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A SHORT INTRODUCTION TO THE
THEORY OF PLASTICITY

Special thanks are given to
my beloved four-years old daughter Klara
who made calculation for example No 3
when she could not fall asleep one night.

LIST OF CONTENT

NOTATION

$a, b, c, \ldots$	- scalars
$\alpha, \beta, \gamma, \ldots$	- scalars
$\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$	- vectors, tensors
$\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \ldots$	- vectors, tensors
σ_0	- limit stress (yield stress) in uniaxial stress state
τ_0	- limit stress (yield stress) in pure shear state
σ_{eq}	- equivalent stress
k_t	- limit stress at tension
k_c	- limit stress at compression
k_s	- limit stress at shear
$\sigma_1, \sigma_2, \sigma_3$	- principal stresses
I_1	- 1st invariant of a tensor
I_2	- 2nd invariant of a tensor
I_3	- 3rd invariant of a tensor
J_2	- 2nd invariant of a deviator of a tensor
J_3	- 3rd invariant o a deviator of a tensor
p	- hydrostatic stress
q	- deviatoric stress
θ	- Lode angle
ϕ	- strain energy density
ϕ_v	- volumetric strain energy density
ϕ_f	- distortional strain energy density
Ψ	- plastic potential
$f(\boldsymbol{\sigma}) = 0$	- limit state condition (yield condition)
E	- Young modulus (longitudinal stiffness modulus)
G	- Kirchhoff rigidity modulus (transverse stiffness modulus)
ν	- Poisson's ratio
λ	- 1st Lame parameter
n	- strain hardening exponent
$\mathrm{d}\lambda$	- strain path parameter

$\boldsymbol{\sigma}$	- stress tensor
$\mathbf{s}$	- deviator of a stress tensor
$\boldsymbol{\varepsilon}$	- total strain tensor
$\boldsymbol{\varepsilon}^{e}$	- elastic strain tensor
$\boldsymbol{\varepsilon}^{pl}$	- plastic strain tensor
$\mathbf{e}$	- deviator of a total strain tensor
$\mathbf{e}^{e}$	- deviator of an elastic strain tensor
$\mathbf{e}^{pl}$	- deviator of a plastic strain tensor
$\mathrm{d}\boldsymbol{\varepsilon}$	- total strain increment tensor
$\mathrm{d}\boldsymbol{\varepsilon}^{e}$	- elastic strain increment tensor
$\mathrm{d}\boldsymbol{\varepsilon}^{pl}$	- plastic strain increment tensor
$\mathrm{d}\boldsymbol{\sigma}$	- stress increment tensor

If in an expression bottom index is repeated, it means summation with respect to that index for all its possible values, e.g.:

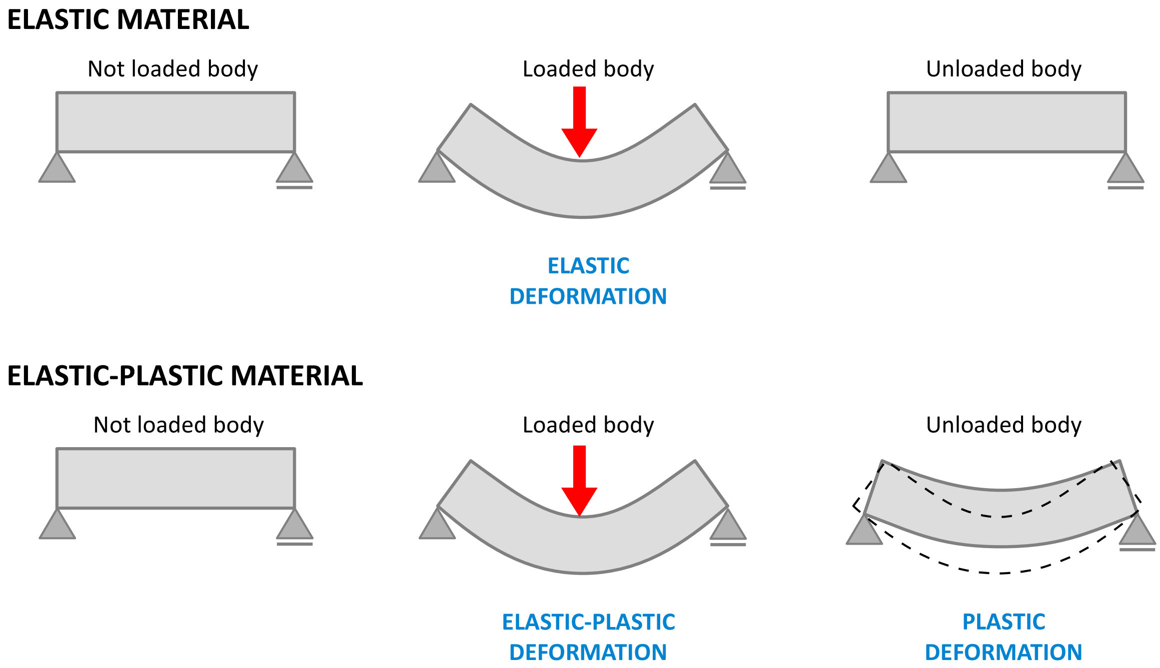
$$\sigma_{ij} n_j = \sum_{j=1}^{3} \sigma_{ij} n_j \quad , \quad \varepsilon_{kk} = \sum_{k=1}^{3} \varepsilon_{kk} \quad , \quad \sigma_{ij}\varepsilon_{ij} = \sum_{i=1}^{3}\sum_{j=1}^{3} \sigma_{ij}\varepsilon_{ij} \quad \text{etc.}$$

Vector addition	$\mathbf{v}+\mathbf{w}$	$v_i + w_i$
Tensor addition	$\boldsymbol{\alpha}+\boldsymbol{\beta}$	$\alpha_{ij}+\beta_{ij}$
Action of a tensor on a vector	$\boldsymbol{\sigma}\,\mathbf{n}$	$\sigma_{ij} n_j$
Dot product of vectors	$\mathbf{v}\cdot\mathbf{w}$	$v_i w_i$
Dot product of tensors	$\boldsymbol{\alpha}\cdot\boldsymbol{\beta}$	$\alpha_{ij}\beta_{ij}$
Trace of a tensor	$\mathrm{tr}(\boldsymbol{\alpha})$	α_{ii}

A SHORT INTRODUCTION TO THE THEORY OF PLASTICITY

1. CHARACTER OF PLASTIC DEFORMATION

Contrary to the **elastic deformation, which is reversible**, **plastic deformation is permanent**:



Notions of **permanent strain** and **plastic strain** are to some extent equivalent. The fact that strain is permanent means that new configuration of a body is in fact a **new state of equilibrium of internal forces**. In general it can be stated that plastic strain occurs when **particles of a body** (in particular: atoms in a crystal lattice) **due to applied load create such a spatial configuration, that interaction forces between all those particles are in equilibrium**.

Some permanent strain occur even when applied stress or strain is very small – it is due to complex and non0uniform structure of material in microscopic scale (granular structure of material) as well as in atomic scale – those deformations are of similar order of magnitude, co macroscopically the deformation is considered elastic. It is important to be aware of the fact that **plastic deformation occurs always simultaneously with elastic deformation** even when magnitude of reversible strain is much smaller than magnitude of permanent strain or when elastic strain is unnoticeable or – due to character of considered problem – negligible.

In general both elastic strain and plastic strain will be present and they should be distinguished is we want to describe arbitrary deformation processes accounting for plastic strain. In case of unloading elastic strains decrease while plastic strain remain permanent. For these reasons the **total strain tensor** $\boldsymbol{\varepsilon}$ is written down as a sum of **elastic strain tensor** $\boldsymbol{\varepsilon}^e$ and

plastic strain tensor $\boldsymbol{\varepsilon}^{pl}$ **:**

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{e} + \boldsymbol{\varepsilon}^{pl}$$

however, relation of each of those components with the stress tensor (constitutive law) is different.

Observable plastic deformation occur only after **limit state condition is satisfied**, what in some cases (simple load cases) is equivalent to the statement that **magnitude of stress exceed certain limit value** which is termed **yield stress** and denoted with σ_0 . In more general cases of complex stress state the condition of initiation of plastic deformation – so called **yield condition** – is written down in the following form:

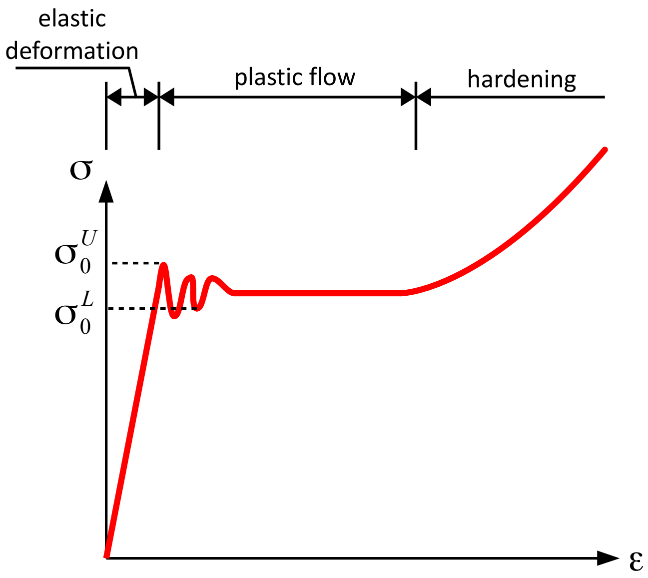
$$f(\boldsymbol{\sigma}) = 0$$

where $f(\boldsymbol{\sigma})$ is a certain function of a stress state.

Let's consider a simple stress state, in which yield condition is equivalent to relation $\sigma = \sigma_0$. It must be noted that this limit value is not „sharp" in the sense that there is always some transient state in which increment of plastic strain becomes gradually larger till it is finally observable and measurable. For this reason various definitions of yield stress are used.

Let these consideration be illustrated with simple graphs of axial stress – linear strain relation in case of quasistatic tensile test. We will consider two such graphs, characteristic for two types of materials.

The first graph – shown below – is typical for **ductile materials with quite a distinct border between elastic strain domain and plastic strain domain** (e.g. Low-carbon steel).



At first we can observe a domain of approximately **linear elastic** deformation and then a region in which a **large increment of strain is observed while stress varies in a very small extent** (so called "**plastic flow**"). An important mathematical aspect of that phenomenon should be emphasized here – it is clear that **relation** $\sigma(\varepsilon)$ **is not a one-to-one relation within this "plastic *plateau*"**. It means that a single value of tress corresponds with many possible values of strain, what results in that some problems of theory of plasticity cannot be solved for only static boundary conditions given. In particular, uniaxial tensile test of a material exhibiting plastic plateau cannot be performed by a force-controlled machine because increasing the value of stress above the value corresponding with the plateau would require an immediate large increase in strain. In fact the process is displacement-controlled – it is the elongation of a sample which increases gradually while the force which is needed to produce such an elongation is measured. In particular a drop in the value of that required force may be sometimes observed.

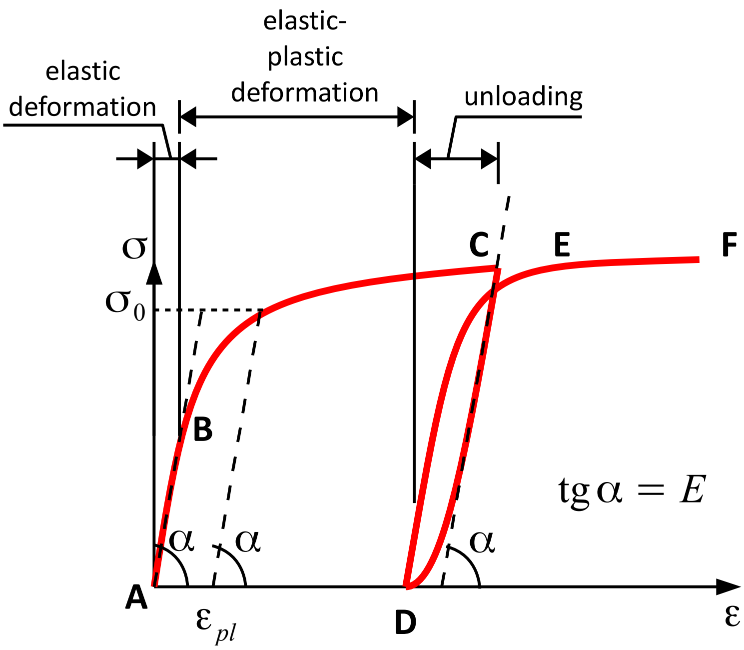
Lack of one-to-one correspondence of constitutive relations of material exhibiting plastic plateau was the reason for which so called incremental theories of plasticity (sometimes called also plastic flow theories) were formulated – these are i.e. Levy-Mises and Prandtl-Reuss models, which will be described later. In those models the stress tensor $\boldsymbol{\sigma}$ **determines uniquely not a strain tensor** $\boldsymbol{\varepsilon}$, but tensor of increment of strain – total one $d\boldsymbol{\varepsilon}$ or plastic one $d\boldsymbol{\varepsilon}^{pl}$ depending on chosen model.

After the region of plastic plateau larger load is required in order to produce further elongation – it is called the **hardening**. In most of cases hardening is exhibited until the material looses its cohesion, namely when it **cracks**, however, it is not a general rule.

Among most commonly used definitions of a yield stress for ductile elastic-plastic materials one should mention:

- **lower yield stress** σ_0^L – it is defined in a number of ways, however, these definitions often correspond with the same value. It may defined as:
  - lowest value of stress which is necessary for further increase of plastic strain after initiation of plastic deformation (after satisfying yield condition),
  - lowest value of stress within the plastic deformation domain, excluding first local minimum after initiation of plastic deformation,
  - last minimum value of stress within the plastic deformation domain before hardening.
- **Upper yield stress** σ_0^U – it is defined in a number of ways, however, these definitions often correspond with the same value. It may defined as:
  - the first local maximum of value of stress after initiation of plastic deformation (satisfying yield condition),
  - maximum value of stress within the plastic deformation domain.

It often happens that yield stress is not „sharp" as it is shown in the figure below. This graph is typical for ductile materials of non-linear material characteristics (e.g. high-carbon steel).



In such situations alternative definition of a yield stress is used:

- **offset yield point (proof stress)** – it is the value of stress which corresponds with a chosen value of permanent strain which is determined with an assumption that unloading process is a totally elastic one and that elastic properties of material do not change due to plastic deformation. A common choice for ductile elastic-plastic materials is $\varepsilon_{pl} = 0{,}2\%$.

In the graph above we may observe a domain of **elastic deformation** (AB), after which plastic strain starts to increase gradually. Simultaneously **we may observe material hardening all the time**. Specific character of plastic deformation may be noticed in case of non-monotonic load process. If we unload the sample, it will emerge, that **unloading process** (CD) **is elastic** – what's more, **elastic properties are almost exactly the same as in the first region of elastic deformation**, even before any considerable plastic strain occurred. In fact, a slight non-linearity of that process is observed, which is due to presence of residual stresses (which in turn are caused by non-uniformity of material's structure), which may lead to some new slips when external load is reduced. **Reloading process has an analogous character as the one in elastic deformation domain** – at first approximately linear elastic deformation occurs and then plastic strain slowly increases so that curve EF becomes to some extent a continuation of BC curve. **CDE loop is a hysteresis loop** and its size is a measure of energy dissipated in the processes of unloading and reloading. It is worth to notice that hardening results in amplification of current yield stress, namely in strengthening. Indeed, plastic deformation of materials exhibiting hardening is one of the ways of amplifying its elastic deformation domain – this treatment is limited by the strength of material (hardening finally ends with cracking) and by residual stresses, which may result in some undesired mechanical properties of a material after unloading.

The above example has one more feature which distinguishes it from the previous one – **there is no plastic plateau**. In such a situation **the stress-strain relation is a one-to-one correspondence**, unless unloading occurs. In such situation a common – yet, from the mathematical point of view, incorrect – approach is the use of so called **deformation theories of plasticity** or – what is perhaps a better name – total strain theories, e.g. Hencky-Ilyushin equation, in which **stress state determines uniquely total strain state** – they will be described later.

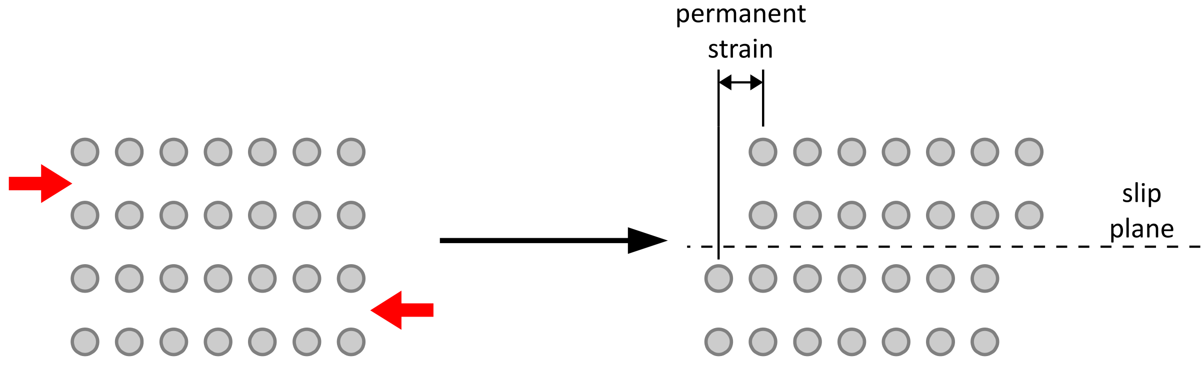
2. MECHANISMS OF PLASTIC DEFORMATION

Mechanism of creation and increase of permanent plastic deformation may be discussed in the most easily understandable way in case of a **monocrystalline** solid – a material in which **all atoms form a regular lattice the geometry of which is repeated in space**. In an ideal crystal lattice there are many **possible equilibrium positions** of atoms – these are positions which **locally minimize potential energy of inter-atomic bonds**. If it happened that due to external load the atoms were moved from one equilibrium position to another one, then these new positions will constitute also an equilibrium configuration, what – macroscopically – results in permanent (plastic) strain.

Two fundamental **mechnisms of plastic deformation in a monocrystalline material** are distinguished:

- **slip**
- **twinning**

Slip in a close-pack plane of atoms – it occurs within a single crystal, this is in a region in which spatial distribution of atoms is regular and repeated. In a structure of that kind such a motion is possible in which – speaking in a simplified way – one **part of that structure is translated as a whole (like a rigid body) with respect to the second one in such a way that particles of that moving part takes position which originally were occupied by their neighbors.** This process may go on further. Such a deformation is termed slip. **A magnitude of load which is required in order to enforce such a motion is the greater the larger are distances between atoms** interchanging their position – for this reason **slip occurs in a plane which is most densely packed with atoms**, so-called **close-pack plane**.



Original configuration of equilibrium and external load

New configuration of equilibrium.

It must be noted that energy which is needed in order to enforce slip in an ideal crystal lattice – this is, translation of a whole plane of atoms – is very large, so also force needed to enforce permanent deformation would be very large – in fact, observed magnitudes of forces resulting in plastic strain are much smaller. This is due to **imperfection of a crystal lattice**, namely existence of some flaws of defects in its structure. **Presence of such an imperfection makes the**

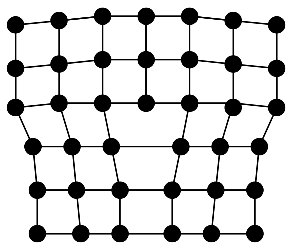
energy required to initiate a slip much smaller than in case of a perfect lattice.

We distinguish three types of crystallographic defects:

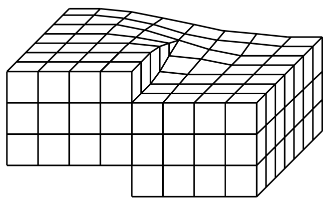
- **point defects**
  - **vacancy** – lack of an atom in a position predicted by a perfect lattice
  - **interstitial defect** – presence of an additional atom in a place which is unoccupied in a perfect lattice. This additional atom may be a native atom (proper for the lattice) or a foreign one (e.g. In alloys).
  - **substitutional defect** – presence of a foreign atom in a position predicted by a perfect lattice (e.g. In alloys).

- **line defects** or **dislocations**
  - **edge dislocation** – a presence of an additional half-plane of atoms in a lattice.
  - **screw dislocation** – deformed structure is of a shape of screw curve.
  - **mixed disloaction** – composition of edge and screw dislocation.

structure with
edge dislocation

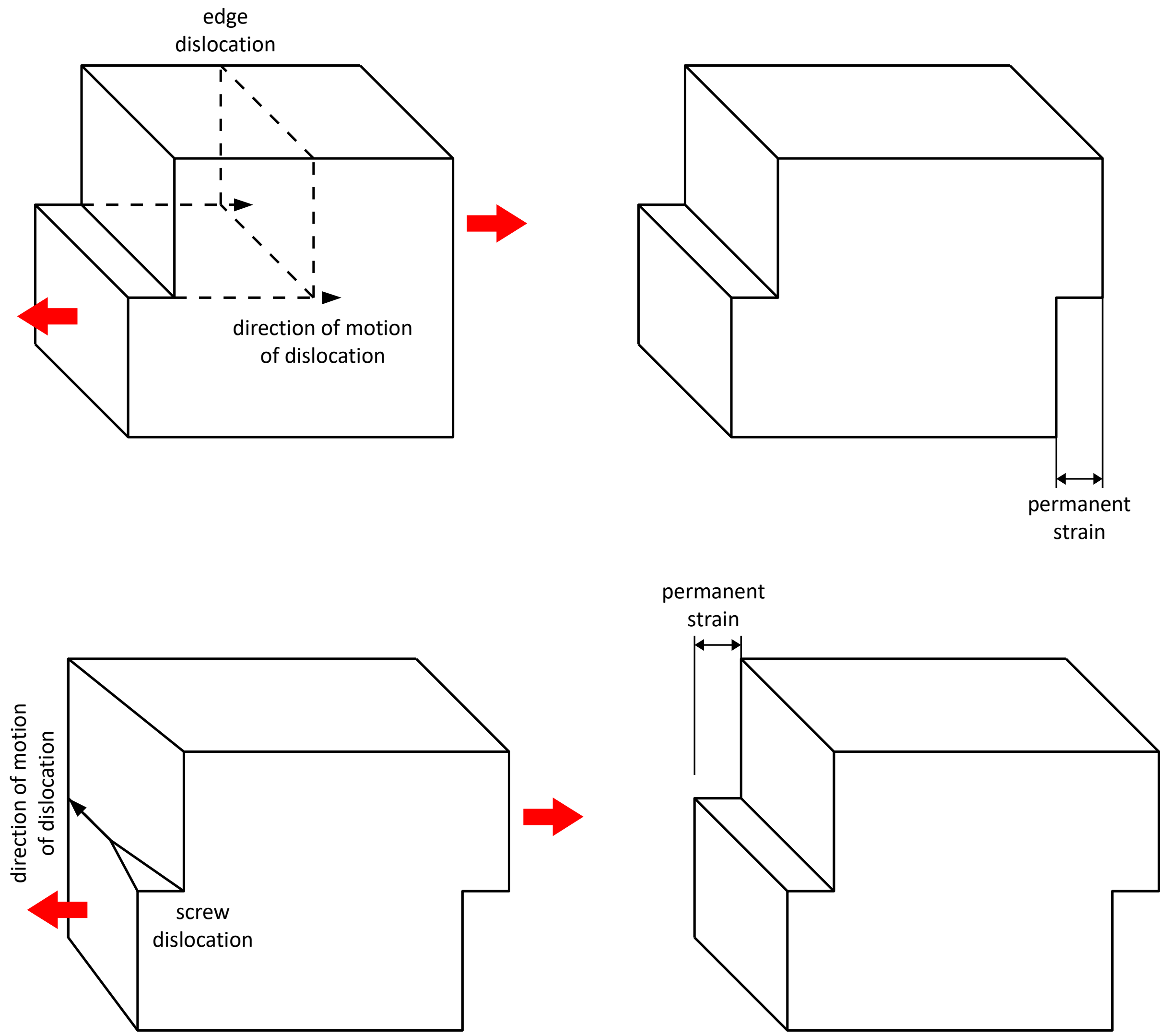


structure with
screw dislocation.



