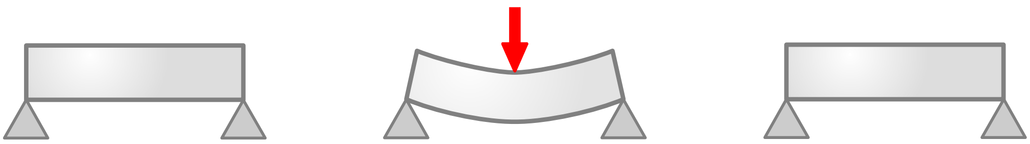
THEORY OF ELASTICITY

1. INTRODUCTION

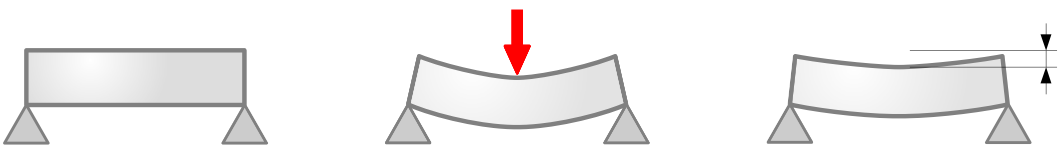
Theory of elasticity as well as **theory of plasticity** are **branches of physics dealing with description of motion of deformable bodies**. It is thus a branch of **mechanics**. As it is based on Newton's laws of motion it should be considered a **classical** theory, in the sense that it do not account for quantun or relativistic phenomena. Contrary to most commonly considered problems of mechanics, which deal with rigid bodies or systems of countable number of material points, theory of elasticity and plasticity deal with a countinuous model of solid – it will be called a **continuum**. From the mathematical point of view a **continuum is a subspace of three-dimensional space on which a continuous and differentiable function may be defined**. A point in this subspace will be called a **particle** however this notion has (almost) nothing to do with its true physical meaning. The shape of this subarea corresponds with shape of the considered body and functions (scalar, vector and tensor fields) defined on it will describe certain mechanical quantities as e.g. pressure, displacement, strain (i.e. relative elongation), stress (internal force density) etc.

Certain types of materials are distinguished – respective for certain theories used for describing them:

- **Elastic materials** – these are materials which deform under applied load, but **restore their initial shape when the load is removed**.



- **Plastic materials** – these are materials which deform under applied load and **restore their initial shape only partially**. The reversible part of deformation is called an elastic one while the **irreversible (permanent) deformation is the plastic one**.



- **Rheological materials** – these are materials in which the factor of time cannot be neglected. This concerns various phenomena, e.g.:
  - greater material stiffness in case of more rapid load (**viscosity**)
  - slow increase of deformation under constant load (**creep**)
  - slow decrease in the level of internal forces under constant deformation (**relaxation**)

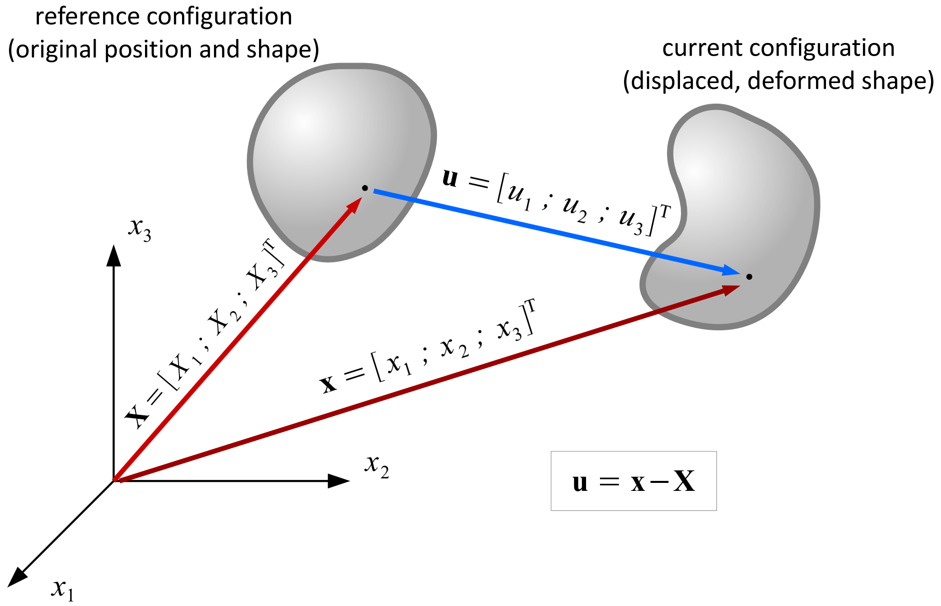
2. KINEMATICS

The deformation of a body will be fully described is position of every particle is known in every time.

- **Reference configuration** is the where the particles of body are at the beginning of the motion.
- **Reference configuration** is the where the the particles of body are in chosen time t .

We introduce following kinematic quantities:

- **position vector in reference configuration**: $\mathbf{X} = [X_1 \; ; \; X_2 \; ; \; X_3]^{\mathrm{T}}$
- **position vector in current configuration**: $\mathbf{x} = [x_1 \; ; \; x_2 \; ; \; x_3]^{\mathrm{T}}$
- **displacement vector**: $u_i = x_i - X_i \quad \Leftrightarrow \quad \mathbf{u} = \mathbf{x} - \mathbf{X}$
- **velocity vector**: $v_i = \dfrac{\mathrm{d}\,u_i}{\mathrm{d}\,t} \quad \Leftrightarrow \quad \mathbf{v} = \dot{\mathbf{u}}$
- **acceleration vector**: $a_i = \dfrac{\mathrm{d}^2 u_i}{\mathrm{d}\,t^2} \quad \Leftrightarrow \quad \mathbf{a} = \ddot{\mathbf{u}}$



- Components of position vector in reference configuration are called **Lagrange** or **material cordinates**.
- Components of position vector in current configuration are called **Euler** or **spatial cordinates**.

TWO POSSIBLE DESCRIPTIONS

Description of deformation may be performed in two ways:

- **MATERIAL DESCRIPTION** (**Lagrangian description**) – material coordinates are considered independent variables while spatial coordinates are considered unknown function of the latter:

$$\mathbf{x} = \mathbf{x}(t, \mathbf{X})$$

In material description, in every moment t we answer the following question:

„Consider a particle which at the beginning of the motion was located in point $\mathbf{X}$ *. In what point of space* $\mathbf{x}$ *is it now located?"*

- **SPATIAL DESCRIPTION** (**Eulerian description**) – spatial coordinates are considered independent variables while material coordinates are considered unknown function of the latter:

$$\mathbf{X} = \mathbf{X}(t, \mathbf{x})$$

In spatial description, in every moment t we answer the following question:

„Consider a point in space $\mathbf{x}$ *. If there is a particle there, what was its initial position* $\mathbf{X}$ *at the beginning of motion"*

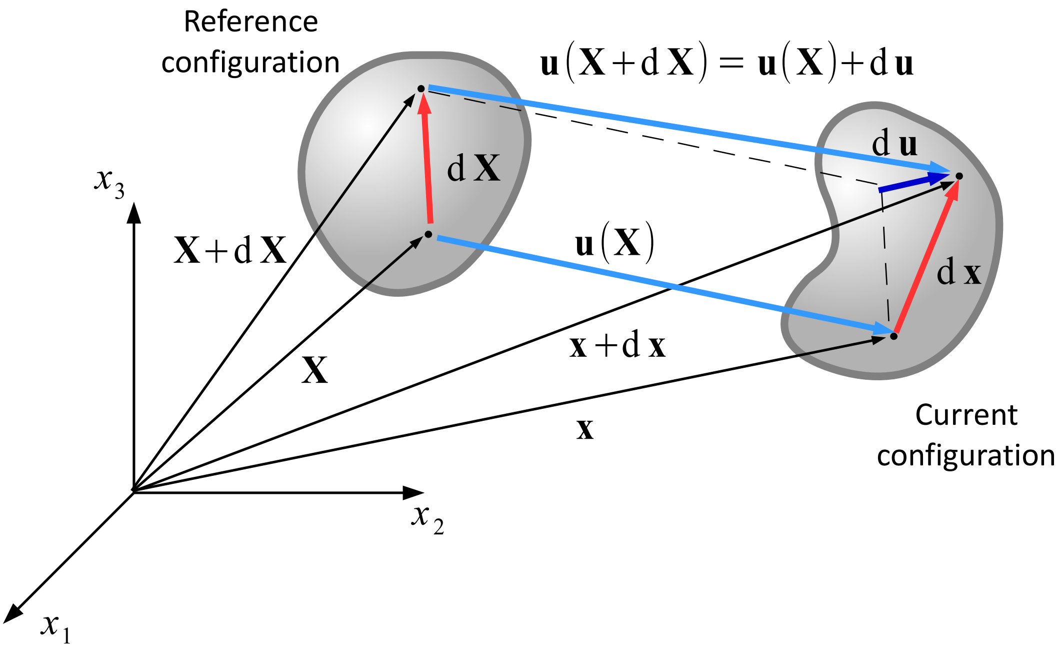
- Both descriptions are equivalent.
- **Material** description is usually used for **solids**.
- **Spatial** description is usually used for **fluids**.

In case of spatial description, when calculating time derivatives of position vector (velocity, acceleration) one must remember that $\mathbf{x}$ depends on time (particles are in motion), so time derivative must account for this implicit time-dependence :

- **material description**: $\dfrac{\mathrm{d}}{\mathrm{d}t} = \dfrac{\partial}{\partial t}$
- **spatial description**: $\dfrac{\mathrm{d}}{\mathrm{d}t} = \dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial x_1}\dfrac{\partial x_1}{\partial t} + \dfrac{\partial}{\partial x_2}\dfrac{\partial x_2}{\partial t} + \dfrac{\partial}{\partial x_3}\dfrac{\partial x_3}{\partial t}$

DEFORMATION GRADIENT

Let's consider a small **material fiber** – let's call it a **differential linear element** - it will be described by an infinitely small vector $\mathrm{d}\mathbf{X}$. After deformation it will change its position, orientation and length and shape. Considering it very small we may assume that it will be still approximately straight and we will describe it with another infinitely small vector $\mathrm{d}\mathbf{x}$.



$$\mathbf{X} = [X_1\,;\,X_2\,;\,X_3]^{\mathrm{T}} \qquad \mathbf{X}+\mathrm{d}\mathbf{X} = [X_1+\mathrm{d}X_1\,;\,X_2+\mathrm{d}X_2\,;\,X_3+\mathrm{d}X_3]^{\mathrm{T}}$$
$$\mathbf{x} = [x_1\,;\,x_2\,;\,x_3]^{\mathrm{T}} \qquad \mathbf{x}+\mathrm{d}\mathbf{x} = [x_1+\mathrm{d}x_1\,;\,x_2+\mathrm{d}x_2\,;\,x_3+\mathrm{d}x_3]^{\mathrm{T}}$$

Let's consider a material description of deformation: $\mathbf{x} = \mathbf{x}(\mathbf{X})$. This function may be expanded into a power series in the neighborhood of $\mathbf{X}=\mathbf{0}$ (assuming that the origin of an auxiliary coordinate system coincides at the tail of $\mathrm{d}\mathbf{X}$):

$$\mathrm{d}x_i = \frac{\partial x_i}{\partial X_1}\mathrm{d}X_1 + \frac{\partial x_i}{\partial X_2}\mathrm{d}X_2 + \frac{\partial x_i}{\partial X_3}\mathrm{d}X_3 + \ldots \qquad i=1,2,3$$

Neglecting higher order terms gives us (using Einstein's notation):

$$\mathrm{d}x_i \approx \sum_{i=1}^{3} \frac{\partial x_i}{\partial X_j}\mathrm{d}X_j = \frac{\partial x_i}{\partial X_j}\mathrm{d}X_j$$

We may define a two-index quantity called a **material deformation gradient** defined as

$$\boxed{F_{ij} = \frac{\partial x_i}{\partial X_j} \quad \Leftrightarrow \quad \mathbf{F}=\mathbf{x}\otimes\nabla_{\mathbf{X}} \quad \Leftrightarrow \quad \mathbf{F} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\mathbf{X}}}$$

so that:

$$\boxed{\mathrm{d}x_i = F_{ij}\,\mathrm{d}X_j \quad \Leftrightarrow \quad \mathrm{d}\mathbf{x}=\mathbf{F}\cdot\mathrm{d}\mathbf{X}}$$

Similarly one may define the **spatial deformation gradient**:

$$f_{ij} = \frac{\partial X_i}{\partial x_j} \quad \Leftrightarrow \quad \mathbf{f} = \mathbf{X} \otimes \nabla_{\mathbf{x}} \quad \Leftrightarrow \quad \mathbf{f} = \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}\mathbf{x}}$$

so that:

$$\mathrm{d}X_i = f_{ij}\,\mathrm{d}x_j \quad \Leftrightarrow \quad \mathrm{d}\mathbf{X} = \mathbf{f} \cdot \mathrm{d}\mathbf{x}$$

We may write down

$$\begin{cases} \mathrm{d}x_i = F_{ij}\,\mathrm{d}X_j \\ \mathrm{d}X_j = f_{jk}\,\mathrm{d}x_k \end{cases} \quad \Rightarrow \quad \mathrm{d}x_i = F_{ij} f_{jk}\,\mathrm{d}x_k \quad \Rightarrow \quad F_{ij} f_{jk} = \delta_{ik}$$

so **the material deformation gradient is an inverse of the spatial deformation gradient and vice versa:**

$$\mathbf{f} = \mathbf{F}^{-1}$$

In order for the description to be invertible we will require that:

- **Determinant of $\mathbf{F}$ is unequal 0** – otherwise our description may result in a situation in which many possible current positions correspond with a single particle or many particles are located in the same point after deformation.
- **Determinant of $\mathbf{F}$ is positive** – otherwise the material behaves as if it was put inside out.

It is sufficient to write, that:

$$\det \mathbf{F} = \frac{1}{\det \mathbf{f}} > 0$$

DISPLACEMENT GRADIENT

We define also:

- **material displacement gradient**: $H_{ij} = \dfrac{\partial u_i}{\partial X_j} = F_{ij} - \delta_{ij} \quad \Leftrightarrow \mathbf{H} = \dfrac{\partial \mathbf{u}}{\partial \mathbf{X}} = \mathbf{F} - \mathbf{1}$
- **spatial displacement gradient**: $h_{ij} = \dfrac{\partial u_i}{\partial x_j} = \delta_{ij} - f_{ij} \quad \Leftrightarrow \mathbf{h} = \dfrac{\partial \mathbf{u}}{\partial \mathbf{x}} = \mathbf{1} - \mathbf{f}$

POLAR DECOMPOSITION OF DEFORMATION GRADIENT

It can be shown that deformation tensor may considered a product of two other second-rank tensors, one of which is a symmetric one, while the other one is an orthogonal one.

$$\mathbf{F} = \mathbf{V}\,\mathbf{R} = \mathbf{R}\,\mathbf{U}$$

$\mathbf{V}$ – left stretch tensor - symmetric tensor: $\mathbf{V}^{\mathrm{T}} = \mathbf{V}$
$\mathbf{U}$ – right stretch tensor – symmetric tensor $\mathbf{U}^{\mathrm{T}} = \mathbf{U}$
$\mathbf{R}$ – rotation tensor – orthogonal tensor $\mathbf{R}^{\mathrm{T}} = \mathbf{R}^{-1}$

Such a decomposition is unique.

