m:

$$L_R = \frac{2\pi R}{4} \approx 0{,}7854 \text{ m}$$

In order to find its length after deformation we will need **deformation gradient**:

$$\mathbf{F} = \begin{bmatrix} \dfrac{\partial x_1}{\partial X_1} & \dfrac{\partial x_1}{\partial X_2} & \dfrac{\partial x_1}{\partial X_3} \\ \dfrac{\partial x_2}{\partial X_1} & \dfrac{\partial x_2}{\partial X_2} & \dfrac{\partial x_2}{\partial X_3} \\ \dfrac{\partial x_3}{\partial X_1} & \dfrac{\partial x_3}{\partial X_2} & \dfrac{\partial x_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \dfrac{X_3}{2} \\ 0 & -\dfrac{3}{4} X_2^2 X_3 & 1 - \dfrac{1}{4} X_2^3 \end{bmatrix}$$

Jacobian determinant $J = 1 + \frac{3}{8} X_2^2 X_3^2 - \frac{1}{4} X_2^3$ is positive in each point of reference configuration.

Deformation tensor:

$$\mathbf{C}=\mathbf{F}^{\mathrm{T}}\cdot\mathbf{F}=\frac{1}{16}\begin{bmatrix}16 & 0 & 0\\ 0 & 16+9X_2^4X_3^2 & X_3(8-12X_2^2+3X_2^5)\\ 0 & X_3(8-12X_2^2+3X_2^5) & 16-8X_2^3+X_2^6+4X_3^2\end{bmatrix}$$

The curve before deformation may be parametrized with the use of angular coordinate of a cylindrical coordinate system:

$$\begin{cases}X_1=r\cos\phi\\ X_2=r\sin\phi\\ X_3=z\end{cases}\Leftrightarrow\begin{cases}r=\sqrt{X_1^2+X_2^2}\\ \phi=\operatorname{arctg}\dfrac{X_2}{X_1}\\ z=X_3\end{cases}\Rightarrow\quad K:\begin{cases}r=0{,}5\\ \phi\in\left(0;\dfrac{\pi}{2}\right)\\ z=2\end{cases}$$

Curve length after deformation:

$$L=\int\mathrm{d}s=\int\sqrt{C_{ij}\,\mathrm{d}X_i\,\mathrm{d}X_j}=\int_0^{\pi/2}\sqrt{C_{ij}\frac{\mathrm{d}X_i}{\mathrm{d}\phi}\frac{\mathrm{d}X_j}{\mathrm{d}\phi}}\,\mathrm{d}\phi$$

Non-zero terms in the sum inside the square root (accounting for symmetry):

$$C_{11}\frac{\mathrm{d}X_1}{\mathrm{d}\phi}\frac{\mathrm{d}X_1}{\mathrm{d}\phi}=(1)\cdot(-r\sin\phi)^2=r^2\sin^2\phi$$

$$C_{22}\frac{\mathrm{d}X_2}{\mathrm{d}\phi}\frac{\mathrm{d}X_2}{\mathrm{d}\phi}=\left[1+\frac{9}{16}(r\sin\phi)^4z^2\right]\cdot(r\cos\phi)^2$$

$$2C_{23}\frac{\mathrm{d}X_2}{\mathrm{d}\phi}\frac{\mathrm{d}X_3}{\mathrm{d}\phi}=\frac{z}{16}\left[8-12(r\sin\phi)^2+3(r\sin\phi)^5\right]\cdot(r\cos\phi)\cdot 0=0$$

$$C_{33}\frac{\mathrm{d}X_3}{\mathrm{d}\phi}\frac{\mathrm{d}X_3}{\mathrm{d}\phi}=\frac{1}{16}\left[16-8(r\sin\phi)^3+(r\sin\phi)^6+4z^2\right]\cdot 0=0$$

The integral is calulated accounting for fixed values of r and z :

$$L=\int_0^{\pi/2}\sqrt{C_{ij}\frac{\mathrm{d}X_i}{\mathrm{d}\phi}\frac{\mathrm{d}X_j}{\mathrm{d}\phi}}\,\mathrm{d}\phi=\int_0^{\pi/2}\sqrt{r^2\sin^2\phi+\left[1+\frac{9}{16}(r\sin\phi)^4z^2\right]r^2\cos^2\phi}\,\mathrm{d}\phi=$$

$$=\int_0^{\pi/2}\sqrt{\frac{1}{4}\sin^2\phi+\frac{1}{4}\left[1+\frac{9}{64}\sin^4\phi\right]\cos^2\phi}\,\mathrm{d}\phi\approx 0{,}7888$$

Length of a fiber after deformation: $L\approx 0{,}7888\ \mathrm{m}$

CHANGE OF SURFACE AREA:

Surface area of top face before deformation is the area of a circe of radius 0,5m:
$A_R = \pi R^2 = 3{,}1416 \text{ m}^2$

Relation between differential surface elements is as follows:

$$\mathrm{d}A = J\sqrt{(\mathbf{N}^{\mathrm{T}}\cdot\mathbf{F}^{-1})\cdot(\mathbf{N}^{\mathrm{T}}\cdot\mathbf{F}^{-1})^{\mathrm{T}}}\,\mathrm{d}A_R$$

Jacobian determinant: $J = 1 + \frac{3}{8}X_2^2X_3^2 - \frac{1}{4}X_2^3$

Unit normal vector for the face before deformation: $\mathbf{N}^{\mathrm{T}} = [0;0;1]$

Spatial deformation gradient:

$$\mathbf{F}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{J}\left(1-\frac{1}{4}X_2^3\right) & \frac{-X_3}{2J} \\ 0 & \frac{3X_2^2X_3}{4J} & \frac{1}{J} \end{bmatrix}$$

$$\mathbf{N}^{\mathrm{T}}\cdot\mathbf{F}^{-1} = \begin{bmatrix} 0 & \frac{3X_2^2X_3}{4J} & \frac{1}{J} \end{bmatrix}$$

$$(\mathbf{N}^{\mathrm{T}}\cdot\mathbf{F}^{-1})\cdot(\mathbf{N}^{\mathrm{T}}\cdot\mathbf{F}^{-1})^{\mathrm{T}} = \frac{1}{J^2}\left(1+\frac{9}{16}X_2^4X_3^2\right)$$

$$J\sqrt{(\mathbf{N}^{\mathrm{T}}\cdot\mathbf{F}^{-1})\cdot(\mathbf{N}^{\mathrm{T}}\cdot\mathbf{F}^{-1})^{\mathrm{T}}} = \sqrt{1+\frac{9}{16}X_2^4X_3^2}$$

We will again use cylindrical coordinates. The integral may be expressed as:

$$A = \iint_A \mathrm{d}A = \iint_{A_R} J\sqrt{(\mathbf{N}^{\mathrm{T}}\cdot\mathbf{F}^{-1})\cdot(\mathbf{N}^{\mathrm{T}}\cdot\mathbf{F}^{-1})^{\mathrm{T}}}\,\mathrm{d}A_R = \int_{r=0}^{R}\int_{\phi=0}^{2\pi} J\sqrt{(\mathbf{N}^{\mathrm{T}}\cdot\mathbf{F}^{-1})\cdot(\mathbf{N}^{\mathrm{T}}\cdot\mathbf{F}^{-1})^{\mathrm{T}}}\,r\,\mathrm{d}\phi\,\mathrm{d}r =$$
$$= \int_{r=0}^{R}\int_{\phi=0}^{2\pi}\sqrt{1+\frac{9}{16}(r\sin\phi)^4z^2}\,r\,\mathrm{d}\phi\,\mathrm{d}r$$

We calculate the integral accounting for fixed value of z :

$$A = \int_{r=0}^{R}\int_{\phi=0}^{2\pi}\sqrt{r^2+\frac{9}{4}r^6\sin^4\phi}\,\mathrm{d}\phi\,\mathrm{d}r \approx 3{,}5136 \text{ m}^2$$

Surface area of top face after deformation: $A \approx 3{,}5136 \text{ m}^2$

EXERCISE 15

Perform polar decomposition of deformation gradient for a deformation given by equations:

$$\begin{cases} x_1 = 1{,}2\,X_1 + 0{,}8\,X_2 \\ x_2 = 0{,}6\,X_2 + 1{,}5\,X_3 \\ x_3 = 1{,}4\,X_1 + X_3 \end{cases}$$

How stretch tensors and rotation tensor deform a material fibre $\mathrm{d}\mathbf{X} = [1{,}0{,}0]$?

SOLUTION:

Deformation gradient: $\mathbf{F} = \begin{bmatrix} 1{,}2 & 0{,}8 & 0 \\ 0 & 0{,}6 & 1{,}5 \\ 1{,}4 & 0 & 1 \end{bmatrix}$

Deformation tensor: $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \begin{bmatrix} 1{,}2 & 0 & 1{,}4 \\ 0{,}8 & 0{,}6 & 0 \\ 0 & 1{,}5 & 1 \end{bmatrix}\begin{bmatrix} 1{,}2 & 0{,}8 & 0 \\ 0 & 0{,}6 & 1{,}5 \\ 1{,}4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3{,}40 & 0{,}96 & 1{,}4 \\ 0{,}96 & 1 & 0{,}90 \\ 1{,}4 & 0{,}90 & 3{,}25 \end{bmatrix}$

Eigenvalues are found numerically:

$$C_1 = 0{,}582 \qquad C_2 = 1{,}923 \qquad C_3 = 5{,}145$$

Eigenvectors of $\mathbf{C}$

Eigenvector corresponding with $C_1 = 0{,}582$

$$(\mathbf{C} - C_1\mathbf{I})\cdot\mathbf{u} = \mathbf{0} \quad \Rightarrow \quad \begin{bmatrix} 2{,}818 & 0{,}96 & 1{,}4 \\ 0{,}96 & 0{,}418 & 0{,}90 \\ 1{,}4 & 0{,}90 & 2{,}668 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{u} = [2{,}818\ ;\ 0{,}96\ ;\ 1{,}4] \times [0{,}96\ ;\ 0{,}418\ ;\ 0{,}90] = [0{,}279\ ;\ -1{,}192\ ;\ 0{,}256]$$

$$\mathbf{c}_1 = \frac{\mathbf{u}}{|\mathbf{u}|} = [0{,}223\ ;\ -0{,}953\ ;\ 0{,}204]$$

Eigenvector corresponding with $C_2 = 1{,}923$

$$(\mathbf{C} - C_2\mathbf{I})\cdot\mathbf{v} = \mathbf{0} \quad \Rightarrow \quad \begin{bmatrix} 1{,}477 & 0{,}96 & 1{,}4 \\ 0{,}96 & -0{,}923 & 0{,}90 \\ 1{,}4 & 0{,}90 & 1{,}327 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{v} = [1{,}477\ ;\ 0{,}96\ ;\ 1{,}4] \times [0{,}96\ ;\ -0{,}923\ ;\ 0{,}90] = [2{,}156\ ;\ 0{,}0147\ ;\ -2{,}284]$$

$$\mathbf{c}_2 = \frac{\mathbf{v}}{|\mathbf{v}|} = [0{,}686\ ;\ 0{,}00469\ ;\ -0{,}727]$$

Eigenvector corresponding with $C_3 = 5{,}145$

$$\mathbf{c}_3 = \mathbf{c}_1 \times \mathbf{c}_2 = [0{,}692\ ;\ 0{,}303\ ;\ 0{,}655]$$

Transformation matrix from original coordinate system to system of eigenvectors of $\mathbf{C}$:

$$\mathbf{A} = \begin{bmatrix} 0{,}223 & -0{,}953 & 0{,}204 \\ 0{,}686 & 0{,}00469 & -0{,}727 \\ 0{,}692 & 0{,}303 & 0{,}655 \end{bmatrix}$$

Stretch tensor in eigenvector coordinate system:

$$\mathbf{U}^2 = \mathbf{C} \quad \Rightarrow \quad \mathbf{U}_{[\omega]} = \begin{bmatrix} \sqrt{0{,}582} & 0 & 0 \\ 0 & \sqrt{1{,}923} & 0 \\ 0 & 0 & \sqrt{5{,}145} \end{bmatrix} = \begin{bmatrix} 0{,}763 & 0 & 0 \\ 0 & 1{,}387 & 0 \\ 0 & 0 & 2{,}268 \end{bmatrix}$$

Inverse of stretch tensor in eigenvector coordinate system:

$$\mathbf{U}^{-1}_{[\omega]} = \begin{bmatrix} 1/0{,}763 & 0 & 0 \\ 0 & 1/1{,}387 & 0 \\ 0 & 0 & 1/2{,}268 \end{bmatrix} = \begin{bmatrix} 1{,}311 & 0 & 0 \\ 0 & 0{,}721 & 0 \\ 0 & 0 & 0{,}441 \end{bmatrix}$$

Right stretch tensor in original coordinate system:

$$\mathbf{U} = \mathbf{A}^{\mathrm{T}} \cdot \mathbf{U}_{[\omega]} \cdot \mathbf{A} = \begin{bmatrix} 1{,}778 & 0{,}317 & 0{,}371 \\ 0{,}317 & 0{,}901 & 0{,}296 \\ 0{,}371 & 0{,}296 & 1{,}739 \end{bmatrix}$$

Inverse of right stretch tensor in original coordinate system:

$$\mathbf{U}^{-1} = \mathbf{A}^{\mathrm{T}} \cdot \mathbf{U}^{-1}_{[\omega]} \cdot \mathbf{A} = \begin{bmatrix} 0{,}616 & -0{,}184 & -0{,}100 \\ -0{,}184 & 1{,}231 & -0{,}170 \\ -0{,}100 & -0{,}170 & 0{,}625 \end{bmatrix}$$

Rotation tensor will be found as $\mathbf{R} = \mathbf{F} \cdot \mathbf{U}^{-1}$, yet $\mathbf{U}^{-1}$ must be expressed in original cordinate system. **Rotation tensor in original coordinate system:**

$$\mathbf{R} = \mathbf{F} \cdot \mathbf{U}^{-1} = \begin{bmatrix} 1{,}2 & 0{,}8 & 0 \\ 0 & 0{,}6 & 1{,}5 \\ -1{,}4 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0{,}616 & -0{,}184 & -0{,}100 \\ -0{,}184 & 1{,}231 & -0{,}170 \\ -0{,}100 & -0{,}170 & 0{,}625 \end{bmatrix} = \begin{bmatrix} 0{,}592 & 0{,}764 & -0{,}257 \\ -0{,}261 & 0{,}483 & 0{,}836 \\ 0{,}762 & -0{,}428 & 0{,}485 \end{bmatrix}$$

Left stretch tensor in original coordinate system:

$$\mathbf{V} = \mathbf{R} \cdot \mathbf{U} \cdot \mathbf{R}^{\mathrm{T}} = \begin{bmatrix} 1{,}322 & 0{,}072 & 0{,}571 \\ 0{,}072 & 1{,}544 & 0{,}470 \\ 0{,}571 & 0{,}470 & 1{,}551 \end{bmatrix}$$

CHECK

- $$\mathbf{R} \cdot \mathbf{U} = \begin{bmatrix} 0{,}592 & 0{,}764 & -0{,}257 \\ -0{,}261 & 0{,}483 & 0{,}836 \\ 0{,}762 & -0{,}428 & 0{,}485 \end{bmatrix} \cdot \begin{bmatrix} 1{,}778 & 0{,}317 & 0{,}371 \\ 0{,}317 & 0{,}901 & 0{,}296 \\ 0{,}371 & 0{,}296 & 1{,}739 \end{bmatrix} = \begin{bmatrix} 1{,}2 & 0{,}8 & 0 \\ 0 & 0{,}6 & 1{,}5 \\ 1{,}4 & 0 & 1 \end{bmatrix} = \mathbf{F}$$
- $\mathbf{U}^{\mathrm{T}} = \mathbf{U} \quad \Rightarrow \quad \mathbf{U}$ is symmetric
- $\det \mathbf{R} = 1 \quad \Rightarrow \quad \mathbf{R}$ is orthogonal

DEFORMATION OF MATERIAL FIBRE $\mathrm{d}\mathbf{X} = [1,0,0]$:

Stretch before rotation:

$$\mathbf{U}\cdot\mathrm{d}\mathbf{X} = \begin{bmatrix} 1{,}778 & 0{,}317 & 0{,}371 \\ 0{,}317 & 0{,}901 & 0{,}296 \\ 0{,}371 & 0{,}296 & 1{,}739 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1{,}778 \\ 0{,}317 \\ 0{,}371 \end{bmatrix}$$

Rotation after stretch:

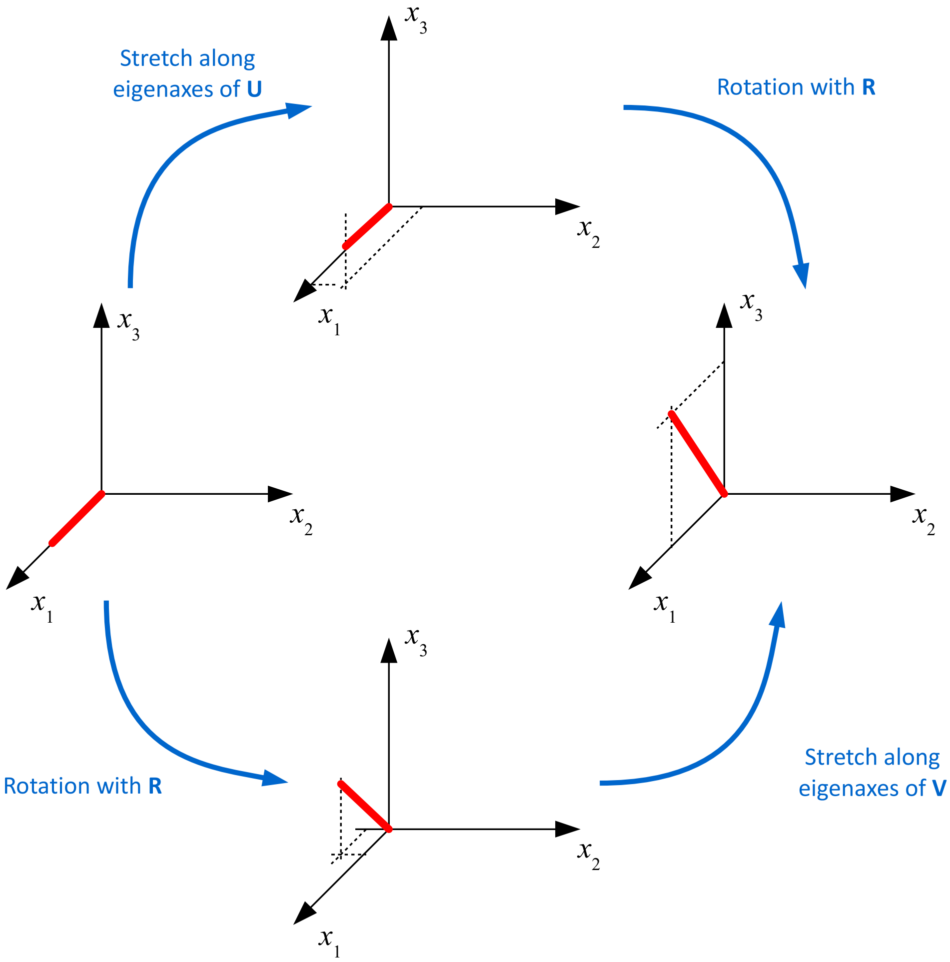
$$\mathbf{F}\cdot\mathrm{d}\mathbf{X} = \mathbf{R}\cdot(\mathbf{U}\cdot\mathrm{d}\mathbf{X}) = \begin{bmatrix} 0{,}592 & 0{,}764 & -0{,}257 \\ -0{,}261 & 0{,}483 & 0{,}836 \\ 0{,}762 & -0{,}428 & 0{,}485 \end{bmatrix}\begin{bmatrix} 1{,}778 \\ 0{,}317 \\ 0{,}371 \end{bmatrix} = \begin{bmatrix} 1{,}2 \\ 0 \\ 1{,}4 \end{bmatrix}$$

Rotation before stretch:

$$\mathbf{R}\cdot\mathrm{d}\mathbf{X} = \begin{bmatrix} 0{,}592 & 0{,}764 & -0{,}257 \\ -0{,}261 & 0{,}483 & 0{,}836 \\ 0{,}762 & -0{,}428 & 0{,}485 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0{,}592 \\ -0{,}261 \\ 0{,}762 \end{bmatrix}$$

Stretch after rotation:

$$\mathbf{F}\cdot\mathrm{d}\mathbf{X} = \mathbf{V}\cdot(\mathbf{R}\cdot\mathrm{d}\mathbf{X}) = \begin{bmatrix} 1{,}322 & 0{,}072 & 0{,}571 \\ 0{,}072 & 1{,}544 & 0{,}470 \\ 0{,}571 & 0{,}470 & 1{,}551 \end{bmatrix}\begin{bmatrix} 0{,}592 \\ -0{,}261 \\ 0{,}762 \end{bmatrix} = \begin{bmatrix} 1{,}2 \\ 0 \\ 1{,}4 \end{bmatrix}$$



EXERCISE 16

Perform polar decomposition of deformation gradient for a deformation given by equations:

$$\begin{cases} x_1 = 2X_1 - X_2 \\ x_2 = 2X_1 + 4X_2 \end{cases}$$

How stretch tensors and rotation tensor deform a material fibre which was initially parallel to the first eigenvector of right stretch tensor?