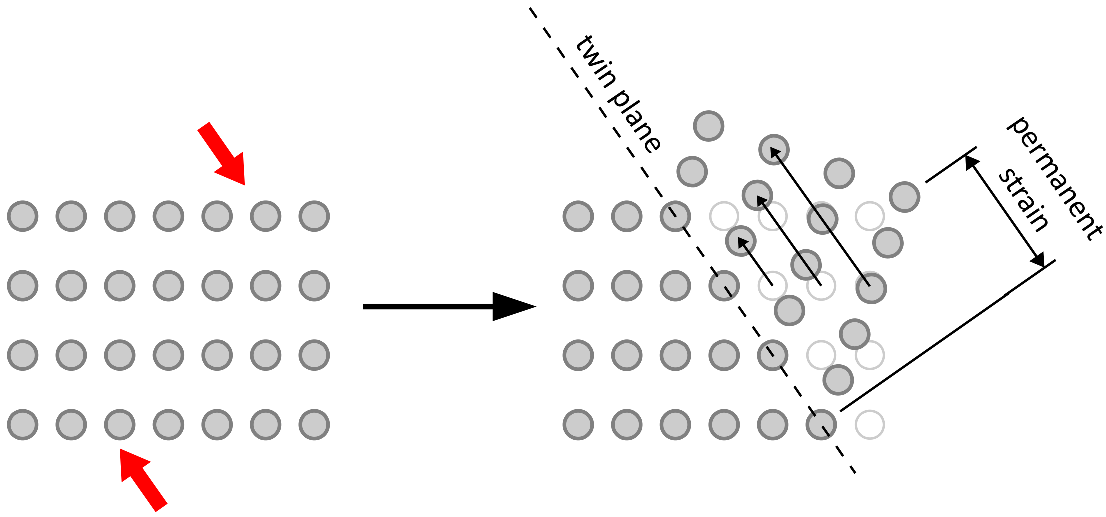
- **Planar defects**
  - **boundaries** – these are surfaces dividing the regions in a lattice, in which orientation of crystallographic structure are different. Depending on the magnitude of this misorientation, we distinguish low-angle and high-angle boundaries. Each of them has its specific structure. An important kind of high-angle boundary is a twin-boundary – plane between twin crystals.
  - **stacking faults** – disturbance in normally repeated sequence of atoms
  - **anti-phase domain boundaries** – they occur in lattices composed of at least two types of atoms, in which there are some regions (domains), in which a place in a perfect lattice is occupied by an improper type of atom..

The defects which are crucial in description of mechanisms of plastic deformation are **dislocations**. It was already mentioned that a slip of a whole plane of atoms would require a great amount of energy, and translation of only a pert of a lattice needs much more energy. If in a structure dislocations are present, then a slip of even a large part of a plane may be performed by a multiple translation of only small parts of it. We speak of **motion of dislocations** – it should not be understood literally. We speak of **motion of a certain geometric disturbance of geometry of a system**, not of a motion of a certain material object. It is the motion of dislocation that is a basic mechanism of plastic deformation. Th figures below depict how motion of dislocations may result with a permanent deformation.



It is clear that the magnitude of plastic strain as well as the resistance against plastic deformation (stiffness in plastic deformation domain, hardness) depends on the number of dislocations in a structure – the more dislocations, the greater permanent deformation may be enforced by a given load.

The second mechanism of plastic deformation that was mention is **twinning** – it is a phenomenon in which **crystallographic structure is changes due to external load in such a way, that part of this structure is transformed with respect to the second part by a symmetry transformation** – it may be inversion (point reflection) or reflection about a line or about a plane. In the latter case new grain becomes a symmetric reflection of the undeformed part about a certain plane, which is termed **twin plane**. As a result we obtain two crystals of the same composition and of the same lattice structure, yet orientation of those structures are different – we term them **twins**. Twin plane is inclined at the same angle to corresponding crystallographic plane in both crystals. Displacements of atoms is proportional to the distance of the atom from the twin plane.



Original equilibrium configuration with external load

New equilibrium configuration.

Everything what has been already told concerned in fact only monocrystalline structures of a single orintation of the lattice. Additional **phenomena at the boundary of grains** must be also accounted for – these concern grains of the same type but of different orientation or grains of different crystals. One should mention two phenomenta. The first one is twinning which was already mentiond. The second thing is that the motion of dislocations described above concerned the motion within a lattice of a single type of a crystal of fixed orientation. Motion of dislocations in not possible at grain boundaries – different phenomena occur there.

One should also notice, that basic mechanisms of plastic deformation allow us to make some conclusions concerning the magnitude of stress which is required to initiate plastic deformation but corresponding only to certain slip planes. **In real bodies, which are composed of multiple crystal lattices of different orientation – polycrystalline bodies** – the same macroscopically determined shear stress act on multiple planes of different orientation. **Depending on relative orientation of a certain plane with respect to the directioin of load, the same load mey produce slip or not**.

Atoms in a close neighborhood of the grain boundary occupy positions which are to some extent „intermediate", between positions predicted by perfect structure of one crystal and of the second one. As a result, **atoms which are close to the grain boundaries have higher potential energy and enforcing their motion requires much lower stress**. Such a **slip at the grain boundary** is another mechanism of plastic deformation.

Despite fact that plastic strain measured macroscopically as an averaged (along a certain gauge length) permanent strain seems to be distributed uniformly in a sample, in fact distribution of plastic strain is characterized by the presence of multiple **localizations**, regions of much higher intensity of plastic strain. Regions of that kind are e.g. **slip bands** – **narrow regions created by multiple dislocations passing along slip planes which are close one to another** – many of those dislocations are then locked or they block each other. Orientation of slip bands is strictly connected with slip planes determined by crystallographic structure of a grain.

Shear bands are different in their nature – **these are narrow regions of large distortional strains that occur only in case of very large plastic strain**. Contrary to the slip bands, shear bands are not contained within a single grain, but **they range through multiple grains and pass through the boundaries**. Orientation of shear bands is dependent not on internal structure of material but rather on orientation of external load – they are oriented in an oblique way to the axis of the greatest plastic deformation.

3. YIELD CONDITION

It was mentioned that plastic deformation occurs only when the eternal load is sufficiently large to initiate processes of motion of dislocations. In simple load cases – e.g. uniaxial tension, pure shear, pure bending, pure torsion – this condition is equivalent to the statement that appropriate stress – normal stress in case of tension and bending, shear stress in case of shear and torsion – exceeds certain limit value. In case of a complex stress state yield condition is written down in the following general form:

$$\boxed{f(\boldsymbol{\sigma}) = 0} \tag{3.1}$$

A function of a tensorial argument is of course a function of all its components of its argument in an assumed coordinate system.

$$f(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{31}, \sigma_{12}) = 0 \tag{3.2}$$

Function f is determined in such a way that **for processes, during which no plastic deformation occurs** $f(\boldsymbol{\sigma}) < 0$. This means that **0 is maximum value of the yield condition**.

We make no additional constraints on the form the yield condition so far. The fact, that yield condition may be expressed in terms of components of a stress tensor, is only an assumption – in fact, yield condition may be (and in some cases it is indeed) formulated as a **function of components of a strain tensor or even as a function of components of both of those tensors**. In case of elastic materials the relation between stress and strain is invertible, so without any loss of generality we may narrow our considerations to the yield conditions expressed as functions of the stress state.

Of course, stress state in a body is in general non-uniform, so the yield condition may be satisfied in one point and at the same time it may be not satisfied in another point. We may consider different levels of yielding:

- **yielding at a point** – **stress exceeds the limit value only locally** and only in that point permanent strains occur, while **in other points deformation is still elastic**. Yielding of edge fibers of a bent cross-section may be an example of such situation. The value of load parameter P which results in first yielding in any point of a system may be termed **limit elastic bearing capacity** and it will be denoted by a bar above the symbol of parameter, e.g. $\bar{P}$.

- **yielding of a cross-section** or more generally: **yielding in a region which do not lead to loss of stability of a system as a whole** – in case of bar structures, in which distribution of cross-sectional forces along bar's axis determines uniquely the distribution of stress tensor components in each cross-section, it may happen that the **yield condition is satisfied in all points of a certain cross-section** – in other cross-sections yielding does not occur or occurs only in part of it. In case of **statically determinate systems** yielding of a cross-section is equivalent to **adding a new degree of freedom** to a system – since it was statically determinate **this additional degree of freedom makes the system unstable**. In case of **statically indeterminate system yielding of a cross-section not necessarily leads to the loss of stability** of a system, however, **statical configuration of the system changes**. The value of load parameter P which results in first yielding of a cross-section as a whole may be termed **limit plastic bearing capacity** and it will be denoted by a double bar above the symbol of parameter, e.g. $\bar{\bar{P}}$. Partial yielding phenomena in bounded regions e.g. In membranes, plates, brick elements etc. have similar character.

- **Yielding of a system** and **loss of stability due to yielding** – sufficiently large magnitude of load or appropriately chosen character of load may lead to situation in which yielding occurs in such regions of a system that **certain parts of it have total freedom of deformation**. We say then that the system has lost its stability due to yielding and the value of load parameter P which caused it is termed **bearing capacity of the system** and it will be denoted with a star, e.g. P^*.

We will deal now only with **local** (at a point) **yield condition**, namely only with the function $f(\boldsymbol{\sigma})$ itself – its properties and porposed forms of that function.

In case of materials with no hardening yield condition remain constant during ongoing plastic deformation. In case of **materials with hardening** plastic strain results in increasing the yield stress what corresponds with a change in the yield condition – **after some plastic deformation a larger** (in the sense of yield condition) **stress state than previously is required** in order to continue the increment of plastic strains.

3.1 YIELD SURFACE

Yield condition (3.1) may be interpreted in terms of analytical geometry as an equation of a surface. It is not any true surface in three-dimensional space – it is an **abstract hypersurface in six-dimensional space of components of a stress tensor**. This hypersurface is termed yield surface. This interpretation is of almost no practical significance or use in the general case, however it becomes a great and useful tool for depicting characteristic properties of considered yield condition in case of isotropic materials

Just as in case of a yield condition for materials with no hardening, yield surface is fixed for them. **In case of materials exhibiting hardening yield surface changes its shape and size during ongoing plastic deformation.**

3.2 YIELD CONDITION FOR ISOTROPIC MATERIALS

It is obvious that yield condition for isotropic materials cannot depend in any extent on spatial orientation of direction of stresses but only on sole values of components of the stress tensor – those components may be determined uniquely with the use of eigenvalues of stress tensor, namely the principal stresses, or by its invariants. Generally, it can be stated that isotropic function of tensorial argument may be expressed in terms of invariants of its argument. So, for isotropic materials, function (3.1) may be expressed in one of the following forms:

- **function of principal strsses** $\sigma_1, \sigma_2, \sigma_3$:

$$\boxed{f(\sigma_1, \sigma_2, \sigma_3) = 0} \tag{3.3}$$

where $\sigma_1, \sigma_2, \sigma_3$ are principal stresses.

- **function of invariants of stress tensor** I_1, I_2, I_3 :

$$\boxed{f(I_1, I_2, I_3)} \tag{3.4}$$

where

$$I_1(\boldsymbol{\sigma}) = \mathrm{tr}(\boldsymbol{\sigma}) = \sigma_1 + \sigma_2 + \sigma_3 = \sigma_{11} + \sigma_{22} + \sigma_{33} \tag{3.5}$$

$$I_2(\boldsymbol{\sigma}) = \frac{1}{2}\left[\mathrm{tr}^2(\boldsymbol{\sigma}) - \mathrm{tr}(\boldsymbol{\sigma}^2)\right] = \begin{vmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{vmatrix} + \begin{vmatrix}\sigma_{11} & \sigma_{31}\\ \sigma_{31} & \sigma_{33}\end{vmatrix} + \begin{vmatrix}\sigma_{22} & \sigma_{23}\\ \sigma_{23} & \sigma_{33}\end{vmatrix} = \tag{3.6}$$

$$= \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} + \sigma_{11}\sigma_{22} - \sigma_{23}^2 - \sigma_{31}^2 - \sigma_{12}^2 \quad = \quad \sigma_2\sigma_3 + \sigma_3\sigma_1 + \sigma_1\sigma_2$$

$$I_3(\boldsymbol{\sigma}) = \det(\boldsymbol{\sigma}) = \frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma}^3) - \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma})\,\mathrm{tr}(\boldsymbol{\sigma}^2) + \frac{1}{6}\mathrm{tr}^3(\boldsymbol{\sigma}) = \begin{vmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ & \sigma_{22} & \sigma_{23}\\ \mathrm{sym} & & \sigma_{33}\end{vmatrix} = \tag{3.7}$$

$$= \sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{23}\sigma_{31}\sigma_{12} - \sigma_{11}\sigma_{23}^2 - \sigma_{22}\sigma_{31}^2 - \sigma_{33}\sigma_{12}^2 \quad = \quad \sigma_1\sigma_2\sigma_3$$

- **function of invariants of stress tensor and its deviator** I_1, J_2, J_3 :

$$\boxed{f(I_1, J_2, J_3)} \tag{3.8}$$

where

$$J_2 = -I_2(\mathbf{s}) = \frac{1}{2}\mathrm{tr}(\mathbf{s}^2) =$$

$$= \frac{1}{6}\left[(\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2 + (\sigma_{11}-\sigma_{22})^2\right] + \left(\sigma_{23}^2 + \sigma_{31}^2 + \sigma_{12}^2\right) = \tag{3.9}$$

$$= \frac{1}{6}\left[(\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2 + (\sigma_1-\sigma_2)^2\right]$$

$$J_3(\boldsymbol{\sigma}) = I_3(\mathbf{s}) = \frac{1}{3}\mathrm{tr}(\mathbf{s}^3) = \left(\sigma_1 - \frac{1}{3}I_1\right)\left(\sigma_2 - \frac{1}{3}I_1\right)\left(\sigma_3 - \frac{1}{3}I_1\right) \tag{3.10}$$

$$\mathbf{s} = \boldsymbol{\sigma} - \frac{1}{3}I_1\mathbf{1}$$

- **function of invariants** p, q, θ :

$$\boxed{f(p, q, \theta)} \tag{3.11}$$

where

hydrostatic stress (pressure): $$p = \frac{1}{3} I_1 \tag{3.12}$$

deviatoric stress: $$q = \sqrt{2 J_2} \tag{3.13}$$

Lode angle: $$\theta = \frac{1}{3} \arccos\left[\frac{3\sqrt{3}}{2} \frac{J_3}{J_2^{3/2}} \right] \tag{3.14}$$

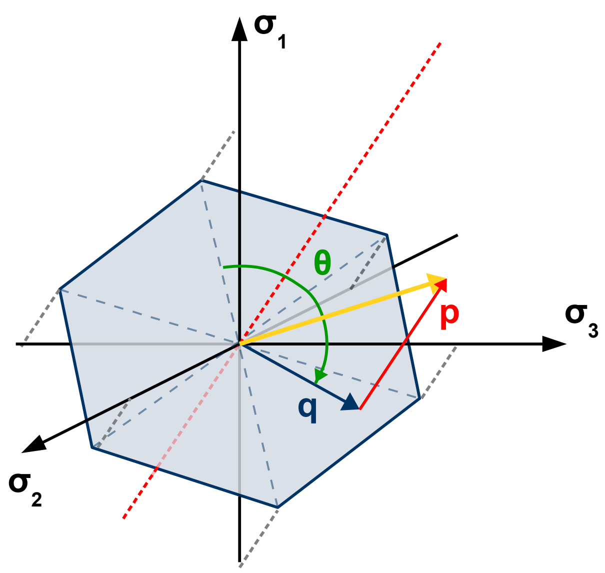
Invariants p, q, θ are closely related with cylindrical coordinates in the space of principal stresses. Cylinder axis of is then equally inclined to axes $\sigma_1, \sigma_2, \sigma_3$ - it corresponds with hydrostatic component of stress tensor. A component which is orthogonal to the hydrostatic one is deviatoric component, which lies in deviatoric (octahedral) plane, perpendicular to the axis of hydrostratic stress. An angle between deviatoric component and a projection of chosen principal stress on octahedral plane is the Lode angle.

Norm of hydrostatic component: $|\mathbf{A}_\sigma| = \frac{1}{\sqrt{3}} I_1 = \sqrt{3}\, p$

Norm of deviatoric component: $|\mathbf{D}_\sigma| = \sqrt{2 J_2} = q$

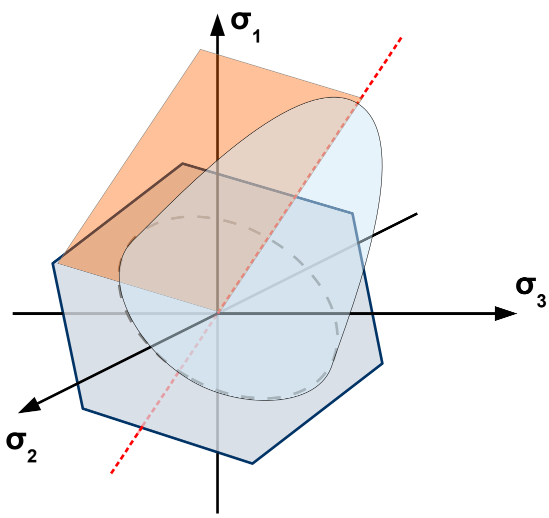
Lode angle: $\theta = \frac{1}{3} \arccos\left[\frac{3\sqrt{3}}{2} \frac{J_3}{J_2^{3/2}} \right]$

- uniaxial tension: $\theta = 0^\circ$
- uniaxial compression: $\theta = 60^\circ$
- pure shear: $\theta = 30^\circ$

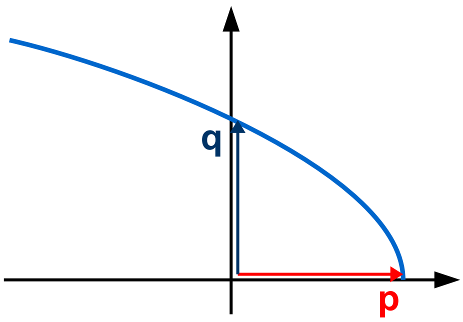


Also other combinations of the above invariants are used, yet there is no common way of denoting them.

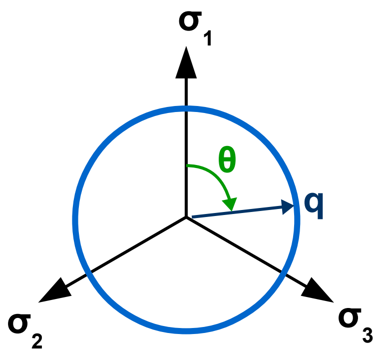
A common feature of all above propositions is that the yield condition is a function of three independent variables – considering them as coordinates in a certain three-dimensional space, yield surface becomes a **curved two-dimensional surface submerged in three-dimensional space**. It is convenient to describe properties of the yield surface with the use of **cross-sections of the yield surface**:



- **hydrostatic cross-section** (section with a plane containing axis of hydrostatic stress)
- **deviatoric cross-section** (section with a plane perpendicular to the axis of hydrostatic stress)



Cross-section with a hydrostatic plane.



Cross-section with a deviatoric plane.

In general, those cross-sections are not constant, however most commonly used yield conditions are independent of the Lode angle (surfaces of revolution with constant hydrostatic cross-section) and yield conditions independent of hydrostatic stress (cylindrical surfaces with constant deviatoric cross-section)

Chosen yield conditions for isotropic material will be described now. They are directly related with the material failure criteria or – more generally – with limit state criteria which were introduced in the science of strength of materials. A limit state may be in particular the yield point. Some classical limit state criteria concern limit states which differ considerably in their nature from the plastic yielding – e.g. Coulomb-Mohr criterion for soils and rocks – so not all propositions in strength of materials are used as yield conditions, despite the fact that description methods in both problems are in fact the same. A final criterion which decides if given proposition may be used as a yield condition is its agreement with results of experiments. A common approach is to formulate a yield condition in form of simple functions of invariants of stress tensor (e.g. polynomial or power functions), in which constant coefficients are determined according to the experimental data.

3.2.1 COULOMB-TRESCA-GUEST YIELD CONDITION

A yield condition which still emerges to be a good estimate in case of some materials and some load cases is a classical limit state condition of **Coulomb** (1776) **– Tresca** (1864) **– Guest** (1899), namely: **maximum shear stress criterion** (CTG):

$$\tau_{max} = \tau_0 = const. \tag{3.15}$$

where τ_0 is a **limit value of shear stress**. This criterion corresponds directly with the fact tha main mechanisms of plastic deformation are slip and twinning, which are initiated by a sufficiently large magnitude of shearing stress. However, this condition makes no account for orientation of that stress with respect to the crystal lattice and it is known that both slip and twinning may occur only in certain planes. It should be admitted that in case of isotropic materials shear stress even in a small region acts in fact on a great number of grains of different orientation of their structures, so in macroscopic description of that problem we make a kind of „averaging" and maximum value of shear stress becomes a good (experimentally proved) measure of a factor enforcing plastic strain.

For a given stress state extreme values of shear stress correspond with **shearing in planes which are perpendicular to the one of the principal stresses and the direction of shearing is inclined at angle 45° to the directions of the remaining two principal stresses**. Maximum shear stress may be calculated with the use of values of principal stresses $\sigma_1, \sigma_2, \sigma_3$ in the following way:

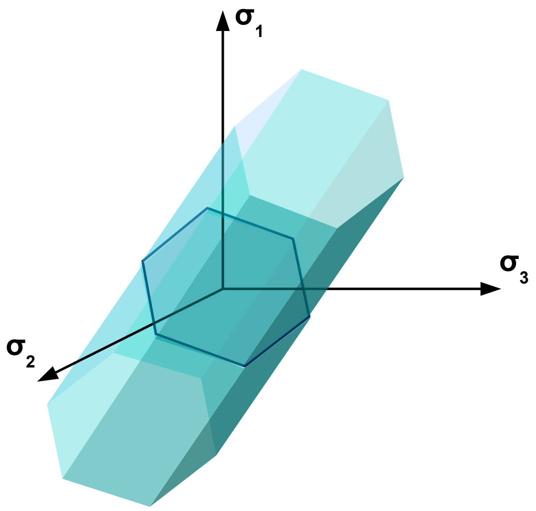
$$\tau_{max} = \max\left(\frac{|\sigma_2-\sigma_3|}{2}\ ;\ \frac{|\sigma_2-\sigma_3|}{2}\ ;\ \frac{|\sigma_2-\sigma_3|}{2}\right) \tag{3.16}$$

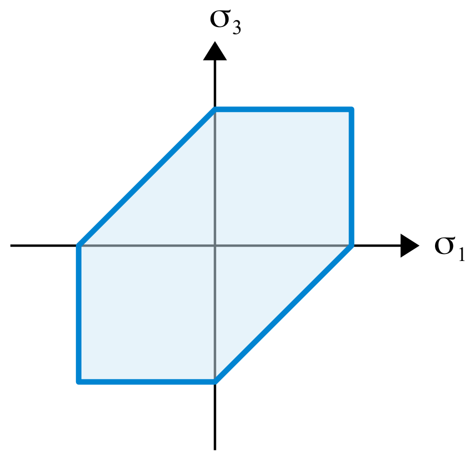
It is common to write down the yield condition in such a way that at the right hand side there is a **limit value of a normal stress** σ_0 (yield stress at uniaxial tension). CTG condition takes then the following form:

$$\boxed{\sigma_{eq} = \max\left(|\sigma_2-\sigma_3|\ ;\ |\sigma_2-\sigma_3|\ ;\ |\sigma_2-\sigma_3|\right) = \sigma_0} \tag{3.17}$$

Value σ_{eq} is termed **equivalent stress according to the CTG condition**.