ROTATION TENSOR

Interpretation of $\mathbf{R}$ as a rotation tensor mat be justified as follows. Consider a material fiber $\mathrm{d}\,\mathbf{X}$ undergoing deformation such that $\mathbf{F} = \mathbf{R}$ ($\mathbf{U} = \mathbf{V} = \mathbf{1}$). Then the length before and after deformation is equal:

$$|\mathrm{d}\,\mathbf{X}| = \sqrt{\mathrm{d}\,\mathbf{X}^{\mathrm{T}}\cdot \mathrm{d}\,\mathbf{X}}$$

$$|\mathbf{R}\cdot \mathrm{d}\,\mathbf{X}| = \sqrt{(\mathbf{R}\cdot \mathrm{d}\,\mathbf{X})^{\mathrm{T}}\cdot(\mathbf{R}\cdot \mathrm{d}\,\mathbf{X})} = \sqrt{\mathrm{d}\,\mathbf{X}^{\mathrm{T}}\cdot\mathbf{R}^{\mathrm{T}}\cdot\mathbf{R}\cdot \mathrm{d}\,\mathbf{X}} = \sqrt{\mathrm{d}\,\mathbf{X}^{\mathrm{T}}\cdot\mathbf{R}^{-1}\cdot\mathbf{R}\cdot \mathrm{d}\,\mathbf{X}} = \sqrt{\mathrm{d}\,\mathbf{X}^{\mathrm{T}}\cdot \mathrm{d}\,\mathbf{X}} = |\mathrm{d}\,\mathbf{X}|$$

So such a deformation do not change the length of the fiber – only orientation is changed, so it can be considered a rotation.

STRETCH TENSOR

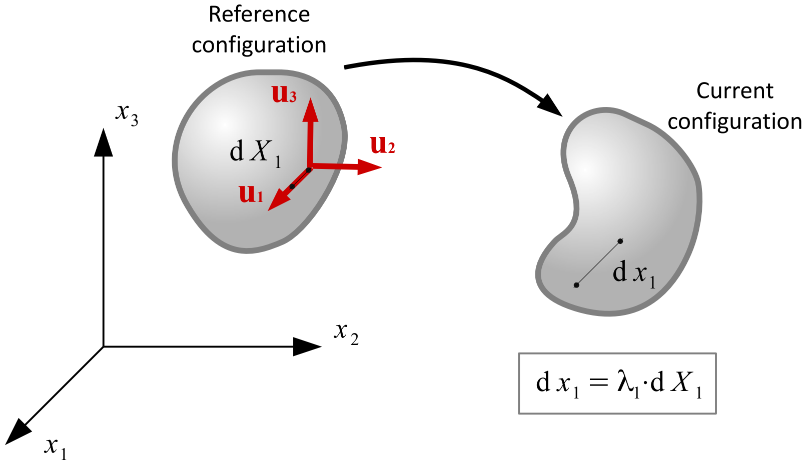
Interpretation of $\mathbf{U}$ and $\mathbf{V}$ as a stretch tensor mat be justified as follows. Consider a material fiber $\mathrm{d}\,\mathbf{X}$ undergoing deformation such that $\mathbf{F} = \mathbf{U} = \mathbf{V}$ ($\mathbf{R} = \mathbf{1}$). Now, we can consider an **eigenvalue problem** for e.g. $\mathbf{U}$, namely looking for such vectors $\mathrm{d}\,\mathbf{X}$ (**eigenvectors** of $\mathbf{U}$) and such scalars λ (**eigenvalues** of $\mathbf{U}$) that:

$$\mathbf{U}\cdot \mathrm{d}\,\mathbf{X} = \lambda\,\mathrm{d}\,\mathbf{X}$$

It can be seen that for such vectors a general deformation is reduced to simple scaling – change of length without change in orientation (stretching or compressing the material fibres). It can be shown that **for symmetric tensors (such as $\mathbf{U}$ and $\mathbf{V}$) there are always 3 real eigenvalues (principal stretches) and corresponding 3 mutually orthogonal eigenvectors**. Any three mutually orthogonal vectors may constitute a basis for three-dimensional vector space. We may also introduce a new Cartesian coordinate system axes of which are parallel to the eigenvectors of $\mathbf{U}$.

In such a coordinate system:

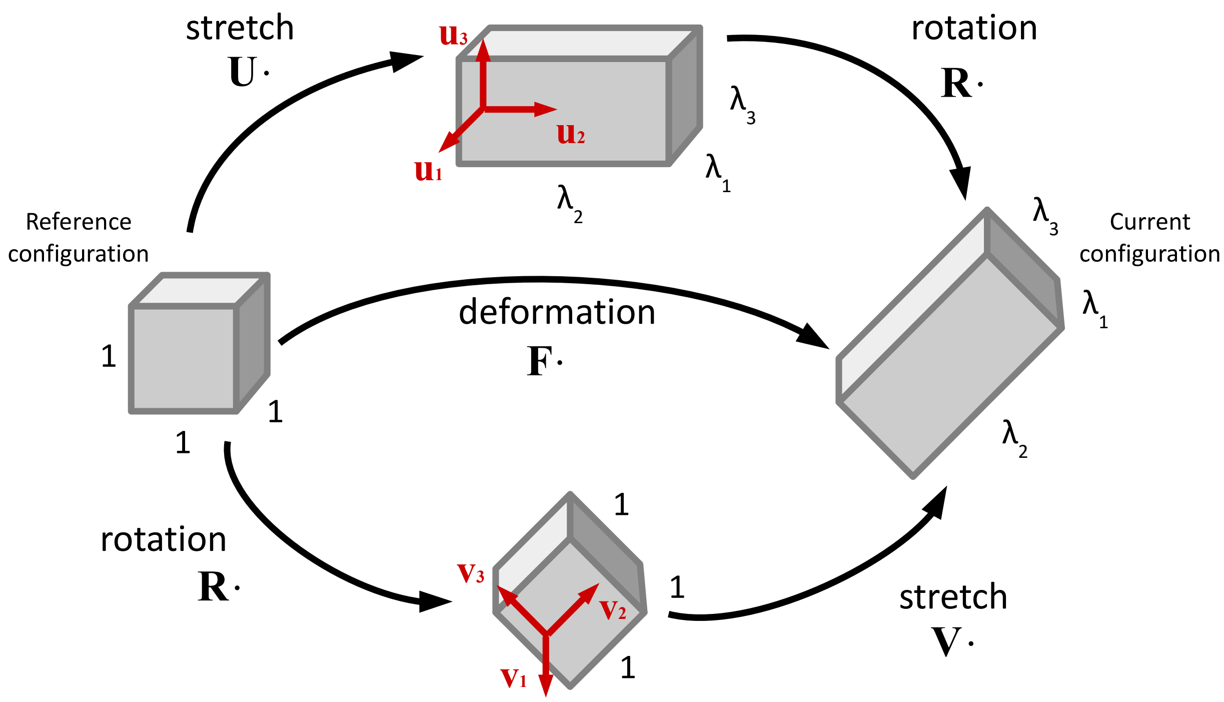
$$\mathbf{U}\cdot\mathrm{d}\,\mathbf{X}_1 = \lambda_1\cdot\mathrm{d}\,\mathbf{X}_1$$
$$\mathbf{U}\cdot\mathrm{d}\,\mathbf{X}_3 = \lambda_3\cdot\mathrm{d}\,\mathbf{X}_2$$
$$\mathbf{U}\cdot\mathrm{d}\,\mathbf{X}_2 = \lambda_2\cdot\mathrm{d}\,\mathbf{X}_3$$



Eigenvalues of $\mathbf{V}$ are the same as of $\mathbf{U}$ - only the corresponding vectors are oriented in a different way.

FUNDAMENTAL THEOREM ON KINEMATICS OF DEFORMABLE SOLID

Deformation of any infinitely small material fiber may is a composition of parallel (rigid) translation, rotation and elongation (stretching or compression) along the directions of eigenvectors of stretch tensor.



DEFORMATION TENSOR

Let's try to express the length of deformed material fiber in terms of its reference dimensions:

Before deformation: $\mathrm{d}\mathbf{X}=[\mathrm{d}X_1\,;\,\mathrm{d}X_2\,;\,\mathrm{d}X_3]^{\mathrm{T}}$ $\quad \mathrm{d}S\approx|\mathrm{d}\mathbf{X}|=\sqrt{\mathrm{d}\mathbf{X}\cdot\mathrm{d}\mathbf{X}}$

After deformation: $\mathrm{d}\mathbf{x}=[\mathrm{d}x_1\,;\,\mathrm{d}x_2\,;\,\mathrm{d}x_3]^{\mathrm{T}}$ $\quad \mathrm{d}s\approx|\mathrm{d}\mathbf{x}|=\sqrt{\mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x}}$

$$\mathrm{d}s=\sqrt{\mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x}}=\sqrt{\mathrm{d}x_i\mathrm{d}x_i}=\sqrt{F_{ij}\mathrm{d}X_j F_{ik}\mathrm{d}X_k}=\sqrt{C_{jk}\mathrm{d}X_j\mathrm{d}X_k}$$

We introduce a quantity defined as follows:

$$C_{jk}=F_{ij}F_{ik}=\frac{\partial x_i}{\partial X_j}\frac{\partial x_i}{\partial X_k}\quad\Leftrightarrow\quad\mathbf{C}=\mathbf{F}^{\mathrm{T}}\cdot\mathbf{F}$$

Quantity $\mathbf{C}$ is called material deformation tensor or **right Cauchy – Green deformation tensor**. It is a **symmetric tensor.** It can be noticed that:

$$\mathbf{C}=\mathbf{F}^{\mathrm{T}}\cdot\mathbf{F}=(\mathbf{R}\cdot\mathbf{U})^{\mathrm{T}}\cdot(\mathbf{R}\cdot\mathbf{U})=\mathbf{U}\cdot(\mathbf{R}^{-1}\cdot\mathbf{R})\cdot\mathbf{U}=\mathbf{U}^2$$

As a conclusion we may state that **eigenvectors of $\mathbf{C}$ are the same as those of $\mathbf{U}$ and corresponding eigenvalues are (algebraic) squares of corresponding principal stretches**. If a body is not deformed (rigid motion) then $\mathbf{C}=\mathbf{1}$. Similarly we may express the length of undeformed material fiber in terms of its deformed dimensions:

$$\mathrm{d}S=\sqrt{\mathrm{d}\mathbf{X}\cdot\mathrm{d}\mathbf{X}}=\sqrt{\mathrm{d}X_i\mathrm{d}X_i}=\sqrt{f_{ij}\mathrm{d}x_j f_{ik}\mathrm{d}x_k}=\sqrt{c_{jk}\mathrm{d}x_j\mathrm{d}x_k}$$

where spatial deformation tensor or **Cauchy deformation tensor** is defined as follows:

$$c_{kl}=f_{ik}f_{il}=\frac{\partial X_i}{\partial x_k}\frac{\partial X_i}{\partial x_l}\quad\Leftrightarrow\quad\mathbf{c}=(\mathbf{F}^{-1})^{\mathrm{T}}\cdot\mathbf{F}^{-1}$$

It is a **symmetric tensor.** Other symmetric deformation tensors may also be defined as follows:

- left Cauchy-Green deformation tensor (material description)

$$B_{jk}=F_{ji}F_{ki}=\frac{\partial x_i}{\partial X_j}\frac{\partial x_k}{\partial X_i}\quad\Leftrightarrow\quad\mathbf{B}=\mathbf{F}\cdot\mathbf{F}^{\mathrm{T}}=\mathbf{c}^{-1}$$
$$\mathbf{B}=\mathbf{F}\cdot\mathbf{F}^{\mathrm{T}}=(\mathbf{V}\cdot\mathbf{R})\cdot(\mathbf{V}\cdot\mathbf{R})^{\mathrm{T}}=\mathbf{V}\cdot(\mathbf{R}^{-1}\cdot\mathbf{R})\cdot\mathbf{V}=\mathbf{V}^2$$

- Finger tensor (spatial description)

$$b_{kl}=f_{ki}f_{li}=\frac{\partial X_k}{\partial x_i}\frac{\partial X_l}{\partial x_i}\quad\Leftrightarrow\quad\mathbf{b}=\mathbf{F}^{-1}\cdot(\mathbf{F}^{-1})^{\mathrm{T}}=\mathbf{C}^{-1}$$

STRAIN TENSOR

The most commonly used measures of deformation are so called strain tensors, which arise when a difference in squares of current and reference lengths of material fibers are caluclated:

$$\mathrm{d}s^2 - \mathrm{d}S^2 = \mathrm{d}x_i \mathrm{d}x_i - \mathrm{d}X_j \mathrm{d}X_j = (C_{jk} - \delta_{jk})\mathrm{d}X_j \mathrm{d}X_k = 2 \cdot \underbrace{\frac{C_{jk} - \delta_{jk}}{2}}_{E_{jk}} \mathrm{d}X_j \mathrm{d}X_k$$

Material strain tensor, or **Green – de Saint-Venant strain tensor** or **Green – Lagrange strain tensor** is defined as follows:

$$\boxed{E_{jk} = \frac{1}{2}(C_{jk} - \delta_{jk}) \quad \Leftrightarrow \quad \mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{1})}$$

It is a **symmetric tensor**. If a body is not deformed (rigid motion) then $\mathbf{E} = \mathbf{0}$. Similarly – in spatial description – we obtain:

$$\mathrm{d}s^2 - \mathrm{d}S^2 = \mathrm{d}x_i \mathrm{d}x_i - \mathrm{d}X_j \mathrm{d}X_j = (\delta_{jk} - c_{jk})\mathrm{d}x_j \mathrm{d}x_k = 2 \cdot \underbrace{\frac{\delta_{jk} - c_{jk}}{2}}_{e_{jk}} \mathrm{d}x_j \mathrm{d}x_k$$

where **spatial strain tensor** or **Almansi-Hamel strain tensor** is defined as follows:

$$\boxed{e_{jk} = \frac{1}{2}(\delta_{jk} - c_{jk}) \quad \Leftrightarrow \quad \mathbf{e} = \frac{1}{2}(\mathbf{1} - \mathbf{c})}$$

INTERPRETATION OF COMPONENTS OF STRAIN TENSOR

A measure of deformation of a material fiber may be its **relative elongation**, namely a **ratio between total elongation and initial length**:

$$\Delta = \frac{\mathrm{d}s - \mathrm{d}S}{\mathrm{d}S} = \frac{\mathrm{d}s}{\mathrm{d}S} - 1 \quad \Leftrightarrow \quad \mathrm{d}s = (1 + \Delta)\mathrm{d}S$$

It is a dimensionless quantity. Let's consider three material fibers, each one of it being parallel to one of the axes of chosen coordinate system. Then, relative elongation of the one parallel to x_1 may be calculated as follows:

Differential linear element:	$\mathrm{d}\mathbf{X} = [\mathrm{d}X_1 \,;\, 0 \,;\, 0]^{\mathrm{T}}$
Reference length:	$\mathrm{d}S = \lvert\mathrm{d}\mathbf{X}\rvert = \mathrm{d}X_1$
Current length:	$\mathrm{d}s = \lvert\mathrm{d}\mathbf{x}\rvert = \sqrt{C_{ij}\mathrm{d}X_i \mathrm{d}X_j} = \sqrt{C_{11}}\mathrm{d}X_1$
Relative elaongation:	$\Delta = \frac{\mathrm{d}s}{\mathrm{d}S} - 1 = \sqrt{C_{11}} - 1$

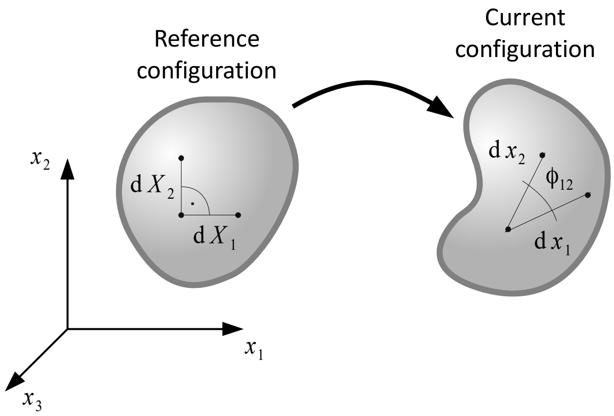
In general, we could write:

$$\begin{cases} \Delta_1 = \sqrt{C_{11}} - 1 = \sqrt{2E_{11}+1} - 1 \\ \Delta_2 = \sqrt{C_{22}} - 1 = \sqrt{2E_{22}+1} - 1 \\ \Delta_3 = \sqrt{C_{33}} - 1 = \sqrt{2E_{33}+1} - 1 \end{cases}$$

It can be seen that relative elongations depend on diagonal components of strain tensor – that's why **we call the diagonal components of $\mathbf{E}$ the linear strains.**