ROZWIĄZANIE:

Deformation gradient: $\mathbf{F} = \begin{bmatrix} 2 & -1 \\ 2 & 4 \end{bmatrix}$

Deformation tensor: $\mathbf{C} = \mathbf{F}^{\mathrm{T}} \cdot \mathbf{F} = \begin{bmatrix} 8 & 6 \\ 6 & 17 \end{bmatrix}$

Eigenvalues of deformation tensor:

$$C_1 = \frac{C_{11}+C_{22}}{2} + \sqrt{\left(\frac{C_{11}-C_{22}}{2}\right)^2 + C_{12}^2} = 20$$

$$C_2 = \frac{C_{11}+C_{22}}{2} - \sqrt{\left(\frac{C_{11}-C_{22}}{2}\right)^2 + C_{12}^2} = 5$$

Angle between horizontal axis and first eigenaxis of $\mathbf{C}$:

$$\phi = \operatorname{arctg} \frac{C_{12}}{C_1 - C_{22}} = \operatorname{arctg} 2 \approx 63{,}43^{\circ}$$

Eigenvectors of deformation tensor:

$$\mathbf{c}_1 = [\cos\phi \ ; \ \sin\phi] = \left[\frac{1}{\sqrt{5}} \ ; \ \frac{2}{\sqrt{5}}\right]$$

$$\mathbf{c}_2 = [-\sin\phi \ ; \ \cos\phi] = \left[-\frac{2}{\sqrt{5}} \ ; \ \frac{1}{\sqrt{5}}\right]$$

Transformation matrix:

$$\mathbf{A} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \approx \begin{bmatrix} 0{,}4472 & 0{,}8944 \\ -0{,}8944 & 0{,}4472 \end{bmatrix}$$

Stretch tensor in the eigenvector coordinate system:

$$\mathbf{U}_{[\omega]} = \begin{bmatrix} \sqrt{C_1} & 0 \\ 0 & \sqrt{C_2} \end{bmatrix} = \begin{bmatrix} 2\sqrt{5} & 0 \\ 0 & \sqrt{5} \end{bmatrix} \approx \begin{bmatrix} 4{,}472 & 0 \\ 0 & 2{,}236 \end{bmatrix}$$

Inverse of stretch tensor in the eigenvector coordinate system:

$$\mathbf{U}_{[\omega]}^{-1} = = \begin{bmatrix} \frac{1}{\sqrt{C_1}} & 0 \\ 0 & \frac{1}{\sqrt{C_2}} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{5}}{10} & 0 \\ 0 & \frac{\sqrt{5}}{5} \end{bmatrix} \approx \begin{bmatrix} 0{,}2236 & 0 \\ 0 & 0{,}4472 \end{bmatrix}$$

Right stretch tensor in original coordinate system:

$$\mathbf{U} = \mathbf{A}^{\mathrm{T}} \cdot \mathbf{U}_{[\omega]} \cdot \mathbf{A} = \begin{bmatrix} \frac{6}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{9}{\sqrt{5}} \end{bmatrix} \approx \begin{bmatrix} 2{,}683 & 0{,}8944 \\ 0{,}8944 & 4{,}025 \end{bmatrix}$$

Inverse of right stretch tensor in original coordinate system:

$$\mathbf{U}^{-1} = \mathbf{A}^{\mathrm{T}} \cdot \mathbf{U}_{[\omega]}^{-1} \cdot \mathbf{A} = \begin{bmatrix} \frac{9\sqrt{5}}{50} & -\frac{\sqrt{5}}{25} \\ -\frac{\sqrt{5}}{25} & \frac{15\sqrt{5}}{25} \end{bmatrix} \approx \begin{bmatrix} 0{,}4025 & -0{,}08944 \\ -0{,}08944 & 1{,}342 \end{bmatrix}$$

Rotation tensor in original coordinate system:

$$\mathbf{R} = \mathbf{F} \cdot \mathbf{U}^{-1} = \begin{bmatrix} 2 & -1 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} \frac{9\sqrt{5}}{50} & -\frac{\sqrt{5}}{25} \\ -\frac{\sqrt{5}}{25} & \frac{3\sqrt{5}}{25} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \cos(26{,}57^{\circ}) & -\sin(26{,}57^{\circ}) \\ \sin(26{,}57^{\circ}) & \cos(26{,}57^{\circ}) \end{bmatrix} \approx$$

$$\approx \begin{bmatrix} 0{,}4472 & 0{,}8944 \\ -0{,}8944 & 0{,}4472 \end{bmatrix}$$

Left stretch tensor in original coordinate system:

$$\mathbf{V} = \mathbf{R} \cdot \mathbf{U} \cdot \mathbf{R}^{\mathrm{T}} = \begin{bmatrix} \sqrt{5} & 0 \\ 0 & 2\sqrt{5} \end{bmatrix} \approx \begin{bmatrix} 2{,}236 & 0 \\ 0 & 4{,}472 \end{bmatrix}$$

CHECK

- $$\mathbf{R} \cdot \mathbf{U} = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \cdot \begin{bmatrix} \frac{6}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{9}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 2 & 4 \end{bmatrix} = \mathbf{F}$$
- $\mathbf{U}^{\mathrm{T}} = \mathbf{U} \quad \Rightarrow \quad \mathbf{U}$ is symmetric
- $\det \mathbf{R} = 1 \quad \Rightarrow \quad \mathbf{R}$ is orthogonal

DEFORMATION OF MATERIAL FIBRE

Material fibre of unit length which is parallel to the first eigenaxis of right stretch tensor is given by first eigenvector of deformation tensor:

$$\mathrm{d}\mathbf{X} = \mathbf{c}_1 = [\cos\phi \,;\, \sin\phi] = \left[\frac{1}{\sqrt{5}} \,;\, \frac{2}{\sqrt{5}}\right] = [0{,}4472 \,;\, 0{,}8944]$$

Stretch before rotation:

$$\mathbf{U}\cdot\mathrm{d}\,\mathbf{X}=\begin{bmatrix}\frac{6}{\sqrt{5}} & \frac{2}{\sqrt{5}}\\ \frac{2}{\sqrt{5}} & \frac{9}{\sqrt{5}}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{5}}\\ \frac{2}{\sqrt{5}}\end{bmatrix}=\begin{bmatrix}2\\4\end{bmatrix}$$

Rotation after stretch:

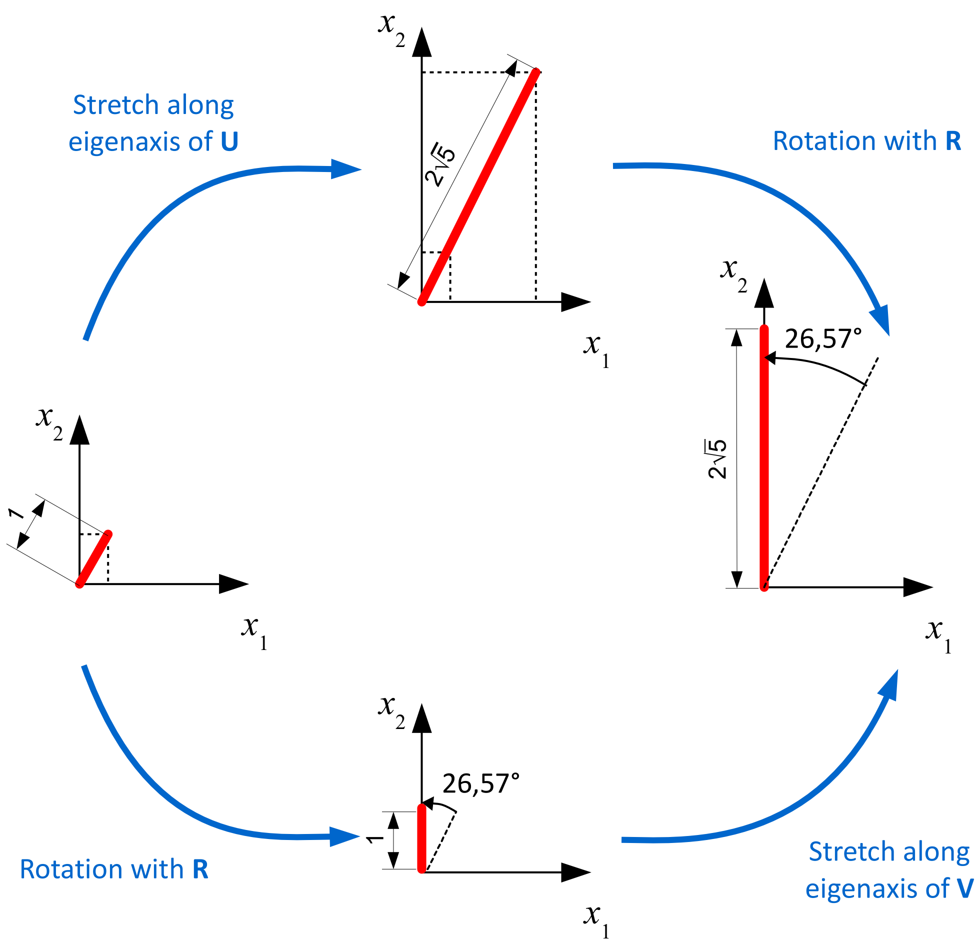
$$\mathbf{F}\cdot\mathrm{d}\,\mathbf{X}=\mathbf{R}\cdot\left(\mathbf{U}\cdot\mathrm{d}\,\mathbf{X}\right)=\begin{bmatrix}\frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}}\\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{bmatrix}\begin{bmatrix}2\\4\end{bmatrix}=\begin{bmatrix}0\\2\sqrt{5}\end{bmatrix}$$

Rotation before stretch:

$$\mathbf{R}\cdot\mathrm{d}\,\mathbf{X}=\begin{bmatrix}\frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}}\\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{5}}\\ \frac{2}{\sqrt{5}}\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$$

Stretch after rotation:

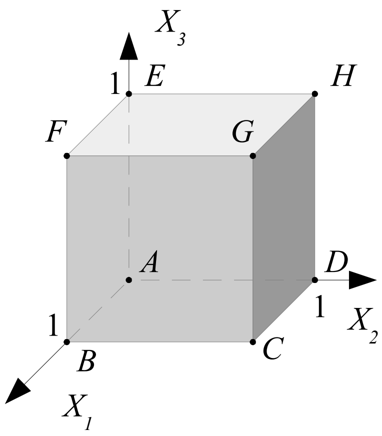
$$\mathbf{F}\cdot\mathrm{d}\,\mathbf{X}=\mathbf{V}\cdot\left(\mathbf{R}\cdot\mathrm{d}\,\mathbf{X}\right)=\begin{bmatrix}\sqrt{5} & 0\\ 0 & 2\sqrt{5}\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\2\sqrt{5}\end{bmatrix}$$



EXERCISE 21

Deformation of a cube of unit side length is give as follows:

$$\begin{cases} x_1 = 1{,}2\,X_1 \\ x_2 = 1{,}2\,X_2 \\ x_3 = 0{,}6\,X_3 - 0{,}2\,X_1 - 0{,}2\,X_2 \end{cases}$$



It is made of isotropic linear elastic material of Poisson ratio $\nu = 0{,}3$ and Young modulus $E = 10\ \mathrm{kPa}$ (these constants correspond with relation $\mathbf{T}_S(\mathbf{E})$). Find current configuration of cube and determine current load of EFGH face in current configuration and referred to reference configuration.

SOLUTION:

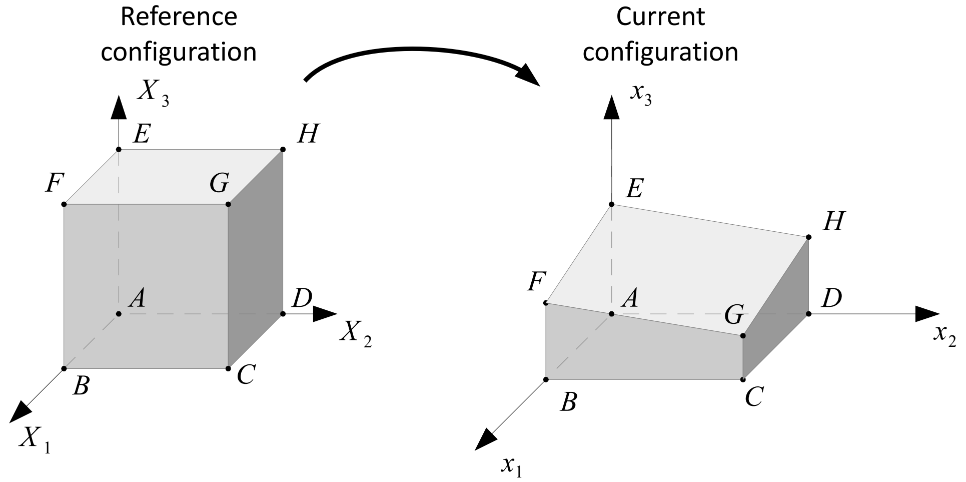
Invertibility of equations is checked:

$$F_{ij} = \frac{\partial x_i}{\partial X_j} \quad \Rightarrow \quad \mathbf{F} = \begin{bmatrix} 1{,}2 & 0 & 0 \\ 0 & 1{,}2 & 0 \\ -0{,}2 & -0{,}2 & 0{,}6 \end{bmatrix} \quad J = \det \mathbf{F} = 0{,}864$$

Jacobian determinant J>0 in all points. Relations are invertible.

ACTUAL CONFIGURATION

$$A:\ \mathbf{x}(0;0;0) = [0;0;0]^{\mathrm{T}} \qquad B:\ \mathbf{x}(1;0;0) = [1{,}2;0;0]^{\mathrm{T}}$$
$$C:\ \mathbf{x}(1;1;0) = [1{,}2;1{,}2;0]^{\mathrm{T}} \qquad D:\ \mathbf{x}(0;1;0) = [0;1{,}2;0]^{\mathrm{T}}$$
$$E:\ \mathbf{x}(0;0;1) = [0;0;0{,}6]^{\mathrm{T}} \qquad F:\ \mathbf{x}(1;0;1) = [1{,}2;0;0{,}4]^{\mathrm{T}}$$
$$G:\ \mathbf{x}(1;1;1) = [1{,}2;1{,}2;0{,}2]^{\mathrm{T}} \qquad H:\ \mathbf{x}(0;1;1) = [0;1{,}2;0{,}4]^{\mathrm{T}}$$



DISPLACEMENT VECTOR

Displacement vector in material description:

$$u_i = x_i - X_i \quad \Rightarrow \quad \begin{cases} u_1 = 0{,}2\,X_1 \\ u_2 = 0{,}2\,X_2 \\ u_3 = -0{,}4\,X_3 - 0{,}2\,X_1 - 0{,}2\,X_2 \end{cases}$$

STRAIN STATE

Green – de Saint-Venant strain tesnor is determined according to **geometric relations**:

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right) \quad \Rightarrow$$

$$E_1 = \frac{1}{2}\left(\frac{\partial u_1}{\partial X_1} + \frac{\partial u_1}{\partial X_1} + \frac{\partial u_1}{\partial X_1}\frac{\partial u_1}{\partial X_1} + \frac{\partial u_2}{\partial X_1}\frac{\partial u_2}{\partial X_1} + \frac{\partial u_3}{\partial X_1}\frac{\partial u_3}{\partial X_1}\right) = 0{,}24$$

$$E_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} + \frac{\partial u_1}{\partial X_1}\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\frac{\partial u_2}{\partial X_2} + \frac{\partial u_3}{\partial X_1}\frac{\partial u_3}{\partial X_2}\right) = 0{,}02$$

...

$$\mathbf{E} = \begin{bmatrix} 0{,}24 & 0{,}02 & -0{,}06 \\ 0{,}02 & 0{,}24 & -0{,}06 \\ -0{,}06 & -0{,}06 & -0{,}32 \end{bmatrix} \; [-]$$

STRESS STATE

Piola–Kirchhoff stress tensor of the 2nd kind is found with the use of the **Hooke's Law**:

Kirchhoff modulus: $G = \dfrac{E}{2(1+\nu)} = 3{,}846\ \text{kPa}$

Lame parameter: $\lambda = \dfrac{E\nu}{(1+\nu)(1-2\nu)} = 5{,}769\ \text{kPa}$

$$S_{ij} = 2G\,E_{ij} + \lambda\,E_{kk} \quad \Rightarrow$$

$$S_{11} = 2G\,E_{11} + \lambda(E_{11}+E_{22}+E_{33}) = 3{,}577\ \text{kPa}$$

$$S_{12} = 2G\,E_{12} = 0{,}154\ \text{kPa}$$

...

$$\mathbf{T}_S = \begin{bmatrix} 2{,}769 & 0{,}154 & -0{,}462 \\ 0{,}154 & 2{,}769 & -0{,}462 \\ -0{,}462 & -0{,}462 & -1{,}538 \end{bmatrix} \; [\text{kPa}]$$

Piola – Kirchhoff stress tensor of the 1st kind:

$$\mathbf{T}_R = \mathbf{F}\cdot\mathbf{T}_S = \begin{bmatrix} 1{,}2 & 0 & 0 \\ 0 & 1{,}2 & 0 \\ -0{,}2 & -0{,}2 & 0{,}6 \end{bmatrix}\begin{bmatrix} 2{,}769 & 0{,}154 & -0{,}462 \\ 0{,}154 & 2{,}769 & -0{,}462 \\ -0{,}462 & -0{,}462 & -1{,}538 \end{bmatrix} = \begin{bmatrix} 3{,}323 & 0{,}185 & -0{,}554 \\ 0{,}185 & 3{,}323 & -0{,}554 \\ -0{,}862 & -0{,}862 & -0{,}738 \end{bmatrix} \; [\text{kPa}]$$