Yield condition corresponding with the CTG condition is a surface of an infinitely long **prism the cross-section of which is a regular hexagon and the axis of which is inclined a the same angles to each axis of a system** $\sigma_1, \sigma_2, \sigma_3$:





3.2.2 MAXWELL-HUBER-MISES-HENKCKY YIELD CONDITION

Most commonly used yield condition is the **Maxwell-Huber-Mises-Hencky limit state condition**, namely the **distortion strain energy criterion** (MHMH). Maxwell (1856) and Mises (1913) have porposed the yield condition as a stress state dependent function of the following form:

$$\sigma_{eq}=\sqrt{\frac{1}{2}\left[(\sigma_{22}-\sigma_{33})^2+(\sigma_{11}-\sigma_{22})^2+(\sigma_{33}-\sigma_{11})^2+6(\sigma_{23}^2+\sigma_{31}^2+\sigma_{12}^2)\right]}=\sigma_0 \tag{3.18}$$

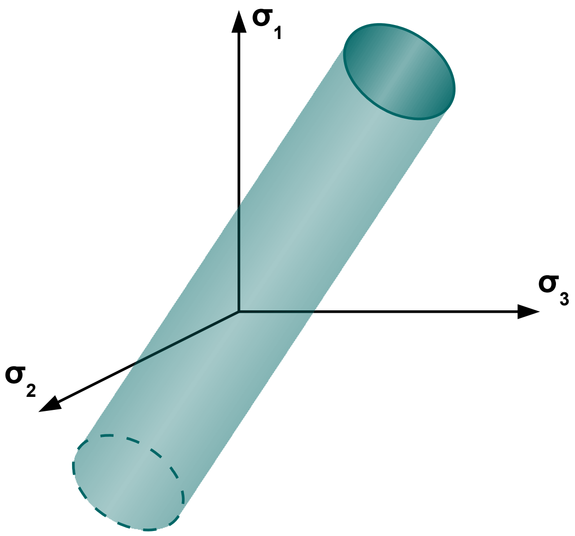
where σ_0 in a **limit normal stress** and σ_{eq} is termed **equivalent stress** according to the MHMH condition. Equivalent stress acoording to MHMH may be also expressed as:

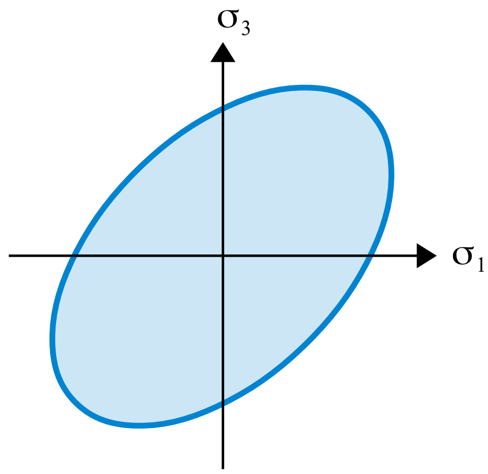
$$\sigma_{eq}=\sqrt{3J_2}=\sqrt{\frac{3}{2}}q=\sqrt{\frac{3}{2}\mathbf{s}\cdot\mathbf{s}}=\sqrt{\frac{3}{2}s_{ij}s_{ij}} \tag{3.19}$$

Similar results was obtained by Maksymilian Tytus Huber (1904) – he has posed a hypothesis that a measure of material effort in case of load cases resulting in hydrostatic compression is the **density of energy of distortional strain** ϕ_f :

$$\phi_f=\frac{1}{12G}\left[(\sigma_{22}-\sigma_{33})^2+(\sigma_{11}-\sigma_{22})^2+(\sigma_{33}-\sigma_{11})^2+6(\sigma_{23}^2+\sigma_{31}^2+\sigma_{12}^2)\right]=h \tag{3.20}$$

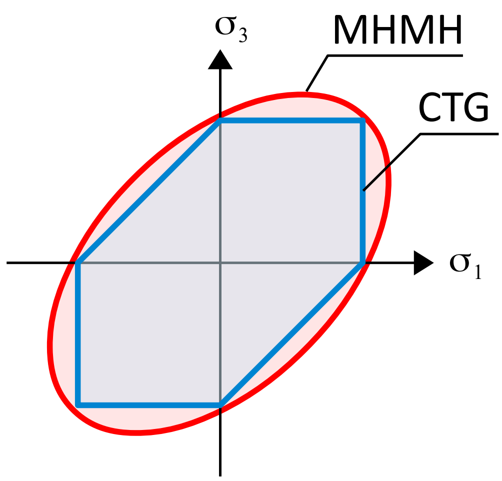
where G is the **Kirchhoff modulus** and h is a **limit value of the energy density of distortional strain**. Hencky (1924) has noticed energetic interpretation of proposition of Mises. Yield surface of the MHMH condition is an **axis-symmetric cylinder of an axis equally inclined to the axes** $\sigma_1,\sigma_2,\sigma_3$.





In case of plane problems $(\sigma_2=0)$ yield surface is an ellipse in plane (σ_1,σ_3) with its semi-axes inclined equally to axes σ_1,σ_3 .

If CTG and MHMH condition predict the same limit normal stress σ_0 , then **CTG condition is a „safer" one – equivalent stress according to MHMH condition is smaller or equal to the corresponding equivalent stress according to CTG condition. MHMH cylinder is circumscribed on CTG prism**.

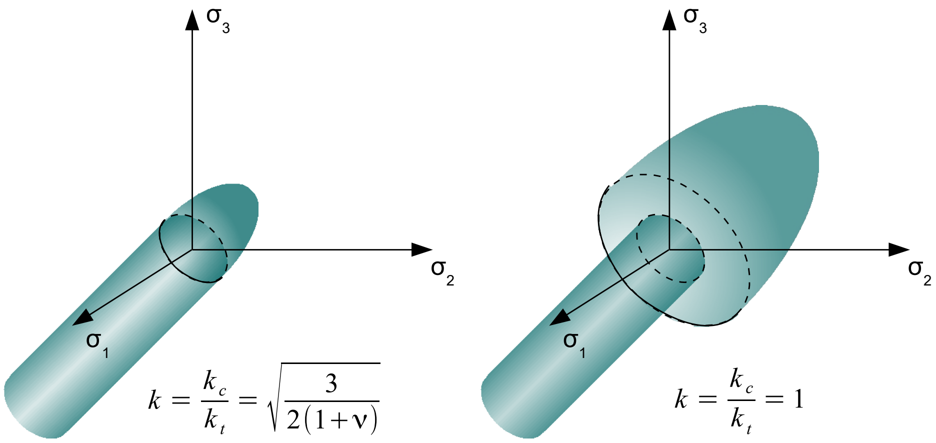


It is worth noting that proposition of Huber was qualitatively distinct from Mises condition. Huber assumed that ϕ_f is a measure of material effort only in case of compression, namely when $p<0$. For th states of tension $(p>0)$ Huber assumed that the measure of material effort is the total strain energy density (as Beltrami assumed for all states):

$$\begin{cases}\phi_f = h & \Leftrightarrow \quad p<0 \\ \phi_f + \phi_v = h & \Leftrightarrow \quad p \geqslant 0\end{cases} \qquad \text{where} \quad \phi_v = \frac{1}{18K}(\sigma_{11}+\sigma_{22}+\sigma_{33})^2 \qquad (3.21)$$

and K is the **bulk modulus**. Some authors have interpreted hypothesis of Huber in such a way, as if it required the same value on limit uniaxial stress both in case of tension and compression $k=k_c/k_t=1$ – this would be possible only if the limit value of energy density was different in those two cases, what would in turn result with discontinuity of the yield surface for $p=0$ - it would be of a shape of a „mushroom" (cylinder of smaller diameter stacked to a half of an ellipsoid of revolution of larger diameter). One should remember that Huber often referred to the limit state as "cracking" what may suggest application of his hypothesis to a brittle destruction – not t yielding of ductile materials. If he considered materials of different compressive strength k_c and tensile strength k_t, then – according to his hypothesis – ration between those strengths would be:

$$k = \frac{k_c}{k_t} = \sqrt{\frac{3}{2(1+\nu)}} \qquad (3.22)$$



There is no experimental evidence for that relation for brittle materials (e.g. rocks, concrete), for which the ratio between compressive and tensile strength $k \in (10\,;30)$, yet for common values of the Poisson's ratio for material of that kind $\nu \in (0{,}15\,;0{,}4)$ we would get, according to (3.22), $k \in (1{,}04\,;\,1{,}14)$. It is worth noting that such values of k are typical for ductile materials having similar values of ν and exhibiting the SD effect – for such materials Huber hypothesis may provide a simple yield condition which has also a convincing physical interpretation.

3.2.3 BURZYŃSKI YIELD CONDITION

Schleicher (1926) and Mises (1928) suggested that right hand side in the MHMH yield condition may be in general a function of hydrostatic stress – limit value of the equivalent stress would be dependent on actual value of hydrostatic stress what was already accounted for in some extent in the original formulation of Huber's hypothesis. Huber's idea has no experimental evidence while Schleicher and Mises did not propose any form of the considered function. It was the pupil of Huber, Włodzimierz Burzyński, who independently of Schleicher and Mises proposed a yield condition of that form, which is mathematically simple and has great capabilities concerning description of yield surfaces of different geometrical properties – it is clearly motivated physically, yet it does not refer to any mechanisms of yielding. In the original formulation **Burzyński yield condition** is as follows:

$$\phi_f + \left(A + \frac{B}{p}\right)\phi_v = h \tag{3.23}$$

where ϕ_f, ϕ_v are **densities of strain energy of distortional and volumetric deformation respectively**, p is the **hydrostatic stress**, h is the **limit value of the strain energy density** and A, B are material constants.

Burzyński yield condition may be written in the following form:

$$\boxed{\begin{aligned}&(\sigma_{11}^2+\sigma_{22}^2+\sigma_{33}^2) - 2\left(\frac{k_c k_r}{2k_s^2}-1\right)(\sigma_{22}\sigma_{33}+\sigma_{33}\sigma_{11}+\sigma_{11}\sigma_{22}) + \\ &+ \frac{k_c k_r}{k_s^2}(\sigma_{23}^2+\sigma_{31}^2+\sigma_{12}^2) + (k_c-k_r)(\sigma_{11}+\sigma_{22}+\sigma_{33})-k_c k_r = 0\end{aligned}} \tag{3.24}$$

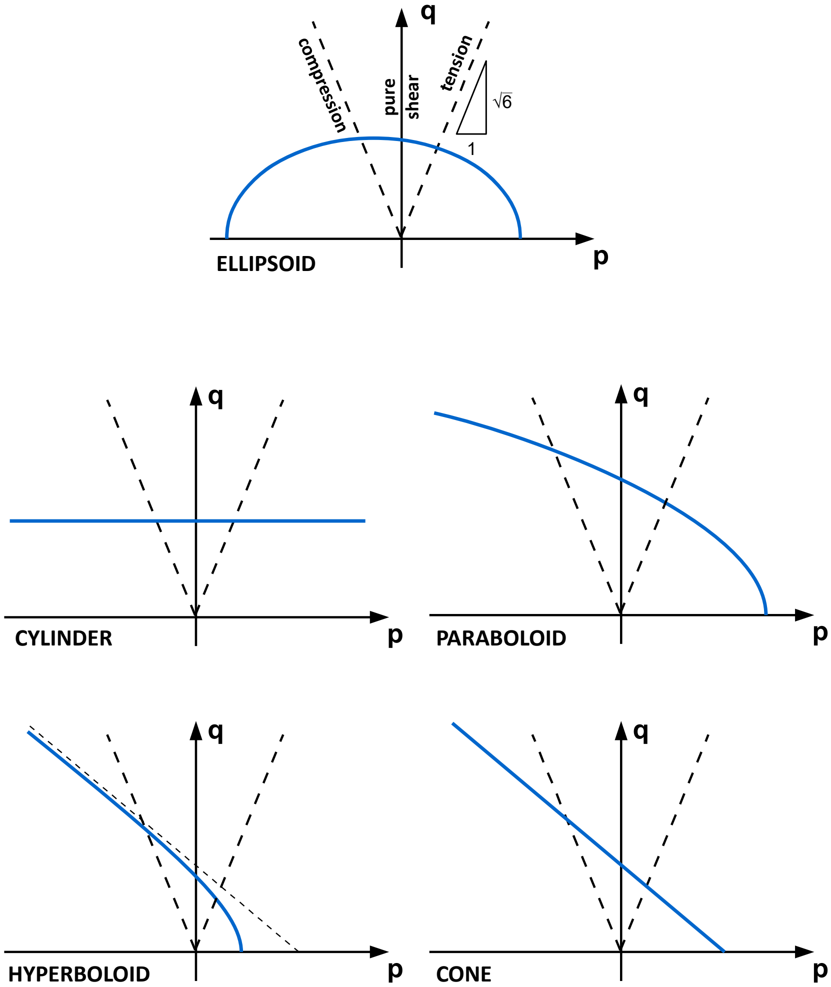
where k_c is the limit uniaxial compressive stress, k_t is the limit uniaxial tensile stress, k_s is the limit shear stress. All those parameters may be determined in relatively simple strength tests. Burzyński's criterion accounts for **asymmetric elastic range, this is, different value of limit uniaxial stress in tension and in compression**.

Equation (3.24) is a general equation of a **quadric surface** – a second degree surface. Depending on mutual relations between values of parameters k_c, k_t, k_s the character of the yield surface may be different. In each case, however, **it is a surface of revolution – its axis of symmetry is** $\sigma_1 = \sigma_2 = \sigma_3$. Let's introduce parameters:

$$\mu = \frac{k_c k_r}{2k_s^2} - 1\,, \quad k = \frac{k_c}{k_t}\,, \quad k_{s,h} = \sqrt{\frac{k_c k_t}{3}}\,, \quad k_{s,min} = \frac{2k_c k_t}{\sqrt{3}(k_c + k_t)}$$

Yield surface corresponding with the Burzyński yield condition is:

- **ellipsoid of revolution** for $\mu < 0{,}5$
- **paraboloid of revolution** for $\mu = 0{,}5\ ,\quad k \neq 1$
- **cylinder** for $\mu = 0{,}5\ ,\quad k = 1$
- **two-sheet hyperboloid of revolution** for $\mu > 0{,}5\ ,\quad k_s > k_{s,\,min}$
- **cone** for $\mu > 0{,}5\ ,\quad k_s = k_{s,\,min}$

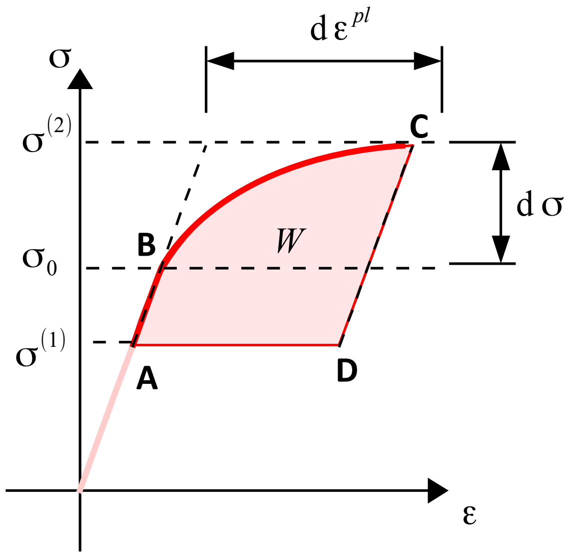


In case of two-sheet hyperboloid of revolution we consider only this sheet, inside which the zero stress state is. The case of one-sheet hyperboloid is rejected since it is against experimental evidence – shear strength would be the smallest for certain fixed value of hhydrostatic stress, and it would increase with any change in the value of hydrostatic stress (independently of its sign) – up to infinity. It is worth to mention two special cases:

- the case of **cylinder** is equivalent to the **MHMH yield condition**,
- the case of **cone** is equivalent with a later proposition of a yield condition by **Drucker** and **Prager** (1952).

3.3 DRUCKER'S POSTULATES

Drucker introduced a **notion of material stability** and formulated two conditions of that stability. Let's consider that certain **material is loaded** (it corresponds with stress state $\boldsymbol{\sigma}^{(1)}$) and then and **additional load** is applied so that final stress state is described by a stress tensor $\boldsymbol{\sigma}^{(2)}$. Then the material is **unloaded** until stress state $\boldsymbol{\sigma}^{(1)}$ is obtained again. Such a process is termed a **stress cycle**.



Material is **considered** stable is:

1. Work performed by an additional load is positive.
2. Total work performed in a cycle is non-negative.

Wrok performed in a cycle is equal:

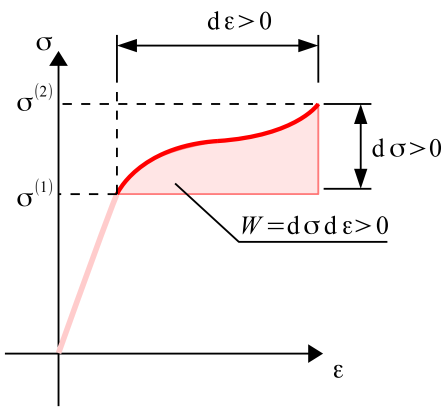
$$W = \int_{ABCD} (\boldsymbol{\sigma}-\boldsymbol{\sigma}^{(1)})(d\boldsymbol{\varepsilon}^{e}+d\boldsymbol{\varepsilon}^{pl}) = \int_{BC} (\boldsymbol{\sigma}-\boldsymbol{\sigma}^{(1)})d\boldsymbol{\varepsilon}^{pl} \tag{3.25}$$

Condition that the total work in a cycle is non-negative is equivalent to

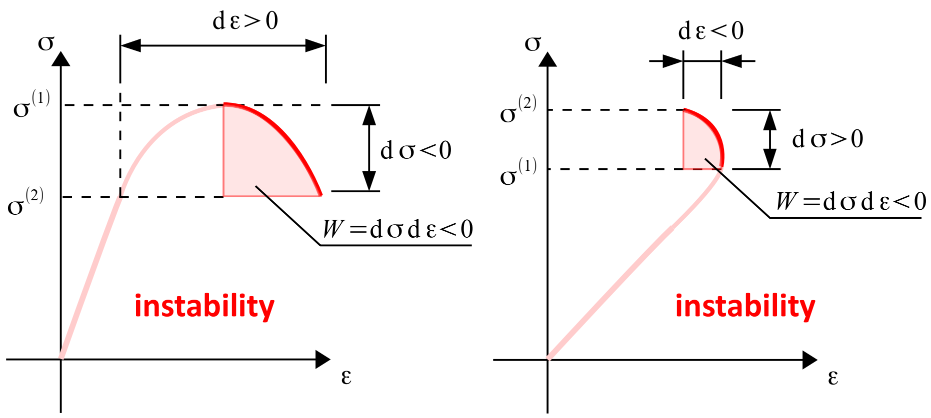
$$\boxed{(\sigma_{ij}-\sigma_{ij}^{(1)})d\varepsilon_{ij}^{pl} \geqslant 0} \tag{3.26}$$

In case of an infinitely small increment of stress we may write:

$$\boxed{d\sigma_{ij} d\varepsilon_{ij}^{pl} \geqslant 0} \tag{3.27}$$



Occurence of the lower yield stress is related with material instability, namely, with a situation in which some additional permanent strain occurs even when stress decreases.

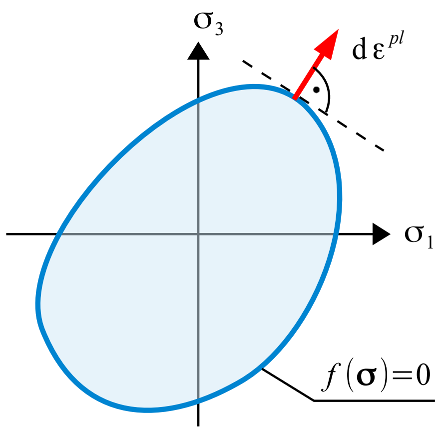


From the Drucker's postulate is follows that:

- **plastic strain increment tensor must be given by a tensor which is orthogonal to the yield surface**:

$$d\varepsilon_{ij}^{pl} = d\lambda \frac{\partial f}{\partial \sigma_{ij}} \qquad (3.28)$$

  where $d\lambda$ is a certain coefficient dependent on material and on the strain path (history of deformation). This relation is called **normality rule**, what corresponds with so called associated flow rule, which will be described later.



- **Yield surface** in each point must be **convex**.

4. CHOSEN PHENOMENA OCCURING DURING PLASTIC DEFORMATION

4.1 PROCESS OF PLASTIC DEFORMATION

In the theory of plasticity we distinguish two types of processes:

- **active processes** – or processes of **loading** – these are **irreversible processes due to dissipation of energy in the process of creation and development of plastic strain**. During an active process **both elastic strain tensor and plastic strain tensor vary**. Simplifying the problem, it may be stated that an active process is a process in which yield condition is satisfied and applied load results in increment of plastic strain. For active processes:

  - yield condition is satisfied:

$$\boxed{f(\boldsymbol{\sigma}) = 0} \qquad (3.29)$$

  - increment of the yield condition corresponding to increment of stress on a yield surface is non-negative:

$$\boxed{\frac{\partial f}{\partial \boldsymbol{\sigma}} \cdot \mathrm{d}\boldsymbol{\sigma} \geqslant 0} \qquad (3.30)$$

- **passive processes** – or processes of **unloading** – these are processes during which **no dissipation of energy occurs**. Simplifying the problem, it can be stated that these are elastic processes – loading in elastic deformation domain and unloading – namely, the processes in which **only elastic strain tensor varies**. For passive processes:

  - yield condition is not satisfied:

$$\boxed{f(\boldsymbol{\sigma}) < 0} \qquad (3.31)$$

  - **or** yield condition is satisfied but increment of the yield condition corresponding to increment of stress on a yield surface is negative:

$$\boxed{f(\boldsymbol{\sigma}) = 0 \quad \wedge \quad \frac{\partial f}{\partial \boldsymbol{\sigma}} \cdot \mathrm{d}\boldsymbol{\sigma} < 0} \qquad (3.32)$$

This second case corresponds with a situation in which yield condition is satisfied but increment of stress would result in unloading. One should mention now that in case of **incremental models of plasticity with associated flow rule**, derivative of the yield condition with respect to the stress tensor is proportional to the plastic strain increment tensor. In such a case a dot product of the derivative and of stress increment tensor may be interpreted as **proportional to a work performed by an increment of stress on corresponding increment of plastic strain**.

4.2 ELASTIC CHARACTER OF UNLOADING

It was already mentioned that the **unloading process has an elastic character**. Indeed, the only strains which may change due to unloading are elastic strains which have developed simultaneously with plastic deformation. **Initiation of additional slips would require an additional energy, so due to decrease in external load plastic strain remains (it is permanent) and only elastic strain decreases**.