Let's consider now a change in an angle between two fibers. Let's consider two linear elements:

$$\mathrm{d}\mathbf{X}^{(1)} = [\mathrm{d}X_1 \,;\, 0 \,;\, 0]^{\mathrm{T}}$$
$$\mathrm{d}\mathbf{X}^{(2)} = [0 \,;\, \mathrm{d}X_2 \,;\, 0]^{\mathrm{T}}$$



After deformation:

$$\mathrm{d}\mathbf{x}^{(1)} = \mathrm{d}X_1[F_{11} \,;\, F_{21} \,;\, F_{31}]^{\mathrm{T}}$$
$$\mathrm{d}\mathbf{x}^{(2)} = \mathrm{d}X_2[F_{12} \,;\, F_{22} \,;\, F_{32}]^{\mathrm{T}}$$

An angle between two vectors may be found with the use of the dot product:

$$\cos\phi_{12} = \frac{\mathrm{d}\mathbf{x}^{(1)} \cdot \mathrm{d}\mathbf{x}^{(2)}}{|\mathrm{d}\mathbf{x}^{(1)}| \cdot |\mathrm{d}\mathbf{x}^{(2)}|} = \frac{\mathrm{d}X_1 \mathrm{d}X_2 (F_{11}F_{12} + F_{21}F_{22} + F_{31}F_{32})}{\mathrm{d}X_1\sqrt{F_{11}^2 + F_{21}^2 + F_{31}^2} \cdot \mathrm{d}X_2\sqrt{F_{12}^2 + F_{22}^2 + F_{32}^2}}$$

Making use of the definition of deformation tensor:

$$C_{ij} = F_{ki}F_{kj} = F_{1i}F_{1j} + F_{2i}F_{2j} + F_{3i}F_{3j} \quad ,$$

we can finally write:

$$\cos\phi_{12} = \frac{C_{12}}{\sqrt{C_{11}} \cdot \sqrt{C_{22}}} = \frac{2E_{12}}{\sqrt{(2E_{11}+1)(2E_{22}+1)}}$$
$$\cos\phi_{13} = \frac{C_{13}}{\sqrt{C_{11}} \cdot \sqrt{C_{33}}} = \frac{2E_{13}}{\sqrt{(2E_{11}+1)(2E_{33}+1)}}$$
$$\cos\phi_{23} = \frac{C_{23}}{\sqrt{C_{22}} \cdot \sqrt{C_{33}}} = \frac{2E_{23}}{\sqrt{(2E_{22}+1)(2E_{33}+1)}}$$

It can be seen that change in angle between material fibres depend on off-diagonal components of strain tensor – **we may call the off-diagonal components of $\mathbf{E}$ the distortion strains.**

KINEMATIC (GEOMETRIC) RELATIONS

The strain tensor is the most commonly used measure of deformation due to its direct correspondence with stress (intensity of internal forces) which in turn result from the applied load, which is usually known, contrary to the displacement which is usually to be determined (this is, of course, a simplification, since there are many situations in which it works exactly the other way round). However, strains themselves do not allow us to determine the position of particle in current configuration, because they are derivatives (in common meaning) of components of deformation tensor and deformation gradient, which in turn are only derivatives (in mathematical meaning) of displacement vector. Only a certain integration of components of deformation gradient / deformation tensor / strain tensor allow us to determine the displacement vector and then the position vector in current configuration. This differential relation between strain and displacement is called **kinematic (geometric) relation**.

Let's start with the definition of strain tensor:

$$E_{ij} = \frac{1}{2}(C_{ij} - \delta_{ij}) = \frac{1}{2}(F_{ki}F_{kj} - \delta_{ij}) = \frac{1}{2}\left(\frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} - \delta_{ij}\right)$$

Basic relations between position vectors and displacement vector gives us:

$$x_k = u_k + X_k$$

$$\frac{\partial x_k}{\partial X_i} = \frac{\partial u_k}{\partial X_i} + \delta_{ki}$$

Substituting it in the definition of $\mathbf{E}$:

$$E_{ij} = \frac{1}{2}\left[(u_{k,i} + \delta_{ki})(u_{k,j} + \delta_{kj}) - \delta_{ij}\right] = \frac{1}{2}\left[u_{k,i}\delta_{kj} + u_{k,j}\delta_{ki} + u_{k,i}u_{k,j} + \delta_{ki}\delta_{kj} - \delta_{ij}\right] =$$

$$= \frac{1}{2}\left[u_{j,i} + u_{i,j} + u_{k,i}u_{k,j} + \delta_{ij} - \delta_{ij}\right] = \frac{1}{2}\left[u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}\right]$$

Finally, **geometric relations** are as follows:

$$\boxed{E_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}\right) \quad \Leftrightarrow \quad \mathbf{E} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^{\mathrm{T}} + \mathbf{H}^{\mathrm{T}}\cdot\mathbf{H}\right)}$$

or, written more explicitly:

$$E_{11} = \frac{1}{2}\left[2\frac{\partial u_1}{\partial X_1} + \left(\frac{\partial u_1}{\partial X_1}\right)^2 + \left(\frac{\partial u_2}{\partial X_1}\right)^2 + \left(\frac{\partial u_3}{\partial X_1}\right)^2\right]$$

$$E_{22} = \frac{1}{2}\left[2\frac{\partial u_2}{\partial X_2} + \left(\frac{\partial u_1}{\partial X_2}\right)^2 + \left(\frac{\partial u_2}{\partial X_2}\right)^2 + \left(\frac{\partial u_3}{\partial X_2}\right)^2\right]$$

$$\ldots$$

$$E_{12} = \frac{1}{2}\left[\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} + \frac{\partial u_1}{\partial X_1}\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\frac{\partial u_2}{\partial X_2} + \frac{\partial u_3}{\partial X_1}\frac{\partial u_3}{\partial X_2}\right]$$

It can be seen that those relations are **non-linear**. Similarly, thing may be done in spatial description:

$$e_{ij} = \frac{1}{2}(\delta_{ij} - c_{ij}) = \frac{1}{2}(\delta_{ij} - f_{ki} f_{kj}) = \frac{1}{2}\left(\delta_{ij} - \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}\right)$$

$$X_k = x_k - u_k \quad \Rightarrow \quad \frac{\partial x_k}{\partial X_i} = \delta_{ki} - \frac{\partial u_k}{\partial X_i}$$

$$e_{ij} = \frac{1}{2}\left[\delta_{ij} - (\delta_{ki} - u_{k,i})(\delta_{kj} - u_{k,j})\right] = \frac{1}{2}\left[u_{k,i}\delta_{kj} + u_{k,j}\delta_{ki} - u_{k,i}u_{k,j} - \delta_{ki}\delta_{kj} + \delta_{ij}\right] =$$
$$= \frac{1}{2}\left[u_{j,i} + u_{i,j} - u_{k,i}u_{k,j} - \delta_{ij} + \delta_{ij}\right] = \frac{1}{2}\left[u_{i,j} + u_{j,i} - u_{k,i}u_{k,j}\right]$$

CHANGE OF LENGTH OF LINEAR ELEMENT IN ELASTIC DEFORMATION

- Before deformation: $\mathrm{d}S = |\mathrm{d}\mathbf{X}| = \sqrt{\mathrm{d}X_i \mathrm{d}X_i}$
- After deformation: $\mathrm{d}s = |\mathrm{d}\mathbf{x}| = \sqrt{\mathrm{d}x_i \mathrm{d}x_i} = \sqrt{C_{ij}\mathrm{d}X_i \mathrm{d}X_j} = \frac{\sqrt{C_{ij}\mathrm{d}X_i \mathrm{d}X_j}}{\sqrt{\mathrm{d}X_k \mathrm{d}X_k}}\mathrm{d}S$

If the above elements are considered elements of a material curve, then the length of the curve may be found by proper integration:

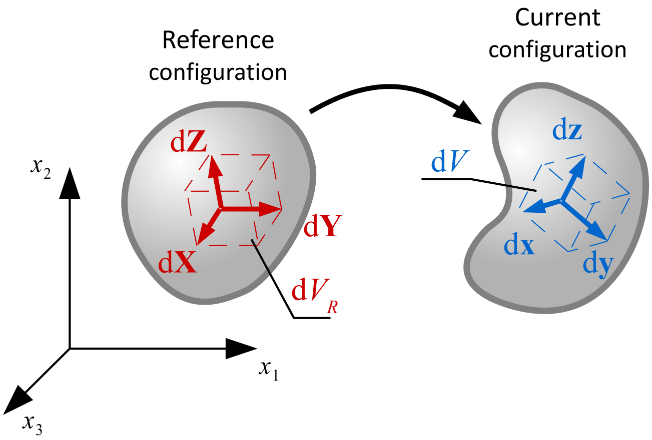
- Before deformation: $L_R = \int_K \mathrm{d}S$
- After deformation: $L = \int_K \mathrm{d}s$

CHANGE OF ANGLE BETWEEN TWO LINEAR ELEMENTS IN ELASTIC DEFORMATION

- Before deformation: $\phi_R = \arccos\left(\frac{\mathrm{d}\mathbf{X}\cdot\mathrm{d}\mathbf{Y}}{|\mathrm{d}\mathbf{X}|\cdot|\mathrm{d}\mathbf{Y}|}\right)$
- After deformation: $\phi = \arccos\left(\frac{(\mathbf{F}\cdot\mathrm{d}\mathbf{X})\cdot(\mathbf{F}\cdot\mathrm{d}\mathbf{Y})}{|\mathbf{F}\cdot\mathrm{d}\mathbf{X}|\cdot|\mathbf{F}\cdot\mathrm{d}\mathbf{Y}|}\right)$

CHANGE OF VOLUME IN ELASTIC DEFORMATION

Volume of a parallelepiped given by three material fibers (not necessarily perpendicular) may be calculated with the use of the triple product:



- Before deformation: $\mathrm{d}V_R = [\mathrm{d}\mathbf{X}\,;\,\mathrm{d}\mathbf{Y}\,;\,\mathrm{d}\mathbf{Z}] = (\mathrm{d}\mathbf{X}\times\mathrm{d}\mathbf{Y})\cdot\mathrm{d}\mathbf{Z}$
- After deformation: $\mathrm{d}V = [\mathrm{d}\mathbf{x}\,;\,\mathrm{d}\mathbf{y}\,;\,\mathrm{d}\mathbf{z}] = (\mathrm{d}\mathbf{x}\times\mathrm{d}\mathbf{y})\cdot\mathrm{d}\mathbf{z}$

Definition of deformation gradient allow us to write:

$$\mathrm{d}V = \left((\mathbf{F}\cdot\mathrm{d}\mathbf{X})\times(\mathbf{F}\cdot\mathrm{d}\mathbf{Y})\right)\cdot(\mathbf{F}\cdot\mathrm{d}\mathbf{Z})$$

It may be shown that

$$\mathrm{d}V = \left((\mathbf{F}\cdot\mathrm{d}\mathbf{X})\times(\mathbf{F}\cdot\mathrm{d}\mathbf{Y})\right)\cdot(\mathbf{F}\cdot\mathrm{d}\mathbf{Z}) = \det\mathbf{F}\cdot(\mathrm{d}\mathbf{X}\times\mathrm{d}\mathbf{Y})\cdot\mathrm{d}\mathbf{Z} = J\,dV_R$$

so finally, we can write:

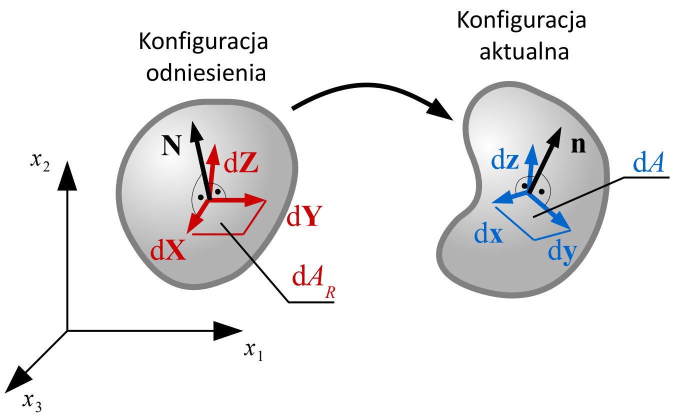
- Before deformation: dV_R
- After deformation: $dV = J\,dV_R$

If the above elements are considered elements of a certain material volume, then the volume after deformation may be found by proper integration:

- Before deformation: $V_R = \iiint\limits_{V_R} \mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z$
- After deformation: $V = \iiint\limits_{V} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint\limits_{V_R} J\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z$

CHANGE OF AREA IN ELASTIC DEFORMATION

Volume of a parallelogram given by two material fibers (not necessarily perpendicular) may be calculated with the use of the cross product. Let's consider volumes of two parallelepipedes as above:



Before deformation: $\mathrm{d}V_R = [\mathrm{d}\mathbf{X}, \mathrm{d}\mathbf{Y}, \mathrm{d}\mathbf{Z}] = (\mathrm{d}\mathbf{X}\times\mathrm{d}\mathbf{Y})^{\mathrm{T}}\cdot\mathrm{d}\mathbf{Z} = \mathrm{d}A_R\cdot\mathbf{N}^{\mathrm{T}}\cdot\mathrm{d}\mathbf{Z}$

After deformation: $\mathrm{d}V = [\mathrm{d}\mathbf{x}, \mathrm{d}\mathbf{y}, \mathrm{d}\mathbf{z}] = (\mathrm{d}\mathbf{x}\times\mathrm{d}\mathbf{y})^{\mathrm{T}}\cdot\mathrm{d}\mathbf{z} = \mathrm{d}A\cdot\mathbf{n}^{\mathrm{T}}\cdot\mathrm{d}\mathbf{z}$,

where $\mathbf{N}$ and $\mathbf{n}$ are unit vectors perpendicular to the corresponding area elements. Making use of the derived relation between reference and current volume gives us:

$$\mathrm{d}A\cdot\mathbf{n}^{\mathrm{T}}\cdot\mathrm{d}\mathbf{z} = J\cdot\mathrm{d}A_R\cdot\mathbf{N}^{\mathrm{T}}\cdot\mathrm{d}\mathbf{Z}$$

Vector $\mathrm{d}\mathbf{z}$ may be expressed as a product of deformation gradient and vector $\mathrm{d}\mathbf{Z}$:

$$\mathrm{d}A\cdot\mathbf{n}^{\mathrm{T}}\cdot(\mathbf{F}\cdot\mathrm{d}\mathbf{Z}) = J\cdot\mathrm{d}A_R\cdot\mathbf{N}^{\mathrm{T}}\cdot\mathrm{d}\mathbf{Z}$$

The above relation may be considered an equality of two operators acting on $\mathrm{d}\mathbf{Z}$. It could be more easily noticed in index notation:

$$(\mathrm{d}A\, n_j\cdot F_{ji})\cdot\mathrm{d}Z_i = (J\cdot\mathrm{d}A_R\cdot N_i)\cdot\mathrm{d}Z_i$$

Expressions in brackets define a certain vector. Since the above relation must be true for any $\mathrm{d}\mathbf{Z}$, then those two vectors must be equal:

$$\mathrm{d}A\cdot\mathbf{n}^{\mathrm{T}}\cdot\mathbf{F} = J\cdot\mathrm{d}A_R\cdot\mathbf{N}^{\mathrm{T}}$$

Multiplication with $\mathbf{F}^{-1}$ gives us so called **Nanson formula**:

$$\boxed{\mathrm{d}A\cdot\mathbf{n}^{\mathrm{T}} = J\cdot\mathrm{d}A_R\cdot\mathbf{N}^{\mathrm{T}}\cdot\mathbf{F}^{-1}}$$

The above relation is againg a relation of equality of two vectors. In particular their lengths must be also equal:

$$\sqrt{\mathrm{d}A^2\,\mathbf{n}^{\mathrm{T}}\cdot\mathbf{n}} = \sqrt{J^2\mathrm{d}A_R^2(\mathbf{N}^{\mathrm{T}}\cdot\mathbf{F}^{-1})\cdot(\mathbf{N}^{\mathrm{T}}\cdot\mathbf{F}^{-1})^{\mathrm{T}}}$$