Cauchy stress tensor:

$$\mathbf{T}_\sigma = \frac{1}{J}\mathbf{T}_R \cdot \mathbf{F}^\mathrm{T} =$$

$$= \frac{1}{0{,}864}\begin{bmatrix} 3{,}323 & 0{,}185 & -0{,}554 \\ 0{,}185 & 3{,}323 & -0{,}554 \\ -0{,}862 & -0{,}862 & -0{,}738 \end{bmatrix}\begin{bmatrix} 1{,}2 & 0 & -0{,}2 \\ 0 & 1{,}2 & -0{,}2 \\ 0 & 0 & 0{,}6 \end{bmatrix} = \begin{bmatrix} 4{,}815 & 0{,}456 & -0{,}598 \\ 0{,}456 & 4{,}815 & -0{,}598 \\ -0{,}598 & -0{,}598 & -0{,}513 \end{bmatrix} \ [\mathrm{kPa}]$$

LOAD ON EFGH FACE

Current load referred to reference configuration:

Equation of plane containing EFGH face before deformation: $F: \quad X_3 - 1 = 0$

Unit normal vector of EFGH face before deformation: $\mathbf{N} = \dfrac{\nabla_X F}{|\nabla_X F|} = [0{,}0{,}1]$

Current load on face EFGH **referred to reference configuration**:

$$\mathbf{Q} = \mathbf{T}_R \cdot \mathbf{N} = \begin{bmatrix} 3{,}323 & 0{,}185 & -0{,}554 \\ 0{,}185 & 3{,}323 & -0{,}554 \\ -0{,}862 & -0{,}862 & -0{,}738 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -0{,}554 \\ -0{,}554 \\ -0{,}738 \end{bmatrix} \ [\mathrm{kPa}]$$

Current load referred to current configuration:

Equation of plane containing EFGH face after deformation are obtained in such a way that in the equations of that plane before deformation we introduce inverted deformation relations:

$$\begin{cases} x_1 = 1{,}2\,X_1 \\ x_2 = 1{,}2\,X_2 \\ x_3 = 0{,}6\,X_3 - 0{,}2\,X_1 - 0{,}2\,X_2 \end{cases} \Leftrightarrow \begin{cases} X_1 = 0{,}833\,x_1 \\ X_2 = 0{,}833\,x_2 \\ X_3 = 1{,}667\,x_3 + 0{,}278\,x_1 + 0{,}278\,x_2 \end{cases}$$

Equation of plane containing face EFGH after deformation:

$$f: \quad X_3 - 1 = 1{,}667\,x_3 + 0{,}278\,x_2 + 0{,}278\,x_1 - 1 = 0$$

Unit normal vector of EFGH face after deformation: $\mathbf{n} = \dfrac{\nabla_x f}{|\nabla_x f|} = [0{,}162\,,\ ;\ 0{,}162\ ;\ 0{,}973]$

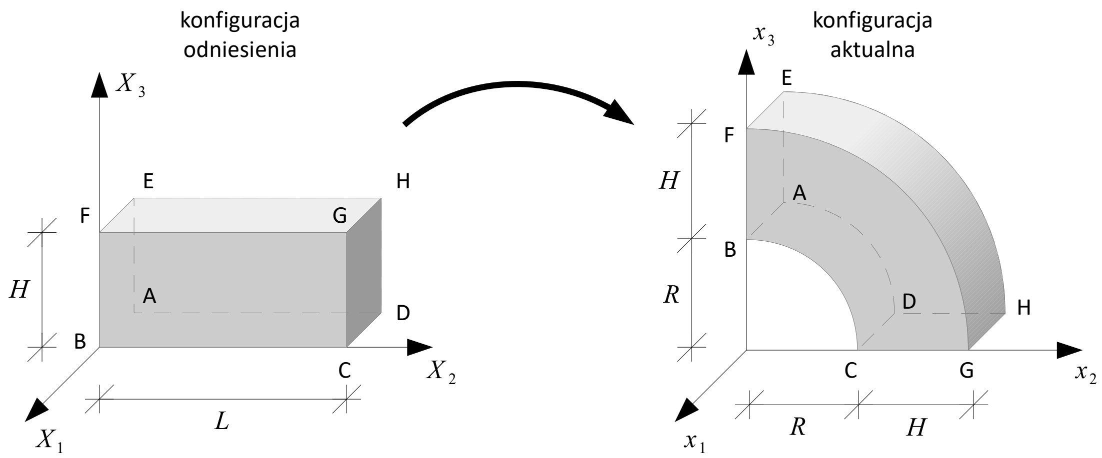
Current load on face EFGH **referred to current configuration**:

$$\mathbf{q} = \mathbf{T}_\sigma \cdot \mathbf{n} = \begin{bmatrix} 4{,}815 & 0{,}456 & -0{,}598 \\ 0{,}456 & 4{,}815 & -0{,}598 \\ -0{,}598 & -0{,}598 & -0{,}513 \end{bmatrix} \cdot \begin{bmatrix} 0{,}162 \\ 0{,}162 \\ 0{,}973 \end{bmatrix} = \begin{bmatrix} 0{,}272 \\ 0{,}272 \\ -0{,}693 \end{bmatrix} \ [\mathrm{kPa}]$$

EXERCISE 24

Deformation in material description is given by following equations

$$\begin{cases} x_1 = X_1 \\ x_2 = (R+X_3)\cdot\sin\left(\dfrac{\pi}{2L}X_2\right) \\ x_3 = (R+X_3)\cdot\cos\left(\dfrac{\pi}{2L}X_2\right) \end{cases} \quad \text{where} \quad \begin{cases} R>0 \\ L>0 \\ H>0 \end{cases}$$



Determine both strain and stress state as well as current load on EFGH face both in material and spatial description. Assume linear constitutive relations of Hooke's Law between Piola-Kirchhoff stress tensor of the 2nd kind and Green – de Saint-Venant strain tensor.

SOLUTION:

DEFORMATION GRADIENT

Material deformation gradient:

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{\pi(R+X_3)}{2L}\cos\left(\dfrac{\pi}{2L}X_2\right) & \sin\left(\dfrac{\pi}{2L}X_2\right) \\ 0 & -\dfrac{\pi(R+X_3)}{2L}\sin\left(\dfrac{\pi}{2L}X_2\right) & \cos\left(\dfrac{\pi}{2L}X_2\right) \end{bmatrix}$$

Jacobian determinant:

$$J = \det\mathbf{F} = \frac{\pi(R+X_3)}{2L} \quad \text{dla} \quad X_3>0 \quad \text{mamy} \quad J>0$$

Jacobian determinant is positive in considered region. Relations are invertible.

SPATIAL DESCRIPTION

Inverted deformation relations:

$$\begin{cases} X_1 = x_1 \\ X_2 = \dfrac{2L}{\pi}\operatorname{arctg}\dfrac{x_2}{x_3} \\ X_3 = \sqrt{x_2^2+x_3^2}-R \end{cases}$$

spatial deformation gradient:

$$\mathbf{f} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{2Lx_3}{\pi(x_2^2+x_3^2)} & -\dfrac{2Lx_2}{\pi(x_2^2+x_3^2)} \\ 0 & \dfrac{x_2}{\sqrt{x_2^2+x_3^2}} & \dfrac{x_3}{\sqrt{x_2^2+x_3^2}} \end{bmatrix}$$

DEFORMATION TENSOR

Material deformation tensor:

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\cdot\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{\pi^2(R+X_3)^2}{4L^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Spatial deformation tensor:

$$\mathbf{c} = \mathbf{f}^{\mathrm{T}}\cdot\mathbf{f} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{\pi^2 x_2^2(x_2^2+x_3^2)+4z^2L^2}{\pi^2(x_2^2+x_3^2)^2} & \dfrac{x_2x_3[\pi^2(x_2^2+x_3^2)-4L^2]}{\pi^2(x_2^2+x_3^2)^2} \\ 0 & \dfrac{x_2x_3[\pi^2(x_2^2+x_3^2)-4L^2]}{\pi^2(x_2^2+x_3^2)^2} & \dfrac{\pi^2 x_3^2(x_2^2+x_3^2)+4y^2L^2}{\pi^2(x_2^2+x_3^2)^2} \end{bmatrix}$$

STRAIN TENSOR

Materialny strain tensor:

$$\mathbf{E} = \frac{1}{2}(\mathbf{C}-\mathbf{1}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \dfrac{\pi^2(R+X_3)^2-4L^2}{8L^2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Spatial strain tensor:

$$\mathbf{e} = \frac{1}{2}(\mathbf{1}-\mathbf{c}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \dfrac{x_3^2[\pi^2(x_2^2+x_3^2)-4L^2]}{2\pi^2(x_2^2+x_3^2)^2} & -\dfrac{x_2x_3[\pi^2(x_2^2+x_3^2)-4L^2]}{2\pi^2(x_2^2+x_3^2)^2} \\ 0 & -\dfrac{x_2x_3[\pi^2(x_2^2+x_3^2)-4L^2]}{2\pi^2(x_2^2+x_3^2)^2} & \dfrac{x_2^2[\pi^2(x_2^2+x_3^2)-4L^2]}{2\pi^2(x_2^2+x_3^2)^2} \end{bmatrix}$$

STRESS TENSOR

Piola-Kirchhoff stress tensor of the 2nd kind:

$$\mathbf{T}_S = 2G\,\mathbf{E}+\lambda\cdot\mathrm{tr}(\mathbf{E})\cdot\mathbf{1} =$$

$$= \begin{bmatrix} \frac{\lambda}{8L^2}\left[\pi^2(R+X_3)^2-4L^2\right] & 0 & 0 \\ 0 & \frac{\lambda+2G}{8L^2}\left[\pi^2(R+X_3)^2-4L^2\right] & 0 \\ 0 & 0 & \frac{\lambda}{8L^2}\left[\pi^2(R+X_3)^2-4L^2\right] \end{bmatrix}$$

Piola-Kirchhoff stress tensor of the 1st kind:

$$\mathbf{T}_R = \mathbf{F}\cdot\mathbf{T}_S =$$

$$= \begin{bmatrix} \frac{\lambda}{8L^2}\left[\pi^2(R+X_3)^2-4L^2\right] & 0 & \dots \\ 0 & \frac{\pi(\lambda+2G)}{16L^3}\left[\pi^2(R+X_3)^2-4L^2\right](R+X_3)\cos\left(\frac{\pi}{2L}X_2\right) & \dots \\ 0 & \frac{\pi(\lambda+2G)}{16L^3}\left[\pi^2(R+X_3)^2-4L^2\right](R+X_3)\sin\left(\frac{\pi}{2L}X_2\right) & \dots \end{bmatrix}$$

$$\begin{matrix} \dots & 0 \\ \dots & \frac{\lambda}{8L^2}\left[\pi^2(R+X_3)^2-4L^2\right]\sin\left(\frac{\pi}{2L}X_2\right) \\ \dots & \frac{\lambda}{8L^2}\left[\pi^2(R+X_3)^2-4L^2\right]\cos\left(\frac{\pi}{2L}X_2\right) \end{matrix}\Bigg]$$

Cauchy stress tensor:

$$\mathbf{T}_\sigma = \frac{1}{J}\mathbf{T}_R\cdot\mathbf{F}^\mathrm{T} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ & \sigma_{22} & \sigma_{23} \\ \mathrm{sym} & & \sigma_{33} \end{bmatrix}$$

$$\sigma_{11} = \frac{\lambda\left[\pi^2(x_2^2+x_3^2)-4L^2\right]}{4\pi L\sqrt{x_3^2+x_2^2}}$$

$$\sigma_{12} = 0$$

$$\sigma_{13} = 0$$

$$\sigma_{22} = \frac{\left[\pi^2(x_2^2+x_3^2)-4L^2\right]\cdot\left[\pi^2(\lambda+2G)x_3^2(x_2^2+x_3^2)+4\lambda x_2^2L^2\right]}{16\pi L^3(x_2^2+x_3^2)^{3/2}}$$

$$\sigma_{23} = -\frac{x_2x_3\left[\pi^2(x_2^2+x_3^2)-4L^2\right]\cdot\left[\pi^2(\lambda+2G)(x_2^2+x_3^2)-4\lambda L^2\right]}{16\pi L^3(x_2^2+x_3^2)^{3/2}}$$

$$\sigma_{33} = \frac{\left[\pi^2(x_2^2+x_3^2)-4L^2\right]\cdot\left[\pi^2(\lambda+2G)x_2^2(x_2^2+x_3^2)+4\lambda x_3^2L^2\right]}{16\pi L^3(x_2^2+x_3^2)^{3/2}}$$

BOUNDARY TRACTIONS

Face EFGH before deformation lies in plane given by equation: $F: \quad X_3 - H = 0$

Unit normal of EFGH face before deformation: $\mathbf{N} = \dfrac{\nabla_{\mathbf{X}} F}{|\nabla_{\mathbf{X}} F|} = [0; 0; 1]$

Current load on EFGH referred to reference configuration:

$$\mathbf{Q} = \mathbf{T}_R \cdot \mathbf{N} = \begin{bmatrix} 0 \\ \dfrac{\lambda}{8L^2}\left[(R+X_3)^2 - 4L^2\right]\sin\left(\dfrac{\pi}{2L}X_2\right) \\ \dfrac{\lambda}{8L^2}\left[(R+X_3)^2 - 4L^2\right]\cos\left(\dfrac{\pi}{2L}X_2\right) \end{bmatrix}$$

Face EFGH before deformation lies in a surface given by equation: $f: \quad \sqrt{x_2^2 + x_3^2} - (R+H) = 0$

Unit normal of EFGH face after deformation: $\mathbf{n} = \dfrac{\nabla_{\mathbf{x}} f}{|\nabla_{\mathbf{x}} f|} = \left[0 \; ; \; \dfrac{x_2}{\sqrt{x_2^2+x_3^2}} \; ; \; \dfrac{x_3}{\sqrt{x_2^2+x_3^2}}\right]$

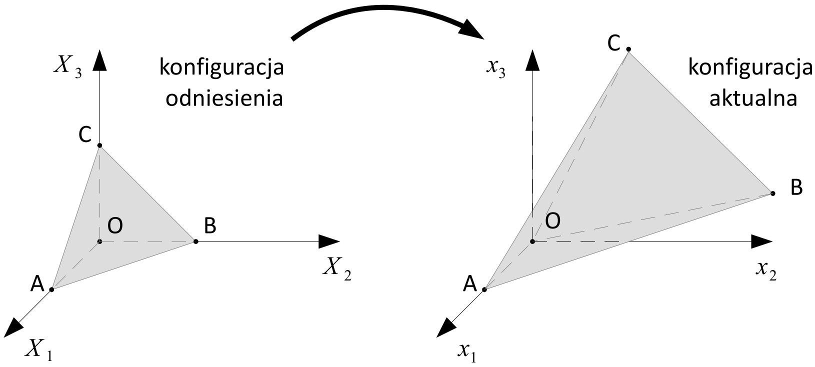
Current load on EFGH referred to current configuration:

$$\mathbf{q} = \mathbf{T}_\sigma \cdot \mathbf{n} = \begin{bmatrix} 0 \\ \dfrac{\lambda y\left[\pi^2 (x_2^2 + x_3^2) - 4L^2\right]}{4\pi L (x_2^2 + x_3^2)} \\ \dfrac{\lambda z\left[\pi^2 (x_2^2 + x_3^2) - 4L^2\right]}{4\pi L (x_2^2 + x_3^2)} \end{bmatrix}$$

EXERCISE 25

Elastic deformation is given by equations:

$$\begin{cases} x_1 = X_1 - X_2 \\ x_2 = 2X_2 + X_3 \\ x_3 = 2X_3 \end{cases}$$



Current stress state is described by the Cauchy stress tensor:

$$\mathbf{T}_\sigma = \begin{bmatrix} x_1 + x_2 & 0 & 0 \\ 0 & x_2 - x_3 & 2x_3 \\ 0 & 2x_3 & x_3^2 \end{bmatrix}$$

Find the body forces as well as load vector on faces OAC and ABC referred both to current and reference configuration.

SOLUTION:

Material deformation gradienti: $\mathbf{F} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Jacobian determinant: $J = \det \mathbf{F} = 4$

Since $J > 0$ the deformation is locally invertible in all points. Deformation relations in spatial description are as follows:

$$\begin{cases} X_1 = x_1 + \dfrac{x_2}{2} - \dfrac{x_3}{4} \\ X_2 = \dfrac{x_2}{2} - \dfrac{x_3}{4} \\ X_3 = \dfrac{x_3}{2} \end{cases}$$

BODY FORCES

Equilibrium equation is spatial description using the Cauchy stress tensor:

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = 0 \quad i=1,2,3$$

According to the above relations we determine **body forces vector** in spatial description:

$$\begin{cases} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + \rho b_1 = 1 + \rho b_1 = 0 \\ \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + \rho b_2 = 1 + 2 + \rho b_2 = 0 \\ \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \rho b_3 = 2 x_3 + \rho b_3 = 0 \end{cases} \Rightarrow \begin{cases} b_1 = -\frac{1}{\rho} \\ b_2 = -\frac{3}{\rho} \\ b_3 = -\frac{2 x_3}{\rho} \end{cases}$$

In order to find body forces in material description we need to find **Piola-Kirchhoff stress tensor of the 1st kind**:

$$\mathbf{T}_R = J \mathbf{T}_\sigma \cdot \mathbf{F}^{-\mathrm{T}} = J \mathbf{T}_\sigma \cdot \mathbf{f}^{\mathrm{T}} =$$

$$= \begin{bmatrix} 4(x_1 + x_2) & 0 & 0 \\ 2(x_2 - 2x_3) & 2(x_2 - 2x_3) & 4x_3 \\ x_3(4 - x_3) & x_3(4 - x_3) & 2x_3^2 \end{bmatrix} = \begin{bmatrix} 4(X_1 + X_2 + X_3) & 0 & 0 \\ 2(2X_2 - 3X_3) & 2(2X_2 - 3X_3) & 8X_3 \\ 4X_3(2 - X_3) & 4X_3(2 - X_3) & 8X_3^2 \end{bmatrix}$$

Equilibrium equation is material description using the Piola-Kirchhoff stress tensor of the 1st kind:

$$\frac{\partial T_{ij}}{\partial X_j} + \rho_R B_i = 0 \quad i=1,2,3$$

According to the above relations we determine **body forces vector** in material description:

$$\begin{cases} \frac{\partial T_{11}}{\partial X_1} + \frac{\partial T_{12}}{\partial X_2} + \frac{\partial T_{13}}{\partial X_3} + \rho_R B_1 = 4 + \rho_R B_1 = 0 \\ \frac{\partial T_{21}}{\partial X_1} + \frac{\partial T_{22}}{\partial X_2} + \frac{\partial T_{23}}{\partial X_3} + \rho_R B_2 = 4 + 8 + \rho_R B_2 = 0 \\ \frac{\partial T_{31}}{\partial X_1} + \frac{\partial T_{32}}{\partial X_2} + \frac{\partial T_{33}}{\partial X_3} + \rho_R B_3 = 16 X_3 + \rho_R B_3 = 0 \end{cases} \Rightarrow \begin{cases} B_1 = -\frac{4}{\rho_R} \\ B_2 = -\frac{12}{\rho_R} \\ B_3 = -\frac{16 X_3}{\rho_R} \end{cases}$$

Reference material density is found according to the principle of conservation of mass:

$$\rho_R = J \rho = 4 \rho$$

LOAD ON FACE OAC

Equation of OAC surface before deformation: $F = X_2 = 0$

Unit normal to OAC before deformation: $\mathbf{N} = -\dfrac{\nabla_X F}{|\nabla_X F|} = [0\ ;\ -1\ ;\ 0]$

Current load referred to reference configuration:

$$\mathbf{Q} = \mathbf{T}_R \cdot \mathbf{N} = \begin{bmatrix} 0 \\ 2(3X_3 - 2X_2) \\ 4X_3(X_3 - 2) \end{bmatrix}$$

Equation of OAC surface after deformation: $f = X_2 = \dfrac{x_2}{2} - \dfrac{x_3}{4} = 0$

Unit normal to OAC after deformation: $\mathbf{n} = -\dfrac{\nabla_x f}{|\nabla_x f|} = \left[0\ ;\ -\dfrac{2}{\sqrt{5}}\ ;\ \dfrac{1}{\sqrt{5}}\right]$

Current load referred to current configuration:

$$\mathbf{q} = \mathbf{T}_\sigma \cdot \mathbf{n} = \begin{bmatrix} 0 \\ \dfrac{2}{\sqrt{5}}(2x_3 - x_2) \\ \dfrac{1}{\sqrt{5}} x_3 (x_3 - 4) \end{bmatrix}$$

LOAD ON FACE ABC

Equation of ABC surface before deformation: $F = X_1 + X_2 + X_3 - 1 = 0$

Unit normal to ABC before deformation: $\mathbf{N} = \dfrac{\nabla_X F}{|\nabla_X F|} = \left[\dfrac{1}{\sqrt{3}}\ ;\ \dfrac{1}{\sqrt{3}}\ ;\ \dfrac{1}{\sqrt{3}}\right]$

Current load referred to reference configuration:

$$\mathbf{Q} = \mathbf{T}_R \cdot \mathbf{N} = \begin{bmatrix} \dfrac{4}{\sqrt{3}}(X_1 + X_2 + X_3) \\ \dfrac{4}{\sqrt{3}}(2X_2 - X_3) \\ \dfrac{16}{\sqrt{3}} X_3 \end{bmatrix}$$

Equation of ABC surface after deformation: $f = X_1 + X_2 + X_3 - 1 = x_1 + x_2 - 1 = 0$

Unit normal to ABC after deformation: $\mathbf{n} = \dfrac{\nabla_x f}{|\nabla_x f|} = \left[\dfrac{1}{\sqrt{2}}\ ;\ \dfrac{1}{\sqrt{2}}\ ;\ 0\right]$

Current load referred to current configuration:

$$\mathbf{q} = \mathbf{T}_\sigma \cdot \mathbf{n} = \begin{bmatrix} \dfrac{x_1 + x_2}{\sqrt{2}} \\ \dfrac{x_2 - x_3}{\sqrt{2}} \\ \sqrt{2}\, x_3 \end{bmatrix}$$

EXERCISE 27

Deformation is given by a displacement field:

$$\begin{cases} u_1 = 0{,}002\, x_1^2 - 0{,}001\, x_1 x_2 \\ u_2 = 0{,}004\, x_2 x_3 \\ u_3 = -0{,}002\, x_3^2 \end{cases}$$

Assuming that both displacements and strain are small and assuming constitutive elations according to Hooke's Law for Young modulus $E = 210\ \mathrm{GPa}$ Poisson ratio $\nu = 0{,}2$ find:

- small strain tensor,
- stress tensor,
- body forces.