It is an interesting fact that elastic characteristics of initial loading (in elastic deformation domain) as well as of later unloading are almost identical (the same Young's modulus E). Plastic strain results in permanent changes in internal structure of material which, determines its mechanical properties (i.e. elastic constants). In fact, **total volume of regions in which dislocations occur is very small** compared to the total volume of a monocrystalline grain. **Beyond those small regions the structure of a crystal remains untouched by the load resulting in plastic deformation** – this structure is deformed in an elastic way and after removing the load it recovers its original configuration. For this reason such mechanical properties as e.g. Elastic constants do not change their values during plastic deformation, unless the strain is sufficiently large.

4.3 INCOMPRESSIBLE CHARACTER OF PLASTIC DEFORMATION AND INDEPENDENCE OF YIELD CONDITION OF HYDROSTATIC PRESSURE

Many experiments indicate that during the plastic deformation **volume of material** – to a first approximation – **does not change**, namely that **plastic volumetric strain is approximately zero**. It is consistent with known mechanisms of plastic deformation, which base on **permanent distortional strains** – slip and twinning – which are caused by shear stresses. Indeed, those mechanisms results in change of spatial distribution of atoms in a lattice in such a way that there is no significant change in the volume of the region occupied by those atoms.

In case of materials in which it is possible to observe plastic „flow”, the process of plastic deformation is similar to the motion of an incompressible fluid – this is the reason for using the word „flow”. A phenomenon which is to some extent related with this is i**ndependence of yield condition of hydrostatic stress**. It can be noticed that in classical yield conditions of CTG and MHMH adding any value of a hydrostatic stress do not change the yield condition or the value of equivalent stress.

Both those phenomena are related within **incremental models of plasticity with associated flow rule**, in which plastic strain increment tensor is determined as a gradient of a plastic potential, which in turn in equal to the yield condition. If the yield condition is independent of hydrostatic stress, then the derivative of **plastic potential** $\Psi(\boldsymbol{\sigma})$ **(given by the same function as the yield condition** $f(\boldsymbol{\sigma})$) with respect to this component is equal 0. This derivative – according to the assumed constitutive law – is a measure of volumetric strain:

$$\mathrm{d}\varepsilon_{ij}^{pl} = \mathrm{d}\lambda \frac{\partial \Psi}{\partial \sigma_{ij}} \,, \quad \Psi(\boldsymbol{\sigma}) = f(\boldsymbol{\sigma})$$

$$\frac{\partial f}{\partial \sigma_{kk}} = 0 \quad \Rightarrow \quad \mathrm{d}\varepsilon_{kk}^{pl} = 0$$

It must be emphasized that this observation is only approximately true – in fact, precise measurements indicate both small changes in the volume of plastically deformed materials as well as dependency of the yield condition on hydrostatic component, what is accounted for in yield conditions of Burzyński, Schleicher-Mises, Drucker-Prager, etc.

4.4 HARDENING

We already know that a basic mechanism of plastic deformation is the motion of dislocations. Since in a certain range of plastic deformation those processes require greater external load (hardening) there must exist some obstacles in motion of dislocations. We may indicate two such obstacles.

The first one is a **locally large density of dislocations**. Each dislocation corresponds with a field of stress. Fields corresponding to few dislocations may interact in such a way that motion of dislocation requires smaller external load („dislocations attract each other") or greated external load („dislocations repulse each other"). A highly plastically strained material has many dislocations. If **dislocations are concentrated in a small region and interact one with another in such a way that they make the motion more difficult** (e.g. Dislocations in the same plane and of the same Burgers vector) the phenomenon of **hardening** occurs, namely, further increments of plastic strain would require greater load.

It may happen also that **in a place of high conscentration of dislocation of different types, they may mutually ease their motion** and further increments of plastic strain would require smaller load – we speak then of **softening** phenomenon.

Another mechanism of hardening is **locking the dislocations on grain boundaries**. The mechanisms of propagation of dislocations concern the motion only in a perfect lattice. Some obstacles may be due to presence of other defects in that structure or by a boundary of a region of fixed structure. Distribution of atoms in the boundary layers and in close neighborhood of it is highly disturbed and motion of dislocation in such a region is impossible. For this reason polycrystalline solids exhibit stronger hardening than monocrystalline solids and – similarly – fine graded materials (small grains – larger total are of boundary surfaces) exhibit stronger hardening then materials with large grain sizes. Reducing the average size of grains is one of the methods of strengthening of material (increasing the yield stress).

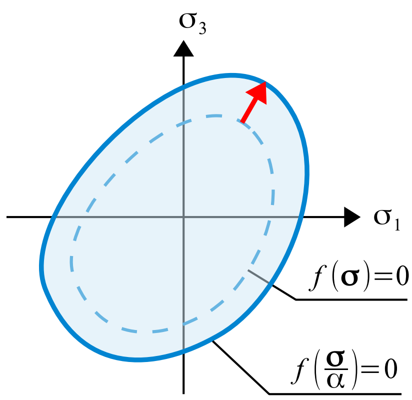
In simple one-dimensional problems, hardening may be describet easily with approximate empirical formulae. For example:

• **Hollomon** formula	$\sigma = K\left(\varepsilon^{pl}\right)^n$	(3.33)
• **Ludwik** formula	$\sigma = \sigma_0 + K\left(\varepsilon^{pl}\right)^n$	(3.34)
• **Ramberg-Osgood** formula	$\varepsilon = \frac{\sigma}{E} + K\left(\frac{\sigma}{E}\right)^n$	(3.35)

In case of formulae of Hollomon and Ludwik only plastic strain is accounted for – elastic strain is considered negligibly small and those formulae correspond in face with a rigid-plastic solid. Parameter n is termed the **strain hardening exponent**, and it is considered a material constant – it is a basic parameter describing hardening of a material. Material constant K is a measure of material's stiffness in plastic deformation domain. This description is a very simplified one.

Problem of hardening is in fact much more complex. It concerns the problem of **evolution of a yield condition** (and also of a **yield surface**) **during ongoing plastic deformation.** In case of isotropic materials following models of hardening are commonly accepted:

- **Isotropic hardening** assume that the **yield surface do not change its shape but only expands uniformly** (in an isotropic way) **in the space of stresses.** If, for instance, hardening during tension in plastic deformation domain resulted in increase in limit tensile stress α times, then in other limit states (e.g. Compression, shear), corresponding limit stress also increases α times. In such a situation the change of a yield condition after hardening may be expressed as follows:



$$f(\boldsymbol{\sigma}) \quad \rightarrow \quad f\left(\frac{\boldsymbol{\sigma}}{\alpha}\right) = 0\,, \quad \alpha \geqslant 1 \tag{3.36}$$

If the yield condition is written down in the form:

$$f(\boldsymbol{\sigma}) = c$$

where c is a certain limit value, then for materials with no hardening c will be constant. For materials with isotropic hardening c may be a function accounting for a past deformation (strain path). It is postulated that the strain path was accounted for by a single scalar parameter.

Two porpositions are commonly accepted:

- **Taylor-Quinney isotropic hardening model** – it assumes that hardening depends on the work of stress on plastic strain W_{pl} :

$$c = c(W_{pl}) \quad \text{, where} \quad W_{pl} = \int \sigma_{ij} \, d\varepsilon_{ij}^{pl} \tag{3.37}$$

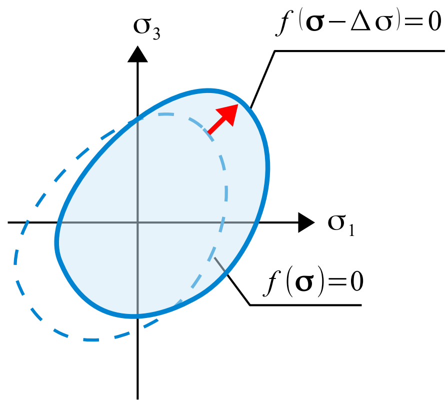
  Integral path corresponds with true history of stress and strain change processes.

- **Odquist-Hill isotropic hardening model** – it assumes that hardening depends on total length of a plastic strain path d_{pl} :

$$c = c(d_{pl}) \quad \text{, where} \quad d_{pl} = \int \sqrt{d\varepsilon_{ij}^{pl} \, d\varepsilon_{ij}^{pl}} \tag{3.38}$$

  Integral path corresponds with true strain history. Original proposition of Odquist concerned the length of total strain.

- **Kinematic hardening** assumes thet the **yield surface does not change neither its shape nor dimensions but it translates in the space of stress in the same manner as rigid body moves**. A property of that model is that if, for instance, hardening during tension in plastic deformation domain resulted in increase of the limit tensile stress with $\Delta\boldsymbol{\sigma}$, then in an opposite stress state – which is compression – corresponding limit stress decreases with $\Delta\boldsymbol{\sigma}$. In such a situation the change of a yield condition after hardening may be expressed as follows:



$$f(\boldsymbol{\sigma}) \quad \rightarrow \quad f(\boldsymbol{\sigma}-\Delta\boldsymbol{\sigma}) = 0 \tag{3.39}$$

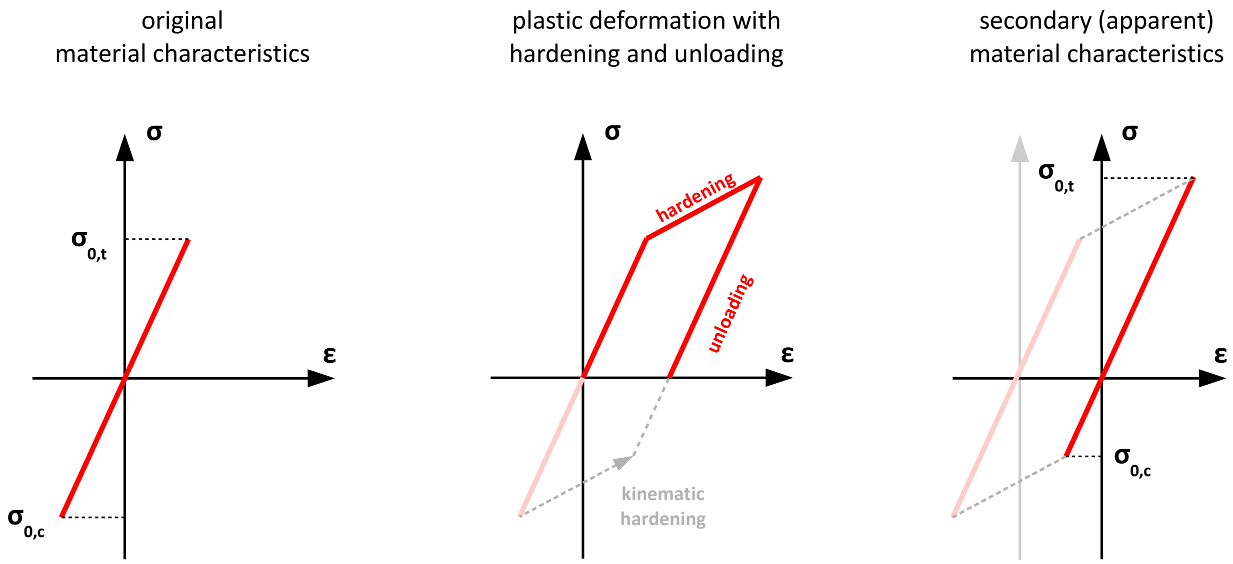
- **Mixed hardening** is a composition of kinematic and isotropic hardening models – **yield surface may change its dimensions and it may move in the space of stress:**

$$f(\boldsymbol{\sigma}) \quad \rightarrow \quad f\left(\frac{\boldsymbol{\sigma}}{\alpha}-\Delta\boldsymbol{\sigma}\right) = 0 \tag{3.40}$$

- **Anisotropic hardening** assumes that the yield surface may change its shape in any direction in the stress space in general in a different way.

4.5 BAUSCHINGER EFFECT

When a material is deformed plastically and then unloaded, inside its structure in microscopic scale some **residual stresses** remain – it is due to inhomogeneity of internal structure of material what results with different stress states in grains of different orientation of their crystallographic structure. Those residual stresses constitute a system in equilibrium. If the material is then loaded in a different manner then originally, those residual stresses influence the initiation of mechanisms of plastic deformation what results in the fact that plastic strain occurs for a different value of the stress than in the original process of loading. Such an effect is called the **Bauschinger effect**. One of the simple ways of accounting for the Bauschinger effect is the use of kinematic hardening model as it is depicted in the figure below:



4.6 INDUCED ANISOTROPY

It was mentioned that plastic deformation changes the internal structure of a monocrystalline grain only in small extent – total volume of regions influenced directly by dislocations is small compared to the total volume of a grain. On the other hand, in a polycrystalline solid a strong plastic deformation may result in change on orientation of all grains. It is obvious that e.g. uniaxial tension is characterized by a **specific direction** – **crystal grains will rotate and elongate along the direction af maximum stretching strain**. Similar phenomena occur also in complex stress states – the target orientation of the crystallographic structure depends on the strain path. In case of polycrystalline solids this target orientation is in general different for each grain as the grain's motion is constrained. As a result, **internal structure of a material changes in such a way, that macroscopic mechanical properties of a material which was initially isotropic develop the properties of anisotropy induced by a plastic deformation** – such properties as stiffness (also in elastic deformation domain) or yield stress take different values, depending on orientation of a load with respect to the characteristic directions of the induced anisotropy.

Even for materials which were initially a complex of grains of totally random distribution of orientation of crystallographic structure of grains – we say, that the structure do not exhibit any noticeable texture – one may observe that **during ongoing plastic deformation the material's** **texture** **develops, namely, that a certain orientation of crystallographic structure of grains becomes gradually more common**.

5. YIELD IN A POINT

5.1 CONSTITUTIVE RELATIONS IN ELASTIC-PLASTIC DEFORMATION

Kinematics and equilibrium equations derived in the theory of elasticity are generally common for all branches of mechanics of deformable solids. Key element in a proper formulation of mathematical theory of plasticity is the choice of **appropriate constitutive relations between the stress state and strain state which accounts for distinct character of elastic and plastic strains**. Three approaches are common:

- **simplified models** – these are in face non-linear elastic models, namely the models of an elastic solid for which non-linear constitutive relations are assumed in such a way, that they fit well the $\sigma-\varepsilon$ curve determined experimentally. Those models assume that the **relation between stress and strain is a one-to-one relation, an invertible one**. Their applicability is **constrained only to the processes of monotonic load – they do not account for elastic character of passive processes**. Models of hardening by Ludwik or Ramberg-Osgood are examples of such models. They are usually formulated only for simple load ceses, e.g. uniaxial state.

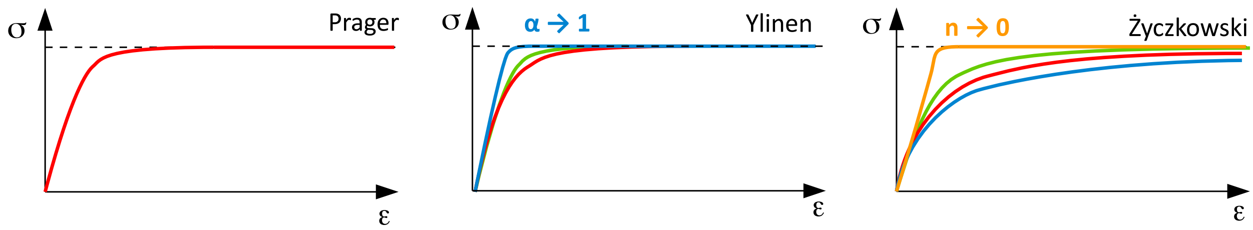
- **Deformation theories of plasticity** or **total strain theories** – they assume, that **the stress state determine the strain state (both elastic strain and plastic strain) in a unique way**. An important flaw of these theories is that a single stress state may correspond with different strain states which were obtained by a different strain path. In particular these models do not describe in a correct way the processes on unloading and subsequent loading.

- **Plastic flow theories** or **incremental theories of plasticity** – they assume, that **the stress state determines in a unique a tensor of increment of plastic strain**. Total plastic strain must be determined by summing up (integration) of known increments.

5.1.1 MODELS OF ASYMPTOTIC PLASTICITY

Some examples of simple models of plasticity are, so called, **models of asymptotic plasticity**:

- **Prager**: $$\varepsilon = \frac{\sigma_0}{E}\operatorname{arctgh}\left(\frac{\sigma}{\sigma_0}\right) \tag{3.41}$$
- **Ylinen**: $$\varepsilon = \frac{1}{E}\left[\alpha\sigma-(1-\alpha)\sigma_0\ln\left(1-\frac{\sigma}{\sigma_0}\right)\right], \quad \alpha\in\langle 0\,;\,1\rangle \tag{3.42}$$
- **Życzkowski**: $$\varepsilon = \frac{\sigma}{E}\left(1-\frac{\sigma}{\sigma_0}\right)^{-n}, \quad n\geqslant 0 \tag{3.43}$$



5.1.2. TOTAL STRAIN THEORIES OF PLASTICITY

5.1.2.1 THEORY OF HENCKY AND ILYUSHIN

Most commonly used total strain theory is the one due to **Hencky and Ilyushin**. According to a commonly accepted assumption that the yield condition does not depend on hydrostatic stress and that permanent volumetric deformation is approximately zero, the relation between stress and strain is formulated separately for deviatoric and hydrostatic components. Let's introduce:

- **deviator of a total strain tensor**: $$\mathbf{e} = \boldsymbol{\varepsilon}-\frac{1}{3}\operatorname{tr}(\boldsymbol{\varepsilon})\cdot\mathbf{1} \tag{3.44}$$
- **deviator of stress tensor**: $$\mathbf{s} = \boldsymbol{\sigma}-\frac{1}{3}\operatorname{tr}(\boldsymbol{\sigma})\cdot\mathbf{1} \tag{3.45}$$

Constitutive relations of the Hencky – Ilyushin theory are as follows:

$$\begin{cases} e_{ij} = \left(\phi+\dfrac{1}{2G}\right)s_{ij} \\ \varepsilon_{kk} = \dfrac{1-2\nu}{E}\sigma_{kk} \end{cases} \tag{3.46}$$

It is easy to separate the elastic and plastic components.

Elastic deformation:

$$\begin{cases} e_{ij}^{e} = \dfrac{s_{ij}}{2G} \\ \varepsilon_{kk}^{e} = \dfrac{1-2\nu}{E}\sigma_{kk} \end{cases} \tag{3.47}$$

Plastic deformation:

$$\varepsilon_{ij}^{pl} = e_{ij}^{pl} = \phi\, s_{ij} \qquad \begin{cases} \phi > 0 & \text{for active processes} \\ \phi = 0 & \text{for passive processes} \end{cases} \tag{3.48}$$

Constitutive relations for elastic components are the same as in case of a linear-elastic solid. Plastic component has only a deviatoric component – its hydrostatic component $\varepsilon_{kk}^{pl} = 0$.

The key flaw of this theory is **lack of possibility of correct description of processes of multiple loading and unloading**. Let's assume that a certain plastic (permanent) strain state was obtained due to original load. We consider then an unloading process in which elastic strain decreases and plastic strain remains constant. Then we load the body again but in a different way than originally, namely deviator s_{ij} is different when the yield condition is again satisfied. It requires the plastic strain state to be changed now according to relation $e_{ij}^{pl} = \phi\, s_{ij}$ which is inconsistent with previous plastic strain state. If the model is used for description of monotonic load processes, it provides sufficiently accurate predictions.

5.1.3 INCREMENTAL THEORIES OF PLASTICITY

An alternative approach are the so called incremental theories of plasticity, which are sometimes termed also **plastic flow theories.** This term refers to the fluid dynamics in which one of the fundamental equations has an analogous form as the assumed constitutive relation. Assumption of lack of volumetric plastic strain corresponds with the mechanics of incompressible fluids. This analogy in not completely correct, so it is more appropriate to use the term of **incremental model**. Constitutive relations in incremental models may be written down in the following form:

$$\boxed{\mathrm{d}\varepsilon_{ij}^{pl} = \mathrm{d}\lambda \frac{\partial \Psi}{\partial \sigma_{ij}}} \tag{3.49}$$

where $\mathrm{d}\varepsilon_{ij}^{pl}$ are the components of the plastic strain increment tensor, $\Psi(\boldsymbol{\sigma})$ is termed plastic potential, and $\mathrm{d}\lambda$ is a parameter accounting for mechanical properties of a material as well as the history of deformation – in particular, this parameter may account also for hardening.

Most common approach in formulating the incremental constitutive relations is assumption of the so called **associated flow rule**, namely an assumption that **plastic potential is expressed by the same function as the yield condition**:

$$\boxed{\Psi(\boldsymbol{\sigma}) = f(\boldsymbol{\sigma})} \tag{3.50}$$

Assumption of the associated flow rule concerns many important aspects of modeling the elastic-plastic materials:

- In case of an associated flow rule, **the stress state in an elastic-plastic solid** (with or without hardening) **for given statical boundary conditions is given uniquely**.
- In case of an associated flow rule, the **normality condition** is satisfied – **plastic strain increment tensor is orthogonal to the yield surface**.
- If the material is **stable in the sense of Drucker**, then the **flow rule must ba associated with the yield condition** and the **yield surface must be convex**.

5.1.3.1 PRANDTL-REUSS THEORY

Most commonly used incremental model is the **Prantl-Reuss model**, in which the flow rule is associated with the MHMH yield condition, namely, when the yield condition is a quadratic function of the stress deviator:

$$f(\boldsymbol{\sigma}) = \sigma_{eq} - \sigma_0 = \sqrt{\frac{3}{2} s_{ij} s_{ij}} - \sigma_0$$

Then **the plastic strain increment tensor is simply proportional to the stress deviator**:

$$\mathrm{d}\varepsilon_{ij}^{pl} = \mathrm{d}\lambda\, s_{ij} \tag{3.51}$$

Constitutive relation for total strain increment tensor is then as follows:

$$\boxed{\begin{cases} \mathrm{d}\varepsilon_{ij} = \mathrm{d}\lambda\, s_{ij} + \dfrac{\mathrm{d}s_{ij}}{2G} \\ \mathrm{d}\varepsilon_{kk} = \dfrac{1-2\nu}{E}\mathrm{d}\sigma_{kk} \end{cases}} \tag{3.52}$$

5.1.3.2 LEVY-MISES THEORY

First historic incremental theory was the proposition of **Levy and Mises**, which is almost the same as the later Prandtl-Reuss theory, the only difference is that it concerned rigid-plastic solid model in which total strain is equal to the plastic strain and elastic component of strain is zero, what leads to relation:

$$\mathrm{d}\varepsilon_{ij} = \mathrm{d}\lambda\, s_{ij} \tag{3.53}$$

6. YIELDING OF A CROSS-SECTION

As it was already mentioned, yielding may occur at various levels in a body. Until now we've considered only yielding in a point, which was equivalent to satisfying the yield condtiion. Yielding in a point resulted in the change of constitutive relations in that point. If the **yielding concerns a set of point (subregion in a body)** we speak then of **partial yielding of a body** or that the body is in an **elastic-plastic state**. The region in which active processes occur is called the plastic region. We shall consider now fundamental cases of loads applied to solid bars, which may result in partial of total yielding in a cross-section. Our considerations will be based on a model of **ideally elastic-plastic material with no hardening**.