Since $\mathbf{n}$ is unit vector, then $|\mathbf{n}| = \sqrt{\mathbf{n}^{\mathrm{T}} \cdot \mathbf{n}} = 1$. Finally the relation between area element before and after deofmration is as follows:

$$\mathrm{d}\,A = \mathrm{d}\,A_R \cdot J \cdot \sqrt{(\mathbf{N}^{\mathrm{T}} \cdot \mathbf{F}^{-1}) \cdot (\mathbf{N}^{\mathrm{T}} \cdot \mathbf{F}^{-1})^{\mathrm{T}}}$$

SMALL STRAIN THEORY

Non-linearity of geometric relations is a considerable difficulty since it is extremely difficult to find any closed-form solution for any non-trivial problem to be solved. Often, one cannot even conclude if the solution exists or is it unique. Fortunately most of engineering problems requiring theory of elasticity are such that the **deformation and strain is very small**. In such a situation **product of any small quantities may be considered even smaller – so small that it could be neglected**. This is the basis for (geometrically) **linear theory of elasticity**:

$$u_{i,j} \ll 1 \qquad \Rightarrow \qquad E_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{k,i} u_{k,j}\right) \approx \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) = \varepsilon_{ij}$$

and quantity $\boldsymbol{\varepsilon}$ called the **small strain tensor** or **Cauchy strain tensor** is defined by the **Cauchy geometric relations**:

$$\varepsilon_{11} = \frac{\partial u_1}{\partial X_1} \qquad \varepsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2}\right)$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial X_2} \qquad \varepsilon_{31} = \frac{1}{2}\left(\frac{\partial u_3}{\partial X_1} + \frac{\partial u_1}{\partial X_3}\right)$$

$$\varepsilon_{33} = \frac{\partial u_3}{\partial X_3} \qquad \varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\right)$$

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^{\mathrm{T}}\right) \quad \Leftrightarrow \quad \varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

Similarly **small rotation tensor** is defined:

$$\boldsymbol{\omega} = \frac{1}{2}\left(\mathbf{H} - \mathbf{H}^{\mathrm{T}}\right) \quad \Leftrightarrow \quad \omega_{ij} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right)$$

So that strain tensor may be expressed as:

$$\mathbf{E} = \boldsymbol{\varepsilon} + \frac{1}{2}\left(\boldsymbol{\varepsilon}^{\mathrm{T}} + \boldsymbol{\omega}^{\mathrm{T}}\right)\left(\boldsymbol{\varepsilon} + \boldsymbol{\omega}\right)$$

INTERPRETATION OF COMPONENTS OF SMALL STRAIN TENSOR

We may interpret the components of $\boldsymbol{\varepsilon}$ by expanding the components of $\mathbf{E}$ into a power series in a neighborhood of 0 – the strains are close to 0.

DIAGONAL COMPONENTS – LINEAR STRAINS – RELATIVE ELONGATIONS

A relative elongation is equal: $\Delta_1 = \sqrt{2E_{11}+1}-1$

Let's analyze function: $f(x) = \sqrt{2x+1}-1$

It's McLaurin series is:

$$f(0+x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}}{n!}x^n + \ldots =$$

$$= \left(\sqrt{2x+1}-1\right)\Big|_{x=0} + \left(\frac{1}{\sqrt{2x+1}}\right)\Bigg|_{x=0} \cdot x + \left(-\frac{1}{(2x+1)^{3/2}}\right)\Bigg|_{x=0} \cdot x^2 + \ldots = 0 + x - x^2 + \ldots \approx x$$

So for small strain, when non-linear part is neglected in the geometric relations, we may write:

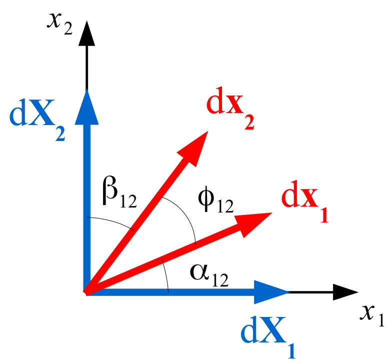
$$\begin{cases} \Delta_1 \approx E_{11} \\ \Delta_2 \approx E_{22} \\ \Delta_3 \approx E_{33} \end{cases}$$

For this reason **diagonal components of small strain tensor are considered approximately equal to relative elongations**.

OFF-DIAGONAL COMPONENTS – DISTORION STRAINS – CHANGE IN ANGLE

Let's consider now an angle between two material fibers. Let's now denote the change in this angle (originally a right angle if the considered coordinate system is a Cartesian one):

$$\gamma_{12} = \alpha_{12} + \beta_{12} = 90^{\circ} - \phi_{12}$$



Trigonometric relations give us:

$$\cos\phi_{12} = \cos(90^{\circ} - \gamma_{12}) = \sin\gamma_{12}$$

so we can write: $\sin\gamma_{12} = \dfrac{2E_{12}}{\sqrt{(2E_{11}+1)(2E_{22}+1)}}$

Let's expand both sides of the above relation into power series. Left hand side:

$$f(x) = \sin(x)$$
$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots = 0 + \frac{\cos(0)}{1!}x - \frac{\sin(0)}{2!}x^2 + \ldots \approx x$$

It is a well known conclusion that a sine of an angle is approximately equal the angle itself if the angle is small.

The right hand side:

$$f(x,y,z) = \frac{2x}{\sqrt{(2y+1)(2z+1)}}$$
$$f(x,y,z) = f(0,0,0) + \frac{1}{1!}\frac{\partial f}{\partial x}x + \frac{1}{1!}\frac{\partial f}{\partial y}y + \frac{1}{1!}\frac{\partial f}{\partial z}z + \frac{1}{2!}\frac{\partial^2 f}{\partial x^2}x^2 + \frac{1}{2!}\frac{\partial^2 f}{\partial x \partial y}xy + \ldots =$$
$$= 0 + 2\cdot x + 0\cdot y + 0\cdot z + \ldots \approx 2x$$

Finally, we can write:

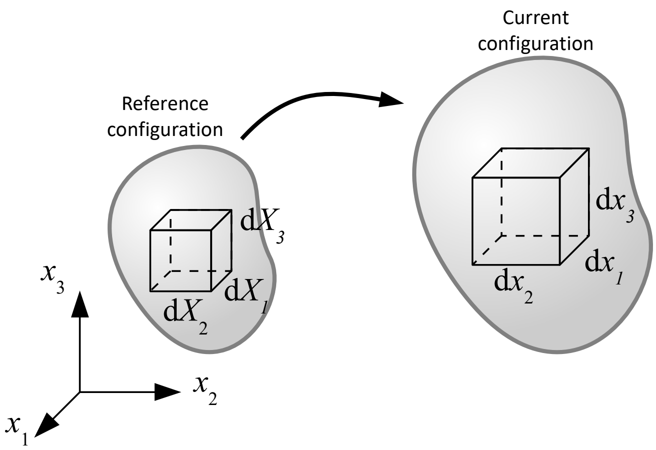
$$\boxed{\begin{cases} \gamma_{23} \approx 2E_{23} \\ \gamma_{31} \approx 2E_{31} \\ \gamma_{12} \approx 2E_{12} \end{cases}}$$

For this reason **off-diagonal components of small strain tensor are considered approximately equal to the half of change in angle between two fibers which were originally orthogonal**.

TRACE OF TENSOR – DILATION – RELATIVE VOLUMETRIC CHANGE

Let's consider a small box given by three mutually perpendicular material fibers of length $\mathrm{d}X_1, \mathrm{d}X_2, \mathrm{d}X_3$. In case of small strain theory, their lengths after deformation can be caluclated as follows:

$$\begin{aligned} \mathrm{d}X_1 &\rightarrow \mathrm{d}x_1 = (1+\varepsilon_{11})\mathrm{d}X_1 \\ \mathrm{d}X_2 &\rightarrow \mathrm{d}x_2 = (1+\varepsilon_{22})\mathrm{d}X_2 \\ \mathrm{d}X_3 &\rightarrow \mathrm{d}x_3 = (1+\varepsilon_{33})\mathrm{d}X_3 \end{aligned}$$



The volume of box before deformation:

$$V_R = \mathrm{d}X_1\mathrm{d}X_2\mathrm{d}X_3$$

The volume of box after deformation:

$$V = \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 =$$
$$= \left[1 + (\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}) + (\varepsilon_{22}\varepsilon_{33} + \varepsilon_{33}\varepsilon_{11} + \varepsilon_{11}\varepsilon_{22} + \varepsilon_{11}\varepsilon_{22}\varepsilon_{33})\right]\mathrm{d}X_2\mathrm{d}X_2\mathrm{d}X_3$$

Relative change in volume:

$$\frac{\Delta V}{V_R} = \frac{V - V_R}{V_R} = \underbrace{(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})}_{\theta} + (\varepsilon_{22}\varepsilon_{33} + \varepsilon_{33}\varepsilon_{11} + \varepsilon_{11}\varepsilon_{22} + \varepsilon_{11}\varepsilon_{22}\varepsilon_{33})$$

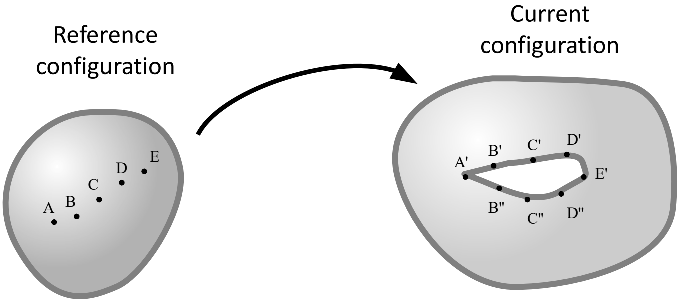
Neglecting the non-linear terms gives us:

$$\frac{\Delta V}{V_R} \approx \theta = \varepsilon_{ij}\delta_{ij} = \varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = u_{k,k} = \nabla \cdot \mathbf{u}$$

The quantity defined above is called **dilation**.

COMPATIBILITY CONDITIONS

Concerning the geometric relations between strain components and displacements a question rises – **is an integration of geometric relations always possible**? Geometric relations are in fact certain system of first order partial differential equations. Especially in case of non-linear theory, solution of such a system may not exist. Furthermore, we have six equations and only three unknowns (displacements). What conditions must be satisfied if such a system is to be solved? These conditions are called **integrability conditions** or **compatibility conditions** and follow from requirement that we look for such a field $\mathbf{x}(\mathbf{X})$ which is a single-valued function, namely a function that associates only a single position in current configuration $\mathbf{x}(\mathbf{X})$ to each particle $\mathbf{X}$. The above image depicts restricted deformation field:



It may be shown that the compatibility conditions for finite strain theory (non-linear geometric relations) are:

$$\nabla_{\mathbf{X}} \times \mathbf{F} = \mathbf{0} \quad \Leftrightarrow \quad \epsilon_{pqi} F_{jq,p} = 0$$

It gives us following conditions:

$$\begin{array}{lll}
\dfrac{\partial F_{13}}{\partial X_2} - \dfrac{\partial F_{12}}{\partial X_3} = 0 & \dfrac{\partial F_{23}}{\partial X_2} - \dfrac{\partial F_{22}}{\partial X_3} = 0 & \dfrac{\partial F_{33}}{\partial X_2} - \dfrac{\partial F_{32}}{\partial X_3} = 0 \\
\dfrac{\partial F_{11}}{\partial X_3} - \dfrac{\partial F_{13}}{\partial X_1} = 0 & \dfrac{\partial F_{21}}{\partial X_3} - \dfrac{\partial F_{23}}{\partial X_1} = 0 & \dfrac{\partial F_{31}}{\partial X_3} - \dfrac{\partial F_{33}}{\partial X_1} = 0 \\
\dfrac{\partial F_{12}}{\partial X_1} - \dfrac{\partial F_{11}}{\partial X_2} = 0 & \dfrac{\partial F_{22}}{\partial X_1} - \dfrac{\partial F_{21}}{\partial X_2} = 0 & \dfrac{\partial F_{32}}{\partial X_1} - \dfrac{\partial F_{31}}{\partial X_2} = 0
\end{array}$$

For small strain theory (linear geometric relations) the compatibility conditions are:

$$\boxed{\nabla_{\mathbf{X}} \times (\nabla_{\mathbf{X}} \times \boldsymbol{\varepsilon}) = \mathbf{0}}$$

In index notation:

$$\epsilon_{pri}\,\epsilon_{qsj}\,\varepsilon_{pq,rs} = 0 \qquad i,\, j = 1,2,3$$

It is equivalent to 81 equations, only 6 of which are independent:

$$\begin{array}{ll}
\dfrac{\partial^2 \varepsilon_{22}}{\partial X_3^2} - 2\dfrac{\partial^2 \varepsilon_{23}}{\partial X_2 \partial X_3} + \dfrac{\partial^2 \varepsilon_{33}}{\partial X_2^2} = 0 & \dfrac{\partial^2 \varepsilon_{11}}{\partial X_2 X_3} - \dfrac{\partial}{\partial X_1}\left[-\dfrac{\partial \varepsilon_{23}}{\partial X_1} + \dfrac{\partial \varepsilon_{31}}{\partial X_2} + \dfrac{\partial \varepsilon_{12}}{\partial X_3} \right] = 0 \\
\dfrac{\partial^2 \varepsilon_{33}}{\partial X_1^2} - 2\dfrac{\partial^2 \varepsilon_{31}}{\partial X_3 \partial X_1} + \dfrac{\partial^2 \varepsilon_{11}}{\partial X_3^2} = 0 & \dfrac{\partial^2 \varepsilon_{22}}{\partial X_3 X_1} - \dfrac{\partial}{\partial X_2}\left[\dfrac{\partial \varepsilon_{23}}{\partial X_1} - \dfrac{\partial \varepsilon_{31}}{\partial X_2} + \dfrac{\partial \varepsilon_{12}}{\partial X_3} \right] = 0 \\
\dfrac{\partial^2 \varepsilon_{11}}{\partial X_2^2} - 2\dfrac{\partial^2 \varepsilon_{12}}{\partial X_1 \partial X_2} + \dfrac{\partial^2 \varepsilon_{22}}{\partial X_1^2} = 0 & \dfrac{\partial^2 \varepsilon_{33}}{\partial X_1 X_2} - \dfrac{\partial}{\partial X_3}\left[\dfrac{\partial \varepsilon_{23}}{\partial X_1} + \dfrac{\partial \varepsilon_{31}}{\partial X_2} - \dfrac{\partial \varepsilon_{12}}{\partial X_3} \right] = 0
\end{array}$$

SUMMARY

Material coordinates (coordinates in reference configuration) $\mathbf{X} \Leftrightarrow X_i$
Spatial coordinates (coordinates in current configuration) $\mathbf{x} \Leftrightarrow x_i$

DISPLACEMENT

- Displacement vector $\mathbf{u} = \mathbf{x} - \mathbf{X}$
- Material displacement gradient $\mathbf{H} = \mathbf{u} \otimes \nabla_{\mathbf{X}} = \boldsymbol{\varepsilon} + \boldsymbol{\omega} = \mathbf{F} - \mathbf{1}$
- Rotation vector (material description) $\mathbf{w} = \nabla_{\mathbf{X}} \times \mathbf{u}$
- Rotation gradient $\mathbf{w} \otimes \nabla_{\mathbf{X}} = \nabla_{\mathbf{X}} \times \boldsymbol{\varepsilon} = -\nabla_{\mathbf{X}} \times \boldsymbol{\omega}$
- Velocity vector $\mathbf{v} = \dfrac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}$
- Acceleration vector $\mathbf{a} = \dfrac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \dfrac{\mathrm{d}^2\mathbf{u}}{\mathrm{d}t^2}$

DEFORMATION

MATERIAL DESCRIPTION:

- Material deformation gradient $\mathbf{F} = \dfrac{\partial \mathbf{x}}{\partial \mathbf{X}} = \mathbf{x} \otimes \nabla_{\mathbf{X}} = \mathbf{R} \cdot \mathbf{U} = \mathbf{V} \cdot \mathbf{R}$
- Right stretch tensor $\mathbf{U}$
- Left stretch tensor $\mathbf{V}$
- Rotation tensor $\mathbf{R}$
- Right Cauchy–Green deformation tensor $\mathbf{C} = \mathbf{F}^{\mathrm{T}} \cdot \mathbf{F} = \mathbf{U}^2 = \mathbf{b}^{-1}$
- LEft Cauchy–Green deformation tensor $\mathbf{B} = \mathbf{F} \cdot \mathbf{F}^{\mathrm{T}} = \mathbf{V}^2 = \mathbf{c}^{-1}$
- Green – de Saint-Venant strain tensor $\mathbf{E} = \dfrac{1}{2}(\mathbf{C} - \mathbf{1})$
- Small strain tensor $\boldsymbol{\varepsilon} = \dfrac{1}{2}(\mathbf{H} + \mathbf{H}^{\mathrm{T}}) = \dfrac{1}{2}\left[\mathbf{u} \otimes \nabla_{\mathbf{X}} + (\mathbf{u} \otimes \nabla_{\mathbf{X}})^{\mathrm{T}}\right]$
- Small rotation tensor $\boldsymbol{\omega} = \dfrac{1}{2}(\mathbf{H} - \mathbf{H}^{\mathrm{T}}) = \dfrac{1}{2}\left[\mathbf{u} \otimes \nabla_{\mathbf{X}} - (\mathbf{u} \otimes \nabla_{\mathbf{X}})^{\mathrm{T}}\right]$

SPATIAL DESCRIPTION

- Spatial deformation gradient $\mathbf{f} = \dfrac{\partial \mathbf{X}}{\partial \mathbf{x}} = \mathbf{X} \otimes \nabla_{\mathbf{x}} = \mathbf{F}^{-1}$
- Cauchy deformation tensor $\mathbf{c} = \mathbf{f}^{\mathrm{T}} \cdot \mathbf{f} = \mathbf{F}^{-\mathrm{T}} \cdot \mathbf{F}^{-1} = \mathbf{B}^{-1}$
- Finger deformation tensor $\mathbf{b} = \mathbf{f} \cdot \mathbf{f}^{\mathrm{T}} = \mathbf{F}^{-1} \cdot \mathbf{F}^{-\mathrm{T}} = \mathbf{C}^{-1}$
- Almansi – Hamel strain tensor $\mathbf{e} = \dfrac{1}{2}(\mathbf{1} - \mathbf{c})$

GEOMETRIC (KINEMATIC) RELATIONS

FINITE STRAIN THEORY:

- **MATERIAL DESCRIPTION**

$$\mathbf{E} = \frac{1}{2}\left[\mathbf{u}\otimes\nabla_{\mathbf{X}} + (\mathbf{u}\otimes\nabla_{\mathbf{X}})^{\mathrm{T}} + (\mathbf{u}\otimes\nabla_{\mathbf{X}})^{\mathrm{T}}\cdot\mathbf{u}\otimes\nabla_{\mathbf{X}}\right] \quad\Leftrightarrow\quad E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \sum_{k=1}^{3}\frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right)$$

- **SPATIAL DESCRIPTION**

$$\mathbf{e} = \frac{1}{2}\left[\mathbf{u}\otimes\nabla_{\mathbf{x}} + (\mathbf{u}\otimes\nabla_{\mathbf{x}})^{\mathrm{T}} - (\mathbf{u}\otimes\nabla_{\mathbf{x}})^{\mathrm{T}}\cdot\mathbf{u}\otimes\nabla_{\mathbf{x}}\right] \quad\Leftrightarrow\quad e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \sum_{k=1}^{3}\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)$$

SMALL STRAIN THEORY:

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left[\mathbf{u}\otimes\nabla_{\mathbf{X}} + (\mathbf{u}\otimes\nabla_{\mathbf{X}})^{\mathrm{T}}\right] \quad\Leftrightarrow\quad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)$$

RELATIONS BETWEEN DEFORMED AND UNDEFORMED DIFFERENTIAL MATERIAL ELEMENTS

- LINEAR ELEMENT $\quad \mathrm{d}s = \dfrac{\sqrt{\mathrm{d}\mathbf{X}\cdot\mathbf{C}\cdot\mathrm{d}\mathbf{X}}}{\sqrt{\mathrm{d}\mathbf{X}\cdot\mathrm{d}\mathbf{X}}}\,\mathrm{d}S$
- SURFACE ELEMENT $\quad \mathrm{d}A = J\sqrt{(\mathbf{N}^{\mathrm{T}}\cdot\mathbf{F}^{-1})\cdot(\mathbf{N}^{\mathrm{T}}\cdot\mathbf{F}^{-1})^{\mathrm{T}}}\,\mathrm{d}A_R$
- VOLUME ELEMENT $\quad \mathrm{d}V = J\,\mathrm{d}V_R$

CONDITION OF LOCAL INVERTIBILITY

$$J = \det\mathbf{F} = \frac{1}{\det\mathbf{f}} > 0$$

COMPATIBILITY CONDITIONS

- **FINITE STRAIN THEORY:** $\quad \nabla_{\mathbf{X}}\times\mathbf{F} = \mathbf{0}$
- **SMALL STRAIN THEORY:** $\quad \nabla_{\mathbf{X}}\times(\nabla_{\mathbf{X}}\times\boldsymbol{\varepsilon}) = \mathbf{0}$

3. STATICS AND DYNAMICS

CONSERVATION OF MASS AND CONTINUITY

In continuum mechanics we assume the **principle of conservation of mass**:

total mass of the system does not change in time.

Material particle cannot vanish and cannot be created. Total mass of the system is constant, however both its volume and density may change:

$$m_{ref} = m_t \qquad \Rightarrow \qquad \iiint_{V_R} \rho_R \, \mathrm{d} V_R = \iiint_{V} \rho \, \mathrm{d} V \quad ,$$

where ρ_R, V_R, m_{ref} are density, volume and mass in reference configuration respectively, and ρ, V, m_t are the same quantities after deformation. According to the relations derived in previous chapter we may write:

$$\iiint_{V} \rho \, \mathrm{d} V = \iiint_{V_R} \rho J \, \mathrm{d} V_R \qquad \Rightarrow \qquad \iiint_{V_R} \rho_R \, \mathrm{d} V_R = \iiint_{V_R} \rho J \, \mathrm{d} V_R$$

As the above relation must hold for any volume V_R , we may write:

$$\boxed{\rho_R = \rho \cdot J}$$

Density may vary in space as jacobian may vary in space and time, however the above relation is true in every point and every time.

FORCES

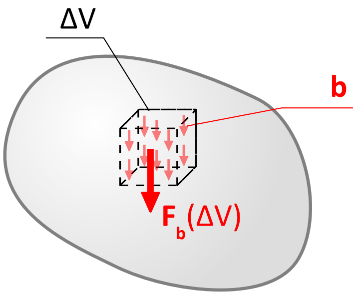
A **force** may be considered a vector measure of mechanical action of a body on another body. In particular a whole continuous field (curve, area, volume) of space-varying forces may be considered. We postulate an existence of two kinds of forces:

- **external forces** – action of environment on particles of the body
- **internal forces** – mutual interaction between particles of the body

Among **external forces** we shall distinguish two kinds of them:

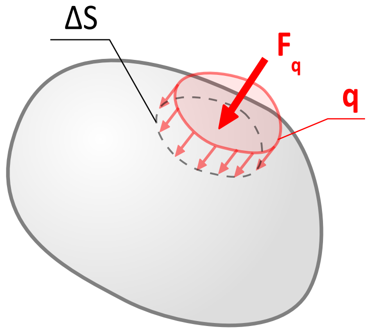
- **external volumetric forces** (**body forces**) – actions on particles inside the body. They are described as a vector of volumetric density of forces.

$$\mathbf{b}(\mathbf{x}) = \lim_{\Delta V \to 0} \frac{\mathbf{F}_b(\Delta V)}{\Delta V} \qquad [\mathbf{b}] = \frac{\mathrm{N}}{\mathrm{m}^3}$$

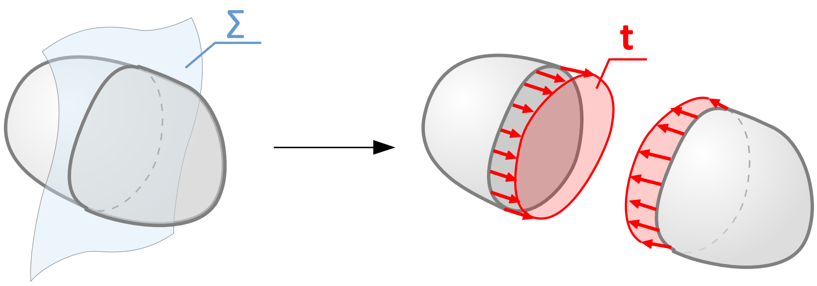


- **external surface forces** (**surface tractions**) – actions on particles on the boundary of the body. They are described as a vector of surface density of forces.

$$\mathbf{q}(\mathbf{x}) = \lim_{\Delta S \to 0} \frac{\mathbf{F}_q(\Delta S)}{\Delta S} \qquad [\mathbf{q}] = \frac{\mathrm{N}}{\mathrm{m}^2} = \mathrm{Pa}$$

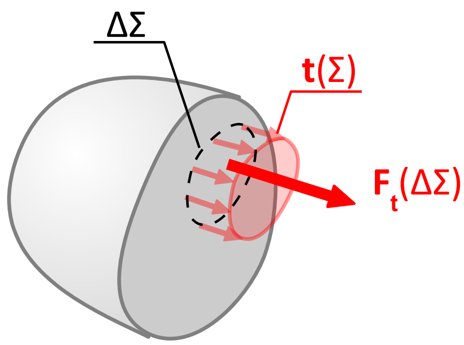


Internal forces represent mutual interactions between particles – they are a macroscopic measure of inter-atomic forces or forces between grains etc. Let's consider a body – we shall perform an imaginary cut-through with a surface Σ .



The system of internal forces will depend both on the point in which they are considered and on the surface of cross-section. We postulate that internal forces can be described with the use of a **stress vector** - vector of **surface density of internal forces** (**internal tractions**)

$$\mathbf{t}(\mathbf{x}_0, \Sigma) = \lim_{\Delta \Sigma \to 0} \frac{\mathbf{F}_t(\Delta \Sigma)}{\Delta \Sigma} \qquad [\mathbf{t}] = \frac{\mathrm{N}}{\mathrm{m}^2} = \mathrm{Pa}$$



LAWS OF MOTION

Continuum mechanics is a classical theory of mechanics, namely it is based on **Newtonian laws of motion**:

1st LAW

*There exist **inertial reference frames** in which if all forces acting on a body are in equilibrium, then the body stands still if it performed no motion, or moves along a straight line with constant velocity is it was in motion.*

2nd LAW

In inertial reference frames, if the forces acting on a body are not in an equilibrium, then the body moves with a non-zero acceleration and:

- *in **tranlational motion**, change of momentum in time is equal the sum of forces acting on the body (**principle of momentum**):*

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_V \rho\,\mathbf{v}\,\mathrm{d}V = \quad \Leftrightarrow \quad \dot{\mathbf{P}} = \mathbf{S} \quad ,$$

- *in rotational motion change of moment of momentum about point O in time is equal the sum of moments of all forces acting on the body about point O (**principle of moment of momentum**):*

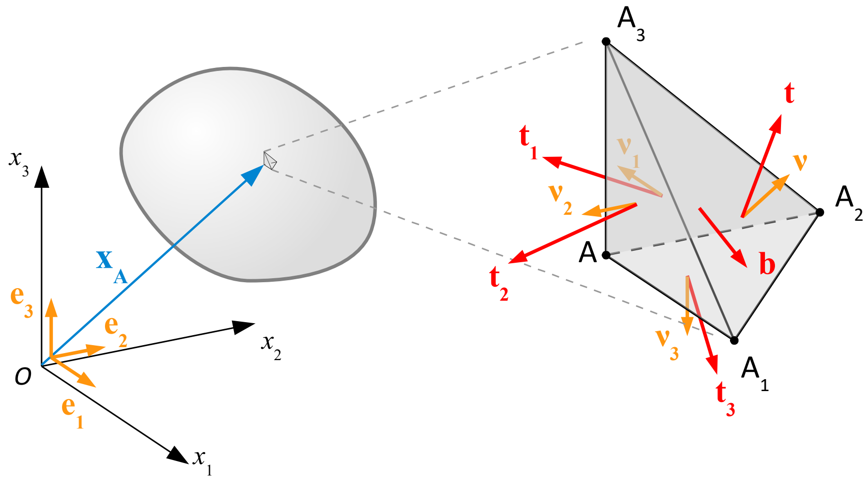
$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_V \rho\,\mathbf{v}\times\mathbf{r}\,\mathrm{d}V \quad \Leftrightarrow \quad \dot{\mathbf{K}}_O = \mathbf{M}_O \quad .$$

3rd LAW

If a body A acts on body B with a certain force, then body B acts on body A with a force which has the same magnitude and direction but opposite orientation.

TETRAHEDRON CONDITIONS

Making the use of the above assumption we shall consider an **equilibrium of an infinitely small part of an elastic solid**. Let's assume that it has a shape of **tetrahedron** three faces of which $\Sigma_i\ (i=1,2,3)$ are perpendicular to the axes of chosen Cartesian coordinate system given by versors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \quad (|\mathbf{e}_i|=1,\ i=1,2,3)$. The fourth one Σ is inclined to them and given by **external unit normal** vector $\boldsymbol{\nu}=[\nu_1\ ;\ \nu_2\ ;\ \nu_3]^{\mathrm{T}},\ (|\boldsymbol{\nu}|=1)$. External unit normals of the rest of faces are $\boldsymbol{\nu}_i \parallel \mathbf{e}_i\ (i=1,2,3)$.



We assume that there is (in general) non-uniform **stress distribution on each face** given by vectors $\mathbf{t}_i=[t_{i1}\ ;\ t_{i2}\ ;\ t_{i3}]$ on $\Sigma_i\ (i=1,2,3)$ and $\mathbf{t}=[t_1\ ;\ t_2\ ;\ t_3]$ on Σ . These are whole fields of stress on faces, so they are in general functions of **x**: $\mathbf{t}_i(\mathbf{x})\ (i=1,2,3)$ and $\mathbf{t}(\mathbf{x})$. They are depicted in the above picture with single vectors. Furthermore we assume that on each point inside the tetrahedron **body forces** $\mathbf{b}=[b_1\ ;\ b_2\ ;\ b_3]$ act.

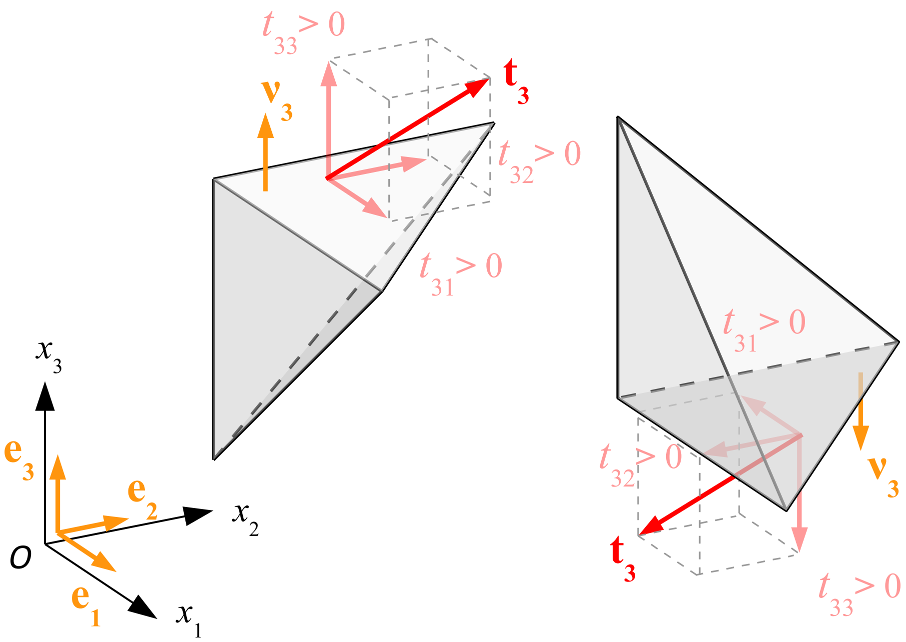
Let's denote the area of triangle $\Delta A_1 A_2 A_3$ with S and area of three other faces $\Delta A A_i A_j$ with S_{ij} . Let h be a distance of point A from $\Delta A_1 A_2 A_3$. Volume of the tetrahedron is $V=\frac{1}{3} h S$.