SOLUTION:

Strain tensor is found according to the **geometric relations**:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \Rightarrow \quad \boldsymbol{\varepsilon} = \begin{bmatrix} 0{,}004\, x_1 - 0{,}001\, x_2 & -0{,}0005\, x_1 & 0 \\ -0{,}0005\, x_1 & 0{,}004\, x_3 & 0{,}002\, x_2 \\ 0 & 0{,}002\, x_2 & -0{,}004\, x_3 \end{bmatrix}$$

Stress tensor is found according to **Hooke's Law**:

$$G = \frac{E}{2(1+\nu)} = 87{,}5\ \mathrm{GPa}$$

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} = 58{,}33\ \mathrm{GPa}$$

$$\sigma_{ij} = 2G\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij} \quad \Rightarrow$$

$$\Rightarrow \begin{bmatrix} 0{,}933\, x_1 - 0{,}233\, x_2 & -0{,}0875\, x_1 & 0 \\ -0{,}0875\, x_1 & 0{,}7\, x_3 - 0{,}0583\, x_2 + 0{,}233\, x_1 & 0{,}350\, x_2 \\ 0 & 0{,}350\, x_2 & -0{,}7\, x_3 - 0{,}0583\, x_2 + 0{,}233\, x_1 \end{bmatrix} \cdot 10^9\ [\mathrm{Pa}]$$

Body forces are found according to the **equilibrium equations**:

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = 0$$

$$i=1: \quad 0{,}933\cdot 10^9 + 0 + 0 + \rho b_1 = 0 \quad \Rightarrow \quad b_1 = -\frac{0{,}933\cdot 10^9}{\rho}$$

$$i=2: \quad -0{,}0875\cdot 10^9 - 0{,}0583\cdot 10^9 + 0 + \rho b_2 = 0 \quad \Rightarrow \quad b_2 = \frac{0{,}1458\cdot 10^9}{\rho}$$

$$i=3: \quad 0 + 0{,}350\cdot 10^9 - 0{,}7\cdot 10^9 + \rho b_3 = 0 \quad \Rightarrow \quad b_3 = \frac{0{,}350\cdot 10^9}{\rho}$$

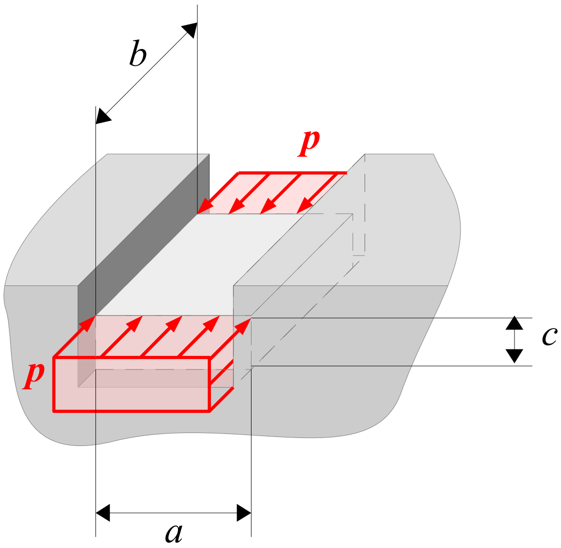
EXERCISE 28

A cuboid of dimensions *a, b, c* is made of isotropic linear elastic material. It is placed in an undeformable (infinitely rigid) trough and compressed with uniform stress *p*. Find stress and strain states (within linear theory), dimensions od cuboid after deformation, volumetric strain (the 1st invariant of small strain tensor) and true volume change. Find extreme shear stress. Assume no friction between faces of the cuboid and surface of trough.

$a = 10 \text{ mm}, \quad b = 15 \text{ mm}, \quad c = 6 \text{ mm}$

Young modulus $E = 70\,\text{GPa}$
Poisson ratio $\nu = 0{,}35$

Load $p = 180 \text{ MPa}$



SOLUTION:

Due to assumptions:

- No constraints on displacements along *x*, uniform compressive load on opposite faces perpendicular to x: $\sigma_{xx} = -p\,, \quad \varepsilon_{xx} \neq 0$
- No load and free deformation along *z* direction: $\sigma_{zz} = 0, \quad \varepsilon_{zz} \neq 0$
- No strain along *y* direction due to undeformable trough what results in compressive reaction stress σ_{yy} : $\varepsilon_{yy} = 0, \quad \sigma_{yy} \neq 0$
- No friction → no shear stress $\sigma_{yz} = \sigma_{zx} = \sigma_{xy} = 0$, so (according to Hooke's Law) no distortion strains $\varepsilon_{yz} = \varepsilon_{zx} = \varepsilon_{xy} = 0$.

We know stresses σ_{xx} , σ_{zz} and strain ε_{yy} - this allows us to find stress σ_{yy} with the use of Hooke's Law:

$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{zz} + \sigma_{xx}\right)\right] \quad \Rightarrow \quad \sigma_{yy} = \left[E\,\varepsilon_{yy} + \nu\left(\sigma_{zz} + \sigma_{xx}\right)\right] = -\nu\, p = -63 \text{ MPa}$$

Stress tresor is completely determined:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ & \sigma_{yy} & \sigma_{yz} \\ \text{sym} & & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -\nu\, p & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -180 & 0 & 0 \\ 0 & -63 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad [\text{MPa}]$$

Stress tensor has a diagonal form if and only if it is described in the coordinate system of its eigenaxes, what means that determined stress components are principal stresses. Extreme shear stress may be found then as::

$$\sigma_{max} = 0 \text{ MPa}$$
$$\sigma_{min} = -180 \text{ MPa}$$
$$|\tau_{max}| = \frac{|\sigma_{max} - \sigma_{min}|}{2} = 90 \text{ MPa}$$

Unknown linear strain ε_{xx} , ε_{zz} are found according to the Hooke's Law:

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right] = \frac{p}{E}\left(\nu^2 - 1\right) = -2{,}256‰$$
$$\varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right] = \frac{p}{E}\nu\left(\nu + 1\right) = 1{,}215‰$$

Strain tensor:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ & \varepsilon_{yy} & \varepsilon_{yz} \\ \text{sym} & & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \frac{p}{E}\left(\nu^2 - 1\right) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{p}{E}\nu\left(\nu + 1\right) \end{bmatrix} = \begin{bmatrix} -2{,}256\cdot 10^{-3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1{,}215\cdot 10^{-3} \end{bmatrix} \; [-]$$

Dimensions of the cuboid after deformation:

$$a' = a + a\,\varepsilon_{yy} = a\left(1 + \varepsilon_{yy}\right) = a = 10 \text{ mm}$$
$$b' = b + b\,\varepsilon_{zz} = b\left(1 + \varepsilon_{xx}\right) = 14{,}96616 \text{ mm}$$
$$c' = c + c\,\varepsilon_{zz} = c\left(1 + \varepsilon_{zz}\right) = 6{,}00729 \text{ mm}$$

Reference volume: $V = abc = 900 \text{ mm}^3$

Current volume: $V' = a'b'c' = 899{,}0606 \text{ mm}^3$

True volume change: $\dfrac{\Delta V}{V} = \dfrac{V' - V}{V} = -1{,}044‰$

1st invariant of strain tensor:

$$\theta = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = -1{,}041‰$$

Comparison of two measures of volumetric strain:

$$\frac{\Delta V}{V} = \frac{V' - V}{V} = \frac{a'b'c' - abc}{abc} = \frac{a\left(1 + \varepsilon_{xx}\right)b\left(1 + \varepsilon_{yy}\right)c\left(1 + \varepsilon_{zz}\right) - abc}{abc} =$$
$$= \left(1 + \varepsilon_{xx}\right)\left(1 + \varepsilon_{yy}\right)\left(1 + \varepsilon_{zz}\right) - 1 = \underbrace{\left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right)}_{\theta} + \varepsilon_{yy}\varepsilon_{zz} + \varepsilon_{zz}\varepsilon_{xx} + \varepsilon_{xx}\varepsilon_{yy} + \varepsilon_{xx}\varepsilon_{yy}\varepsilon_{zz} =$$
$$-1{,}041‰ - 0{,}00274‰ = -1{,}044‰$$

EXERCISE 29

Stress tensor in a certain plane system is determined as:

$$\boldsymbol{\sigma} = \begin{bmatrix} 70 & -30 \\ -30 & 140 \end{bmatrix} [\mathrm{MPa}]$$

What would be the spatial distribution of stress and strain assuming plane stress state, and what would it be in case of plane strain state? Assume Hooke's Law with $E = 70\,\mathrm{GPa}$, $\nu = 0{,}3$.

SOLUTION:

PLANE STRESS STATE $(\sigma_{13}=\sigma_{23}=\sigma_{33}=0)$:

Stress tensor:
$$\boldsymbol{\sigma} = \begin{bmatrix} 70 & -30 & 0 \\ -30 & 140 & 0 \\ 0 & 0 & 0 \end{bmatrix} [\mathrm{MPa}]$$

Strain tensor:
$$\varepsilon_{ij} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\right] \quad \Rightarrow$$
$$\boldsymbol{\varepsilon} = \begin{bmatrix} 4\cdot 10^{-4} & -5{,}571\cdot 10^{-4} & 0 \\ -5{,}571\cdot 10^{-4} & 1{,}7\cdot 10^{-3} & 0 \\ 0 & 0 & -9\cdot 10^{-4} \end{bmatrix} [\mathrm{MPa}]$$

PLANE STRAIN STATE $(\varepsilon_{13}=\varepsilon_{23}=\varepsilon_{33}=0)$:

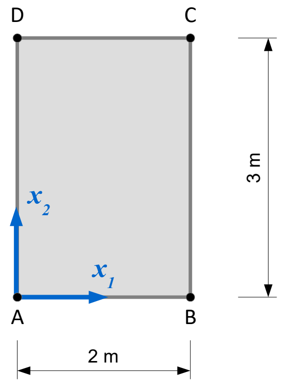
$$\varepsilon_{13} = 0 \quad \Rightarrow \quad \varepsilon_{13} = \frac{\sigma_{13}}{2G} = 0 \quad \Rightarrow \quad \sigma_{13} = 0$$
$$\varepsilon_{12} = 0 \quad \Rightarrow \quad \varepsilon_{12} = \frac{\sigma_{12}}{2G} = 0 \quad \Rightarrow \quad \sigma_{12} = 0$$
$$\varepsilon_{33} = 0 \quad \Rightarrow \quad \varepsilon_{33} = \frac{1}{E}\left[(1+\nu)\sigma_{33} - \nu(\sigma_{11}+\sigma_{22}+\sigma_{33})\right] = 0 \quad \Rightarrow \quad \sigma_{33} = \nu(\sigma_{11}+\sigma_{22}) = 63\ \mathrm{MPa}$$

Stress tensor:
$$\boldsymbol{\sigma} = \begin{bmatrix} 70 & -30 & 0 \\ -30 & 140 & 0 \\ 0 & 0 & 63 \end{bmatrix} [\mathrm{MPa}]$$

Strain tensor:
$$\varepsilon_{ij} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\right] \quad \Rightarrow$$
$$\boldsymbol{\varepsilon} = \begin{bmatrix} 1{,}3\cdot 10^{-4} & -5{,}571\cdot 10^{-4} & 0 \\ -5{,}571\cdot 10^{-4} & 1{,}43\cdot 10^{-3} & 0 \\ 0 & 0 & 0 \end{bmatrix} [\mathrm{MPa}]$$

EXERCISE 30

A rectangular membrane of dimensions $2\,\mathrm{m}\times 3\,\mathrm{m}$ is given. Stress state is given by a stress tensor:



$$\boldsymbol{\sigma}(x_1\ ;\ x_2)=\begin{bmatrix} 3x_1x_2-1 & -2x_1 \\ -2x_1 & 2x_1+3x_2 \end{bmatrix}\ \text{[Pa]}$$

Find the body forces and surface tractions.

SOLUTION:

BODY FORCES:

Body forces are found with the use of **equilibrium equations**:

$$\frac{\partial\sigma_{11}}{\partial x_1}+\frac{\partial\sigma_{12}}{\partial x_2}+b_1=0 \quad\Rightarrow\quad 3x_2+0+b_1=0 \quad\Rightarrow\quad b_1=-3x_2\ \left[\frac{\mathrm{N}}{\mathrm{m}^3}\right]$$

$$\frac{\partial\sigma_{21}}{\partial x_1}+\frac{\partial\sigma_{22}}{\partial x_2}+b_2=0 \quad\Rightarrow\quad -2+3+b_2=0 \quad\Rightarrow\quad b_2=-1\ \left[\frac{\mathrm{N}}{\mathrm{m}^3}\right]$$

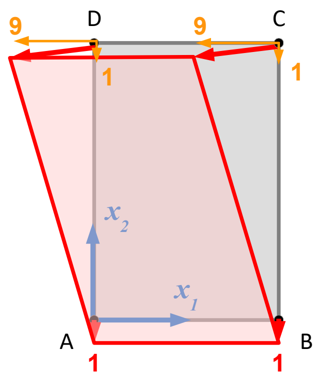
Values in corners of the membrane:

$$A=(0{,}0)$$
$$b_1(A)=0,\quad b_2(A)=-1 \quad\Rightarrow\quad b(A)=\sqrt{b_1^2(A)+b_2^2(A)}=1$$

$$B=(2{,}0)$$
$$b_1(B)=0,\quad b_2(B)=-1 \quad\Rightarrow\quad b(B)=\sqrt{b_1^2(B)+b_2^2(B)}=1$$

$$C=(2{,}3)$$
$$b_1(C)=-9,\quad b_2(C)=-1 \quad\Rightarrow\quad b(C)=\sqrt{b_1^2(C)+b_2^2(C)}=9{,}055$$

$$D=(0{,}3)$$
$$b_1(D)=-9,\quad b_2(D)=-1 \quad\Rightarrow\quad b(D)=\sqrt{b_1^2(D)+b_2^2(D)}=9{,}055$$

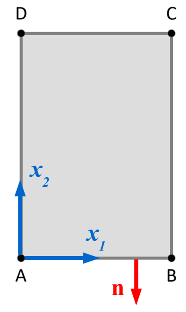


body forces

SURFACE TRACTIONS :

For each boundary segment we find a relation between x_1 and x_2 as well as (external) unit normal **n**. Surface tractions vector **q** is found according to relation $\mathbf{q}=\boldsymbol{\sigma}\cdot\mathbf{n}$. Sense (orientation) of the determined load vector is respective to the assumed global coordinate system.