6.1 YIELDING OF AXIALLY LOADED MEMBERS

Within the **linear theory of elasticity** the solution of the problem of **pure tension** of a prismatic bar given us:

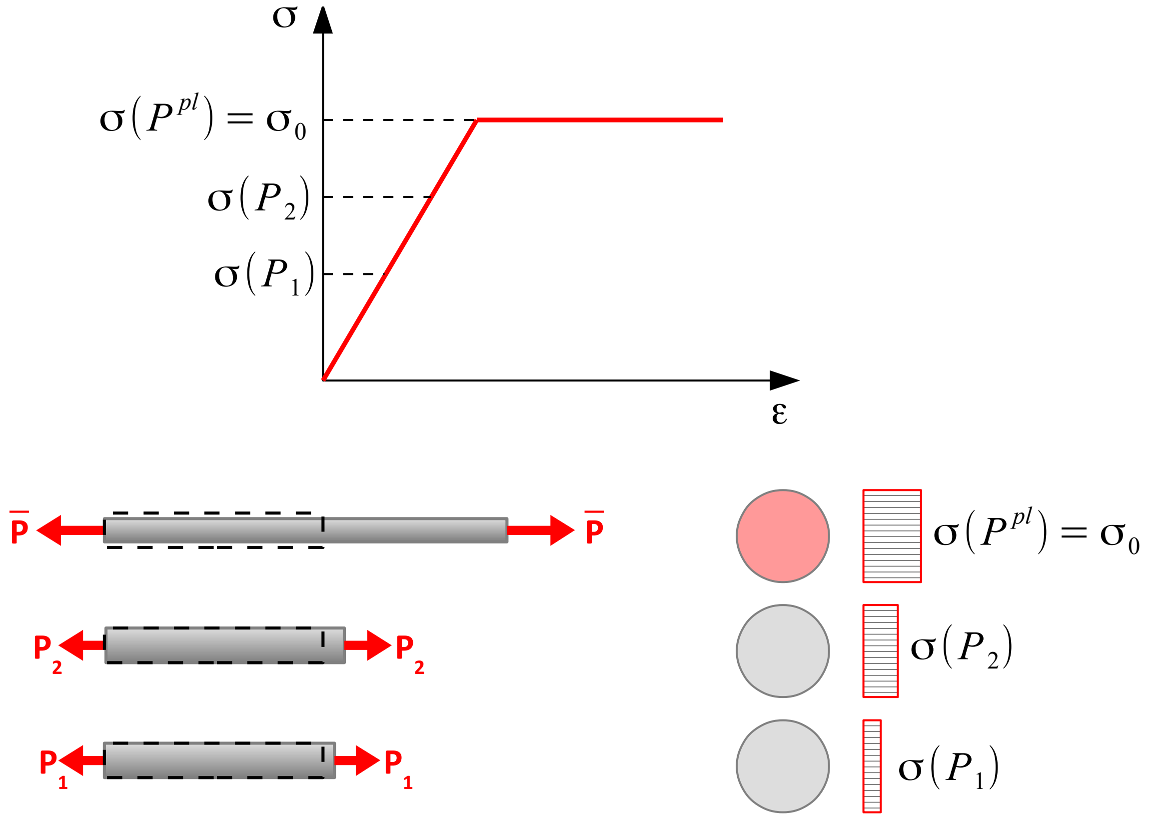
$$\boldsymbol{\sigma}=\begin{bmatrix}\sigma & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\quad,\quad \boldsymbol{\varepsilon}^e=\frac{\sigma}{E}\begin{bmatrix}1 & 0 & 0\\ 0 & -\nu & 0\\ 0 & 0 & -\nu\end{bmatrix}\quad,\quad \mathbf{u}=\frac{\sigma}{E}\begin{bmatrix}x_1\\ -\nu x_2\\ -\nu x_3\end{bmatrix}\quad,\quad \sigma=\frac{P}{A}=const. \tag{3.54}$$

where P is the external tensile force, A is the surface area of the bar's cross-section, E is the Young modulus, and ν is the Poisson's ratio. **Uniaxial stress state assume uniform distribution of stress state in the whole cross-section** – the only non-zero component is the normal stress along the load direction.

Starting with the linear-elastic solution and assuming the model of ideally elastic-plastic material with no hardening, we may state that:

- **the yield condition is satisfied simultaneously in all points of the cross-section**.
- **Normal stress** in the cross-section during ongoing load **dues not exceed the limit stress** $\sigma=\sigma_0$ (no hardening).
- The greatest magnitude of a force, which may applied to a bar, corresponds with its limit plastic bearing capacity $P^{pl}=\sigma_0 A$. Strain increases up to infinity (the bar „flows”).

In the figure below plastic region is marked with pink color.



Of course, in true case finally hardening or cracking occurs, however this second stage of deformation in elastic-plastic analysis in engineering problems is often disregarded due to large discrepancies between theoretical predictions and observed material's behavior.

6.2 YIELDING OF BENT MEMBERS

Within the **linear theory of elasticity** the solution of the problem of **pure bending** of a prismatic bar given us:

$$\boldsymbol{\sigma} = \frac{M}{I}\begin{bmatrix} x_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ , \quad \boldsymbol{\varepsilon}^e = \frac{M}{EI}\begin{bmatrix} x_3 & 0 & 0 \\ 0 & -\nu x_3 & 0 \\ 0 & 0 & -\nu x_3 \end{bmatrix} \ , \quad \mathbf{u} = \frac{M}{EI}\begin{bmatrix} x_1 x_3 \\ -\nu x_2 x_3 \\ \frac{1}{2}\left(-x_1^2 + \nu\left(x_2^2 - x_3^2\right)\right) \end{bmatrix} \ , \tag{3.55}$$

where M is the magnitude of bending moment, I is the principal central second moment of area (moment of inertia) of the bar's cross-section with respect to the axis which is parallel to the vector of bending moment, E is the Young modulus, and ν is the Poisson's ratio. The **state of pure bending within linear-elastic range assumed a linear distribution of a normal stress and corresponding linear strain** in whole cross-scection of a bar.

Starting with this solution, we will assume that the distribution of strain will remain linear. Contrary to the case of uniaxial stress state, yielding will occur in different points depending on the magnitude of load. We may distinguish two limiting situations:

- **yielding in a first fiber of a cross-section** – it corresponds with the bending moment M^e which is a l**imit elastic bearing capacity** of the cross-section.
- **yielding in all points of a cross-section** – it corresponds with the bending moment M^{pl} which is a **limit plastic bearing capacity** of the cross-section.

It is obvious that the first fibers that will undergo yielding will be those, in which the normal stress is the greatest, namely the edge fibers. Limit elastic bearing capacity is equal:

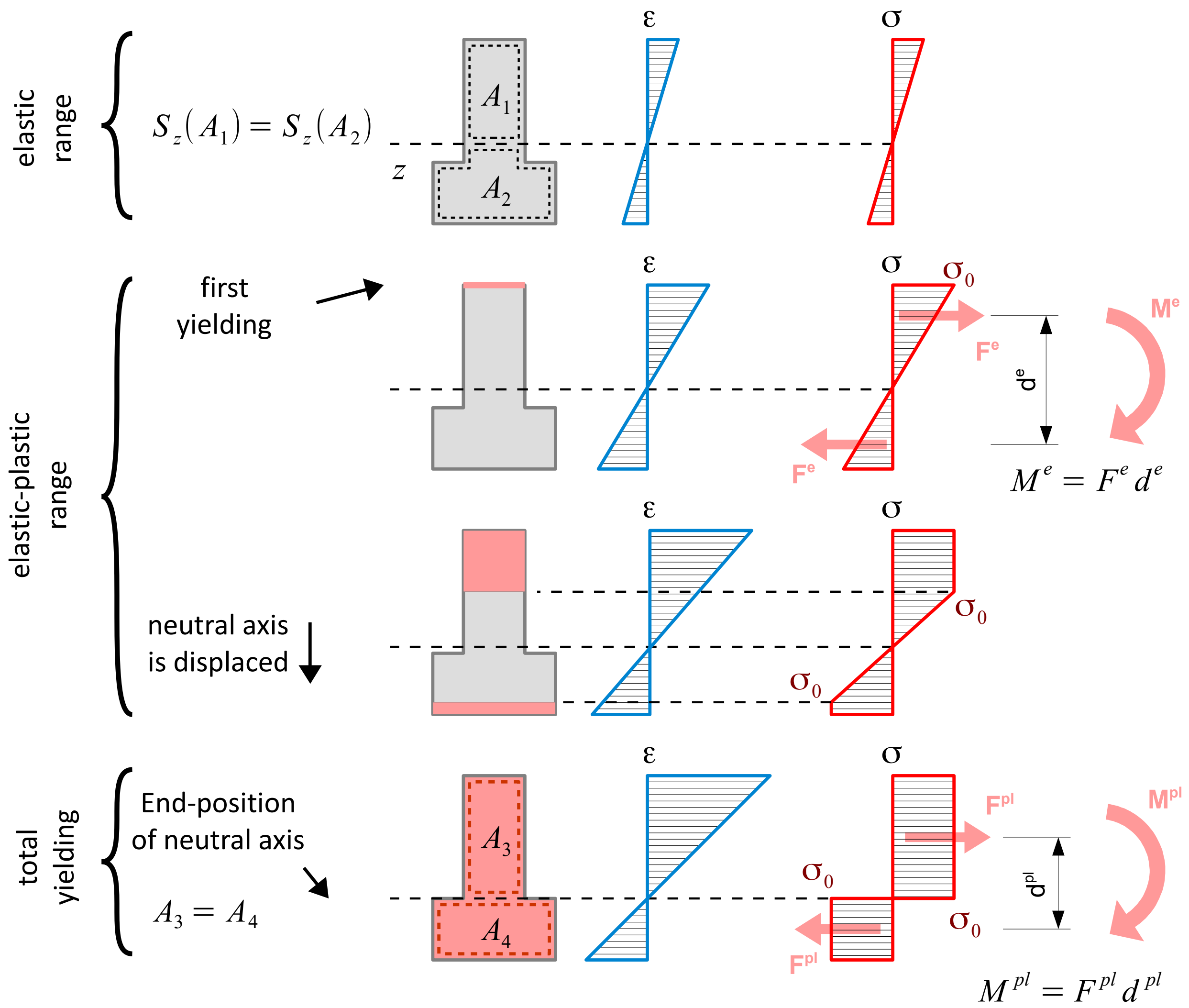
$$M^e = W\,\sigma_0 \tag{3.56}$$

where W is the resistance moment (strength index in bending).

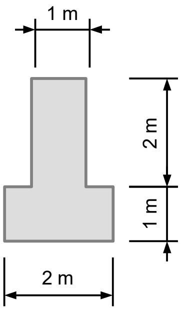
Plastic region consisting of two subregions at opposite edges of the cross-section expands towards the neutral axis, however**, location of the neutral axis in case of non-symmetric cross-section varies** with the ongoing plastic deformation.

In a limit case – **yielding of a cross-section** – condition of equilibrium of internal forces, which correspond to two regions of constant normal stress (tension – compression), requires that the location of the neutral axis is such that the **surface area of the region above the axis (in case of yielding) of a cross-section is the same as the surface area of the region below the axis**, which is different from the situation in elastic range, in which equilibrium required equality of the first moment of area (statical moments) of the regions on opposite sides of this axis. Limit plastic bearing capacity is determined as a moment corresponding with a couple of forces, and each of the forces in that couple is a sum of a uniform system of normal stress on one and on the other side of the neutral axis. Location of those forces is found in a usual way with the use of statical moment.

In the figure below a character of distribution of stress and strain in a bent cross-section if shown in various stages of loading. Plastic region is marked with pink color.



EXAMPLE:



Elastic range:

Surface area of cross-section: $A = [2\cdot 1]+[1\cdot 2] = 4\ \text{m}^2$

1st moment of area wrt bottom edge: $S_Y = [2\cdot 1\cdot 0{,}5]+[1\cdot 2\cdot 2] = 5\ \text{m}^3$

Location of neutral axis: $Z_O = \dfrac{S_Y}{A} = 1{,}25\,\text{m}$

Principal central axis of inertia:

$$I_y = \left[\frac{2\cdot 1^3}{12} + 2\cdot 1\cdot (0{,}5-1{,}25)^2\right]+\left[\frac{1\cdot 2^3}{12} + 1\cdot 2\cdot (2-1{,}25)^2\right] = 4{,}163\ \text{m}^4$$

Distance to the edge fiber: $z_{max} = 1{,}75\ \text{m}$

Resistance moment: $W_y = \dfrac{I_y}{z_{max}} = 2{,}379\ \text{m}^3$

Limit elastic bearing capacity: $M^e = \sigma_0 \cdot W_y = 2{,}379\,\sigma_0$

Yielding of cross-section:

Location of neutral axis: $Z_{O,\,pl} = 1\,\text{m}$

Sum of system of stresses above/below the neutral axis: $F^{pl} = \sigma_0 \cdot 1\cdot 2 = 2\,\sigma_0$

Location of sum of the top system: $Z_{F,g} = 2\ \text{m}$

Location of sum of the bottom system: $Z_{F,d} = 0{,}5\ \text{m}$

Arm of forces in a couple: $d^{pl} = Z_{F,g} - Z_{F,d} = 1{,}5\ \text{m}$

Limit plastic bearing capacity: $M^{pl} = F^{pl} d^{pl} = 3\,\sigma_0$

Ratio of limit plastic and elastic bearing capacity: $\dfrac{M^{pl}}{M^e} = 1{,}261$

The case of a bent cross-section is a nice way of depicting of problems of permanent strains, elastic character of unloading as well as residual stresses. Let's consider a rectangular cross-section for simplicity. Let's denote the location of neutral axis with O, A – boundary of plastic region, B – edge fibers. Let the cross-section be bent with a moment M_1, such that is results in a plastic region ranging up to the one third of the total surface area of the cross-section. Its value is:

$$M_1 = 2\left[\left(\sigma_0 \cdot b \cdot \frac{h}{6} \cdot \left(\frac{h}{3} + \frac{1}{2} \cdot \frac{h}{6}\right)\right) + \frac{1}{2}\sigma_0 \cdot b \cdot \frac{h}{3} \cdot \frac{2}{3} \cdot \frac{h}{3}\right] = \frac{23}{108} b h^2 \sigma_0 \approx 0{,}213\, b h^2 \sigma_0$$

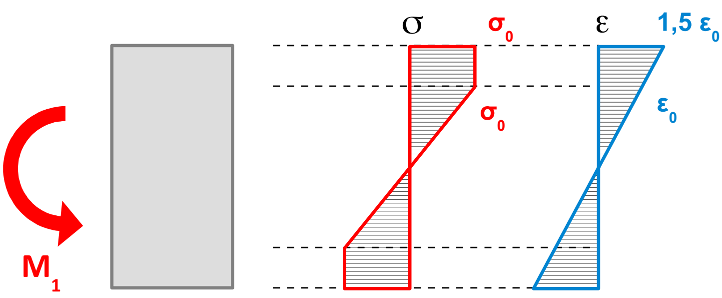
Limit elastic bearing capacity of the cross-section: $M^e = \frac{1}{6} b h^2 \sigma_0 \approx 0{,}167\, b h^2 \sigma_0$

Limit plastic bearing capacity of the cross-section: $M^{pl} = bh = \frac{1}{4} b h^2 \sigma_0 \approx 0{,}250\, b h^2 \sigma_0$

$M^e < M_1 < M^{pl} \quad \Rightarrow$ elastic-plastic deformation domain.

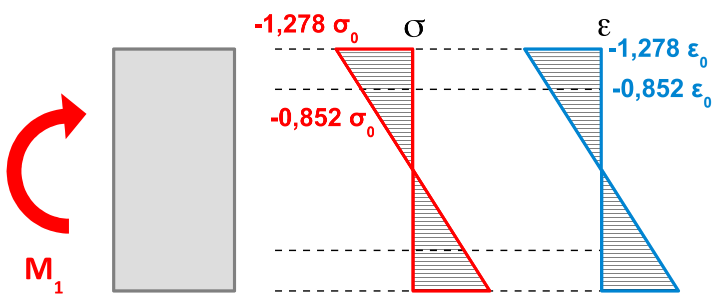
Stress and strain in points O, A, B after loading:

$$\sigma_O = 0 \qquad \varepsilon_O = 0$$
$$\sigma_A = \sigma_0 \qquad \varepsilon_A = \varepsilon_0 = \frac{\sigma_0}{E}$$
$$\sigma_B = \sigma_0 \qquad \varepsilon_B = \frac{z_B}{z_A}\varepsilon_A = 1{,}5\frac{\sigma_0}{E}$$



Let's now unload the cross-section, namely, let's load it with a bending moment of the same value but opposite orientation, remembering that **unloading is an elastic process**, so **corresponding stress and strain distribution is linear**. Additional stress and strain due to unloading in points O, A, B:

$$\sigma_O = 0 \qquad \varepsilon_O = -\frac{M_1}{EI} z_O = 0$$
$$\sigma_A = -\frac{M_1}{I} z_A = -0{,}852\,\sigma_0 \qquad \varepsilon_A = -\frac{M_1}{EI} z_A \approx -0{,}852\frac{\sigma_0}{E}$$
$$\sigma_B = -\frac{M_1}{I} z_B = -1{,}278\,\sigma_0 \qquad \varepsilon_B = -\frac{M_1}{EI} z_B \approx -1{,}278\frac{\sigma_0}{E}$$

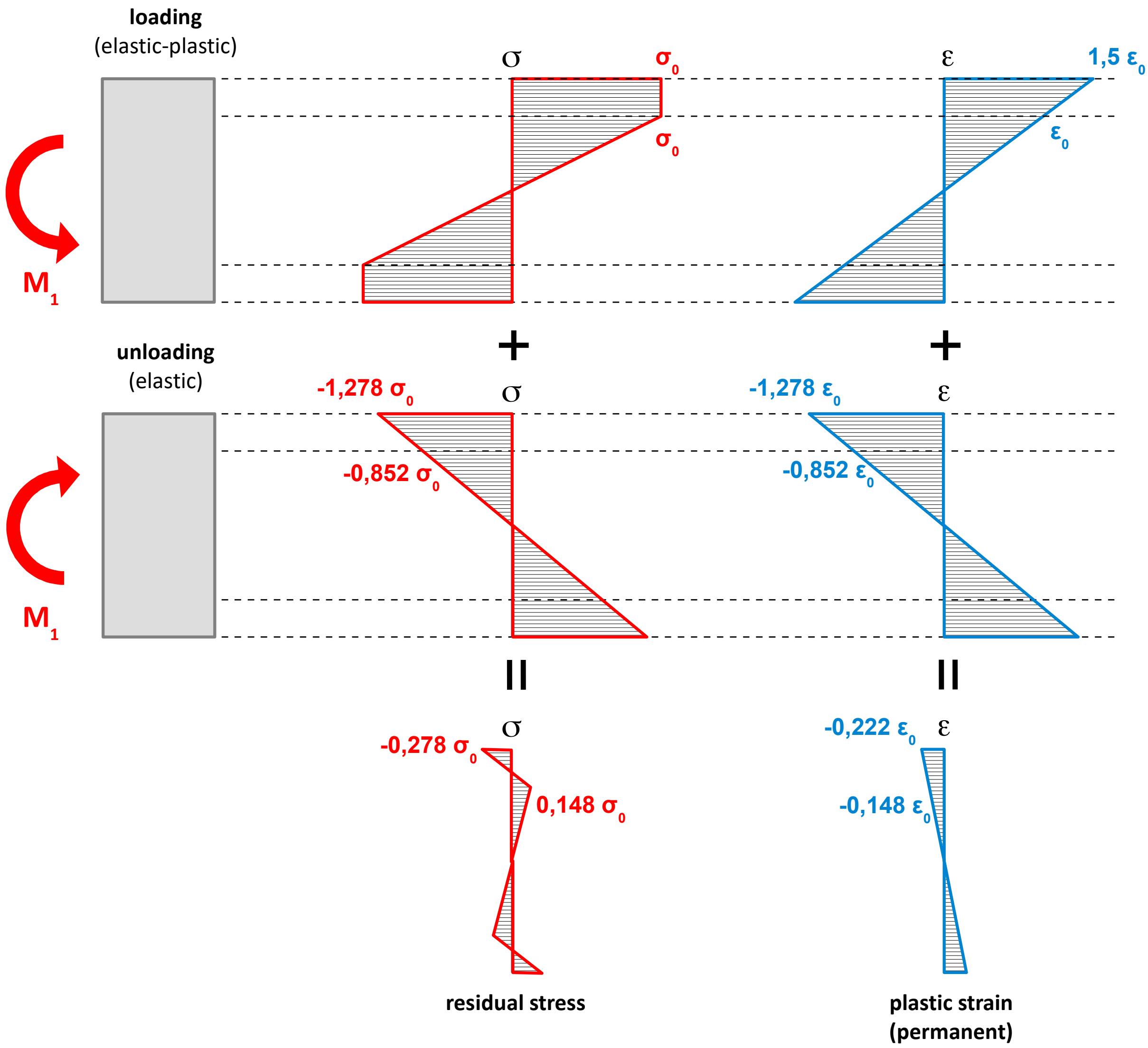


Stress and strain in points O, A, B after unloading:

$$\sigma_O = 0 - 0 = 0 \qquad \varepsilon_O = 0 - 0 = 0$$

$$\sigma_A = \sigma_0 - \frac{M_1}{I} z_A \approx 0{,}148\,\sigma_0 \qquad \varepsilon_A = \varepsilon_0 - \frac{M_1}{EI} z_A \approx 0{,}148\,\frac{\sigma_0}{E}$$

$$\sigma_B = \sigma_0 - \frac{M_1}{I} z_B \approx -0{,}278\,\sigma_0 \qquad \varepsilon_B = 1{,}5\,\varepsilon_0 - \frac{M_1}{EI} z_B \approx 0{,}222\,\frac{\sigma_0}{E}$$



It may be noticed that **the system of residual stresses is in equilibrium**, so the cross-section is indeed fully unloaded. We may also observe that after unloading there is a remaining linear distribution of stain which corresponds with a certain permanent (plastic) curvature of bent bar. Subsequent loading will be again an elastic process unless the value of the bending moment exceeds M_1 . Reaching M_1 again results in the same stress and strain state as after original loading.

6.2.1 CLASSES OF STEEL SECTIONS

Standards for desiging steel structure distinguish 4 classes of steel sections, according to their ability of plastic deformation:

- **CLASS 1** – **the cross-section may reach its limit plastic bearing capacity**, and after full yielding it **preserves the ability of free deformation**, so it is possible that a **plastic hinge is created**. In case of statically-indeterminate systems the presence of such a hinge results in the change of statical diagram of the system, what – in turn – changes the distribution of internal forces (**redistribution of internal forces**). As a result the system may exhibit greater bearing capacity than the limit plastic capacity of its cross-section. Bearing capacity of the system corresponds with the load which results in so great number of plastic hinges that the system looses its stability and becomes mechanism.

- **CLASS 2** – **the cross-section may reach its limit plastic bearing capacity** but its deformation is constrained, so the assumption that the plastic hinge is created is not allowed.

- **CLASS 3** – **the cross-section may reach its limit elastic bearing capacity**, so the bearing capacity of the cross-section corresponds with a load which results in yielding of its first fiber.

- **CLASS 4** – **the cross-section locally looses its stability** due to too great slenderness of its elements (e.g. buckling of a compressed web or flange) **even before yield occurs in any fiber** of the cross-section.