We shall now introduce a convention of signs for the stress components. Stress vector $\mathbf{t}_i(\mathbf{x})\ (i=1,2,3)$ is described with components $\mathbf{t}_i=[t_{i1}\ ;\ t_{i2}\ ;\ t_{i3}]$. Then:

stress component t_{ij} denotes the j-th component of a stress vector applied to the face which is perpendicular to the i-th axis of the chosen Cartesian coordinate system.

If the **external unit normal** ν_i **is oriented in the same way as versor** e_i of respective coordinate axis, then a **positive components of stress will be those oriented in the same way as versors of the respective coordinate axes**.

If the **external unit normal** ν_i **is oriented in an opposite way as versor** e_i of respective coordinate axis, then a **positive components of stress will be those oriented in an opposite way as versors of the respective coordinate axes**.



Let's make use of the principle of momentum. Sum of all forces acting on tetrahedron is equal:

$$\mathbf{S} = \iiint_V \mathbf{b}\,\mathrm{d}V + \iint_\Sigma \mathbf{t}\,\mathrm{d}S + \iint_{\Sigma_1} \mathbf{t}_1\,\mathrm{d}S + \iint_{\Sigma_2} \mathbf{t}_2\,\mathrm{d}S + \iint_{\Sigma_3} \mathbf{t}_3\,\mathrm{d}S$$

Time derivative of sum of momenta of all particles is equal:

$$\dot{\mathbf{P}} = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_V \rho\,\mathbf{v}\,\mathrm{d}V$$

Note, that integration takes place over a time-dependent volume $V = V(t)$, so time derivative must account for that. Let's change the integration domain to the reference configuration:

$$\dot{\mathbf{P}} = \frac{\mathrm{d}}{\mathrm{d}t}\iiint_V \rho\,\mathbf{v}\,\mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V_R} \rho\,\mathbf{v}\,J\,\mathrm{d}V_R = \iiint_{V_R} \frac{\mathrm{d}}{\mathrm{d}t}[\rho\,\mathbf{v}\,J]\,\mathrm{d}V_R =$$

$$= \iiint_{V_R}\left[\frac{\mathrm{d}}{\mathrm{d}t}(\rho J)\cdot\mathbf{v} + (\rho J)\cdot\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}\right]\mathrm{d}V_R$$

According to the conservation of mass $\rho J = \rho_R$. Nota that reference density is time-independent $\rho_R(t) = const.$, what results in:

$$\iiint_{V_R} \frac{\mathrm{d}}{\mathrm{d}t}(\rho J)\cdot\mathbf{v}\,\mathrm{d}V_R = \iiint_{V_R} \frac{\mathrm{d}}{\mathrm{d}t}\rho_R\cdot\mathbf{v}\,\mathrm{d}V_R = 0$$

Derivative of momentum is thus equal:

$$\dot{\mathbf{P}} = \iiint_{V_R} (\rho J)\cdot\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}\,\mathrm{d}V_R = \iiint_{V_R} \rho\,\mathbf{a}\,J\,\mathrm{d}V_R = \iiint_V \rho\,\mathbf{a}\,\mathrm{d}V$$

Let's write down **principle of momentum**:

$$\dot{\mathbf{P}} = \mathbf{S} \quad \Leftrightarrow \quad \iiint_V \rho\,\mathbf{a}\,\mathrm{d}V = \iiint_V \mathbf{b}\,\mathrm{d}V + \iint_\Sigma \mathbf{t}\,\mathrm{d}S + \iint_{\Sigma_1} \mathbf{t_1}\,\mathrm{d}S + \iint_{\Sigma_2} \mathbf{t_2}\,\mathrm{d}S + \iint_{\Sigma_3} \mathbf{t_3}\,\mathrm{d}S$$

Each integral may be replaced with a product of the measure of integration domain and a value in a certain point in that domain (different for each face) according to the **mean value theorem for integrals**:

$$\rho\,\mathbf{a}(\hat{\mathbf{x}})\cdot V = \mathbf{b}(\tilde{\mathbf{x}})\cdot V + \mathbf{t}(\check{\mathbf{x}})\cdot S + \mathbf{t_1}(\mathbf{x}')\cdot S_{23} + \mathbf{t_2}(\mathbf{x}'')\cdot S_{31} + \mathbf{t_3}(\mathbf{x}''')\cdot S_{12}$$

Let's express the volume with the use of area of face Σ and divide whole equation by S :

$$\rho\,\mathbf{a}(\hat{\mathbf{x}})\cdot\frac{h}{3} = \mathbf{b}(\tilde{\mathbf{x}})\cdot\frac{h}{3} + \mathbf{t}(\check{\mathbf{x}}) + \mathbf{t_1}(\mathbf{x}')\cdot\frac{S_{23}}{S} + \mathbf{t_2}(\mathbf{x}'')\cdot\frac{S_{31}}{S} + \mathbf{t_3}(\mathbf{x}''')\cdot\frac{S_{12}}{S}$$

It can be shown (by some geometric considerations) that:

$$\nu_1 = \frac{S_{23}}{S}, \qquad \nu_2 = \frac{S_{31}}{S}, \qquad \nu_3 = \frac{S_{12}}{S}$$

So the principle of momentum takes form:

$$\rho\,\mathbf{a}(\hat{\mathbf{x}})\cdot\frac{h}{3} = \mathbf{b}(\tilde{\mathbf{x}})\cdot\frac{h}{3} + \mathbf{t}(\check{\mathbf{x}}) + \mathbf{t_1}(\mathbf{x}')\cdot\nu_1 + \mathbf{t_2}(\mathbf{x}'')\cdot\nu_2 + \mathbf{t_3}(\mathbf{x}''')\cdot\nu_3$$

and in components (remembering that due to fact that external unit normals of faces $\Sigma_i\ (i=1,2,3)$ are oriented in an opposite way to the versors of respective coordinate axes):

$$\rho a_i(\hat{\mathbf{x}})\cdot\frac{h}{3}=b_i(\tilde{\mathbf{x}})\cdot\frac{h}{3}+t_i(\check{\mathbf{x}})-t_{1i}(\mathbf{x}')\cdot\nu_1-t_{2i}(\mathbf{x}'')\cdot\nu_2-t_{3i}(\mathbf{x}''')\cdot\nu_3 \qquad i=1,2,3$$

Let's consider a sequence of tetrahedrons in which the face Σ moves towards point A. In a limit case when $h\to 0$ we have

$$\lim_{h\to 0}\rho a_i(\hat{\mathbf{x}})\cdot\frac{h}{3}=\lim_{h\to 0}\left[b_i(\tilde{\mathbf{x}})\cdot\frac{h}{3}+t_i(\check{\mathbf{x}})-t_{1i}(\mathbf{x}')\cdot\nu_1-t_{2i}(\mathbf{x}'')\cdot\nu_2-t_{3i}(\mathbf{x}''')\cdot\nu_3\right] \qquad i=1,2,3$$

For each value of *h* mean value point $\hat{\mathbf{x}},\tilde{\mathbf{x}},\check{\mathbf{x}},\mathbf{x}',\mathbf{x}'',\mathbf{x}'''$ will be in general different but for $h\to 0$ they all tend toward A:

$$\hat{\mathbf{x}},\tilde{\mathbf{x}},\check{\mathbf{x}},\mathbf{x}',\mathbf{x}'',\mathbf{x}''' \quad \overset{h\to 0}{\rightarrow} \quad \mathbf{x}_A$$

For $h\to 0$ the volume also tends to 0, so the influence of inertial forces and body forces vanishes. Finally we can write down:

$$t_i(\mathbf{x}_A)-t_{1i}(\mathbf{x}_A)\nu_1-t_{2i}(\mathbf{x}_A)\nu_2-t_{3i}(\mathbf{x}_A)\nu_3=0$$

what can be rewritten inthe following form:

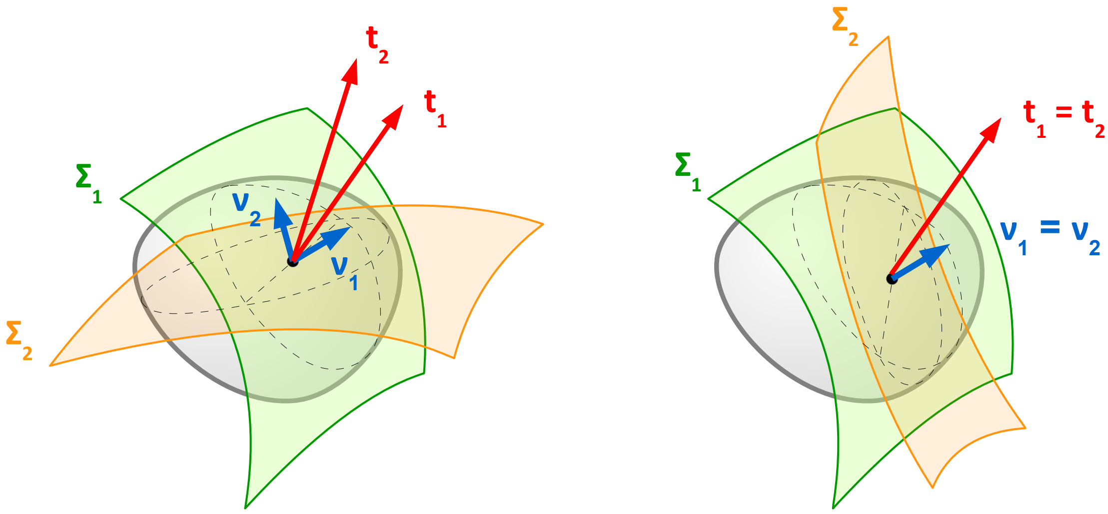
$$\boxed{t_i(\mathbf{x})=t_{ji}(\mathbf{x})\nu_j(\mathbf{x}), \qquad i,j=1,2,3}$$

The above relations are called **tetrahedron conditions**. They can be written down in an absolute form or a matrix form as follows:

$$\mathbf{t}=(\mathbf{T}_\sigma)^{\mathrm{T}}\cdot\boldsymbol{\nu} \qquad \Leftrightarrow \qquad \begin{bmatrix}t_1\\t_2\\t_3\end{bmatrix}=\begin{bmatrix}t_{11}&t_{21}&t_{31}\\t_{12}&t_{22}&t_{32}\\t_{13}&t_{23}&t_{33}\end{bmatrix}\cdot\begin{bmatrix}\nu_1\\\nu_2\\\nu_3\end{bmatrix}$$

CONCLUSIONS FROM TETRAHEDRON CONDITIONS

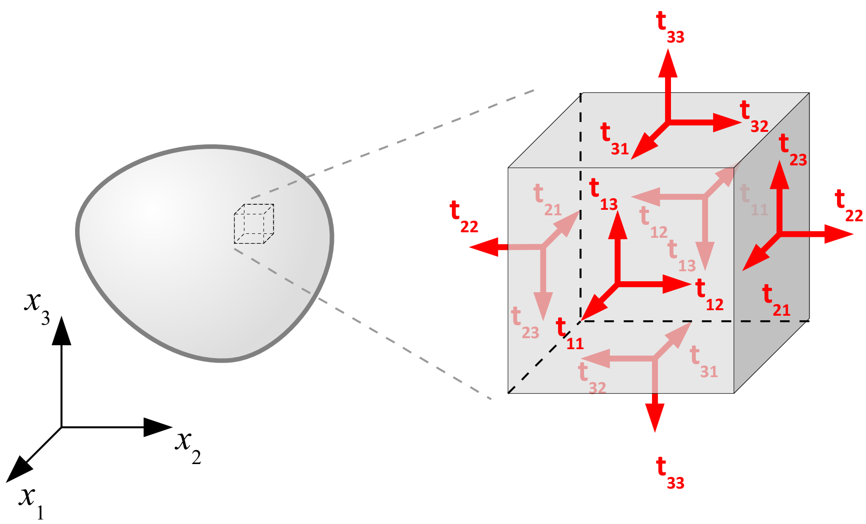
1. In a fixed point the **stress vector depends only on the external unit normal** of the surface of imaginary cut **not on the shape of the surface itself**.



2. **In order to find components of stress vector corresponsind to an arbitrary chosen unit normal it is enough to know the components of stress vectors corresponding to versors of the coordinate systems.**

$$t_i = t_{ji}\nu_j \quad \Leftrightarrow \quad \mathbf{t} = \mathbf{t}_j \cdot \nu_j = \mathbf{t}_1\nu_1 + \mathbf{t}_2\nu_2 + \mathbf{t}_3\nu_3$$

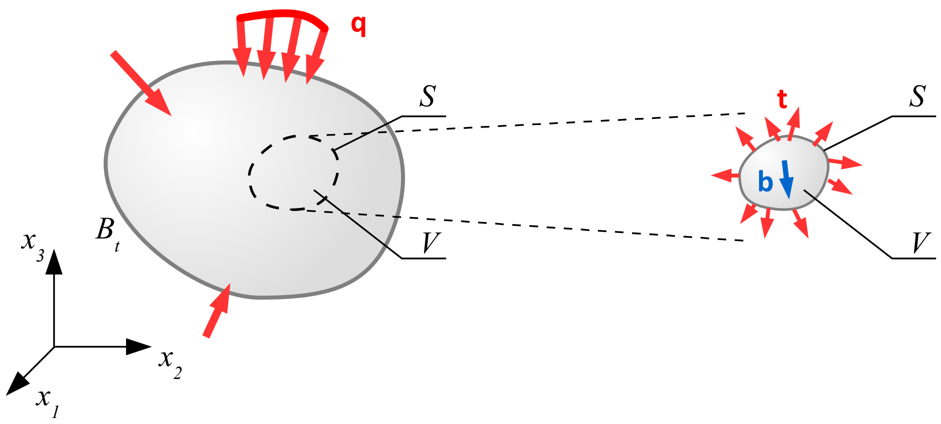
3. Tetrahedron conditions state that **to a given external unit normal a stress vector is assigned in a unique way**. In particular we can conclude that matrix $\mathbf{T}_\sigma$ is a representation of a tensor. It will be called **Cauchy stress tensor** or **true stress tensor**. Its component t_{ij} denotes the j-th component of a stress vector applied to the face which is perpendicular to the i-th axis of the chosen Cartesian coordinate system.



Diagonal components are termed **normal stresses** (**tensile** stress if **positive**, **compressive** stress in **negative**), while the **off-diagonal components** are termed **shear stresses**.

EQUATIONS OF MOTION

Let's write down the **principle of momentum for an arbitrary chosen subregion** of an current configuration. Let it be a volume V bounded with surface S:



$$\dot{\mathbf{P}} = \mathbf{S} \quad \Leftrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} \iiint_V \rho \, \mathbf{v} \, \mathrm{d}V = \iiint_V \mathbf{b} \, \mathrm{d}V + \iint_S \mathbf{t} \, \mathrm{d}S$$

The integral on the left hand side may be transformed in the same way as in derivation of tetrahedron conditions. The internal tractions on the right hand side may be rewritten with the use of tetrahedron conditions:

$$\iiint_V \rho a_i \, \mathrm{d}V = \iiint_V b_i \, \mathrm{d}V + \iint_S t_{ji} \nu_j \, \mathrm{d}S \qquad i=1,2,3$$

According to the **divergence theorem** (also called **Green-Gauss-Ostrogradsky theorem**) we may transform a surface integral of a given vector function into a volume integral of its divergence:

$$\boxed{\iint_S F_i \nu_i \, \mathrm{d}S = \iiint_V F_{i,i} \, \mathrm{d}V}$$

We obtain:

$$\iiint_V \rho a_i \, \mathrm{d}V = \iiint_V b_i \, \mathrm{d}V + \iiint_V t_{ji,j} \, \mathrm{d}V \qquad i=1,2,3$$

All integrals have the same domain of integration, so we may add the integrands. Expressing the acceletaion as the second time derivative of displacement gives us:

$$\iiint_V \left(\rho \ddot{u}_i - b_i - t_{ji,j} \right) \mathrm{d}V = 0 \qquad i=1,2,3$$

Since the volume V may be chosen arbitrarily, the above relation holds if the integrand is equal zero itself. As a conclusion we obtain an **equation of motion**:

$$t_{ji,j} + b_i = \rho \ddot{u}_i \qquad i=1,2,3$$

In case of **statics** we **neglect the influence of inertial forces**:

$$t_{ji,j} + b_i = 0 \qquad i=1,2,3$$

The above equations are termed **equilibrium equations** or **Navier equations**.