Segment AB: $x_1 \in (0;2),\quad x_2 = 0$ **- unit normal:** $\mathbf{n}_{AB} = [0;-1]$

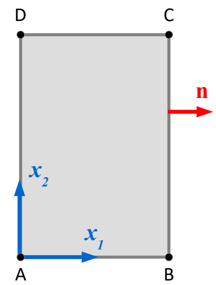


$$\boldsymbol{\sigma} \cdot \mathbf{n}_{AB} = \mathbf{q}_{AB} \quad \Rightarrow \quad \begin{bmatrix} -1 & -2x_1 \\ -2x_1 & 2x_1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ -2x_1 \end{bmatrix} = \begin{bmatrix} q_1^{(AB)} \\ q_2^{(AB)} \end{bmatrix}$$

- ***x₁* component** of load vector is perpendicular to the unit normal so it is a **tangent** component.
- ***x₂* component** of load vector is parallel to the unit normal so it is a **normal** component.

$$A = (0{,}0):\quad q_1^{(AB)}(A) = 0,\quad q_2^{(AB)}(A) = 0$$
$$B = (2{,}0):\quad q_1^{(AB)}(B) = 4\quad q_2^{(AB)}(B) = -4$$

Segment BC: $x_1 = 2,\quad x_2 \in (0;3)$ **- unit normal:** $\mathbf{n}_{BC} = [1;0]$

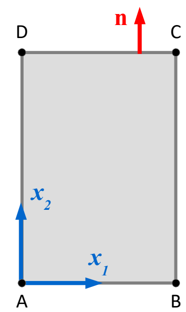


$$\boldsymbol{\sigma} \cdot \mathbf{n}_{BC} = \mathbf{q}_{BC} \quad \Rightarrow \quad \begin{bmatrix} 6x_2 - 1 & -4 \\ -4 & 4 + 3x_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6x_2 - 1 \\ -4 \end{bmatrix} = \begin{bmatrix} q_1^{(BC)} \\ q_2^{(BC)} \end{bmatrix}$$

- ***x₁* component** of load vector is parallel to the unit normal so it is a **normal** component.
- ***x₂* component** of load vector is perpendicular to the unit normal so it is a **tangent** component.

$$B = (2{,}0):\quad q_1^{(BC)}(B) = -1,\quad q_2^{(BC)}(B) = -4$$
$$C = (2{,}3):\quad q_1^{(BC)}(C) = 17\quad q_2^{(BC)}(C) = -4$$

Segment CD: $x_1 \in (0;2),\quad x_2 = 3$ **- unit normal:** $\mathbf{n}_{CD} = [0;1]$

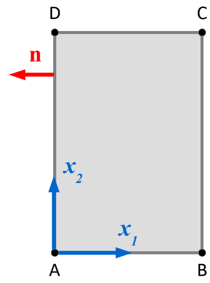


$$\boldsymbol{\sigma} \cdot \mathbf{n}_{CD} = \mathbf{q}_{CD} \quad \Rightarrow \quad \begin{bmatrix} 9x_1 - 1 & -2x_1 \\ -2x_1 & 2x_1 + 9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2x_1 \\ 2x_1 + 9 \end{bmatrix} = \begin{bmatrix} q_1^{(CD)} \\ q_2^{(CD)} \end{bmatrix}$$

- ***x₁* component** of load vector is perpendicular to the unit normal so it is a **tangent** component.
- ***x₂* component** of load vector is parallel to the unit normal so it is a **normal** component.

$$C = (2{,}3):\quad q_1^{(CD)}(C) = -4,\quad q_2^{(CD)}(C) = 13$$
$$D = (0{,}3):\quad q_1^{(CD)}(D) = 0\quad q_2^{(CD)}(D) = 9$$

Segment DA: $x_1=0,\ \ x_2\in(0;3)$ **- unit normal:** $\mathbf{n}_{DA}=[-1;0]$



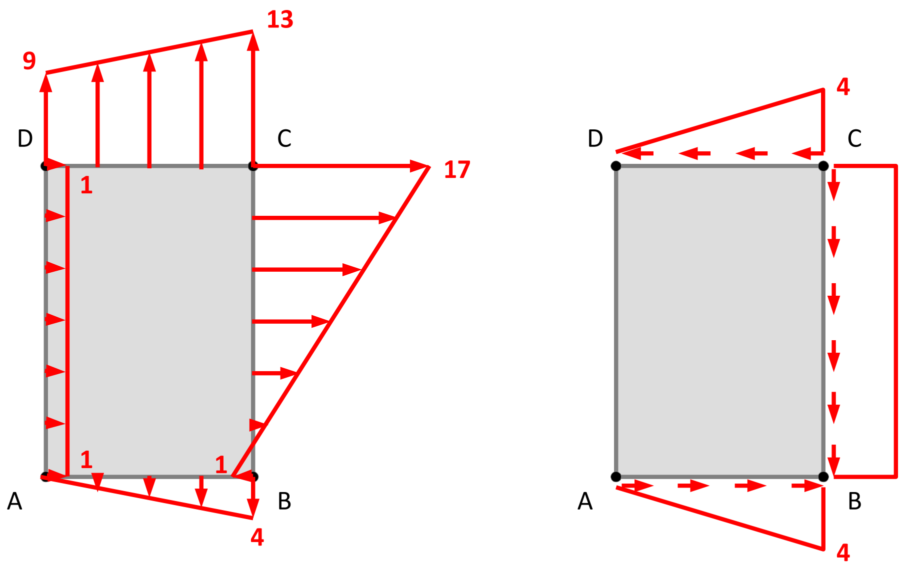
$$\boldsymbol{\sigma}\cdot\mathbf{n}_{DA}=\mathbf{q}_{DA}\quad\Rightarrow\quad\begin{bmatrix}-1 & 0\\ 0 & 3x_2\end{bmatrix}\begin{bmatrix}-1\\ 0\end{bmatrix}=\begin{bmatrix}1\\ 0\end{bmatrix}=\begin{bmatrix}q_1^{(DA)}\\ q_2^{(DA)}\end{bmatrix}$$

- ***x_1* component** of load vector is parallel to the unit normal so it is a **normal** component.

- ***x_2* component** of load vector is perpendicular to the unit normal so it is a **tangent** component.

$$D=(0,3):\quad q_1^{(DA)}(D)=1,\ \ q_2^{(DA)}(D)=0$$
$$A=(0,0):\quad q_1^{(DA)}(A)=1\ \ q_2^{(DA)}(A)=0$$

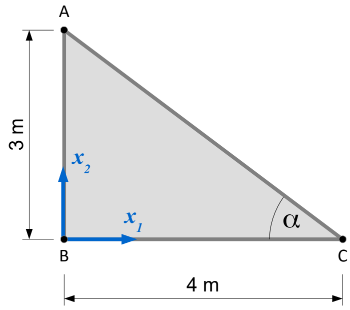
Summary of the obtained results may be presented in the form of graph of normal and tangent surface tractions:



EXERCISE 31

A plane triangular membrane is given. Find the body forces and surface tractions knowing that the stress tensor is as follows:

$$\boldsymbol{\sigma}(x_1;x_2)=\begin{bmatrix}3x_1-4x_2 & 2x_1x_2+1\\ 2x_1x_2+1 & 4-2x_1\end{bmatrix}\qquad [\text{Pa}]$$



SOLUTION:

BODY FORCES:

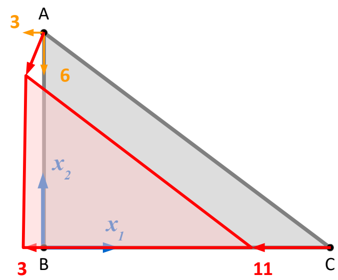
Body forces are found with the use of **equilibrium equations**:

$$\frac{\partial\sigma_{11}}{\partial x_1}+\frac{\partial\sigma_{12}}{\partial x_2}+b_1=0 \quad\Rightarrow\quad 3+2x_1+b_1=0 \quad\Rightarrow\quad b_1=-3-2x_1 \quad \left[\frac{\text{N}}{\text{m}^3}\right]$$

$$\frac{\partial\sigma_{21}}{\partial x_1}+\frac{\partial\sigma_{22}}{\partial x_2}+b_2=0 \quad\Rightarrow\quad 2x_2+b_2=0 \quad\Rightarrow\quad b_2=-2x_2 \quad \left[\frac{\text{N}}{\text{m}^3}\right]$$

Values in corners of the membrane:

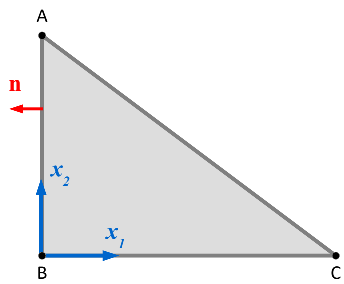
$$\begin{array}{lll} B=(0;0) & C=(4;0) & A=(0;3)\\ b_1(0;0)=-3 & b_1(4;0)=-3 & b_1(0;3)=-11\\ b_2(0;0)=-6 & b_2(4;0)=0 & b_2(0;3)=0\end{array}$$



SURFACE TRACTIONS:

Segment AB: $x_1=0,\ \ x_2\in\langle 0;3\rangle$ **- unit normal:** $\mathbf{n}_{AB}=[-1;0]$

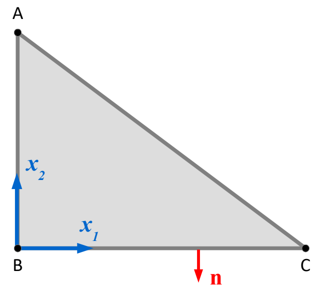
$$\boldsymbol{\sigma}\cdot\mathbf{n}_{AB}=\mathbf{q}_{AB}\quad\Rightarrow\quad\begin{bmatrix}-4x_2 & 1\\ 1 & 4\end{bmatrix}\begin{bmatrix}-1\\ 0\end{bmatrix}=\begin{bmatrix}4x_2\\ -1\end{bmatrix}=\begin{bmatrix}q_1^{(AB)}\\ q_2^{(AB)}\end{bmatrix}$$



- ***x₁* component** of load vector is parallel to the unit normal so it is a **normal** component.
- ***x₂* component** of load vector is perpendicular to the unit normal so it is a **tangent** component.

$$q^{(x)}_{(AB)}(0;3)=12,\quad q^{(x)}_{(AB)}(0;0)=0$$
$$q^{(y)}_{(AB)}(0;3)=-1\quad q^{(y)}_{(AB)}(0;0)=-1$$

Segment BC: $x_1 \in \langle 0;4 \rangle, \quad x_2 = 0$ **- unit normal:** $\mathbf{n}_{BC} = [0;-1]$



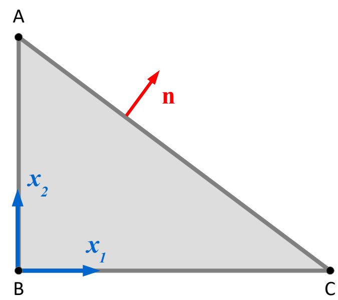
$$\boldsymbol{\sigma} \cdot \mathbf{n}_{BC} = \mathbf{q}_{BC} \quad \Rightarrow \quad \begin{bmatrix} 3x_1 & 1 \\ 1 & 4-2x_1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2x_1 - 4 \end{bmatrix} = \begin{bmatrix} q_1^{(BC)} \\ q_2^{(BC)} \end{bmatrix}$$

- ***x₁* component** of load vector is perpendicular to the unit normal so it is a **tangent** component.
- ***x₂* component** of load vector is parallel to the unit normal so it is a **normal** component.

$$q_{(BC)}^{(x)}(0;0) = -1, \quad q_{(BC)}^{(x)}(4;0) = -1$$
$$q_{(BC)}^{(y)}(0;0) = -4 \quad q_{(BC)}^{(y)}(4;0) = 4$$

Segment CA: $x_1 \in \langle 0;4 \rangle, \quad x_2 = 3 - \frac{3}{4} x_1$ **- unit normal:** $\mathbf{n}_{CA} = [\sin\alpha;\cos\alpha] = \left[\frac{3}{5};\frac{4}{5}\right]$

$$\boldsymbol{\sigma} \cdot \mathbf{n}_{CA} = \mathbf{q}_{CA} \quad \Rightarrow \quad \begin{bmatrix} 3x_1 - 4\left(3 - \frac{3}{4}x_1\right) & 2x_1\left(3 - \frac{3}{4}x_1\right) + 1 \\ 2x_1\left(3 - \frac{3}{4}x_1\right) + 1 & 4 - 2x_1 \end{bmatrix} \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} = \begin{bmatrix} \frac{2}{5}(-3x_1^2 + 21x_1 - 16) \\ \frac{1}{10}(-9x_1^2 + 20x_1 + 38) \end{bmatrix} = \begin{bmatrix} q_1^{(CA)} \\ q_2^{(CA)} \end{bmatrix}$$



Normal component:
(*positive values corresponding with the sense of unit normal*)

$$q_n^{(CA)} = \mathbf{q}_{CA} \cdot \mathbf{n}_{CA} = \frac{2}{25}(-18x_1^2 + 83x_1 - 10)$$

End values: $q_{CA}^{(n)}(4;0) = 2{,}72, \quad q_{CA}^{(n)}(0;3) = -0{,}8$

Zeros: $q_n^{(CA)} = 0 \quad \Rightarrow \quad x_1 = 0{,}124 \ \vee \ x_1 = 4{,}487 \qquad 4{,}487 \notin CA$

Search for stationary points:

$$\frac{d q_n^{(CA)}}{d x_1} = \frac{2}{25}(-36x_1 + 83) = 0 \quad \Rightarrow \quad x_1 = 2{,}306, \quad q_n^{(CA)}(2{,}306) = 6{,}854$$

Tangent component: $\mathbf{q}_s^{(CA)} = \mathbf{q}_{CA} - q_n^{(CA)} \mathbf{n}_{CA} = \frac{21x_1^2 - 276x_1 + 370}{250} \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

End values: $\mathbf{q}_s^{(CA)}(4;0) = \begin{bmatrix} 6{,}368 \\ -4{,}776 \end{bmatrix}, \quad |\mathbf{q}_s^{(CA)}| = 7{,}96$

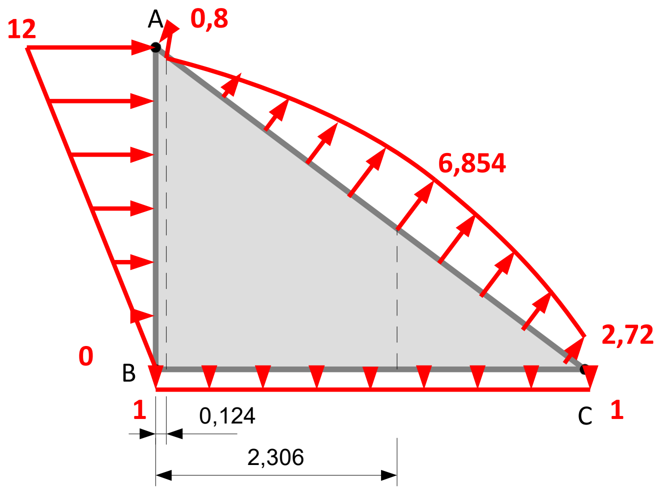
(*sense according to global coordinate system*) $\mathbf{q}_{CA}^{(s)}(x_A;y_A) = \begin{bmatrix} -5{,}98 \\ 4{,}44 \end{bmatrix}, \quad |\mathbf{q}_s^{(CA)}| = 7{,}4$

Zeros:

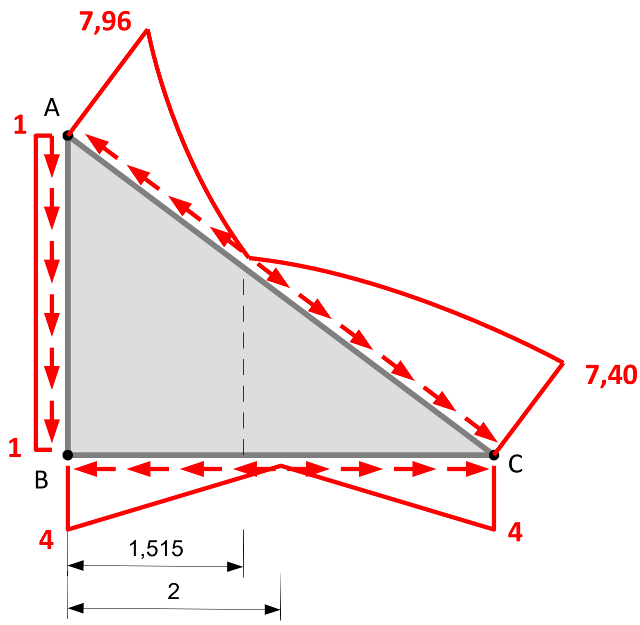
$$\mathbf{q}_s^{(CA)}=\mathbf{0} \Leftrightarrow 21x_1^2-276x_1+370=0 \quad \Rightarrow \quad x_1=1{,}515 \ \vee \ x_1=11{,}628 \quad 11{,}628\notin CA$$

Search for stationary points: $\dfrac{\mathrm{d}\,\mathbf{q}_s^{(CA)}}{\mathrm{d}\,x_1}=0 \Leftrightarrow \dfrac{2}{25}(42x_1-276)=0 \quad \Rightarrow \quad x_1=6{,}571\notin CA$

Summary of obtained results in form of graphs of normal and tangent tractions



normal surface traction



tangent surface traction

EXAMPLE 32

An elastic plane is given. Dimensions: 6 m x 8 m. It is simply supported at circumference. Its thickness is 30 cm. It is made of a material of Young modulus 32 GPa and Poisson's ratio 0,2. It is loaded with a uniform surface load of density (accounting for dead load) 10 kN/m^2.

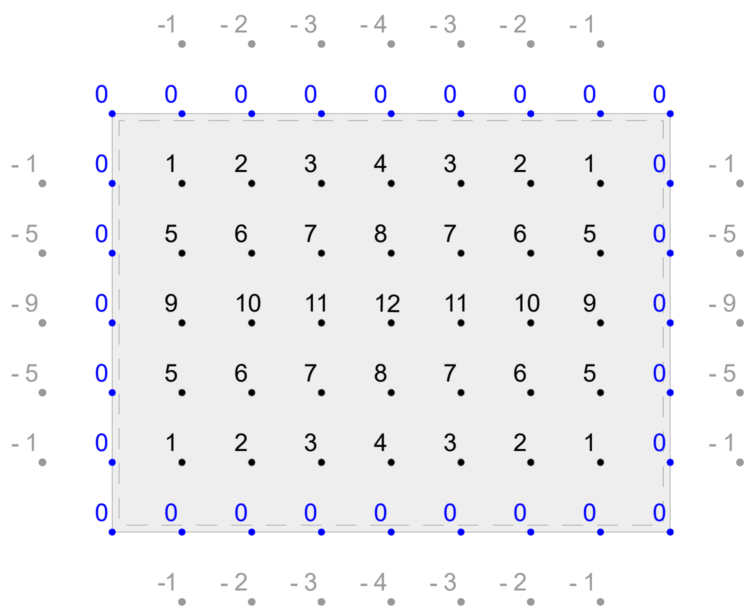
- Find maximum deflection with the use of:
  - Finite difference Method
  - trigonometric series expansion
- Compare obtained results

SOLUTION:

Flexural rigidity: $$D=\frac{Eh^3}{12(1-\nu^2)}=75000\ \text{kNm}$$

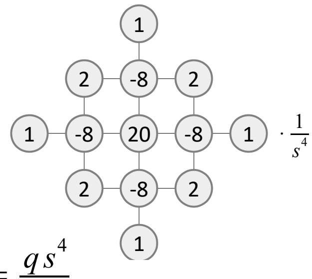
FINITE DIFFERENCE METHOD

- FMD mesh size 1m x 1m. $s=\Delta x_1=\Delta x_2=1\ \text{m}.$
- Boundary nodes has zero deflection.
- Fictious nodes are introduced outside the plate – they account for boundary conditions. Simple support on an edge requires zero bending moment which corresponds with the second derivative – deflection values in the fictious points must be opposite to the values of corresponding points inside the plate.
- The system has two axes of symmetry – deflections in corresponding points must be the same.
- As a results we obtain 12 independent nodes.