6.3 YIELDING OF MEMBERS SUBJECTED TO TORSION

Within the **linear theory of elasticity** the solution of the problem of **pure torsion** of a prismatic bar **with unconstrained warping** given us:

$$\boldsymbol{\sigma} = \Theta G \begin{bmatrix} 0 & \left(\frac{\partial \psi}{\partial x_2} - x_3\right) & \left(\frac{\partial \psi}{\partial x_3} + x_2\right) \\ & 0 & 0 \\ \text{sym} & & 0 \end{bmatrix} \quad , \quad \boldsymbol{\varepsilon}^e = \frac{\Theta}{2} \begin{bmatrix} 0 & \left(\frac{\partial \psi}{\partial x_2} - x_3\right) & \left(\frac{\partial \psi}{\partial x_3} + x_2\right) \\ & 0 & 0 \\ \text{sym} & & 0 \end{bmatrix} \quad , \quad \mathbf{u} = \Theta \begin{bmatrix} \psi(x_1, x_2) \\ -x_1 x_3 \\ x_1 x_2 \end{bmatrix} \quad , \tag{3.57}$$

where Θ is the unit torsion angle, G is the Kirchhoff modulis, a funkcja ψ is the warping function describing the out-of-plane deformation of a cross-section. One of the methods of description of the problem of pure bending it introduction of the **Prandtl stress function** ϕ such that:

$$\sigma_{12} = \frac{\partial \phi}{\partial x_3} \; , \quad \sigma_{31} = -\frac{\partial \phi}{\partial x_2} \tag{3.58}$$

It is easy to check, that for assumed stress state expressing its components with the use of the Prandtl stress function guarantees that all equilibrium conditions are automatically satisfied. The only remaining condition to be fulfilled is the strain compatibility consdition. Such a consdition may be obtained by calculation of the following expression

$$\frac{\partial \sigma_{31}}{\partial x_2} - \frac{\partial \sigma_{12}}{\partial x_3} = \Theta G \left[\left(\frac{\partial^2 \psi}{\partial x_2 \partial x_3} + 1 \right) - \left(\frac{\partial^2 \psi}{\partial x_2 \partial x_3} - 1 \right) \right] = 2 \Theta G \tag{3.59}$$

Accounting for the dependency between stress state components and the strrss function gives us:

$$\nabla^2 \phi = \frac{\partial \phi^2}{\partial x_2^2} + \frac{\partial \phi^2}{\partial x_3^2} = -2 G \Theta \tag{3.60}$$

so the **Prandtl stress function** ϕ **satisfies the Poisson equation**.

Directional derivative of ϕ **is equal to the shear stress in a direction which is perpendicular to the direction for which the derivative is calculated**. Statical boundary condition for a bar subjected to torsion requires that the **shear stress perpendicular to the boundary is zero**, what corresponds with zero value of the directional derivative along the tangent direction – this condition is satisfied when **function** ϕ **is constant along boundary**, e.g. it is equal 0. The above equation, together with the boundary condition $\phi = 0$ on a boundary of the cross-section, enables to find ϕ uniquely, what in turn gives as the distribution of stress in a twisted bar.

The above equation describes also a different problem in mechanics – deformation of a slender membrane or film fixed along a boundary of a shape of the edge of cross-section and loaded with uniform pressure. When the stiffness of a membrane is denoted with D, and the pressure load is denoted with q then the equation which describes the deflection of membrane

w is of the following form:

$$\nabla^2 w = \frac{q}{D} \tag{3.61}$$

So, if the **function ϕ was interpreted as a distribution of deflection of a membrane**, then its partial derivatives with respect to variables x_2, x_3, namely the **tangent of an angle of inclination of deformed membrane along direction of x_2, x_3 is a measure of corresponding shear stress**. This observation is sometimes termed the **Prandtl membrane analogy**. In order to find the total twisting moment which corresponds with a whole system of shear stress, we need to calculate following integral:

$$M = \iint_A (\sigma_{12} x_3 - \sigma_{31} x_2) \, \mathrm{d}A = \iint_A \left(\frac{\partial \phi}{\partial x_3} x_3 + \frac{\partial \phi}{\partial x_2} x_2 \right) \mathrm{d}A \tag{3.62}$$

It may be integrated by parts:

$$M = \iint_A \left(\frac{\partial \phi}{\partial x_3} x_3 + \frac{\partial \phi}{\partial x_2} x_2 \right) \mathrm{d}A = \oint_{\partial A} \left[\phi (x_2 n_2 + x_3 n_3) \right] \mathrm{d}A - \iint_A \phi \left[\frac{\partial}{\partial x_2}(x_2) + \frac{\partial}{\partial x_3}(x_3) \right] \mathrm{d}A \tag{3.63}$$

where $\mathbf{n} = [n_2, n_3]$ is external normal at the boundary. Since along the whole boundary ∂A the function $\phi = 0$, so the contour integral is equal 0. We get:

$$M = -2 \iint_A \phi \, \mathrm{d}A$$

The integral at the right hand side is a volume contained between plane (x_2, x_3) and the graph of ϕ. So the **volume contained between the surface of the cross-section and deformed membrane is a measure of a twisting moment**:

$$M = 2V \tag{3.64}$$

In a limit state (yielding of a first fiber) it is equal to the limit elastic bearing capacity of the twisted cross-section. Let's consider a situation in which the **twisting moment exceeds the limit elastic bearing capacity** and let's assume that still the only non-zero stress state components in the cross-section are shear stresses. If the material exhibits no hardening, resultant shear stress in the plastic region must be everywhere equal τ_0 - limit shear stress – which for material with no hardening is constant:

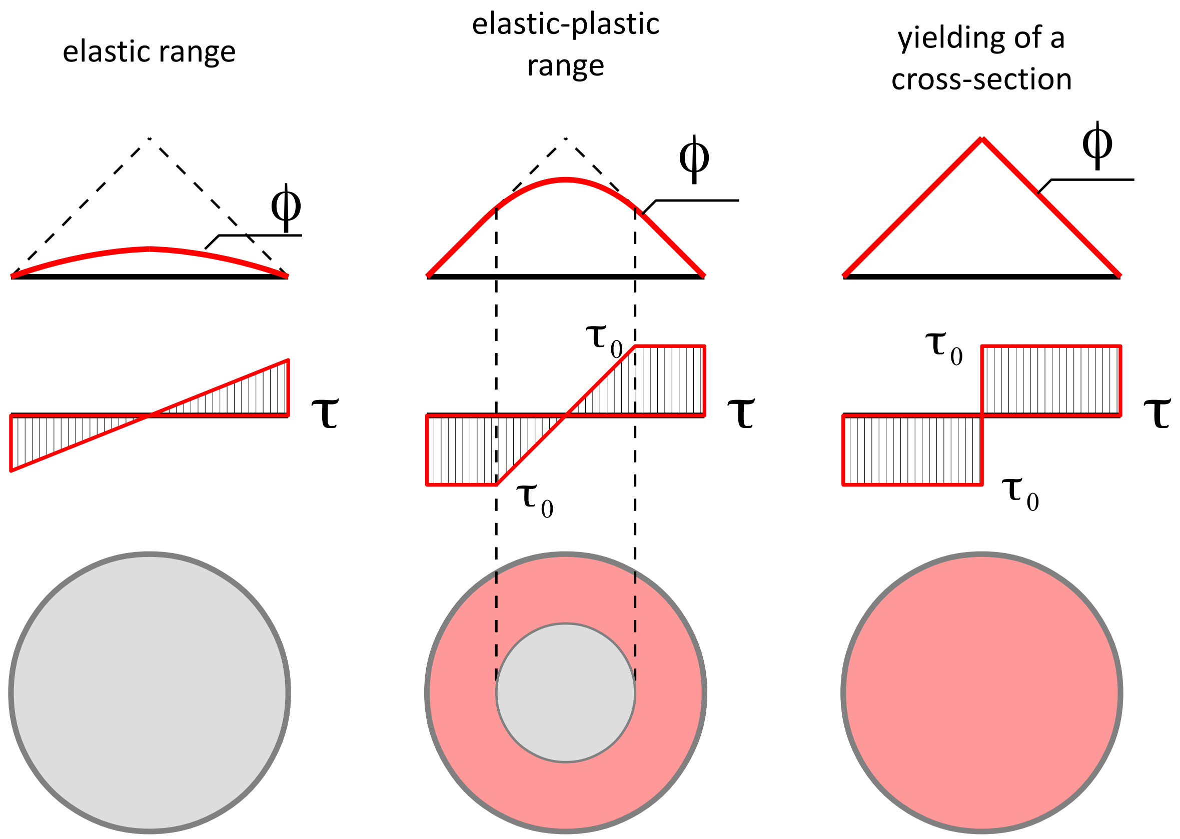
$$\sqrt{\sigma_{12}^2 + \sigma_{31}^2} = \tau_0 \quad \Rightarrow \quad \left(\frac{\partial \phi}{\partial x_2} \right)^2 + \left(\frac{\partial \phi}{\partial x_3} \right)^2 = |\mathrm{grad}\, \phi|^2 = \tau_0^2 = const. \quad , \tag{3.65}$$

Referring to the membrane analogy, the above equation may be interpreted in the following way: **maximum inclination angle** (length of a vector of a gradient of deformed membrane) **in the plastic region must be constant**, equal α . This angle corresponds with limit shear stress τ_0 :

$$\tau_0^2 = \mathrm{tg}^2\alpha = const. \tag{3.66}$$

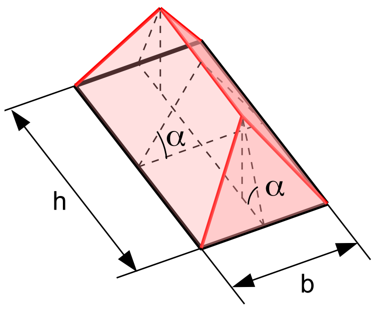
In the elastic region still equation (3.61) holds true and an angle of inclination is maller than the constant value corresponding with τ_0 . We may extend the membrane analogy in such a way, that **above the membrane in the plastic region there is an infinitely rigid roof of slope** τ_0 . The membrane may deform due to applied pressure freely in the elastic region – in the plastic region in sticks to the roof. This observation is sometimes called the **Nadai roof analogy**. The volume contained under the membrane is a measure of twisting moment

In case of yielding of a whole cross-section, the membrane sticks the roof everywhere and the limit plastic bearing capacity is calculated according to a volume under the roof. The roof starts at the contour of the cross-section at it is inclined with a constant angle to the surface of that cross-section – this is a shape of an ideally cohesiveless material with fixed angle of internal friction, which is poured uniformly on the area of cross-section. Its top surface is termed the constant slope surface. The volume of the pile is a measure of limit plastic bearing capacity. This observation is sometimes termed as **Nadai sand hill analogy**. In the figure below the character of distribution of shear stress in a circular twisted cross-section is shown for various stages of loading. Plastic region is marked with pink color.



EXAMPLES:

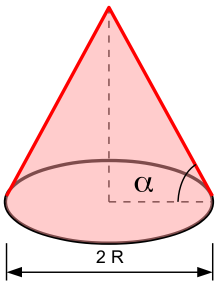
- **limit plastic bearing capacity of a twisted rectangular cross-section**



Volume: $$V = \frac{1}{12} b^2 (3h - b)\ \mathrm{tg}\,\alpha$$

Bearing capacity: $$M^{pl} = \frac{1}{6} b^2 (3h - b)\,\tau_0$$

- **limit plastic bearing capacity of a twisted circular cross-section**

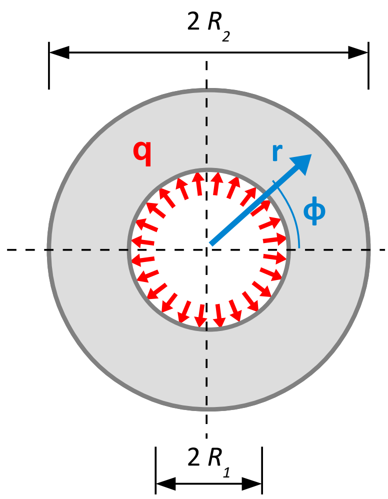


Volume: $$V = \frac{1}{3} \pi r^3\ \mathrm{tg}\,\alpha$$

Bearing capacity: $$M^{pl} = \frac{2}{3} \pi r^3\,\tau_0$$

6.4 YIELDING OF A THICK - WALLED PIPE LOADED WITH INTERNAL PRESSURE

Within the **linear theory of elasticity** the solution of the **problem of a thick-walled pipe of internal radius** R_1 **and external radius** R_2 **, loaded with uniform internal pressure** q gives is (in polar coordinates):



$$u_r(r) = C_1 r + \frac{C_2}{r}$$

$$\varepsilon^e_{rr}(r) = C_1 - \frac{C_2}{r^2} \quad , \qquad \varepsilon^e_{\phi\phi}(r) = C_1 + \frac{C_2}{r^2}$$

$$\sigma_{rr}(r) = 2C_1(G+\lambda) - \frac{2C_2G}{r^2}$$
$$\sigma_{\phi\phi}(r) = 2C_1(G+\lambda) + \frac{2C_2G}{r^2} \qquad (3.67)$$

where: $$C_1 = \frac{qR_1^2}{(2G+\lambda)(R_2^2-R_1^2)} \quad , \qquad C_2 = \frac{qR_1^2R_2^2}{2G(R_2^2-R_1^2)} \qquad (3.68)$$

G is the Kirchhoff modulus, λ is the Lame parameter. We assume that the pipe may deform freele along its axis, namely: $\sigma_{zz}=0$. Let us consider a case in which an internal pressure leads to yielding. **Character of solution elastic region will be the same as above**. Let's write down the expressions for the distribution of stress components:

$$\sigma^I_{rr}(r) = D_1 - \frac{D_2}{r^2} \qquad \sigma^I_{\phi\phi}(r) = D_1 + \frac{D_2}{r^2} \qquad (3.69)$$

in which superscript I denotes the elastic region and II denotes the plastic region.

Constants of integration will change because **boundary conditions are different**:

- Boundary condition for free boundary remains unchanged $\sigma^I_{rr}(R_2) = 0$.
- Next condition is the continuity of radial stress at the boundary of plastic region: $\sigma^{II}_{rr}(r^{pl}) = \sigma^I_{rr}(r_{pl})$
- Condition of continuity of shear stress at the boundary of plastic region is satisfied since $\sigma_{r\phi} = 0$.
- Boundary condition for loaded boundary $\sigma^{II}_{rr}(R_1) = -q$ will be used in order to determine the distribution of stress in plastic region.

Contrary to the previously discussed cases, the stress state now is complex and it is necessary to assume a certain yield condition, for which the analysis is performed. Let it be the **Coulomb-Tresca-Guest condition**. Since **components** σ_{rr} **and** $\sigma_{\phi\phi}$ **are the only non-zero components** of the stress tensor, so they are **principal stresses** and the **difference of their values is a value of maximum shear stress**:

$$\boxed{\tau_{max}=\frac{\sigma_{\phi\phi}-\sigma_{rr}}{2}=\frac{4C_2G}{r^2}} \tag{3.70}$$

The above expression takes its greatest value for minimum value of r, so **yielding starts at internal surface**. Minimum value of internal pressure which results with yielding is equal:

$$2\tau_{max}=\sigma_{\phi\phi}-\sigma_{rr}=\frac{4GC_2}{R_1^2}=\frac{2q_0R_2^2}{(R_2^2-R_1^2)}=\sigma_0 \quad\Rightarrow\quad \boxed{q_0=\frac{\sigma_0}{2}\left(1-\frac{R_1^2}{R_2^2}\right)}$$

Boundary of plastic region is found according to the condition:

$$2\tau_{max}^I(R_{pl})=\sigma_{\phi\phi}^I(R_{pl})-\sigma_{rr}^I(R_{pl})=\frac{2D_2}{R_{pl}^2}=\sigma_0$$

what gives us:

$$\boxed{R_{pl}=\sqrt{\frac{2D_2}{\sigma_0}} \quad\Leftrightarrow\quad D_2=\frac{1}{2}\sigma_0R_{pl}^2} \tag{3.71}$$

In case of a material exhibiting no hardening, in each point of the plastic region following condition must be fulfilled:

$$\boxed{2\tau_{max}^{II}=\sigma_0 \quad\Rightarrow\quad \sigma_{\phi\phi}^{II}-\sigma_{rr}^{II}=\sigma_0} \tag{3.72}$$

Equilibrium equation for an axis-symmeric problem (both in elastic and in plastic range) is of the form:

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{\sigma_{rr}-\sigma_{\phi\phi}}{r}=0 \tag{3.73}$$

what – after accounting for a constraint (3.72) – gives us an **equilibrium condition in plastic region**:

$$\boxed{\frac{\partial\sigma_{rr}^{II}}{\partial r}=\frac{\sigma_0}{r}} \tag{3.74}$$

The solution of that differential equation gives us the distribution of radial stress:

$$\sigma_{rr}^{II} = \sigma_0 \ln r + D_3 \tag{3.75}$$

Relation (3.72) enable us to determine circumferential stress:

$$\sigma_{\phi\phi}^{II} = \sigma_0 + \sigma_{rr} = \sigma_0 (1 + \ln r) + D_3 \tag{3.76}$$

Elastic constants D_1, D_2, D_3 are found with the use of boundary conditions:

$$\begin{cases} \sigma_{rr}^{I}(R_2) = 0 & \Rightarrow \quad D_1 - \dfrac{D_2}{R_2^2} = 0 \\ \sigma_{rr}^{II}(R_1) = -q & \Rightarrow \quad D_3 + \sigma_0 \ln R_1 = -q \\ \sigma_{rr}^{I}(R_{pl}^2) - \sigma_{rr}^{II}(R_{pl}^2) = 0 & \Rightarrow \quad D_1 - \dfrac{\sigma_0}{2} - D_3 - \sigma_0 \ln R_{pl} = 0 \end{cases} \tag{3.77}$$

Accounting for relation (3.71) gives us:

$$D_1 = \frac{\sigma_0 R_{pl}^2}{2 R_2^2} \quad ,$$

hence:

$$\boxed{\sigma_{rr}^{I}(r) = \frac{\sigma_0 R_{pl}^2}{2 R_2^2}\left(1 - \frac{R_2^2}{r^2}\right) \qquad \sigma_{\phi\phi}^{I}(r) = \frac{\sigma_0 R_{pl}^2}{2 R_2^2}\left(1 + \frac{R_2^2}{r^2}\right)} \tag{3.78}$$

Condition of equality of radial stress at the boundary of elastic and plastic region gives us:

$$\frac{\sigma_0 R_{pl}^2}{2 R_2^2}\left(1 - \frac{R_2^2}{R_{pl}^2}\right) = D_3 + \sigma_0 \ln R_{pl} \quad \Rightarrow \quad D_3 = \frac{\sigma_0 R_{pl}^2}{2 R_2^2}\left(1 - \frac{R_2^2}{R_{pl}^2}\right) - \sigma_0 \ln R_{pl}$$

and finally:

$$\boxed{\sigma_{rr}^{II} = \sigma_0 \left[\frac{R_{pl}^2}{2 R_2^2} - \frac{1}{2} - \ln \frac{R_{pl}}{r}\right] \qquad \sigma_{\phi\phi}^{II} = \sigma_0 \left[\frac{R_{pl}^2}{2 R_2^2} + \frac{1}{2} - \ln \frac{R_{pl}}{r}\right]} \tag{3.79}$$

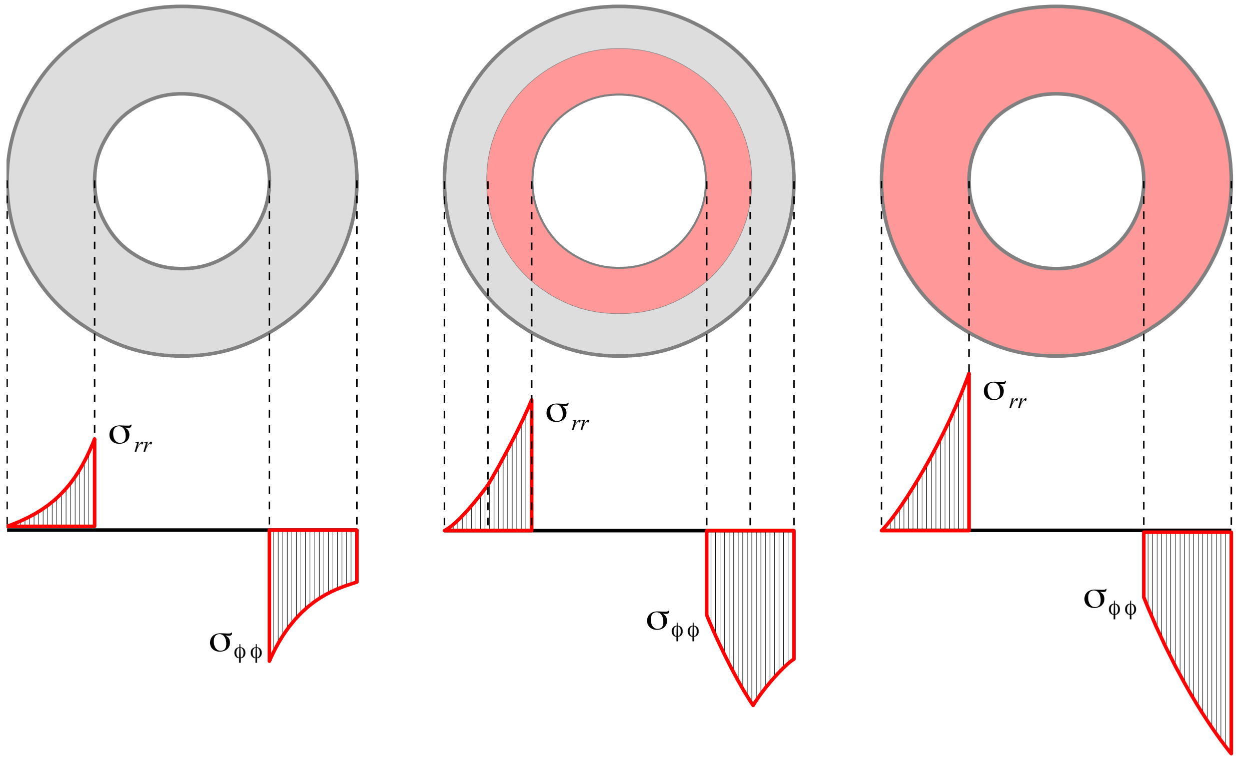
It may be easily noticed that in such a situation also distribution of circumferential stress is countinuous at the boundary fo aplstic region. In the above solution parameter R_{pl} is used. Its value has not been determined yet. It is found according to the boundary condition fo internal surface:

$$\sigma_{rr}^{II}(R_1) = -q \qquad \Rightarrow \qquad \ln \frac{R_{pl}}{R_1} + \frac{1}{2}\left(1 - \frac{R_{pl}^2}{R_2^2}\right) = \frac{q}{\sigma_0}$$

The above equation is a non-linear equation with respect to R_{pl} and it must be solved numerically. The solution will depend on the ratio $\beta = R_1/R_2$ as well as on the ratio $\gamma = q/\sigma_0$. Parameter $\beta \in (0;1)$. Minimum value of parameter γ , for which yielding occurs is the one corresponding with yielding pressure q_0 ,and its maximum value γ_{max} corresponds with situation in which $R_{pl} = R_2$:

$$\gamma_{min} = \frac{q_0}{\sigma_0} = \frac{1}{2}(1-\beta^2) \qquad \gamma_{max} = \ln\frac{1}{\beta}$$

In the figure below a character of distribution of radial and circumferential stress in a whick-walled pipe loaded with internal pressure is shown for various stages of loading. Plastic region is marked with pink color.



7. PLASTIC BEARING CAPACITY OF A STRUCTURAL SYSTEM

Yielding in a point and even yielding of a whole cross-section of one of the elements of a bar structure, even in case of material exhibiting no hardening (total freedom of deformation after yielding) do not necessarily mean that the structure becomes a mechanism, namely that it looses its stability. **In case of statically-indeterminate systems the internal structure and distribution of supports may still provide geometrical invariance even after part of the system deforms fully plastically** and the system still bears some additional load. Maximum load that the system is capable to bear is termed the **plastic bearing capacity of the system** and it may be greater than the limit bearing capacity of a cross-section. One should mention, however, that despite its geometrical invariance, the system may not satisfy e.g. serviceability requirements due to large deformation.