SYMMETRY OF STRESS TENSOR

Let's write down the principle of moment of momentum. Let's calculate the moment of momentum and sum of moments of forces about origin of the chosen coordinate system, so that $\mathbf{r}_O = -\mathbf{x}$.

$$\dot{\mathbf{K}}_O = \mathbf{M}_O \quad \Leftrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} \iiint_V \rho \mathbf{v} \times (-\mathbf{x}) \mathrm{d}V = \iiint_V \mathbf{b} \times (-\mathbf{x}) \mathrm{d}V + \iint_S \mathbf{t} \times (-\mathbf{x}) \mathrm{d}S$$

Calulation of the time derivative of moment of momentum is performed as previously – nothing changes if we subtitute $\mathbf{v} \times (-\mathbf{x})$ in the place of $\mathbf{v}$ in derivative of momentum.

$$\dot{\mathbf{K}}_O = -\frac{\mathrm{d}}{\mathrm{d}t} \iiint_V \rho \mathbf{v} \times \mathbf{x} \mathrm{d}V = -\iiint_V \rho \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v} \times \mathbf{x}) \mathrm{d}V$$

Time derivative of a vector product is calculated in the same way as a derivative of product:

$$\dot{\mathbf{K}}_O = -\iiint_V \rho(\dot{\mathbf{v}} \times \mathbf{x} + \underbrace{\mathbf{v} \times \dot{\mathbf{x}}}_{=\mathbf{v} \times \mathbf{v} = \mathbf{0}}) \mathrm{d}V = -\iiint_V \rho \mathbf{a} \times \mathbf{x} \mathrm{d}V$$

Principle of moment of momentum takes then form:

$$\iiint_V \rho \mathbf{a} \times \mathbf{x} \mathrm{d}V = \iiint_V \mathbf{b} \times \mathbf{x} \mathrm{d}V + \iint_S \mathbf{t} \times \mathbf{x} \mathrm{d}S$$

In index notation:

$$\iiint_V \rho \epsilon_{ijk} a_j x_k \mathrm{d}V = \iiint_V \epsilon_{ijk} b_j x_k \mathrm{d}V + \iint_S \epsilon_{ijk} t_j x_k \mathrm{d}S \qquad i=1,2,3$$

Let's express the stress vector according to the tetrahedron conditions:

$$\iiint_V \rho \epsilon_{ijk} a_j x_k \mathrm{d}V = \iiint_V \epsilon_{ijk} b_j x_k \mathrm{d}V + \iint_S \epsilon_{ijk} t_{lj} \nu_l x_k \mathrm{d}S$$

According to the **divergence theorem**:

$$\iiint_V \rho \epsilon_{ijk} a_j x_k \,\mathrm{d}V = \iiint_V \epsilon_{ijk} b_j x_k \,\mathrm{d}V + \iiint_V \left(\epsilon_{ijk} t_{lj} x_k\right)_{,l} \mathrm{d}V$$

Let's deal with integrand in the second integral on the right hand side:

$$\left(\epsilon_{ijk} t_{lj} x_k\right)_{,l} = \epsilon_{ijk}\left(t_{lj} x_k\right)_{,l} = \epsilon_{ijk}\left(t_{lj,l} x_k + t_{lj} x_{k,l}\right) = \epsilon_{ijk}\left(t_{lj,l} x_k + t_{lj} \delta_{kl}\right) = \epsilon_{ijk}\left(t_{lj,l} x_k + t_{kj}\right)$$

Let's substitute it in the principle of moment of momentum:

$$\iiint_V \rho \epsilon_{ijk} a_j x_k \,\mathrm{d}V = \iiint_V \epsilon_{ijk} b_j x_k \,\mathrm{d}V + \iiint_V \epsilon_{ijk} t_{lj,l} x_k + \epsilon_{ijk} t_{kj} \,\mathrm{d}V$$

Summing the integrals over tha same domain gives us:

$$\iiint_V \epsilon_{ijk} x_k \underbrace{\left(\rho a_j - b_j - t_{lj,l}\right)}_{0} \mathrm{d}V = \iiint_V \epsilon_{ijk} t_{kj} \,\mathrm{d}V$$

In the relation above we made use of the equations of motion. Since the principle of moment of momentum must hold for any V, then we may write:

$$0 = \iiint_V \epsilon_{ijk} t_{kj} \,\mathrm{d}V \quad \underset{V}{\forall} \quad \Rightarrow \quad \epsilon_{ijk} t_{kj} = 0 \quad i=1,2,3$$

The above system of three equations may be written explicitly:

$$i=1: \quad \epsilon_{1jk} t_{kj} = t_{32} - t_{23} = 0 \quad \Rightarrow \quad t_{23} = t_{32}$$

$$i=2: \quad \epsilon_{2jk} t_{kj} = t_{13} - t_{31} = 0 \quad \Rightarrow \quad t_{31} = t_{13}$$

$$i=1: \quad \epsilon_{3jk} t_{kj} = t_{21} - t_{12} = 0 \quad \Rightarrow \quad t_{12} = t_{21}$$

We may conclude that **Cauchy stress tensor is symmetric**:

$$\boxed{t_{ij} = t_{ji}}$$

INITIAL AND BOUNDARY CONDITIONS

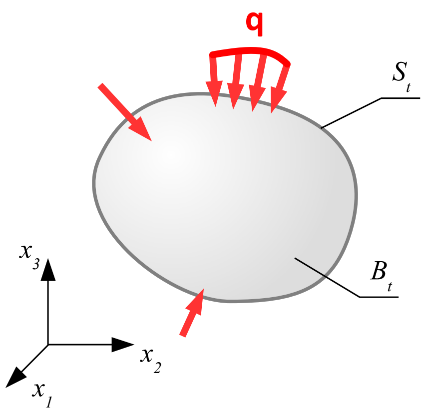
Equations of motion are a system of linear partial differential equations of the second order. They must be equipped with proper initial conditions (due to differentiation with respect to time) and boundary conditions (spatial differentiation).

The highest order of differentiation with respect to time is two, so we need two initial conditions – for position and for velocity:

INITIAL CONDITIONS:
- **initial position of each point** in time t_0 : $\mathbf{u}(\mathbf{x}, t_0) = \mathbf{u_0}(\mathbf{x})$
- **initial velocity of each point** in time t_0 $\dot{\mathbf{u}}(\mathbf{x}, t_0) = \mathbf{v_0}(\mathbf{x})$

Boundary conditions will be determined for both stress and displacements. If the **motion of the body is constrained** in any way, then the **constraint equation becomes a kinematic boundary condition** for a displacement and - as a consequence of constitutive relations that will be discussed later – for stress. Let's assume that such a constraint is applied to a certain part of external surface $S_u \subset S$:

KINEMATIC BOUNDARY CONDITIONS:
- **fixed position of points** on boundary S_u $\mathbf{u}(\mathbf{x}, t) = \hat{\mathbf{u}}_0(\mathbf{x}, t) \qquad \mathbf{x} \in S_u$



Boundary conditions for stresses will be found by application of the principle of momentum to the whole body – until now it was only applied to a part of it:

$$\dot{\mathbf{P}} = \mathbf{S} \quad \Leftrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} \iiint_{B_t} \rho \mathbf{v} \,\mathrm{d}V = \iiint_{B_t} \mathbf{b} \,\mathrm{d}V + \iint_{S_t} \mathbf{q} \,\mathrm{d}S$$

Time derivative is calculated as previously. The body forces vector may be expressed by a transformed equation of motion. In an index notation:

$$\iiint_{B_t} \rho \ddot{u}_i \,\mathrm{d}V = \iiint_{B_t} (\rho \ddot{u}_i - t_{ji,j}) \,\mathrm{d}V + \iint_{S_t} q_i \,\mathrm{d}S$$

$$\iiint_{B_t} t_{ji,j} \,\mathrm{d}V = \iint_{S_t} q_i \,\mathrm{d}S$$

Application of the divergence theorem gives us:

$$\iint_{S_t} t_{ji} \nu_j \,\mathrm{d}S = \iint_{S_t} q_i \,\mathrm{d}S$$

As this relation must hold independently of the shape of the body (its external surface S_t), we may write:

$$\boxed{t_{ji} \nu_j = q_i}$$

STATIC BOUNDARY CONDITIONS:
- **fixed load** on boundary S_q $\mathbf{T}_{\boldsymbol{\sigma}}(\mathbf{x})^{\mathrm{T}} \cdot \boldsymbol{\nu}(\mathbf{x}) = \mathbf{q}(\mathbf{x}) \qquad \mathbf{x} \in S_q$

EQUATIONS OF MOTION AND MEASURES OF INTERNAL TRACTIONS IN MATERIAL DESCRIPTION

Equations of motion derived in the previous chapter are formulated in the spatial description – the independent variables are the spatial (Eulerian) coordinates (differentiation is performed with respect to them). We shall now derive these equations in material description which is more convenient for description of deformation of solids. What must be noted is that in material description it is the reference configuration which is the domain of all fields. In particular stress distribution or external forces distribution is defined in the reference configuration what results in different values of those quantities than in spatial description because **the reference areas (for surface forces) and volumes (for body forces) are different – undeformed instead of deformed ones**.

While **in case of spatial description the true (internal or external) forces are referred to true (deformed) areas and volumes – that's why the Cauchy stress tensor is termed the true stress tensor** – in material description the same true forces are referred to undeformed areas and volumes. As a result those artificial tractions and body forces do not give us true information on their values. **True forces referred to undeformed shape will be termed nominal** instead of being true ones. Knowing true stresses and making use of the relation between deformed and undeformed surface areas derived in previous chapter we will define the **nominal stress tensor**.

NOMINAL STRESS – PIOLA-KIRCHHOFF STRESS TENSOR OF THE 1st KIND

We define the **nominal stress vector** $\mathbf{t}_R$ such that: $\mathbf{t}\,\mathrm{d}A = \mathbf{t}_R \mathrm{d}A_R$

According to the **tetrahedron conditions**: $\mathbf{T}_\sigma^\mathrm{T}\cdot\boldsymbol{\nu}\,\mathrm{d}A = \mathbf{t}_R\mathrm{d}A_R$

Let's introduce the relation between $\mathrm{d}A$ and $\mathrm{d}A_R$:

$$\mathrm{d}A\,\boldsymbol{\nu}^\mathrm{T} = J\;\mathrm{d}A_R\,\mathbf{N}^\mathrm{T}\cdot\mathbf{F}^{-1} \quad\Rightarrow\quad \mathrm{d}A\,\boldsymbol{\nu} = J\;\mathrm{d}A_R\mathbf{F}^{-\mathrm{T}}\cdot\mathbf{N}\ ,$$

where $\mathbf{N}$ is the unit normal to $\mathrm{d}A$ before deformation. We obtain:

$$J\,\mathbf{T}_\sigma^\mathrm{T}\cdot\mathbf{F}^{-\mathrm{T}}\cdot\mathbf{N}\;\mathrm{d}A_R = \mathbf{t}_R\mathrm{d}A_R$$

This must hold for any $\mathrm{d}A_R$, so:

$$\underbrace{J\,\mathbf{T}_\sigma^\mathrm{T}\cdot\mathbf{F}^{-\mathrm{T}}}_{\mathbf{T}_R}\cdot\mathbf{N} = \mathbf{t}_R$$

In the definition of $\mathbf{T}_R$ we may make use of the symmetry of the Cauchy stress tensor $\mathbf{T}_\sigma^\mathrm{T}=\mathbf{T}_\sigma$

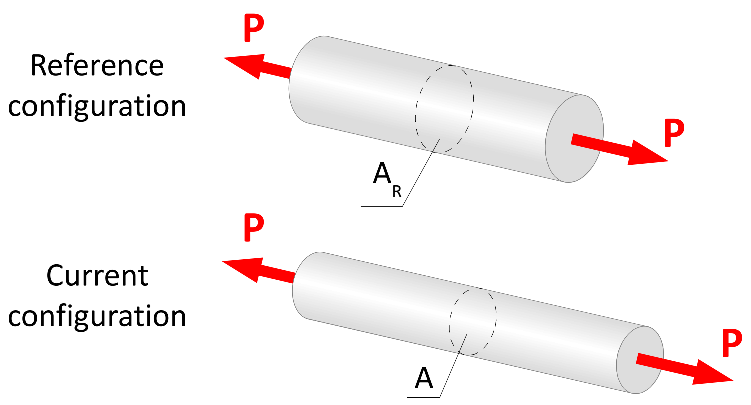
$$\boxed{\mathbf{T}_R = J\,\mathbf{T}_\sigma\cdot\mathbf{F}^{-\mathrm{T}} \quad\Leftrightarrow\quad T_{ik} = J\,t_{ij}\,X_{k,j}}$$

Quantity $\mathbf{T}_R$ will be termed the **Piola-Kirchhoff stress tensor of the 1st kind** (PK1) or **nominal stress tensor**, such that:

$$\mathbf{T}_R \cdot \mathbf{N} = \mathbf{t}_R \qquad T_{ij} N_j = T_i$$

It is important that $\mathbf{T}_R$ is **not symmetric** because in general the deformation gradient is not a symmetric tensor – this is a kind of inconvenience, since we have to know all 9 components of $\mathbf{T}_R$ instead of 6 components of $\mathbf{T}_\sigma$.