Governing equation is written down for each internal node:

$$\nabla^4 w = \frac{\partial^4 w}{\partial x_1^4} + 2\frac{\partial^4 w}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 w}{\partial x_2^4} = \frac{q(x_1, x_2)}{D} \qquad \nabla^4 \approx$$



$$20\,w_1 - 8\cdot(0+0+w_2+w_5) + 2\cdot(0+0+0+w_6) + 1\cdot(-w_1-w_1+w_3+w_9) = \frac{q s^4}{D}$$

$$20\,w_2 - 8\cdot(0+w_1+w_3+w_6) + 2\cdot(0+0+w_5+w_7) + 1\cdot(-w_2+0+w_4+w_{10}) = \frac{q s^4}{D}$$

$$20\,w_3 - 8\cdot(0+w_2+w_4+w_7) + 2\cdot(0+0+w_6+w_8) + 1\cdot(-w_3+w_1+w_3+w_{11}) = \frac{q s^4}{D}$$

$$20\,w_4 - 8\cdot(0+w_3+w_3+w_8) + 2\cdot(0+0+w_7+w_7) + 1\cdot(-w_4+w_2+w_2+w_{12}) = \frac{q s^4}{D}$$

$$20\,w_5 - 8\cdot(w_1+0+w_6+w_9) + 2\cdot(0+w_2+0+w_{10}) + 1\cdot(0-w_5+w_7+w_5) = \frac{q s^4}{D}$$

$$20\,w_6 - 8\cdot(w_2+w_5+w_7+w_{10}) + 2\cdot(w_1+w_3+w_9+w_{11}) + 1\cdot(0+0+w_8+w_6) = \frac{q s^4}{D}$$

$$20\,w_7 - 8\cdot(w_3+w_6+w_8+w_{11}) + 2\cdot(w_2+w_4+w_{10}+w_{12}) + 1\cdot(0+w_5+w_7+w_7) = \frac{q s^4}{D}$$

$$20\,w_8 - 8\cdot(w_4+w_7+w_7+w_{12}) + 2\cdot(w_3+w_3+w_{11}+w_{11}) + 1\cdot(0+w_6+w_6+w_8) = \frac{q s^4}{D}$$

$$20\,w_9 - 8\cdot(w_5+0+w_{10}+w_5) + 2\cdot(0+w_6+0+w_6) + 1\cdot(w_1-w_9+w_{11}+w_1) = \frac{q s^4}{D}$$

$$20\,w_{10} - 8\cdot(w_6+w_9+w_{11}+w_6) + 2\cdot(w_5+w_7+w_5+w_7) + 1\cdot(w_2+0+w_{12}+w_2) = \frac{q s^4}{D}$$

$$20\,w_{11} - 8\cdot(w_7+w_{10}+w_{12}+w_7) + 2\cdot(w_6+w_8+w_6+w_8) + 1\cdot(w_3+w_9+w_{11}+w_3) = \frac{q s^4}{D}$$

$$20\,w_{12} - 8\cdot(w_8+w_{11}+w_{11}+w_8) + 2\cdot(w_7+w_7+w_7+w_7) + 1\cdot(w_4+w_{10}+w_{10}+w_4) = \frac{q s^4}{D}$$

$$\begin{bmatrix} 18 & -8 & 1 & 0 & -8 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ -8 & 19 & -8 & 1 & 2 & -8 & 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & -8 & 20 & -8 & 0 & 2 & -8 & 2 & 0 & 0 & 1 & 0 \\ 0 & 2 & -16 & 19 & 0 & 0 & 4 & -8 & 0 & 0 & 0 & 1 \\ -8 & 2 & 0 & 0 & 20 & -8 & 1 & 0 & -8 & 2 & 0 & 0 \\ 2 & -8 & 2 & 0 & -8 & 21 & -8 & 1 & 2 & -8 & 2 & 0 \\ 0 & 2 & -8 & 2 & 1 & -8 & 22 & -8 & 0 & 2 & -8 & 2 \\ 0 & 0 & 4 & -8 & 0 & 2 & -16 & 21 & 0 & 0 & 4 & -8 \\ 2 & 0 & 0 & 0 & -16 & 4 & 0 & 0 & 19 & -8 & 1 & 0 \\ 0 & 2 & 0 & 0 & 4 & -16 & 4 & 0 & -8 & 20 & -8 & 1 \\ 0 & 0 & 2 & 0 & 0 & 4 & -16 & 4 & 1 & -8 & 21 & -8 \\ 0 & 0 & 0 & 2 & 0 & 0 & 8 & -16 & 0 & 2 & -16 & 20 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \\ w_9 \\ w_{10} \\ w_{11} \\ w_{12} \end{bmatrix} = \frac{q s^4}{D} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \mathbf{w} = \frac{q s^4}{D} \begin{bmatrix} 1{,}851 \\ 3{,}268 \\ 4{,}130 \\ 4{,}418 \\ 3{,}115 \\ 5{,}528 \\ 7{,}007 \\ 7{,}501 \\ 3{,}5580 \\ 6{,}325 \\ 8{,}025 \\ 8{,}594 \end{bmatrix}$$

Maximum deflection: $w_{max} = 8{,}594 \dfrac{q s^4}{D} = 1{,}146 \text{ mm}$

TRIGONOMETRIC SERIES EXPANSION

Deflection of a rectangular plate of dimensions $L_1 \times L_2$ and flexural rigidity D, simply supported at edges may be expressed with an infinite trigonometric series. For a plate loaded with uniform surface load, the result is:

$$w(x_1, x_2) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \frac{16q \left[\frac{(2m-1)^2}{L_1^2} + \frac{(2n-1)^2}{L_2^2} \right]^{-2}}{(2m-1)(2n-1)\pi^6 D} \cdot \sin \frac{(2m-1)\pi x_1}{L_1} \sin \frac{(2n-1)\pi x_2}{L_2} \right\}$$

Convergence of this series may be checked by summation up to a certain chosen maximum value of indices m, n. Deflection will be determined in the points corresponding with nodes of the FDM mesh:

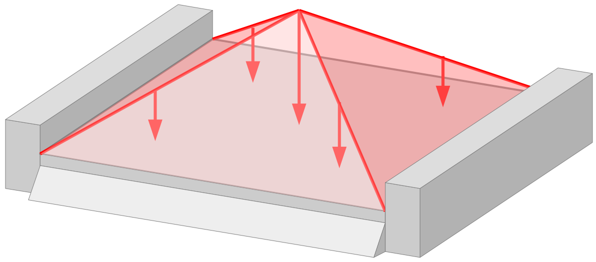
nodes			$(u \cdot D/q/s^4)$				
			trigonometric series				FDM
N	x_1	x_2	n,m=1	n,m=1,...,3	n,m=1,...,5	n,m=1,...,100	
1	1	1	1,690	1,836	1,836	1,835	1,851
2	2	1	3,124	3,251	3,249	3,249	3,268
3	3	1	4,081	4,110	4,111	4,111	4,13
4	4	1	4,417	4,401	4,398	4,398	4,418
5	1	2	2,928	3,096	3,097	3,097	3,115
6	2	2	5,410	5,514	5,513	5,513	5,528
7	3	2	7,069	6,993	6,996	6,996	7,007
8	4	2	7,651	7,493	7,492	7,492	7,501
9	1	3	3,381	3,540	3,540	3,540	3,558
10	2	3	6,247	6,315	6,313	6,313	6,325
11	3	3	8,162	8,018	8,020	8,020	8,025
12	4	3	8,835	8,594	8,591	8,591	8,594

Assuming that an account for 10000 terms of the series gives us a solution which is sufficiently close to the strict solution, we may notice that:

- FDM solution is close to the strict one – relative error is not greater than 1%
- An account for only 9 terms in trigonometric series (m,n=1,...,3) gives us an estimate with a maximum relative error of 0,07%.

EXAMPLE 33

There is an square elastic plate of dimensions $L \times L \ (L=2\,\mathrm{m})$, clamped along two opposite edges and simply supported along the othe two. It is loaded with a load of density given by function

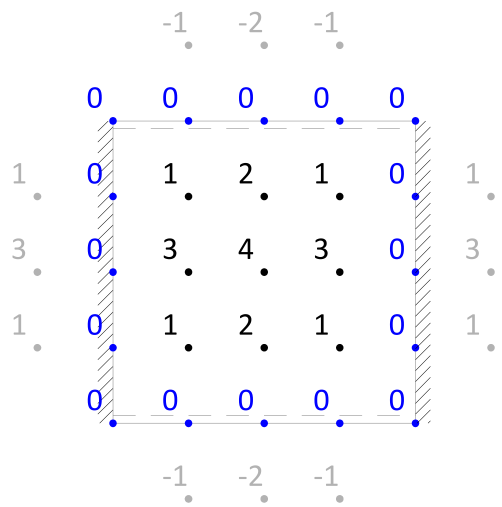


$$q(x_1, x_2) = \frac{4 q_0}{L^2}\left(\frac{L}{2} - |x_1|\right)\left(\frac{L}{2} - |x_2|\right)$$

where $q_0 = 2\,\mathrm{kN/m^2}$, and the beginning of the assumed coordinate system is in the centroid of the plate. Find the distribution of deflection and bending moments m_{xx}, m_{yy}, m_{xy} in the plate with the use of Finite Difference Method. Assume the grid space $s = 0{,}5\,\mathrm{m}$. Thickness of the plate $h = 1\,\mathrm{cm}$, elastic constants: Young's modulus $E = 29$ GPa, Poisson's ratio $\nu = 0{,}2$.

SOLUTION:

- Flexural rigidity: $D = \dfrac{Eh^3}{12(1-\nu^2)} = 2517$ Nm
- Grid line spacing: $s = \Delta x_1 = \Delta x_2 = 0{,}5$ m
- We introduce fictious nodes outside the plate:
  - nodal values in fictious points in case of clamped edge are the same as values in corresponding point inside the plate.
  - nodal values in fictious points in case of simply supported edge are opposite to the values in corresponding point inside the plate.
- After accounting for the symmetry of the problem, the FDM mesh is as below:



Load density values in nodes:

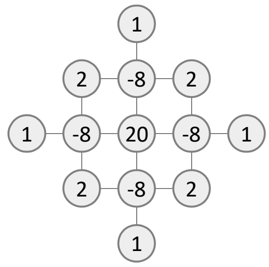
$q_1 = q(0{,}5\,\mathrm{m}\ ;\ 0{,}5\,\mathrm{m}) = 500\ \mathrm{N/m^2}$ $\qquad q_2 = q(0\,\mathrm{m}\ ;\ 0{,}5\,\mathrm{m}) = 1000\ \mathrm{N/m^2}$

$q_3 = q(0{,}5\,\mathrm{m}\ ;\ 0\,\mathrm{m}) = 1000\ \mathrm{N/m^2}$ $\qquad q_4 = q(0\,\mathrm{m}\ ;\ 0\,\mathrm{m}) = 2000\ \mathrm{N/m^2}$

DEFLECTION

Governing equation is written down for each internal node:

$$\nabla^4 w = \frac{\partial^4 w}{\partial x_1^4} + 2\frac{\partial^4 w}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 w}{\partial x_2^4} = \frac{q(x_1, x_2)}{D} \qquad \nabla^4 \approx$$



$$\cdot \frac{1}{s^4}$$

$$20\,w_1 - 8\cdot(0+0+w_2+w_3)+2\cdot(0+0+0+w_4)+1\cdot(-w_1+w_1+w_1+w_1) = \frac{q_1 s^4}{D}$$

$$20\,w_2 - 8\cdot(0+w_1+w_1+w_4)+2\cdot(0+0+w_3+w_3)+1\cdot(-w_2+w_2+0+0) = \frac{q_2 s^4}{D}$$

$$20\,w_3 - 8\cdot(w_1+0+w_4+w_1)+2\cdot(0+w_2+0+w_2)+1\cdot(0+w_3+w_3+0) = \frac{q_3 s^4}{D}$$

$$20\,w_4 - 8\cdot(w_2+w_3+w_3+w_2)+2\cdot(w_1+w_1+w_1+w_1)+1\cdot(0+0+0+0) = \frac{q_4 s^4}{D}$$

We obtain a linear system of equations:

$$\begin{bmatrix} 22 & -8 & -8 & 2 \\ -16 & 20 & 4 & -8 \\ -16 & 4 & 22 & -8 \\ 8 & -16 & -16 & 20 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} 12{,}416\cdot 10^{-3} \\ 24{,}831\cdot 10^{-3} \\ 24{,}831\cdot 10^{-3} \\ 49{,}662\cdot 10^{-3} \end{bmatrix} \mathrm{m} \quad \Rightarrow \quad \mathbf{w} = \begin{bmatrix} 7{,}082\cdot 10^{-3} \\ 11{,}804\cdot 10^{-3} \\ 10{,}492\cdot 10^{-3} \\ 17{,}487\cdot 10^{-3} \end{bmatrix}, m$$

Maximum deflection: $w_{max} \approx w_4 = 17{,}487\cdot 10^{-3}\ \mathrm{m}$. Maximum deflection determined with the use of FEM for a plate modeled with shell elements of maximum size of finite element equal 10 cm (444 elements, 1171 equations) is equal $13{,}480\cdot 10^{-3}\ \mathrm{m}$.

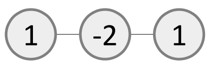
BENDING MOMENTS

Bending moments are found with the use of relations:

$$m_{xx} = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) \quad , \quad m_{yy} = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right) \quad , \quad m_{xy} = -D(1-\nu)\frac{\partial^2 w}{\partial x \partial y} \quad ,$$

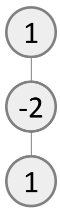
Second derivatives are approximated as follows:

$$\frac{\partial^2}{\partial x_1^2} \approx$$



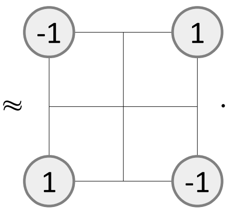
$$\cdot \frac{1}{s^2}$$

$$\frac{\partial^2}{\partial x_2^2} \approx$$



$$\cdot \frac{1}{s^2}$$

$$\frac{\partial^2}{\partial x_1 \partial x_2} \approx$$



$$\cdot \frac{1}{4 s^2}$$

Bending moments in node 1:

$$m_{xx,1}=-\frac{D}{s^2}\left[\left(0-2w_1+w_2\right)+\nu\left(w_3-2w_1+0\right)\right]=1{,}571\,\mathrm{kNm/m}$$

$$m_{yy,1}=-\frac{D}{s^2}\left[\left(w_3-2w_1+0\right)+\nu\left(0-2w_1+w_2\right)\right]=1{,}581\,\mathrm{kNm/m}$$

$$m_{xy,1}=-\frac{D}{4s^2}\left(1-\nu\right)\left[0-0+0-w_4\right]=0{,}035\,\mathrm{kNm/m}$$

Bending moments in node 2:

$$m_{xx,2}=-\frac{D}{s^2}\left[\left(w_1-2w_2+w_1\right)+\nu\left(w_4-2w_2+0\right)\right]=-1{,}176\,\mathrm{kNm/m}$$

$$m_{yy,1}=-\frac{D}{s^2}\left[\left(w_4-2w_2+0\right)+\nu\left(w_1-2w_2+w_1\right)\right]=-0{,}176\,\mathrm{kNm/m}$$

$$m_{xy,1}=-\frac{D}{4s^2}\left(1-\nu\right)\left[0-0+w_3-w_3\right]=0\,\mathrm{kNm/m}$$

Bending moments in node 3:

$$m_{xx,3}=-\frac{D}{s^2}\left[\left(0-2w_3+w_4\right)+\nu\left(w_1-2w_3+w_1\right)\right]=-0{,}208\,\mathrm{kNm/m}$$

$$m_{yy,3}=-\frac{D}{s^2}\left[\left(w_1-2w_3+w_1\right)+\nu\left(0-2w_3+w_4\right)\right]=-1{,}208\,\mathrm{kNm/m}$$

$$m_{xy,1}=-\frac{D}{4s^2}\left(1-\nu\right)\left[w_2-0+0-w_2\right]=0\,\mathrm{kNm/m}$$

Bending moments in node 4:

$$m_{xx,4}=-\frac{D}{s^2}\left[\left(w_3-2w_4+w_3\right)+\nu\left(w_2-2w_4+w_2\right)\right]=0{,}164\,\mathrm{kNm/m}$$

$$m_{yy,4}=-\frac{D}{s^2}\left[\left(w_2-2w_4+w_2\right)+\nu\left(w_3-2w_4+w_3\right)\right]=0{,}143\,\mathrm{kNm/m}$$

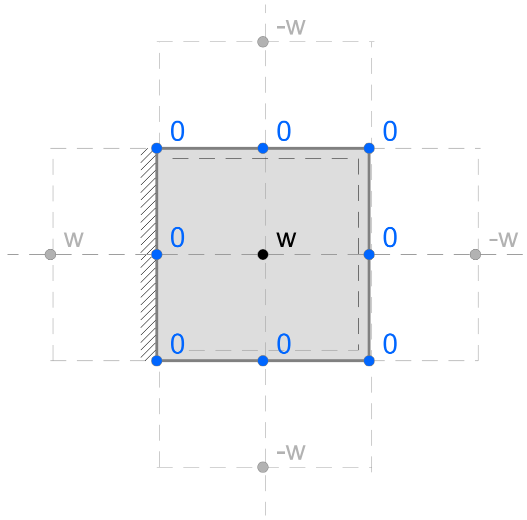
$$m_{xy,1}=-\frac{D}{4s^2}\left(1-\nu\right)\left[w_1-w_1+w_1-w_1\right]=0\,\mathrm{kNm/m}$$

EXAMPLE 34

There is a this square elastic plate of dimensions $1\ \mathrm{m}\times 1\,\mathrm{m}$ and thickness $5\,\mathrm{mm}$,made of steel of Young's modulus $E = 210\ \mathrm{GPa}$ and Poisson's ratio $\nu=0{,}2$. It is clamped along a single edge and simply supported along other edges. It is loaded with a uniform load of density $q = 5\ \mathrm{kN/m^2}$. Find the deflection as well as bending moments= $m_{xx},\ m_{yy},\ m_{xy}$ in the middle of the plate with the use of Finite Difference Method. Assume the grid line spacing $s=0{,}5\,\mathrm{m}$.

SOLUTION:

- Flexural rigidity: $D=\dfrac{Eh^3}{12(1-\nu^2)} = 2278{,}65\ \mathrm{Nm}$
- Grid line spacing: $s = \Delta x_1 = \Delta x_2 = 0{,}5\ \mathrm{m}$
- FDM mesh after accounting for boundary conditions:



Governing equation written down for internal node:

$$20\,w-8(0+0+0+0)+2(0+0+0+0)+(-w+w-w-w) = \frac{q\,s^4}{D}$$

Deflection of a middle point:

$$18\,w = \frac{q\,s^4}{D} \quad\Rightarrow\quad w = \frac{q\,s^4}{18\,D} = 0{,}007619\ \mathrm{m}$$

Bending moments:

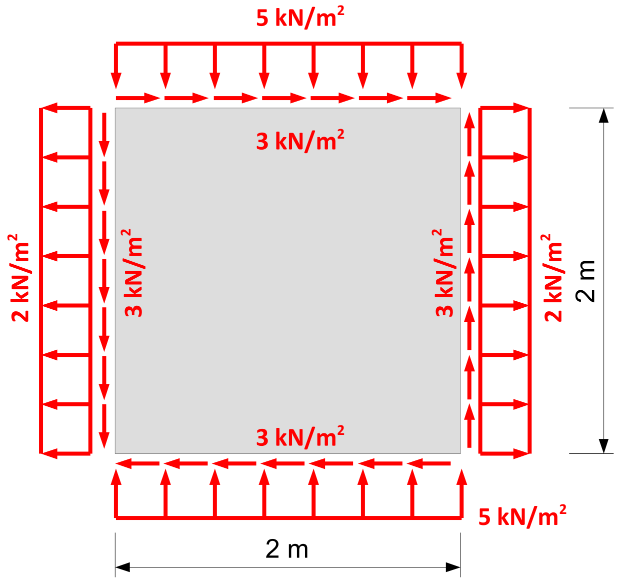
$$m_{xx} = -\frac{D}{s^2}\left[(0-2\,w+0)+\nu(0-2\,w+0)\right] = \frac{2\,D(1+\nu)\,w}{s^2} = 166{,}666\,\mathrm{Nm/m}$$

$$m_{yy} = -\frac{D}{s^2}\left[(0-2\,w+0)+\nu(0-2\,w+0)\right] = \frac{2\,D(1+\nu)\,w}{s^2} = 166{,}666\,\mathrm{Nm/m}$$

$$m_{xy} = -\frac{D}{4\,s^2}(1-\nu)\left[0+0+0+0\right] = 0\,\mathrm{Nm/m}$$

EXAMPLE 37

There is a square elastic membrane of thickness $h=10$ cm, loaded as depicted in the figure. Find the plane stress state in the middle of the mmebrane with the use od Finit Difference Method assuming grid spacing $s=\Delta x_1=\Delta x_2=1\text{ m}$.