There are two approaches used in finding the plastic bearing capacity of a structural system:

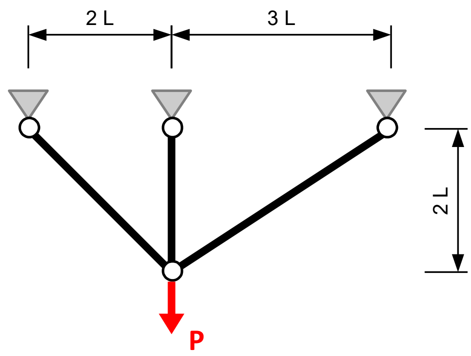
- **elastic-plastic analysis** – it requires an analysis of each stage of loading and finding the plastic region for each of that stage. It given full information on the process of loading, yet it is often troublesome from the mathematical point of view.

- **Determining the bearing capacity with the use of lower-bound and upper-bound estimates** – this approach base on an assumption that the system is a **rigid-ideally plastic system** (**with no hardening**). It makes use of two theorems on lower-bound and upper-bound estimates of the plastic bearing capacity. This approach enables to determine the conditions for the limit state to be reached and gives no information on the state of structure in intermediate stages of loading.

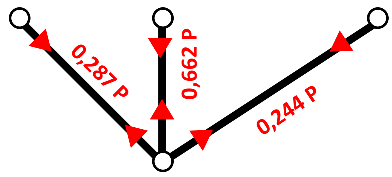
7.1 ELASTIC-PLASTIC ANALYSIS

EXAMPLE 1

Let's consider a simple statically indeterminate truss system.



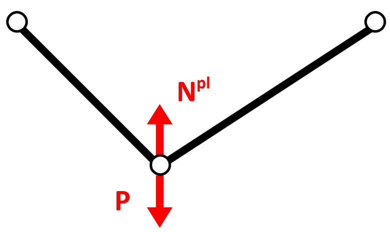
Our task is to find the maximum value of load parameter P , for which the system will collapse. We shall perform an elastic-plastic analysis. At the beginning the system works in an elastic range. In order to get the internal forces, we need to solve a statically indeterminate system:



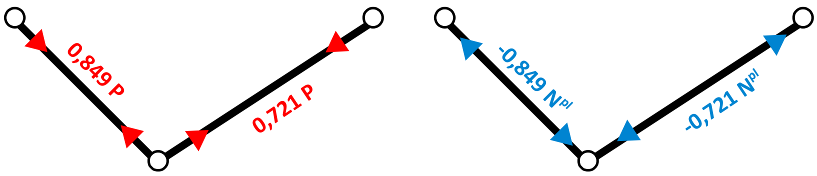
The greates force is present in the middle bar – it will yield as the first one. It will happen when:

$$0{,}662\,P = N^{pl} = \sigma_0 A \qquad \Rightarrow \qquad \bar{P} = 1{,}511\,N^{pl}$$

Yielding bar may deform freely (plastic flow) and the force in that bar is constantly equal N^{pl} . This situation may be modeled in such a way that the bar will be replaced by a system of two opposite force N^{pl} applied to end nodes of that bar.



The system above is still stable. Let's determine the internal forces in this new system. We may make use of the principle of superposition and determine the internal forces separately for external load and for the force N^{pl} and then we may simply add the results together.



We must check now for what value of P The force in other bars will reach N^{pl} .

Bar	Force in a bar N_i	$P:\ N_i = N^{pl}$
1	0,849 P – 0,849 N^{pl}	2,178 N^{pl}
3	0,721 P – 0,721 N^{pl}	2,387 N^{pl}

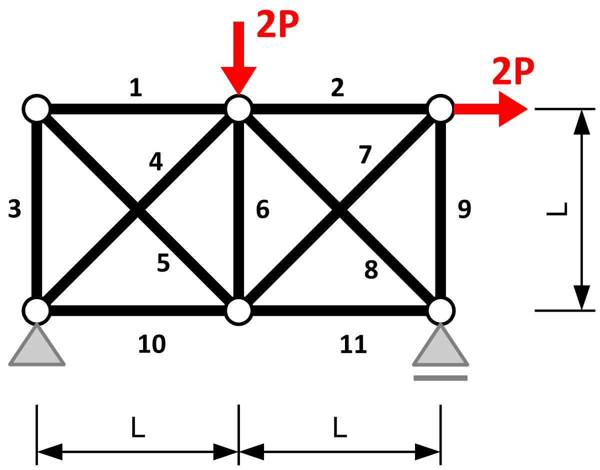
So the next bar which will yield in bar No 1. Summing it up:

- for $\bar{P} = \bar{P} = 1{,}511\sigma_0 A$ the first bar yield – it is bar No 2. The system is stable. The magnitude of external load corresponds with **limit elastic bearing capacity** (first yielding in a point) as well as **limit plastic bearing capacity** (first yielding of a cross-section) **of a system**.

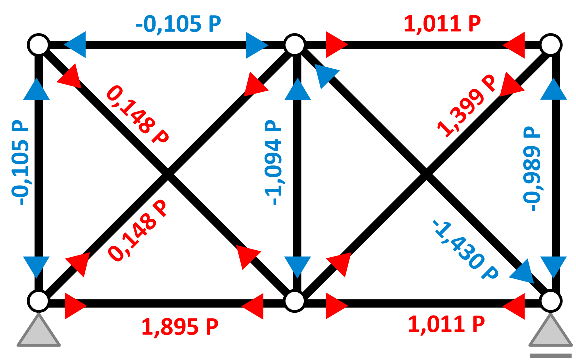
- for $P^* = 2{,}178\sigma_0 A$ next bar yield – it is bar No 1. The system collapses. Corresponding magnitude of load is the **bearing capacity of the system**.

EXAMPLE 2

Let's consider a statically indeterminate truss, in which all bars have the same cross-section.



Our task is to find the maximum value of load parameter P, for which the system will collapse. We shall perform an elastic-plastic analysis. At the beginning the system works in an elastic range. In order to get the internal forces, we need to solve a statically indeterminate system:

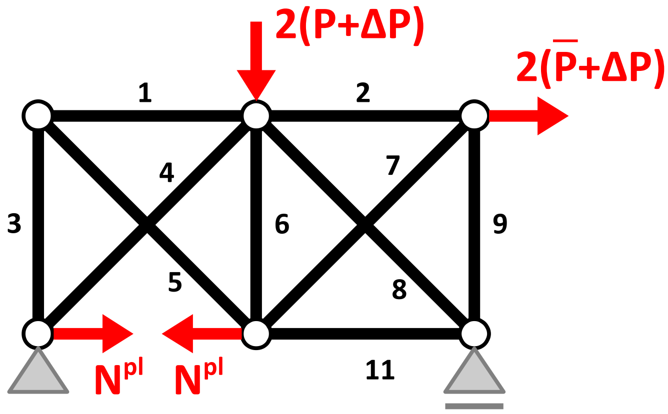


We can notice that the greatest axial force will be in bar No 10 – this is the place of first yielding. Due to yielding force $N^{pl} = \sigma_0 A$ will be present there and this value will remain unchanged (no hardening). Yielding will occur when:

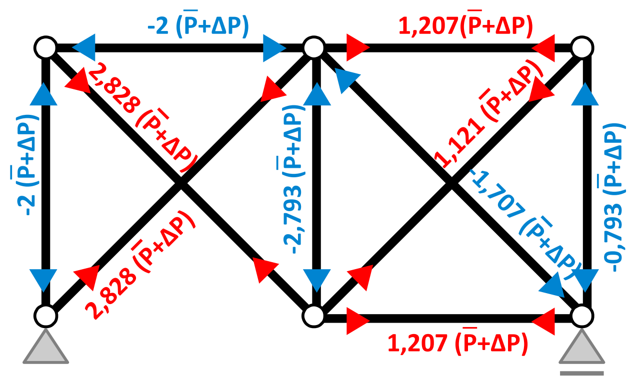
$$1{,}895\,P = \sigma_0 A \qquad \Rightarrow \qquad \bar{P} = 0{,}528\ N^{pl}$$

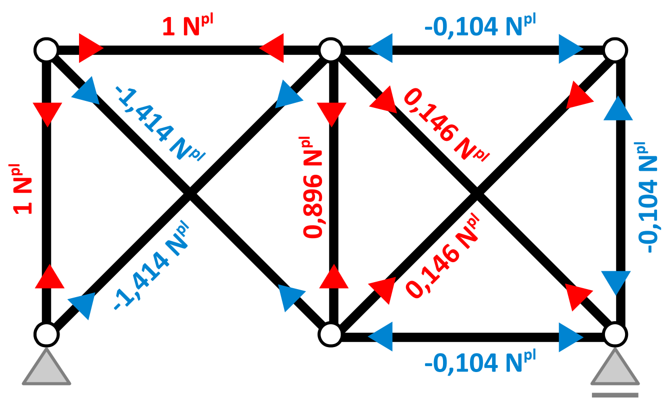
It must be remembered that the system may collapse even earlier due to e.g. Buckling of slender members heavily compressed.

We replace the bar No 10 with a system of two opposite forces N^{pl} applied to the and nodes of that bar – it is a tensile force.



The system is still stable. Let's determine the cross-sectional forces in this new system. We may make use of the principle of superposition and determine the forces separately due to external load and due to force in a yielding bar and then we may add the results together.





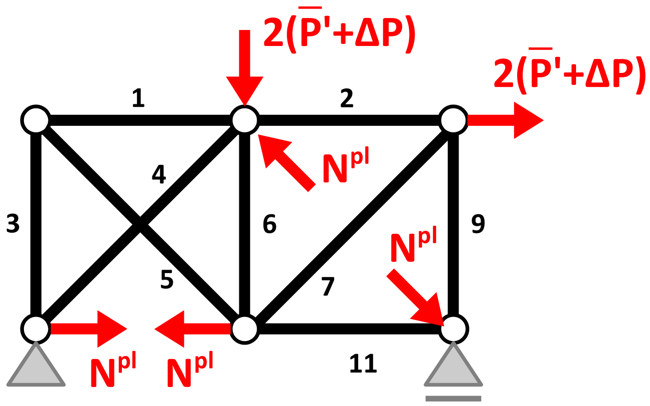
We must check what value of parameter P will result with the presence of force $\pm N^{pl}$ in each bar – the change of statical diagram of a system may lead not only to the change of value of force in the bars but also to the change of sign of force in some bars.

Bar	Force in bar N_i	$N_i(\Delta P=0)$	$\Delta P:\ N_i=-N^{pl}$	$\Delta P:\ N_i=N^{pl}$
1	$-2\,(\bar{P}+\Delta P) + N^{pl}$	$-0{,}056\,N^{pl}$	$0{,}472\,N^{pl}$	$-0{,}528\,N^{pl}$
2	$1{,}207\,(\bar{P}+\Delta P) - 0{,}104\,N^{pl}$	$0{,}533\,N^{pl}$	$-1{,}270\,N^{pl}$	$0{,}387\,N^{pl}$
3	$-2\,(\bar{P}+\Delta P) + N^{pl}$	$-0{,}056\,N^{pl}$	$0{,}472\,N^{pl}$	$-0{,}528\,N^{pl}$
4	$2{,}828\,(\bar{P}+\Delta P) - 1{,}414\,N^{pl}$	$0{,}079\,N^{pl}$	$-0{,}382\,N^{pl}$	$0{,}326\,N^{pl}$
5	$2{,}828\,(\bar{P}+\Delta P) - 1{,}414\,N^{pl}$	$0{,}079\,N^{pl}$	$-0{,}382\,N^{pl}$	$0{,}326\,N^{pl}$
6	$-2{,}793\,(\bar{P}+\Delta P) + 0{,}896\,N^{pl}$	$-0{,}579\,N^{pl}$	$0{,}151\,N^{pl}$	$-0{,}565\,N^{pl}$
7	$1{,}121\,(\bar{P}+\Delta P) + 0{,}146\,N^{pl}$	$0{,}738\,N^{pl}$	$-1{,}550\,N^{pl}$	$0{,}234\,N^{pl}$
8	$-1{,}707\,(\bar{P}+\Delta P) + 0{,}146\,N^{pl}$	$-0{,}755\,N^{pl}$	$0{,}143\,N^{pl}$	$-1{,}028\,N^{pl}$
9	$-0{,}793\,(\bar{P}+\Delta P) - 0{,}104\,N^{pl}$	$-0{,}523\,N^{pl}$	$0{,}602\,N^{pl}$	$-1{,}920\,N^{pl}$
10	N^{pl}	N^{pl}	-	-
11	$1{,}207\,(\bar{P}+\Delta P) - 0{,}104\,N^{pl}$	$0{,}533\,N^{pl}$	$-1{,}270\,N^{pl}$	$0{,}387\,N^{pl}$

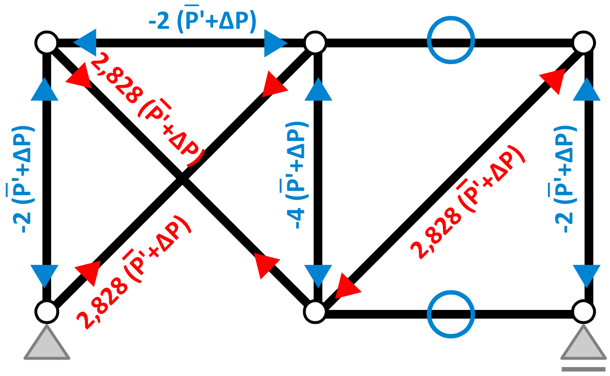
The least increment of load parameter which will result yielding of a bar is $\Delta P = 0{,}143\,N^{pl}$ - it will be bar No 8 that will yield. Load parameter is equal then:

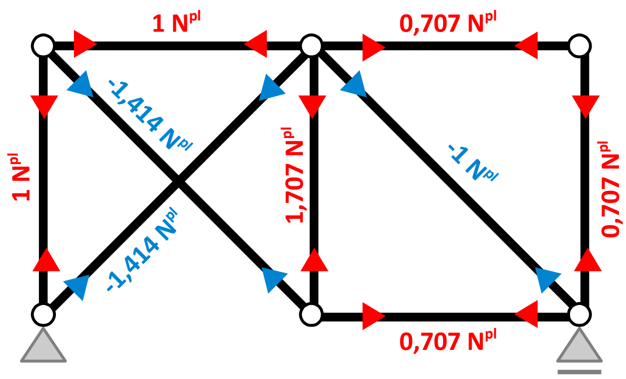
$$\bar{P}' = \bar{P} + \Delta P = 0{,}671\,N^{pl}$$

We replace the bar No 8 with a system of two opposite forces N^{pl} applied to the and nodes of that bar – this time it is a compressive force. Please note, that we neglect all buckling effects.



The system is still stable. Let's determine the cross-sectional forces in this new system. We may make use of the principle of superposition and determine the forces separately due to external load and due to force in a yielding bar and then we may add the results together.





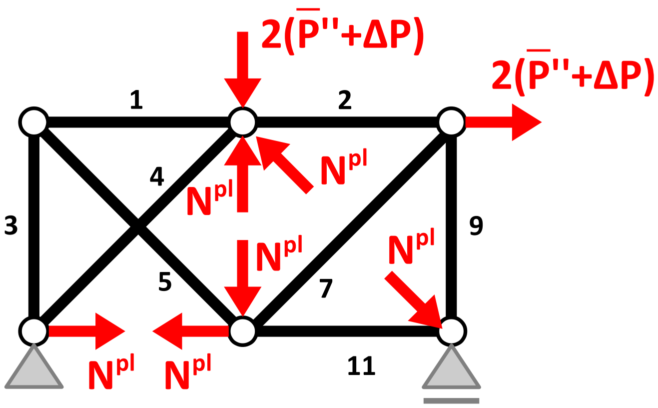
We must check what value of parameter P will result with the presence of force $\pm N^{pl}$ in each bar:

Bar	Force in bar N_i	$N_i(\Delta P=0)$	$\Delta P:\ N_i=-N^{pl}$	$\Delta P:\ N_i=N^{pl}$
1	$-2(\bar{P}'+\Delta P)+N^{pl}$	$-0{,}342N^{pl}$	$0{,}329\,N^{pl}$	$-0{,}671\,N^{pl}$
2	$-0{,}707\,N^{pl}$	$0{,}707\,N^{pl}$	-	-
3	$-2(\bar{P}+\Delta P)+N^{pl}$	$-0{,}342\,N^{pl}$	$0{,}329\,N^{pl}$	$-0{,}671\,N^{pl}$
4	$2{,}828(\bar{P}'+\Delta P)-1{,}414\,N^{pl}$	$0{,}484\,N^{pl}$	$-0{,}525\,N^{pl}$	$0{,}183\,N^{pl}$
5	$2{,}828(\bar{P}'+\Delta P)-1{,}414\,N^{pl}$	$0{,}484\,N^{pl}$	$-0{,}525\,N^{pl}$	$0{,}183\,N^{pl}$
6	$-4(\bar{P}'+\Delta P)+1{,}707\,N^{pl}$	$-0{,}977\,N^{pl}$	$0{,}00575\,N^{pl}$	$-0{,}494\,N^{pl}$
7	$2{,}828(\bar{P}'+\Delta P)-N^{pl}$	$0{,}898\,N^{pl}$	$-0{,}671\,N^{pl}$	$0{,}0362\,N^{pl}$
8	$-N^{pl}$	$-N^{pl}$	-	-
9	$-2(\bar{P}'+\Delta P)+0{,}707\,N^{pl}$	$-0{,}635\,N^{pl}$	$-0{,}183\,N^{pl}$	$-0{,}818\,N^{pl}$
10	N^{pl}	N^{pl}	-	-
11	$0{,}707\,N^{pl}$	$0{,}707\,N^{pl}$	-	-

The least increment of load parameter which will result yielding of a bar is- it $\Delta P=0{,}00575\,N^{pl}$ will be bar No 6 that will yield. Load parameter is equal then:

$$\bar{P}''=\bar{P}'+\Delta P\approx 0{,}676\,N^{pl}$$

We replace the bar No 6 with a system of two opposite forces N^{pl} applied to the and nodes of that bar – it is a compressive force.

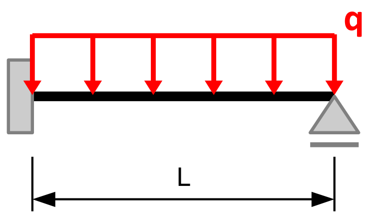


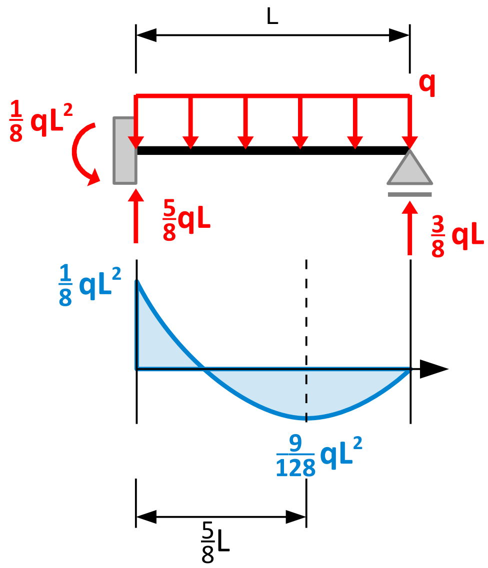
The system above is a mechanism – yielding of bars 10, 8 and 6 leads to a collapse of a system. Baring capacity of a system is equal:

$$\bar{P}^*\approx 0{,}676\,N^{pl}$$

EXAMPLE 3

A statically indeterminate beam is given. It is loaded with a uniform load q . Our taks it to find the value q , for which the system will collapse due to yielding. Elastic-plastic analysis will be performed. At the beginning the system works in elastic range – let's determine the distribution of cross-sectional forces.



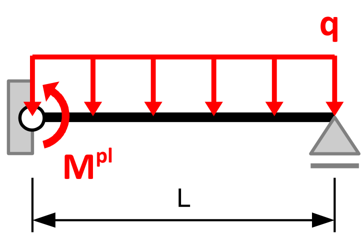


Greatest value of the bending moment occurs in the fixed support to the left – it is the place where the yielding will stars. It the limit plastic bearing capacity of a cross-section is denoted with M^{pl} , then yielding of that cross-section will correspond with value the **limit plastic bearing capacity of the system**:

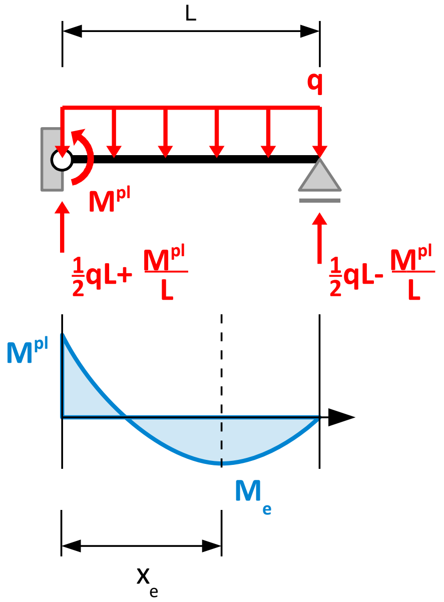
$$\bar{\bar{q}} = 8\frac{M^{pl}}{L^2}$$

Yielding of that cross-section results with creation of a plastic hinge. Plastic hinge for class 1 steel sections may deform freely, so it provides an additional degree of freedom for the system. The difference is that in the hinge there is constant value of bending moment M^{pl} , while in case of a regular hinge the value of a moment in a hinge must be zero.

Yielding of a cross-section will be accounted for by **introduction of a hinge at the left support** (fixed support is replaced with a pinned support) and of a **moment load** M^{pl} in neighboring cross-sections (in general, at both sides. In case of an end cross-section of a beam – only at one side). The system is still stable.



Let's determine the distribution of cross-sectional forces. We are looking for a cross-section in which yielding will occur next – it will be located in the place where the distribution of bending moments in the middle of the beam span reaches its local maximum. This second yielding will surely lead to collapse of the system.



Bending moment:

$$M(x) = -M^{pl} + \left(\frac{qL}{2} + \frac{M^{pl}}{L}\right)x - q\frac{x^2}{2}$$

Shear force

$$Q(x) = \frac{\mathrm{d}M}{\mathrm{d}x} = \left(\frac{qL}{2} + \frac{M^{pl}}{L}\right) - qx$$

Local maximum of moment distribution:

$$Q(x_e) = 0 \quad \Rightarrow \quad x_e = \frac{L}{2} + \frac{M^{pl}}{qL}$$

$$M(x_e) = \frac{(M^{pl})^2}{2qL^2} - \frac{M^{pl}}{2} + \frac{qL^2}{2}$$

When yielding occurs $M(x_e) = +M^{pl}$, what is satisfied when:

$$4(M^{pl})^2 - 12qL^2M^{pl} + q^2L^4 = 0 \quad \Rightarrow \quad \vee \begin{array}{l} q_1 = (6-4\sqrt{2})\dfrac{M^{pl}}{L^2} \approx 0{,}34315\dfrac{M^{pl}}{L^2} \\ q_2 = (6+4\sqrt{2})\dfrac{M^{pl}}{L^2} \approx 11{,}657\dfrac{M^{pl}}{L^2} \end{array}$$

Location x_e corresponding with the above values:

$$x_{e,1} = \left[\frac{1}{2} + \frac{1}{(6-4\sqrt{2})}\right]L \approx 3{,}4142\,L \quad , \quad x_{e2} = \left[\frac{1}{2} + \frac{1}{(6+4\sqrt{2})}\right]L \approx 0{,}58579\,L$$

Value q_1 is smaller than $\bar{\bar{q}}$, what would suggest an immediate collapse of the mid-span cross-section just after yielding of the support cross-section. It is visible however that for $\bar{\bar{q}}$ new statical diagram gives us exactly the same distribution of moments as the original diagram and the mid-span moment is smaller than the limit value. Another thing is that $x_{e,1}$ is beyond domain. Finally **bearing capacity of the beam** is equal:

$$q^* = (6+4\sqrt{2})\frac{M^{pl}}{L^2} \approx 11{,}657\frac{M^{pl}}{L^2}$$

It is worth to mention now, that presence of the plastic hinge does not change the distribution of bending moments in the moment of yielding, yet the distribution of deflection differs substantially – in an elastic range angle of deflection in fixed support is zero and after yielding a free rotation is allowed there, what results in increase of deflection in the span. It is difficult to determine this increment of deflection precisely. We know that **distribution of deflection depends on spatial distribution of flexural rigidity** $EI(x)$ **- and the rigidity depends on the range of plastic region in each bent cross-section**. Strict way of finding the deflection in the elastic-plastic range require determining the plastic range for each cross-section and on the basis of that, the distribution $EI(x)$ should be determined. After getting the distribution of flexural rigidity, the beam equation should be integrated:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[EI(x)\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right] = q(x)$$

For statically indeterminate system the distribution of cross-sectional forces depend on distribution of flexural rigidity and in elastic-plastic range the distribution of rigidity depends on the distribution of cross-sectional forces. We can see that those functions are coupled and **finding the deflection of a statically indeterminate system in elastic-plastic range is a non-linear problem**.

7.2 UPPER AND LOWER BOUND ESTIMATES OF BEARING CAPACITY

Estimate of the bearing capacity of the system is made according to the following theorems:

LOWER BOUND ESTIMATE THEOREM

The structure won't collapse or it will reach a limit equilibrium state due to external load, **if only it is possible to find a statically admissible stress field** corresponding with that load. In such a situation the **bearing capacity is at least the same as corresponding load or even higher** – it is a lower bound estimate.

Statical admissibility of the stress field requires that:

- stress field **is in equilibrium with external load,**
- stress field **satisfied the internal equilibrium conditions,**
- stress field **satisfies statical boundary conditions,**
- stress **do not exceed the limit value** σ_0 .

If for a given external load we find such a system of reactions and cross-sectional forces, that satisfies equilibrium conditions and boundary conditions and that corresponding stresses do not exceed the limit value, then the system will indeed bear that load, or even a greater load.

UPPER BOUND ESTIMATE THEOREM

The **structure will collapse, if only it is possible to find a kinematically admissible velocity field such that total power of external load is not lass than total power of internal forces**. In such a situation the **bearing capacity is at most the same as the one corresponding with that load, but it may lower** – it is upper bound estimate.

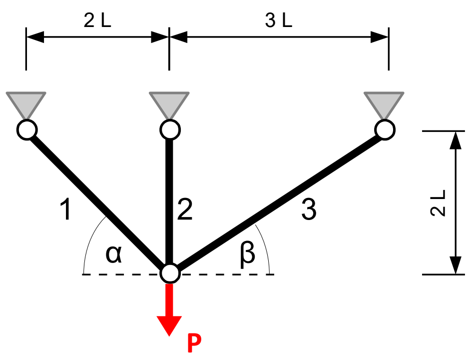
Kinematic admissibility of the velocity field requires that:

- velocity field **satisfies the kinematic boundary conditions** (it is consistent with applied constraints),
- velocity field is such that **displacement is continuous**,
- total power of external load on velocities is positive.

If for a given external load we find such a way of collapse that it is consistent with the applied supports and rules on distribution of velocities in a rigid body and the power of external load is not less than the power of internal forces, then the system will collapse indeed, however it is possible that it will collapse ever for smaller load.

EXAMPLE 4

Fin the bearing capacity of the structure presented in the picture with the use of the lower und upper bound theorems

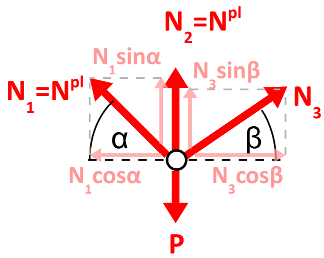


$$\sin\alpha = \frac{1}{\sqrt{2}} \qquad \cos\alpha = \frac{1}{\sqrt{2}}$$
$$\sin\beta = \frac{2}{\sqrt{13}} \qquad \cos\beta = \frac{3}{\sqrt{13}}$$

LOWER BOUND ESTIMATE – STATICAL APPROACH

The system will collapse if two of three bars yield. We have three possibilities:

1) Yielding of bars 1 and 2



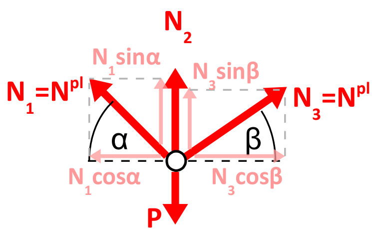
$$\Sigma X = -N^{pl}\cos\alpha + N_3\cos\beta = 0 \quad \Rightarrow$$
$$\Rightarrow \quad N_3 = \frac{\sqrt{26}}{6} N^{pl} \approx 0{,}84984\, N^{pl}$$

$N_3 < N^{pl}$ → the stress field is **statically admissible**.

$$\Sigma Y = N^{pl}\sin\alpha + N^{pl} - P + N_3\sin\beta = 0 \quad \Rightarrow \quad P \approx 2{,}1785\, N^{pl}$$

Bearing capacity is $\bar{\bar{P}} \approx 2{,}11785\, N^{pl}$ **or more.**

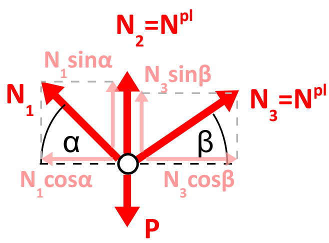
2) Yielding of bars 1 and 3



$$\Sigma X = -N^{pl}\cos\alpha + N^{pl}\cos\beta = 0 \quad \Rightarrow$$
$$\Rightarrow \quad \frac{1}{\sqrt{2}} N^{pl} \neq \frac{3}{\sqrt{13}} N^{pl}$$

equilibrium equation is not satisfied → the stress field is **statically inadmissible**.

3) Yielding of bars 2 and 3



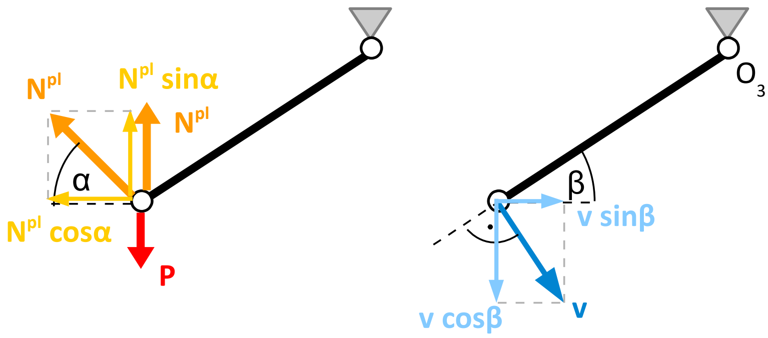
$$\Sigma X = -N_1\cos\alpha + N^{pl}\cos\beta = 0 \quad \Rightarrow$$
$$\Rightarrow \quad N_1 = \frac{3\sqrt{26}}{13} \approx 1{,}1767\, N^{pl}$$

$N_1 > N^{pl}$ → the stress field is **statically inadmissible**.

UPPER BOUND ESTIMATE – KINEMATIC APPROACH

The system will collapse if two of three bars yield. We have three possibilities:

1) **Yielding of bars 1 and 2**



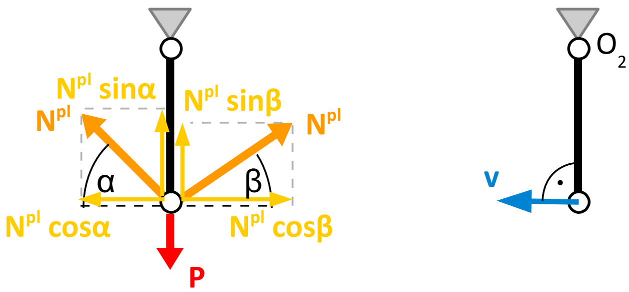
Let's compare power of external load and power of internal forces:

$$\dot{\Phi}_w = \dot{\Phi}_z \quad \Rightarrow$$
$$\Rightarrow \quad (N^{pl}\cos\alpha)\cdot(v\sin\beta) + (N^{pl}\sin\alpha)\cdot(v\cos\beta) + N^{pl}\cdot(v\cos\beta) = P\cdot(v\cos\beta)$$

$$P \approx 2{,}1785\,N^{pl}$$

Bearing capacity is $P^* \approx 2{,}11785\,N^{pl}$ **or less.**

2) **Yielding of bars 1 and 3**

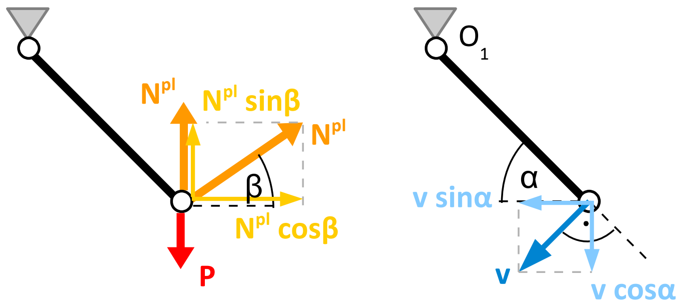


Let's compare power of external load and power of internal forces:

$$\dot{\Phi}_w = \dot{\Phi}_z \quad \Rightarrow$$
$$\Rightarrow \quad (N^{pl}\cos\alpha)\cdot v - (N^{pl}\cos\beta)\cdot v = 0$$

Total power of external load on a velocity field is non-positive →
the velocity field is **kinematically inadmissible**.

3) Yielding of bars 2 and 3



Let's compare power of external load and power of internal forces:

$$\dot{\Phi}_w = \dot{\Phi}_z \quad \Rightarrow$$
$$\Rightarrow \quad (N^{pl}\cos\beta)\cdot(v\sin\alpha) + (N^{pl}\sin\beta)\cdot(v\cos\alpha) + N^{pl}\cdot(v\cos\alpha) = P\cdot(v\cos\alpha)$$

$$P \approx 2{,}3868\, N^{pl}$$

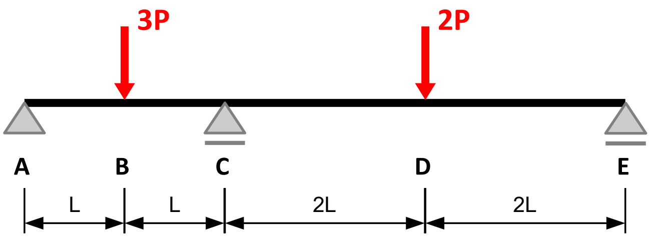
Bearing capacity is $P^* \approx 2{,}3868\, N^{pl}$ **or less.**

Let's summarize our results:

- **Lower bound estimate** indicate that **bearing capacity is at least** $$P^* \approx 2{,}11785\, N^{pl} \quad .$$
- **Upper bound estimate** indicate that **bearing capacity is at most** $$P^* \approx 2{,}11785\, N^{pl} \quad .$$
- **Bearing capacity** is then **equal** $P^* \approx 2{,}11785\, N^{pl}$

PRZYKŁAD 5

Find the bearing capacity of the system presented in the figure with the use of lower and upper bound theorems.

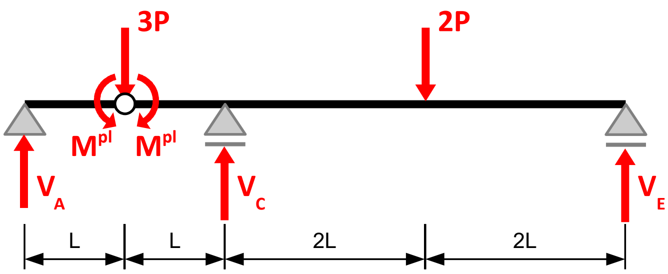


The system is **statically indeterminate with a single hyperstatic quantity – two plastic** hinges must be created in order to make the structure unstable. They will be created in the places where **maximum values of bending moments** are present. Since the structure is **loaded with point forces**, so the **distribution of bending moments is piece-wise linear and maximum values are reached at the boundaries of characteristic intervals** – either at the supported points or at the points of application of a point load. Plastic hinges will be created in those places.

LOWER BOUND ESTIMATE – STATICAL APPROACH

First yielding in cross-section B

We consider now a statically determinate beam:



$$\Sigma M_B^{\leftarrow} = 0: \quad -V_A L + M^{pl} = 0 \quad \Rightarrow \quad V_A = \frac{M^{pl}}{L}$$

$$\begin{cases} \Sigma M_B^{\rightarrow} = 0 \\ \Sigma Y = 0 \end{cases} \quad \begin{cases} -M^{pl} + V_C L - 6PL + 5V_E L = 0 \\ V_A + V_C + V_E - 5P = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} V_C = \dfrac{19}{4}P - \dfrac{3}{2}\dfrac{M^{pl}}{L} \\ V_E = \dfrac{1}{4}P - \dfrac{1}{2}\dfrac{M^{pl}}{L} \end{cases}$$

Bending moments in cross-sections C and D:

$$M(C) = 4V_E L - 4PL = 2M^{pl} - 3PL$$
$$M(D) = 2V_E L = M^{pl} + \frac{PL}{2}$$

Next yielding will occur in cross-section C when $M(C) = -M^{pl} \quad \Rightarrow \quad P = \frac{M}{L}$

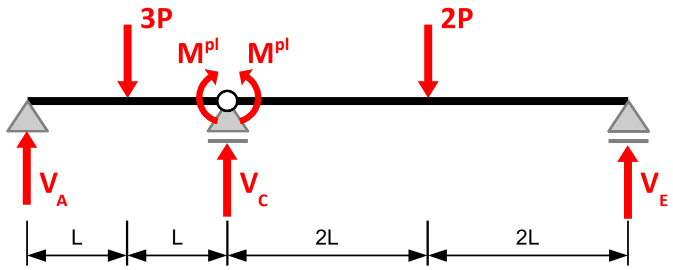
Then: $M(D) = \frac{3}{2} M^{pl} > M^{pl} \quad \Rightarrow$ the stress field is **statically inadmissible**.

Next yielding will occur in cross-section D when $M(D) = M^{pl} \quad \Rightarrow \quad P = 0$

Then: $M(C) = 2M^{pl} > M^{pl} \quad \Rightarrow$ the stress field is **statically inadmissible**.

First yielding in cross-section C

We consider now a statically determinate beam:



$$\Sigma M_C^{\leftarrow} = 0: \quad -M^{pl} + 3PL + 2V_A L = 0 \quad \Rightarrow \quad V_A = \frac{3}{2}P - \frac{M^{pl}}{2L}$$
$$\Sigma M_C^{\rightarrow} = 0: \quad M^{pl} - 4PL + 4V_E L = 0 \quad \Rightarrow \quad V_E = P - \frac{M^{pl}}{4L}$$

Bending moments in cross-sections B and D:

$$M(B) = V_A L = -\frac{1}{2}M^{pl} + \frac{3}{2}PL$$
$$M(D) = 2V_E L = -\frac{1}{2}M^{pl} + 2PL$$

Next yielding will occur in cross-section B when $M(B) = M^{pl} \quad \Rightarrow \quad P = \frac{M}{L}$

Then: $M(D) = \frac{3}{2}M^{pl} > M^{pl} \quad \Rightarrow$ the stress field is **statically inadmissible**.

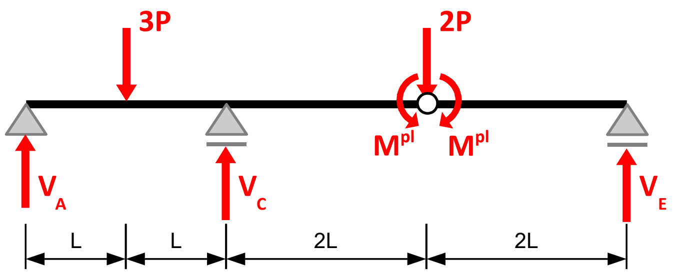
Next yielding will occur in cross-section D when $M(D) = M^{pl} \quad \Rightarrow \quad P = \frac{3}{4}\frac{M}{L}$

Then: $M(B) = \frac{5}{8}M^{pl} \quad \Rightarrow$ the stress field is **statically admissible**.

Bearing capacity is $P^* = \frac{3}{4}\frac{M}{L}$ **or more.**

First yielding in cross-section D

We consider now a statically determinate beam:



$$\Sigma M_D^{\rightarrow} = 0: \quad 2V_E L - M^{pl} = 0 \quad \Rightarrow \quad V_E = \frac{M^{pl}}{2L}$$

$$\begin{cases} \Sigma M_D^{\leftarrow} = 0 \\ \Sigma Y = 0 \end{cases} \quad \begin{cases} M^{pl} - 2V_C L + 9PL - 4V_A L = 0 \\ V_A + V_C + V_E - 5P = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} V_A = -\frac{1}{2}P + \frac{M^{pl}}{L} \\ V_B = \frac{11}{2}P - \frac{3}{2}\frac{M^{pl}}{L} \end{cases}$$

Bending moments in cross-sections B and C:

$$M(B) = V_A L = M^{pl} - \frac{PL}{2}$$
$$M(C) = 2V_A L - 3PL = 2M^{pl} + 4PL$$

Next yielding will occur in cross-section B when $M(B) = M^{pl} \quad \Rightarrow \quad P = 0$

Then: $M(C) = 2M^{pl} > M^{pl} \quad \Rightarrow$ the stress field is **statically inadmissible**.

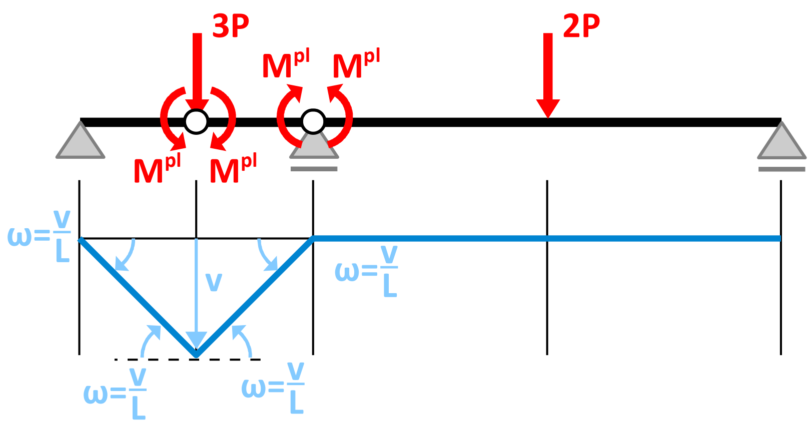
Next yielding will occur in cross-section C when $M(C) = -M^{pl} \quad \Rightarrow \quad P = \frac{3}{4}\frac{M}{L}$

Then: $M(B) = \frac{5}{8}M^{pl} \quad \Rightarrow$ the stress field is **statically admissible**.

Bearing capacity is $P^* = \frac{3}{4}\frac{M}{L}$ **or more.**

UPPER BOUND ESTIMATE – KINEMATIC APPROACH

Yielding in cross-sections B and C

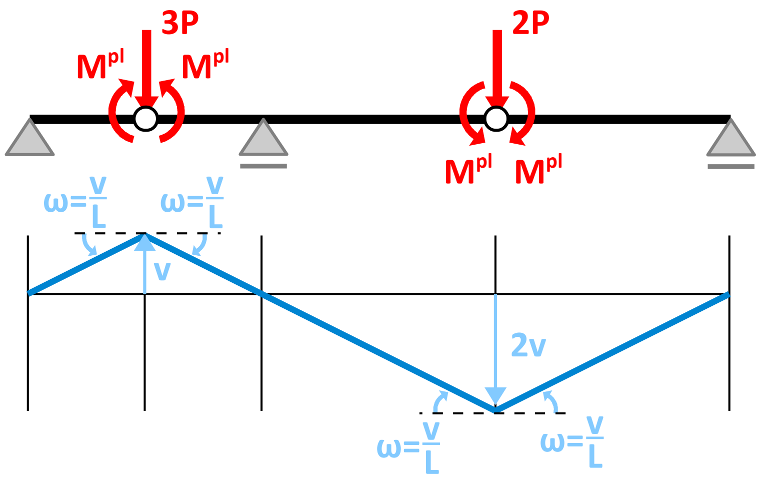


Let's compare power of external load and power of internal forces:

$$\dot{\Phi}_w = \dot{\Phi}_z \quad \Rightarrow \quad M^{pl}\cdot\frac{v}{L} + M^{pl}\cdot\frac{v}{L} + M^{pl}\cdot\frac{v}{L} = 3P\cdot v \quad \Rightarrow \quad P = \frac{M^{pl}}{L}$$

Bearing capacity is $P^* = \frac{M^{pl}}{L}$ **or less.**

Yielding in cross-sections B and D

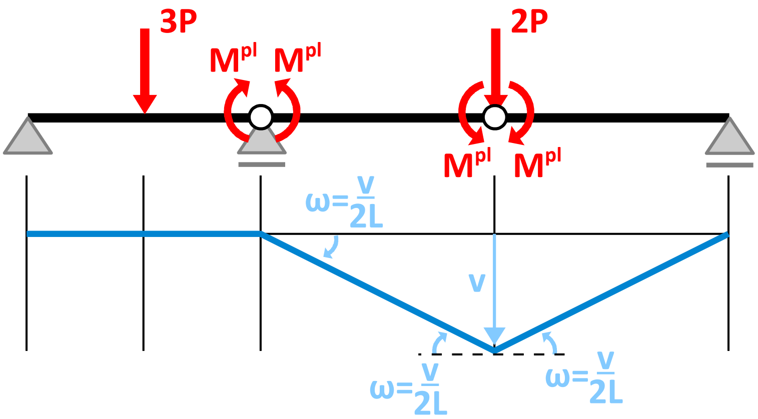


Let's compare power of external load and power of internal forces:

$$\dot{\Phi}_w = \dot{\Phi}_z \quad \Rightarrow \quad M^{pl}\cdot\frac{v}{L} + M^{pl}\cdot\frac{v}{L} + M^{pl}\cdot\frac{v}{L} + M^{pl}\cdot\frac{v}{L} = 2P\cdot 2v - 3P\cdot v \quad \Rightarrow \quad P = 4\frac{M^{pl}}{L}$$

Bearing capacity is $P^* = 4\frac{M^{pl}}{L}$ **or less.**

<u>**Yielding in cross-sections C and D**</u>



Let's compare power of external load and power of internal forces:

$$\dot{\Phi}_w = \dot{\Phi}_z \quad \Rightarrow \quad M^{pl} \cdot \frac{v}{2L} + M^{pl} \cdot \frac{v}{2L} + M^{pl} \cdot \frac{v}{2L} = 2P \cdot v \quad \Rightarrow \quad P = \frac{3}{4} \frac{M^{pl}}{L}$$

Bearing capacity is $P^* = \frac{3}{4} \frac{M^{pl}}{L}$ **or less.**

Let's summarize the obtained results:

- **Lower bound estimate** indicate that the **bearing capacity is at least**
$$P^* = \frac{3}{4} \frac{M}{L} \quad .$$
- **Upper bound estimate indicate that the bearing capacity is at most**
$$P^* = \frac{3}{4} \frac{M}{L} \quad .$$
- **Bearing capacity** is then **equal** $P^* = \frac{3}{4} \frac{M}{L}$