SOLUTION:

Distribution of stresses in plane membrane is found with the use of Airy stress function, which is defined as follows:

$$\frac{\partial^2 F}{\partial x_1^2}=\sigma_{22} \qquad \frac{\partial^2 F}{\partial x_2^2}=\sigma_{11} \qquad \frac{\partial^2 F}{\partial x_1 \partial x_2}=-b_1 x_2-b_2 x_1-\sigma_{12}$$

In our case, there are no body forces, so $b_1=b_2=0$. Airy stress function satisfies biharmonic equation:

$$\nabla^4 F=0$$

General outline of our approach is as follows:

- In any point on the boundary we chose fixed point P_0.
- We chose another point P – its location varies.
- In point P a local coordinate system (n,s) is introduced – its 1st axis (n) direction is normal (perpendicular) to the boundary and oriented towards exterior of the membrane. Its 2nd axis (s) is tangent to the boundary and it is oriented in the same way as oriented curve starting in P_0 and ended in P.

- Consider all boundary load applied to the segment of boundary between point P_0 and P.
  - Calculate sum of all forces in that load which are parallel to *n* axis defined in point P. Let's denote it with $Q_n|_P$ (normal force).
  - Calculate sum of all forces in that load which are parallel to *s* axis defined in point P. Let's denote it with $Q_s|_P$ (tangential force).
  - Calculate total moment due to that load about point P. Let's denote it with $M|_P$.

- Boundary conditions may be now written down as follows :

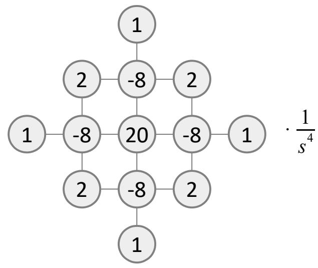
$$F\big|_P = \frac{1}{h} M\big|_P \qquad \left.\frac{\partial F}{\partial s}\right|_P = \frac{1}{h} Q_n\big|_P \qquad \left.\frac{\partial F}{\partial n}\right|_P = -\frac{1}{h} Q_s\big|_P$$

- Sign of Q_s and Q_n is determined according to the orientation of axes (n, s) - contrary to the methods used when finding cross-sectional forces in strength of materials, sign does not depend on the orientation of „imaginary cut surface”.

- Since boundary values of the Airy stress functions are known (calculated according to moments), then also its directional derivatives along the direction tangent to boundary is known and normal load makes no additional contribution in determining the values of F. As a result we need to formulate the boundary conditions only for directional derivatives along a direction normal to the boundary (corresponding with tangential forces). Assuming that (n, s, z) is a right-handed system, positive moment corresponds with positive orientation of *z* axis.

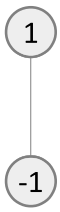
- We introduce a mesh of internal and boundary nodes as well as outside fictious nodes. We consider a rectangular mesh of equal spacing in both directions.

- We write down the governing equation (biharmonic equation) for internal nodes with the use of finite difference pattern as shown to the right.



- We write down the boundary conditions for normal derivative with the use of finite difference patters for first derivatives as below:

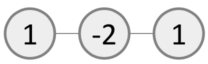
$\frac{\partial}{\partial x_1} \approx$  $\cdot \frac{1}{2s}$

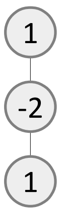
$\frac{\partial}{\partial x_2} \approx$  $\cdot \frac{1}{2s}$

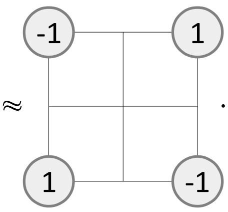
$$\frac{\partial}{\partial n} = n_1 \frac{\partial}{\partial x_1} + n_2 \frac{\partial}{\partial x_2}$$

- Equations which were written down constitute a linear system of algebraic equations for all unknown nodal values of Airy stress function.

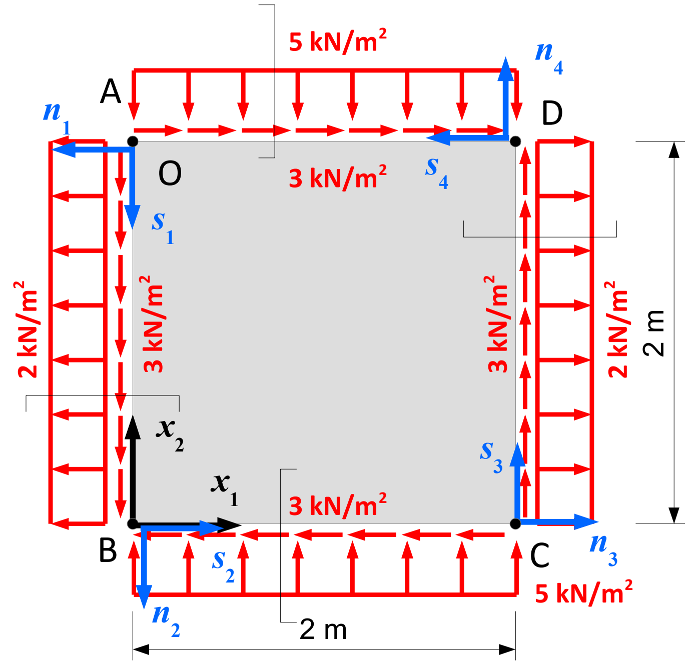
- In order to determine the stress components we make use of the definition of the Airy stress function and of finite difference patterns for the second derivatives:

$\frac{\partial^2}{\partial x_1^2} \approx$  $\cdot \frac{1}{s^2}$

$\frac{\partial^2}{\partial x_2^2} \approx$  $\cdot \frac{1}{s^2}$

$\frac{\partial^2}{\partial x_1 \partial x_2} \approx$  $\cdot \frac{1}{4s^2}$

Initial point P_0=A and local coordinate systems $(n\,,s)$ are assumed as shown below:



BOUNDARY AB: $x_1=0\ ,\ x_2\in\langle 0\ ;\ 2\rangle$ $\qquad s_1=2-x_2\ ,\ n_1=-x_1$

Sum of tangent forces: $Q_{s1}(s_1)=h[3\,s_1]$ [kN]

Sum of normal forces: $Q_{n1}(s_1)=h[2\,s_1]$ [kN]

Moment of forces: $M_1(s_1)=h[s_1^2]$ [kNm]

BOUNDARY BC: $x_1=\text{¿}\langle 0\,;2\rangle\ ,\ x_2=0$ $\qquad s_2=x_1\ ,\ n_2=-x_2$

Sum of tangent forces: $Q_{s2}(s_2)=h[-2\cdot 2-3\cdot s_2]=h[-4-3\,s_2]$ [kN]

Sum of normal forces: $Q_{n2}(s_2)=h[3\cdot 2-5\cdot s_2]=h[6-5\,s_2]$ [kN]

Moment of forces: $M_2(s_2)=h\left[2\cdot 2\cdot 1+3\cdot 2\cdot s_2-\frac{5}{2}s_2^2\right]=$

$=h[4+6\,s_2-2{,}5\,s_2^2]$ [kN]

BOUNDARY CD: $x_1=2\ ,\ x_2\in\langle 0\ ;\ 2\rangle$ $\qquad s_3=x_2\ ,\ n_3=2+x_1$

Sum of tangent forces: $Q_{s3}(s_3)=h[-3\cdot 2+5\cdot 2+3\cdot s_3]=h[4+3\,s_3]$ [kN]

Sum of normal forces: $Q_{n3}(s_3)=h[-2\cdot 2-3\cdot 2+2\cdot s_3]=h[-10+2\,s_3]$ [kN]

Moment of forces: $M_3(s_3)=h[2\cdot 2\cdot(1-s_3)+3\cdot 2\cdot 2-5\cdot 2\cdot 1-3\cdot 2\cdot s_3+s_3^2]=$

$=h[6-10\,s_3+s_3^2]$ [kNm]

BOUNDARY BC: $x_1 \in \langle 0\ ;\ 2 \rangle\ ,\ x_2 = 2 \qquad s_4 = 2 - x_1\ ,\ n_4 = 2 + x_2$

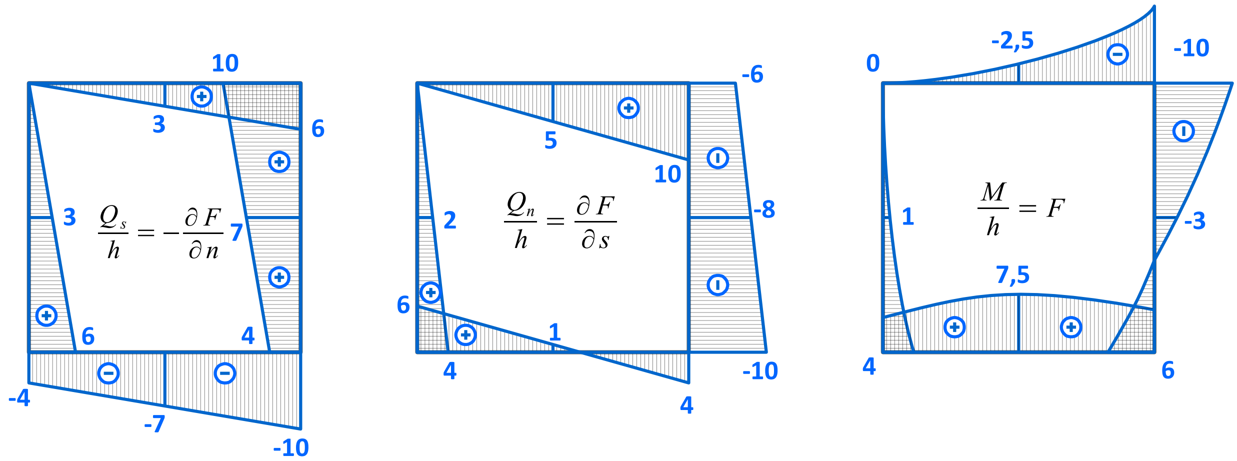
Sum of tangent forces: $Q_{s4}(s_4) = h\left[2\cdot 2 + 3\cdot 2 - 2\cdot 2 - 3\cdot s_4\right] = h\left[6 - 3\,s_4\right]$ [kN]

Sum of normal forces: $Q_{n3}(s_3) = h\left[-3\cdot 2 + 5\cdot 2 + 3\cdot 2 - 5\cdot s_4\right] = h\left[10 - 5\,s_4\right]$ [kN]

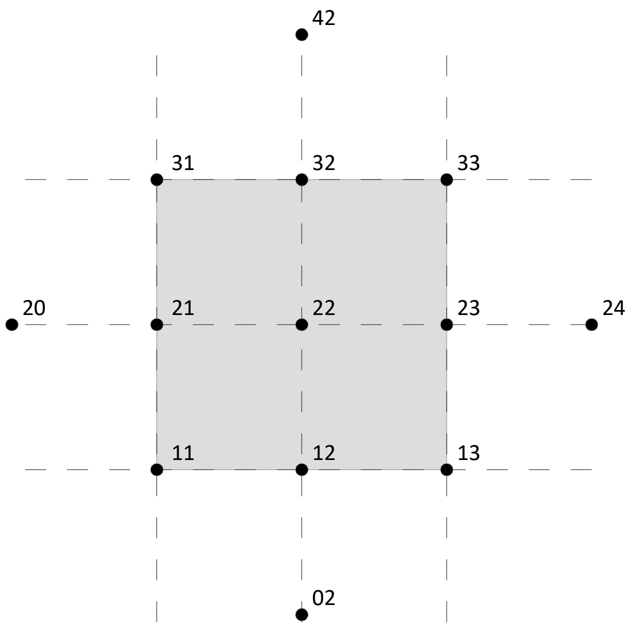
Moment of forces:

$$M_4(s_4) = h\left[-2\cdot 2\cdot 1 + 3\cdot 2\cdot(2 - s_4) - 3\cdot 2\cdot 2 - 5\cdot 2\cdot(1 - s_4) + 3\cdot 2\cdot s_4 + 2\cdot 2\cdot 1 - \frac{5}{2}s_4^2\right] =$$

$$= h\left[-10 + 10\,s_4 - \frac{5}{2}s_4^2\right] \text{ [kNm]}$$



FDM mesh is assumed as shown below:



EQUATION FOR INTERNAL NODES:

$$20\,F_{22}\;-\;8\left(F_{12}+F_{21}+F_{23}+F_{32}\right)\;+\;2\left(F_{11}+F_{13}+F_{31}+F_{33}\right)\;+\;\left(F_{02}+F_{20}+F_{24}+F_{42}\right)=0$$

EQUATIONS FOR BOUNDARY NODES:

BOUNDARY AB

$$F_{11}=\frac{1}{h}M_{11}\qquad\Rightarrow\qquad F_{11}=0\quad\text{[kN]}$$

$$F_{12}=\frac{1}{h}M_{12}\qquad\Rightarrow\qquad F_{33}=1\quad\text{[kN]}$$

$$F_{13}=\frac{1}{h}M_{13}\qquad\Rightarrow\qquad F_{13}=4\quad\text{[kN]}$$

$$\left.\frac{\partial F}{\partial n_1}\right|_{12}=-\left.\frac{\partial F}{\partial x_1}\right|_{12}=-\frac{1}{h}Q_s\big|_{12}\qquad\Rightarrow\qquad -\frac{1}{2s}\left(F_{22}-F_{20}\right)=-3\quad\text{[kN/m]}$$

BOUNDARY BC

$$F_{23}=\frac{1}{h}M_{23}\qquad\Rightarrow\qquad F_{23}=7{,}5\quad\text{[kN]}$$

$$F_{33}=\frac{1}{h}M_{33}\qquad\Rightarrow\qquad F_{33}=6\quad\text{[kN]}$$

$$\left.\frac{\partial F}{\partial n_2}\right|_{23}=-\left.\frac{\partial F}{\partial x_2}\right|_{23}=-\frac{1}{h}Q_s\big|_{23}\qquad\Rightarrow\qquad -\frac{1}{2s}\left(F_{22}-F_{02}\right)=-(-7)\quad\text{[kN/m]}$$

BOUNDARY CD

$$F_{32}=\frac{1}{h}M_{32}\qquad\Rightarrow\qquad F_{32}=-3\quad\text{[kN]}$$

$$F_{31}=\frac{1}{h}M_{31}\qquad\Rightarrow\qquad F_{31}=-10\quad\text{[kN]}$$

$$\left.\frac{\partial F}{\partial n_3}\right|_{32}=\left.\frac{\partial F}{\partial x_1}\right|_{32}=-\frac{1}{h}Q_s\big|_{32}\qquad\Rightarrow\qquad \frac{1}{2s}\left(F_{24}-F_{22}\right)=-7\quad\text{[kN/m]}$$

BOUNDARY DA

$$F_{21}=\frac{1}{h}M_{21}\qquad\Rightarrow\qquad F_{21}=-2{,}5\quad\text{[kN]}$$

$$\left.\frac{\partial F}{\partial n_4}\right|_{21}=\left.\frac{\partial F}{\partial x_2}\right|_{21}=-\frac{1}{h}Q_s\big|_{21}\qquad\Rightarrow\qquad \frac{1}{2s}\left(F_{42}-F_{22}\right)=-3\quad\text{[kN/m]}$$

KNOWN BOUNDARY VALUES

F_{11}	F_{12}	F_{13}	F_{23}	F_{33}	F_{32}	F_{31}	F_{21}
0	1	4	7,5	6	-3	-10	-2,5

We obtain a linear system for unknown nodal values of the Airy stress function:

$$
\begin{aligned}
&20\,F_{22}\; - \;8\left(7{,}5+1-3-2{,}5\right)\; + \;2\left(4+6+0-10\right)\; + \;\left(F_{02}+F_{20}+F_{24}+F_{42}\right)=0\\
&-\frac{1}{2\cdot 1}\left(F_{22}-F_{20}\right)=-3\\
&-\frac{1}{2\cdot 1}\left(F_{22}-F_{02}\right)=7\\
&\frac{1}{2\cdot 1}\left(F_{24}-F_{22}\right)=-7\\
&\frac{1}{2\cdot 1}\left(F_{42}-F_{22}\right)=-3
\end{aligned}
$$

In a matrix form:

$$
\begin{bmatrix} 1 & 1 & 20 & 1 & 1\\ 0 & 1 & -1 & 0 & 0\\ 1 & 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 1 & 0\\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}
\begin{bmatrix} F_{02}\\ F_{20}\\ F_{22}\\ F_{24}\\ F_{42} \end{bmatrix}
=
\begin{bmatrix} 24\\ -6\\ 14\\ -14\\ -6 \end{bmatrix}
\qquad \Rightarrow \qquad
\begin{cases} F_{02}=15{,}5\\ F_{20}=-4{,}5\\ F_{22}=1{,}5\\ F_{24}=-12{,}5\\ F_{42}=-4{,}5 \end{cases}
$$

Stress state in the middle of the membrane:

$$
\begin{aligned}
&\sigma_{11}=\frac{\partial^2 F}{\partial x_2^2}=\frac{1}{s^2}\left[F_{12}-2F_{22}+F_{32}\right]=2\quad \text{[kPa]}\\
&\sigma_{22}=\frac{\partial^2 F}{\partial x_1^2}=\frac{1}{s^2}\left[F_{21}-2F_{22}+F_{23}\right]=-5\quad \text{[kPa]}\\
&\sigma_{12}=-\frac{\partial^2 F}{\partial x_1\partial x_2}=\frac{1}{4s^2}\left[F_{11}-F_{13}-F_{31}+F_{33}\right]=3\quad \text{[kPa]}
\end{aligned}
$$

For the above problem a strict solution may be found. Since there are no body forces, the load is symmetric and uniform, then stress state distribution must be equal – it is given by a stress tensor:

$$
\boldsymbol{\sigma}=\begin{bmatrix} 2 & 3\\ 3 & -5 \end{bmatrix}\ [\text{kPa}]
$$

It satisfies both equilibrium equations as well as boundary conditions. Such a stress state corresponds with the Airy function of the form:

$$
F(x_1,x_2)=-\frac{5}{2}x_1^2+x_2^2-3\,x_1x_2
$$

Biharmonic equation is an equation of the 4^{th} order. The above solution is determined with the use of statical boundary conditions which are conditions for the 2^{nd} order derivatives – in order to get a unique solution we would need also boundary conditions for the function itself or also for 1^{st}

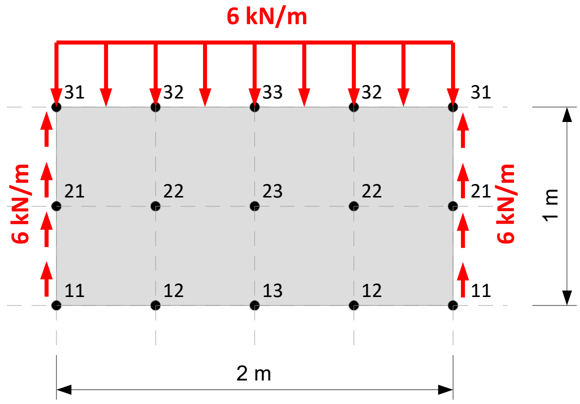
derivatives. We should note that adding to the function F terms which depend at most on the 1st powers of independent variables also satisfies the governing equation and result in the same stress state. Those terms may be chosen in such, that the values of the Airy stress function were the same as those determined by FDM. Finally we may write:

$$F(x_1, x_2) = -\frac{5}{2}x_1^2 + x_2^2 - 3x_1x_2 + 6x_1 - 4x_2 + 4$$

Values of the strict solution are the same as those found by the use of FDM:

$$\begin{cases} F_{02} = F(1;-1) = 15{,}5 \\ F_{20} = F(-1;1) = -4{,}5 \\ F_{22} = F(1;1) = 1{,}5 \\ F_{24} = F(3{,}1) = -12{,}5 \\ F_{42} = F(1{,}3) = -4{,}5 \end{cases}$$

EXAMPLE 38



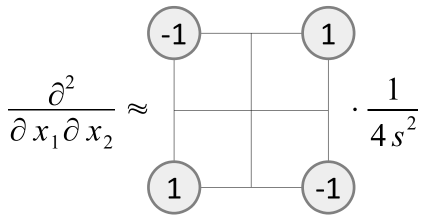
There is a rectangular elastic membrane of thickness $h=20\,\mathrm{cm}$, loaded as shown in the figure. Find the shear stress in node 22 with the use of Finite Difference Method and assumed mesh ($\Delta x_1=\Delta x_2=0{,}5\ \mathrm{m}$).

SOLUTION:

Shear stress is determined by the Airy stress function according to relation:

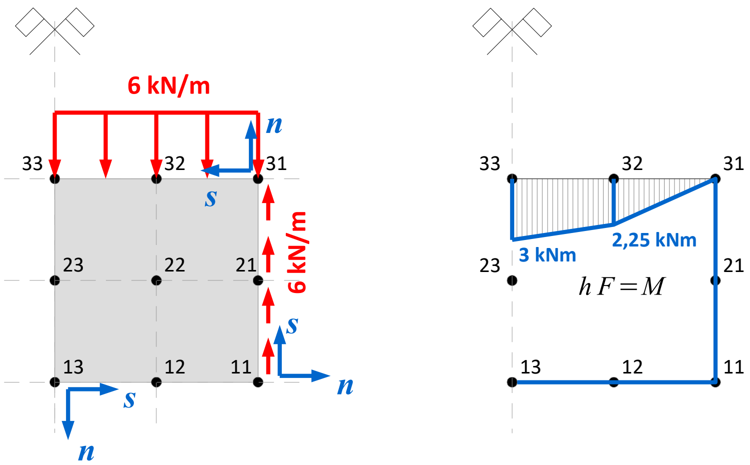
$$\sigma_{12}=-\frac{\partial^2 F}{\partial x_1 \partial x_2}$$

Mixed derivative is approximated by a finite difference pattern as below:



$$[\sigma_{12}]_{22}\approx-\frac{1}{4s^2}\left[F_{13}+F_{31}-F_{11}-F_{33}\right]$$

According to the assumed mesh it can be noted that shear stress in node 22 may be determined with the use of boundary values only, which in turn are determined by moments of external load about chosen boundary points. Accounting for symmetry of the system we obtain:



Boundary values of Airy stress function:

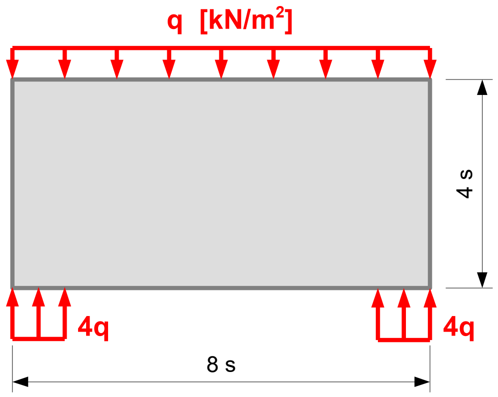
$$F_{11}=\frac{M_{11}}{h}=0\quad[\mathrm{kN}]$$
$$F_{13}=\frac{M_{13}}{h}=0\quad[\mathrm{kN}]$$
$$F_{31}=\frac{M_{31}}{h}=0\quad[\mathrm{kN}]$$
$$F_{33}=\frac{M_{33}}{h}=15\quad[\mathrm{kN}]$$

Shear stress in node 22:

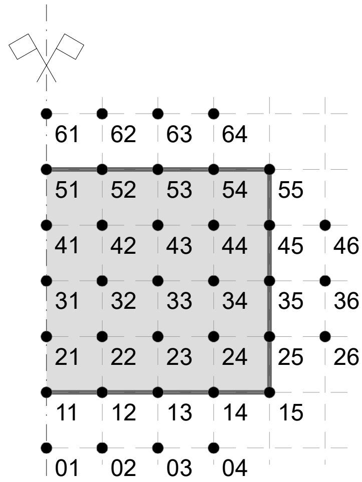
$$[\sigma_{12}]_{22}\approx-\frac{1}{4s^2}\left[F_{13}+F_{31}-F_{11}-F_{33}\right]=15\ \mathrm{kPa}$$

EXAMPLE 39

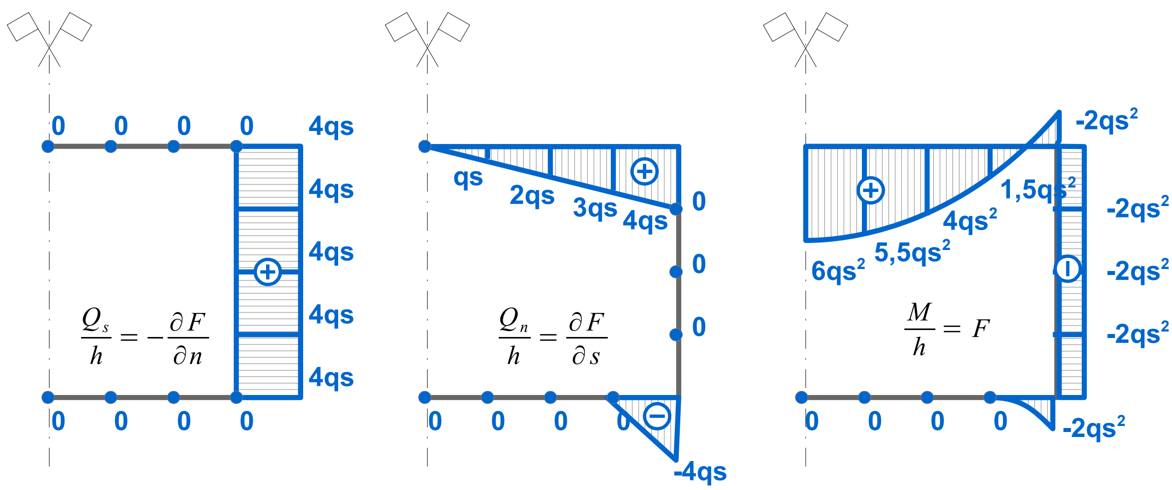
There is a rectangular elastic membrane of thickness h , loaded as shown in the figure. Find the distribution of horizontal normal stress in the middle cross-section of the membrane with the use of Finite Difference Method assuming $s = \Delta x_1 = \Delta x_2$.



SOLUTION:
FDM mesh accounting for the symmetry is as follows.



Boundary values of the Airy stress function as well as of its derivatives are found assuming the initial point in the middle of bottom edge of the membrane:



We may reduce the number of unknown nodal values by accounting for the condition of zero normal derivatives:

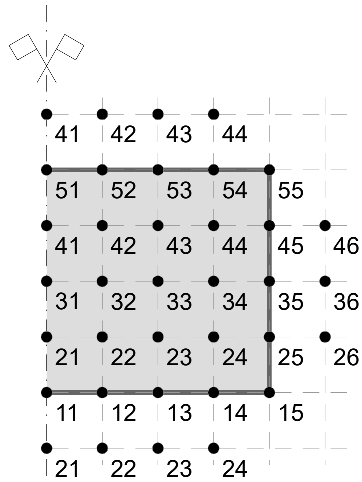
$$\frac{Q_s}{h} = -\left.\frac{\partial F}{\partial n}\right|_{11} = \frac{1}{2s}\left(F_{01}-F_{21}\right) = 0 \quad \Rightarrow \quad F_{01} = F_{21}$$

$$\frac{Q_s}{h} = -\left.\frac{\partial F}{\partial n}\right|_{12} = \frac{1}{2s}\left(F_{02}-F_{22}\right) = 0 \quad \Rightarrow \quad F_{02} = F_{22}$$

$$\ldots$$

$$\frac{Q_s}{h} = -\left.\frac{\partial F}{\partial n}\right|_{51} = \frac{1}{2s}\left(F_{61}-F_{41}\right) = 0 \quad \Rightarrow \quad F_{61} = F_{41}$$

Simplified FDM mesh is as follows:



INTERNAL POINT:

$$20F_{21}-8\left(F_{11}+F_{22}+F_{22}+F_{31}\right)+2\left(F_{12}+F_{12}+F_{32}+F_{32}\right)+\left(F_{21}+F_{23}+F_{23}+F_{41}\right)=0$$
$$20F_{22}-8\left(F_{12}+F_{21}+F_{23}+F_{32}\right)+2\left(F_{11}+F_{13}+F_{31}+F_{33}\right)+\left(F_{22}+F_{22}+F_{24}+F_{42}\right)=0$$
$$20F_{23}-8\left(F_{13}+F_{22}+F_{24}+F_{33}\right)+2\left(F_{12}+F_{14}+F_{32}+F_{34}\right)+\left(F_{23}+F_{21}+F_{25}+F_{43}\right)=0$$
$$20F_{24}-8\left(F_{14}+F_{23}+F_{25}+F_{34}\right)+2\left(F_{13}+F_{15}+F_{33}+F_{35}\right)+\left(F_{24}+F_{22}+F_{26}+F_{44}\right)=0$$
$$20F_{31}-8\left(F_{21}+F_{32}+F_{32}+F_{41}\right)+2\left(F_{22}+F_{22}+F_{42}+F_{42}\right)+\left(F_{11}+F_{33}+F_{33}+F_{51}\right)=0$$
$$20F_{32}-8\left(F_{22}+F_{31}+F_{33}+F_{42}\right)+2\left(F_{21}+F_{23}+F_{41}+F_{43}\right)+\left(F_{12}+F_{32}+F_{34}+F_{52}\right)=0$$
$$20F_{33}-8\left(F_{23}+F_{32}+F_{34}+F_{43}\right)+2\left(F_{22}+F_{24}+F_{42}+F_{44}\right)+\left(F_{13}+F_{31}+F_{35}+F_{53}\right)=0$$
$$20F_{34}-8\left(F_{24}+F_{33}+F_{35}+F_{44}\right)+2\left(F_{23}+F_{25}+F_{43}+F_{45}\right)+\left(F_{14}+F_{32}+F_{36}+F_{54}\right)=0$$
$$20F_{41}-8\left(F_{31}+F_{42}+F_{42}+F_{51}\right)+2\left(F_{32}+F_{32}+F_{52}+F_{52}\right)+\left(F_{21}+F_{43}+F_{43}+F_{41}\right)=0$$
$$20F_{42}-8\left(F_{32}+F_{41}+F_{43}+F_{52}\right)+2\left(F_{31}+F_{33}+F_{51}+F_{53}\right)+\left(F_{22}+F_{42}+F_{44}+F_{42}\right)=0$$
$$20F_{43}-8\left(F_{33}+F_{42}+F_{44}+F_{53}\right)+2\left(F_{32}+F_{34}+F_{52}+F_{54}\right)+\left(F_{23}+F_{41}+F_{45}+F_{43}\right)=0$$
$$20F_{44}-8\left(F_{34}+F_{43}+F_{45}+F_{54}\right)+2\left(F_{33}+F_{35}+F_{53}+F_{55}\right)+\left(F_{24}+F_{42}+F_{46}+F_{44}\right)=0$$

BOUNDARY POINTS:

$$\left.\frac{\partial F}{\partial n}\right|_{25}=\left.\frac{\partial F}{\partial x_1}\right|_{25}=-\frac{Q_s|_{25}}{h}\quad\Rightarrow\quad\frac{1}{2s}\left(F_{26}-F_{24}\right)=-4qs$$

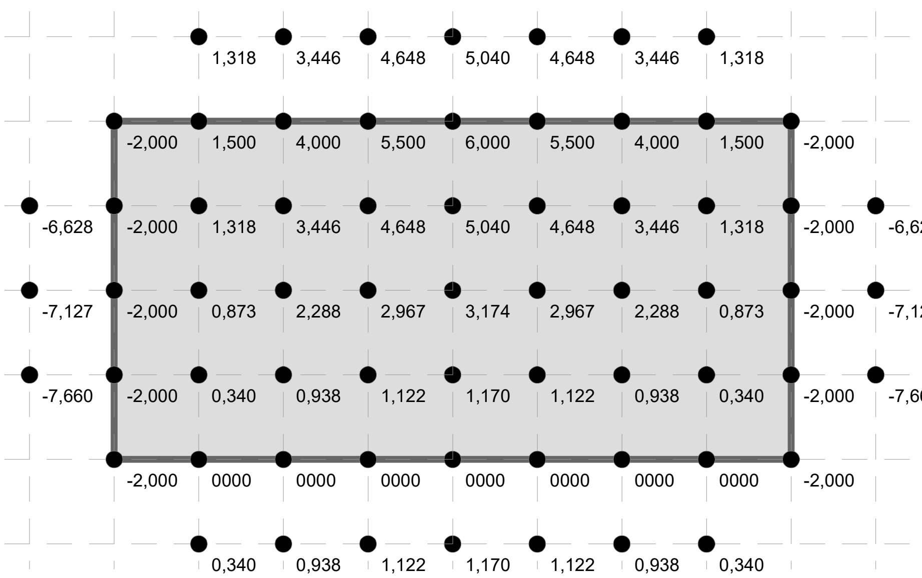
$$\left.\frac{\partial F}{\partial n}\right|_{35}=\left.\frac{\partial F}{\partial x_1}\right|_{35}=-\frac{Q_s|_{35}}{h}\quad\Rightarrow\quad\frac{1}{2s}\left(F_{36}-F_{34}\right)=-4qs$$

$$\left.\frac{\partial F}{\partial n}\right|_{45}=\left.\frac{\partial F}{\partial x_1}\right|_{45}=-\frac{Q_s|_{45}}{h}\quad\Rightarrow\quad\frac{1}{2s}\left(F_{46}-F_{44}\right)=-4qs$$

Governing system of linear equations:

$$\begin{bmatrix}
21 & -16 & 2 & 0 & 0 & -8 & 4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
-8 & 22 & -8 & 1 & 0 & 2 & -8 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
1 & -8 & 21 & -8 & 0 & 0 & 2 & -8 & 2 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 1 & -8 & 21 & 1 & 0 & 0 & 2 & -8 & 0 & 0 & 0 & 0 & 1 & 0\\
-8 & 4 & 0 & 0 & 0 & 20 & -16 & 2 & 0 & 0 & -8 & 4 & 0 & 0 & 0\\
2 & -8 & 2 & 0 & 0 & -8 & 21 & -8 & 1 & 0 & 2 & -8 & 2 & 0 & 0\\
0 & 2 & -8 & 2 & 0 & 1 & -8 & 20 & -8 & 0 & 0 & 2 & -8 & 2 & 0\\
0 & 0 & 2 & -8 & 0 & 0 & 1 & -8 & 20 & 1 & 0 & 0 & 2 & -8 & 0\\
1 & 0 & 0 & 0 & 0 & -8 & 4 & 0 & 0 & 0 & 21 & -16 & 2 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 2 & -8 & 2 & 0 & 0 & -8 & 22 & -8 & 1 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 2 & -8 & 2 & 0 & 1 & -8 & 21 & -8 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & -8 & 0 & 0 & 1 & -8 & 21 & 1\\
0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1
\end{bmatrix}\cdot\begin{bmatrix}F_{21}\\F_{22}\\F_{23}\\F_{24}\\F_{26}\\F_{31}\\F_{32}\\F_{33}\\F_{34}\\F_{36}\\F_{41}\\F_{42}\\F_{43}\\F_{44}\\F_{46}\end{bmatrix}=qs^2\begin{bmatrix}0\\0\\2\\-8\\-6\\-5{,}5\\-2\\-9{,}5\\26\\24\\20\\-4\\-8\\-8\\-8\end{bmatrix}$$

The results are depicted below (values of the Airy stress function are divided by qs^2):



Stresses in the middle cross-section of the membrane:

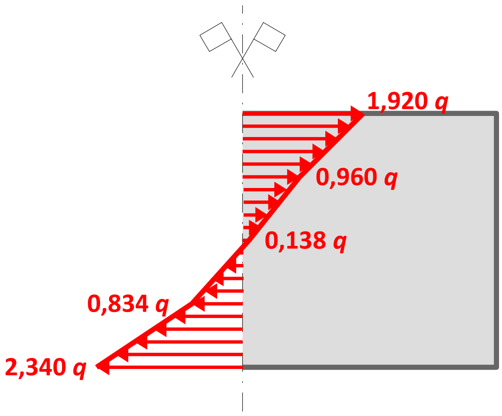
$$[\sigma_{11}]_{11} = \frac{1}{s^2}(F_{21} - 2F_{11} - F_{21}) = 2{,}340\, q$$

$$[\sigma_{11}]_{21} = \frac{1}{s^2}(F_{11} - 2F_{21} - F_{31}) = 0{,}834\, q$$

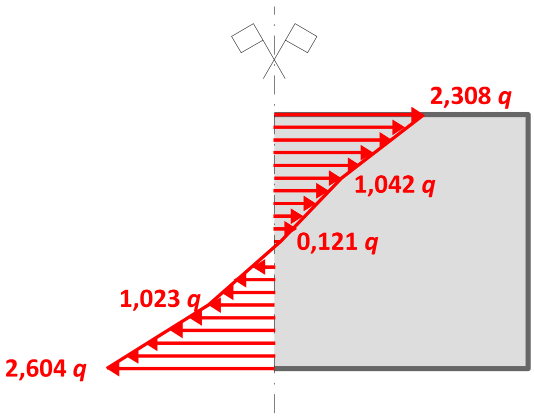
$$[\sigma_{11}]_{31} = \frac{1}{s^2}(F_{21} - 2F_{31} - F_{41}) = -0{,}138\, q$$

$$[\sigma_{11}]_{41} = \frac{1}{s^2}(F_{31} - 2F_{41} - F_{51}) = -0{,}906\, q$$

$$[\sigma_{11}]_{51} = \frac{1}{s^2}(F_{41} - 2F_{51} - F_{41}) = -1{,}92\, q$$



The results may be compared with FEM solution – a membrane of dimensions 8 m x 4 m, was divided into 3200 square membrane elements of maximum element size equal 10 cm (6639 equations). Obtained distribution of stresses in the middle cross-section of the membrane is as follows:

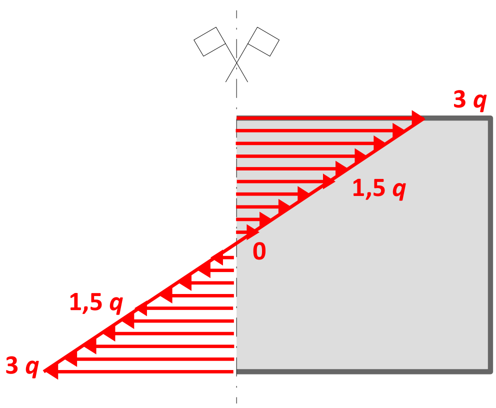


One may also compare it with estimate provided by assumption that the membrane is a simply-supported Bernoulli-Euler beam. Beam of length L and rectangular cross-section $b \times h$, where $h = L/2$, gives us a symmetric and linear distribution of stress:

$$M_{max} = \frac{q b L^2}{8}$$

$$W = \frac{b(L/2)^2}{6}$$

$$\sigma_{max} = \frac{M_{max}}{W} = 3q$$



Better estimate may be obtained with the use of formulas known from course of strength of materials by assuming different span length (e.g. Distance between resultants of the loads applied to the bottom edge):

$$M_{max}=\frac{q\,b\left(7/8\,L\right)^2}{8} \qquad W=\frac{b\left(L/2\right)^2}{6} \qquad \Rightarrow \qquad \sigma_{max}=\frac{M_{max}}{W}=2{,}297\,q$$

Generally, the Bernoulli-Euler beam model is not recommended to be used for beams for which $L:h<10$. For beams for which $4<L:h<10$ the Timoshenko beam model may be used. For even higher beams one should use membrane models. However, it can be noticed that in some cases even a beam model provides a sufficiently good estimate.