Writing down the tetrahedron condition we've always used transposition of $\mathbf{T}_\sigma$ what – due to symmetry of that tensor – was in fact redundant. Since now we will omit this operation in order to make the relations between stress tensor and stress vector look the same in both cases of true and nominal stresses. The difference between true and nominal stresses may be illustrated in the following way



Nominal stress $(\mathbf{T}_R)$:

$$T = \frac{P}{A_0}$$

True stress $(\mathbf{T}_\sigma)$:

$$t = \frac{P}{A}$$

PRINCIPLE OF MOMENTUM AND EQUATIONS OF MOTION IN MATERIAL DESCRIPTION

Let's write down the principle of momentum in material description. Body forces and surface tractions are assumed to be given by vector fields $\mathbf{B}(\mathbf{X})$ i $\mathbf{Q}(\mathbf{X})$ (these are not true loads). Let's chose any subregion V_R within a reference configuration B_{ref} . Its boundary will be denoted with S_R :

$$\dot{\mathbf{P}} = \mathbf{S} \quad \Leftrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} \iiint_{V_R} \rho_R v_i \,\mathrm{d}V_R = \iiint_{V_R} B_i \,\mathrm{d}V_R + \iint_{S_R} T_i \,\mathrm{d}S \qquad i=1,2,3$$

In material description domain of integration is time independent so we may differentiate the integrand directly. Also the reference density is constant in time. Making use of the relation between the PK1 stress tensor and nominal stress vector, we may write:

$$\iiint_{V_R} \rho_R a_i \,\mathrm{d}V_R = \iiint_{V_R} B_i \,\mathrm{d}V_R + \iint_{S_R} T_{ij} N_j \,\mathrm{d}S$$

Application of the divergence theorem gives us:

$$\iiint_{V_R} \rho_R a_i \,\mathrm{d}V_R = \iiint_{V_R} B_i \,\mathrm{d}V_R + \iiint_{V_R} T_{ij,j} \,\mathrm{d}V$$

Adding the integrals together results in:

$$\iiint_{V_R} \left(\rho_R a_i - B_i - T_{ij,j}\right) \mathrm{d}V_R = 0$$

Since V_R may be chosen arbitrary we may write down **equations of motion in material description** in the following form:

$$\boxed{T_{ij,j} + B_i = \rho_R \ddot{u}_i \qquad i=1,2,3}$$

We have to remember that $\mathbf{T}_R$ is not symmetric – it has 9 independent components. Additional required equations may be obtained from the symmetry of Cauchy stress tensor (derived from the principle of moment of momentum). Let's express $\mathbf{T}_\sigma$ by $\mathbf{T}_R$:

$$\mathbf{T}_R = J\,\mathbf{T}_\sigma \cdot \mathbf{F}^{-\mathrm{T}} \quad \Rightarrow \quad \boxed{\mathbf{T}_\sigma = \frac{1}{J}\mathbf{T}_R \cdot \mathbf{F}^{\mathrm{T}}}$$

Symmetry of $\mathbf{T}_\sigma$ gives us:

$$\mathbf{T}_\sigma^{\mathrm{T}} = \mathbf{T}_\sigma \quad \Rightarrow \quad \frac{1}{J}\mathbf{F}\cdot\mathbf{T}_R^{\mathrm{T}} = \frac{1}{J}\mathbf{T}_R \cdot \mathbf{F}^{\mathrm{T}}$$

and in index notation:

$$\boxed{x_{i,j} T_{kj} = T_{ij} x_{k,j} \qquad i,k=1,2,3}$$

PIOLA-KIRCHHOFF STRESS TENSOR OF THE 2nd KIND

Lack of symmetry of the PK1 stress tensor is a problem in this sense, that total number of unknown function to be determined in order to describe the deformation of a body is raised with 3. A solution to than inconvenience is introduction another measure of stress, namely the **Piola-Kirchhoff stress tensor of the 2nd kind** $\mathbf{T}_S$ (PK2) defined as follows:

$$\boxed{\mathbf{T}_S = \mathbf{F}^{-1}\cdot\mathbf{T}_R \quad \Leftrightarrow \quad S_{ik} = X_{i,j} T_{jk}}$$

If we express the PK1 in terms of $\mathbf{T}_\sigma$, we may write

$$S_{ik} = J X_{i,j} t_{jl} X_{k,l}$$
$$S_{ki} = J X_{k,j} t_{jl} X_{i,l} = J X_{k,l} t_{lj} X_{i,j} = J X_{i,j} t_{jl} X_{k,l} = S_{ik}$$

so the **Piola-Kirchhoff stress tensor of the 2nd kind is symmetric**. An inverted relation between $\mathbf{T}_S$ and $\mathbf{T}_R$ is:

$$\mathbf{T}_R=\mathbf{F}\cdot\mathbf{T}_S \qquad \Leftrightarrow \qquad T_{ik}=x_{i,j}S_{jk}$$

Introducing it in the equations of motion gives us an **alternative equations of motion in material description**:

$$\boxed{\left(S_{kj}x_{i,k}\right)_{,j}+B_i=\rho_R\ddot{u}_i \qquad i=1,2,3}$$

A shortcoming of such an approach is that the PK2 stress tensor has no clear physical interpretation. It is sometimes termed as the **material stress tensor**.

SUMMARY

STRESS VECTORS

True stress vector: $\mathbf{t}$: $\mathbf{T}_\sigma\cdot\boldsymbol{\nu}=\mathbf{t}$

- **true stress** – ratio of current force and current surface area

Nominal stress vector: $\mathbf{t}_R$: $\mathbf{T}_R\cdot\mathbf{N}=\mathbf{t}_R$

- **nominal stress** – ratio of current force and reference surface area

STRESS TENSORS

Cauchy stress tensor $\mathbf{T}_\sigma$

- **symmetric** tensor: $t_{ij}=t_{ji}$

Piola – Kirchhoff stress tensor of the 1st kind $\mathbf{T}_R=J\,\mathbf{T}_\sigma\cdot\mathbf{F}^{-\mathrm{T}}$

- **non-symmetric** tensor: $T_{ij}\neq T_{ji}$

Piola – Kirchhoff stress tensor of the 2nd kind $\mathbf{T}_S=\mathbf{F}^{-1}\cdot\mathbf{T}_R$

- **symmetric** tensor: $S_{ij}=S_{ji}$

Cauchy stress tensor $\mathbf{T}_\sigma$	-	$J^{-1}\mathbf{T}_R\cdot\mathbf{F}^{\mathrm{T}}$	$J^{-1}\mathbf{F}\cdot\mathbf{T}_S\cdot\mathbf{F}^{\mathrm{T}}$
PK1 stress tensor $\mathbf{T}_R$	$J\,\mathbf{T}_\sigma\cdot\mathbf{F}^{-\mathrm{T}}$	-	$\mathbf{F}\cdot\mathbf{T}_S$
PK2 stress tensor $\mathbf{T}_S$	$J\,\mathbf{F}^{-1}\cdot\mathbf{T}_\sigma\cdot\mathbf{F}^{-\mathrm{T}}$	$\mathbf{F}^{-1}\cdot\mathbf{T}_R$	-

EQUATIONS OF MOTION

SPATIAL DESCRIPTION:

$$t_{ji,j} + b_i = \rho \ddot{u}_i \qquad i=1,2,3$$

$$\frac{\partial \rho}{\partial t} - (\rho v_k)_{,k} = 0$$ **– mass continuity equation**

MATERIAL DESCRIPTION:

$$\begin{cases} T_{ij,j} + B_i = \rho_R \ddot{u}_i & i=1,2,3 \\ x_{i,j} T_{kj} = T_{ij} x_{k,j} & i,k=1,2,3 \end{cases}$$

or

$$\left(S_{kj} x_{i,k}\right)_{,j} + B_i = \rho_R \ddot{u}_i \qquad i=1,2,3$$

where

$$\rho_R = const.$$

initial conditions: $\mathbf{u} = \mathbf{u_0}$ at $t=t_0$

$\dot{\mathbf{u}} = \mathbf{v_0}$ at $t=t_0$

kinematic boundary conditions: $\mathbf{u} = \hat{\mathbf{u}}$ on S_u

static boundary conditions: $\mathbf{q} = \mathbf{T_\sigma} \cdot \boldsymbol{\nu}$ on S_q **- spatial description**

$\mathbf{Q} = \mathbf{T}_R \cdot \mathbf{N}$ on S_q **- material description**

4. CONSTITUTIVE RELATIONS

Until now, we've determined following equations governing the problems of theory of elasticity:

- 3 equations of motion
- 6 geometric (kinematic) relations

Following unknown functions should be determined with the use of the equations mentioned above:

- 3 unknown components of the displacement vector
- 6 unknown components of the strain tensor
- 6 unknown components of the stress tensor

It looks like we're lacking 6 more equations. In the equations of motion stress is related with displacement while the geometric relations are between strain and displacement. We still don't know how the stress is related to strain. Such relations will be termed **constitutive relations** or **physical relations**. In the most general case we can assume that a **certain measure of stress** $\mathbf{T}$ (one of the stress tensors that we've introduced) **is a function of chosen particle and its current position**:

$$\mathbf{T} = f(\mathbf{X}, \mathbf{x})$$

POSTULATED PRINCIPLES REGARDING CONSTITUTIVE RELATIONS

We postulate that following principles hold:

- **Principle of determinism**
- **Principle of locality**
- **Principle of material objectivity**

PRINCIPLE OF DETERMINISM

The stress in given particle $\mathbf{X}$ *and given time* t *is* ***determined by the choice of the particle and by past motion*** *of all other particles in the body:*

$$\mathbf{T}(\mathbf{X};t) = f(\mathbf{X}, \mathbf{x}(\boldsymbol{\xi}, t-\tau)), \quad \boldsymbol{\xi} \in B_{ref}, \quad \tau \in \langle 0 ; \infty)$$

PRINCIPLE OF LOCALITY

The stress in given particle $\mathbf{X}$ *and given time* t *depends on past motion of* ***particles in arbitrary small neighborhood of the chosen particle****:*

$$|\boldsymbol{\xi} - \mathbf{X}| < \varepsilon \to 0$$

PRINCIPLE OF MATERIAL OBJECTIVITY

> *Constitutive relations describing internal conditions of a physical system and interactions between all its parts must be independent of the choice of the reference frame.*

FIRST GRADIENT THEORY

The most commonly used theory (but not the only one which is used) of the constitutive relation is the so called first gradient theory. According to the principle of locality, the depedndecy of the unknown function in constitutive relation on motion of other particles $\mathbf{x}(\boldsymbol{\xi}, t-\tau)$ may be expanded into a power series. Let $\mathbf{X}$ be a chosen particle and $\boldsymbol{\xi}=\mathbf{X}+\mathrm{d}\mathbf{X}$ a particle in its neighbourhood. Neglecting (only in notation) the time-dependency, we may write:

$$x_i(\xi_j) = x_i(X_j + \mathrm{d}X_j) = x_i(X_j) + \frac{1}{1!}\frac{\partial x_i}{\partial X_k}\bigg|_{X_j}\mathrm{d}X_k + \frac{1}{2!}\frac{\partial x_i}{\partial X_k \partial X_l}\bigg|_{X_j}\mathrm{d}X_k\,\mathrm{d}X_l + \ldots$$

Please, note that $\frac{\partial x_i}{\partial X_k}\Big|_{X_j}$ is simply the value of the material deformation gradient in X_j .
Further derivatives are further (tensorial) gradients of the deformation. General constitutive relation could be written as follows:

$$\mathbf{T}(\mathbf{X}) = f(\mathbf{X};\mathbf{x}(\boldsymbol{\xi})) \quad \Leftrightarrow \quad \mathbf{T}(\mathbf{X}) = f\left(\mathbf{X}\,;\,\mathbf{x}(\mathbf{X}) + \mathbf{F}(\mathbf{X})\cdot\mathrm{d}\mathbf{X} + \frac{1}{2}\nabla\mathbf{F}(\mathbf{X})\cdot\mathrm{d}\mathbf{X}\cdot\mathrm{d}\mathbf{X} + \ldots\right)$$

Or, more generally:

$$\mathbf{T}(\mathbf{X};t) = f(\mathbf{X}\,;\,\mathbf{x}(\mathbf{X};t-\tau)\,;\,\mathbf{F}(\mathbf{X};t-\tau)\,;\,\nabla\mathbf{F}(\mathbf{X};t-\tau);\ldots)$$

In fact we cannot account for $\mathbf{x}(\mathbf{X})$ in expression for $\mathbf{T}$. The stress state indeed depend on the choice of the particle and on deformation in that point (gradients of deformation), yet it must be independent on the location of the particle in space. According to the principle of material objectivity we can write:

$$\mathbf{T}(\mathbf{X};t) = f(\mathbf{X}\,;\,\mathbf{F}(\mathbf{X};t-\tau)\,;\,\nabla\mathbf{F}(\mathbf{X};t-\tau);\ldots)$$

It emerges that a very wide class of materials may be quite precisely described with the use of a function of the following form

$$\mathbf{T}(\mathbf{X};t) = f(\mathbf{X}\,;\,\mathbf{F}(\mathbf{X};t-\tau))$$

namely, accounting only for the first gradient of deformation and neglecting higher derivatives. Theories based on that assumption are called the **first gradient theories**.

We know that the material deformation gradient may be subject of the polar decomposition $\mathbf{F}=\mathbf{R}\cdot\mathbf{U}$ so that it can be expressed as a product of rotation tensor $\mathbf{R}$ and stretch tensor $\mathbf{U}$. Again - according to the principle of material objectivity – just as location in space cannot influence the form of constitutive relations so the rigid rotation can't. The first gradient constitutive relation should depend then on the stretch tensor $\mathbf{U}$ only, or on any function of it, e.g. deformation tensor $\mathbf{C}$ or strain tensor $\mathbf{E}$. As it is the Green - de Saint-Venant strain tensor which is most commonly used, we will try to find the constitutive relation based on this measure of deformation. Among all measures of stress, we shall focus on the PK2 stress tensor, as it is the most convenient in use, being a symmetric tensor applicable in material description. Another feature is that those two quantities may be shown to be energetic conjugates – a feature that will be discussed later. We will consider following relation:

$$\boxed{\mathbf{T}_S(\mathbf{X};t)=f(\mathbf{X};\,\mathbf{E}(\mathbf{X};t-\tau))}$$

HOMOGENEITY

A **homogeneous material** is the one, in which constitutive relation is the same in each point of the body, so it depends on the strain in that point and not on the choice of

$$\mathbf{T}_S(\mathbf{X};t)=f(\mathbf{E}(\mathbf{X};t-\tau))$$

Many materials may be considered approximately homogeneous (steel, alloys, plastics etc.). In other cases inhomogeneity is an important factor – e.g. In case of concrete, wood, soil, etc. Even then homogeneous models may be sometimes applied.

ISOTROPY

An **isotropic material** is the one, in which material properties (longitudinal stifness, tensile strength etc.) are the same in all directions, namely they do not depend on the direction along which they are examined. Many materials cannot be considered isotropic – these are e.g. wood, composited with oriented inclusions, crystals etc.

ELASTIC MATERIAL

In the introduction it was stated that an elastic material is the one, that **recovers its original shape after being deformed, when the load is removed**, so that **current stress state depends only on current deformation and not on past deformation** (load path). This can be accounted for in the constitutive relation as follows:

An elastic material is the one in which:

- current stress state depends on current deformation and not on past deformation:

$$\mathbf{T}_S(\mathbf{X};t)=f(\mathbf{E}(\mathbf{X};t))$$

- deformation is reversible, so constitutive relation must be a one-to-one function so that a unique inverse of f existed.

HYPERELASTIC MATERIAL

A hyperelastic material (an elastic material in the sense of Green) is such an elastic material for which a scalar-valued function W of tensor argument exists, termed an elastic potential, that the constitutive relation may be written in the following form:

$$\mathbf{T}_S=\frac{\partial W}{\partial \mathbf{E}} \quad \Leftrightarrow \quad S_{ij}=\frac{\partial W}{\partial E_{ij}} \quad i,j=1,2,3$$

COMPLETE SET OF EQUATIONS GOVERNING THE NON-LINEAR THEORY OF ELASTICITY FOR HYPERELASTIC MATERIALS:

We have **15 unknowns:**

- 3 components of the **displacement vector** $\mathbf{u}$,
- 6 components of the **strain tensor** $\mathbf{E}$
- 6 components of the **stress tensor** $\mathbf{S}$

and **15 equations**:

- 3 **equations of motions** (non-linear partial differential equations)
- 6 **geometric relations**(non-linear partial differential equations)
- 6 **physical relations** (non-linear algebraic equations)

ISOTROPIC HYPERELASTIC MATERIAL

Elastic potential in given coordinate system may be interpreted as a function of components of the strain tensor:

$$W(\mathbf{E}) = W(E_{11}; E_{22}; E_{33}; E_{23}; E_{31}; E_{12})$$

In case of an isotropic material the elastic potential must provide an isotropic relation – it must be an isotropic function itself. An important feature of an isotropic function is that it can be expressed in terms of invariants of its argument only:

$$W(\mathbf{E}) = W(I_1 ; I_2 ; I_3)$$

so the constitutive relation may be written as follows:

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{E}} = \frac{\partial W}{\partial I_1}\frac{\partial I_1}{\partial \mathbf{E}} + \frac{\partial W}{\partial I_2}\frac{\partial I_2}{\partial \mathbf{E}} + \frac{\partial W}{\partial I_3}\frac{\partial I_3}{\partial \mathbf{E}}$$

Let's introduce following notation:

$$\begin{aligned}
\phi_1 &= \phi_1(I_1; I_2; I_3) = \frac{\partial W}{\partial I_1} \\
\phi_2 &= \phi_2(I_1; I_2; I_3) = \frac{\partial W}{\partial I_2} \\
\phi_3 &= \phi_3(I_1; I_2; I_3) = \frac{\partial W}{\partial I_3}
\end{aligned}$$

Invariants of a tensor are certain scalar-valued functions of a tensor argument. Their derivatives may be calculated with the use of so called **Gâteaux derivative.** Then the **constitutive relation for an isotropic hyperelastic material** may be written as follows:

$$\boxed{\mathbf{T}_S = \phi_1 \mathbf{1} + \phi_2 \left[I_1 \mathbf{1} - \mathbf{E} \right] + \phi_3 \left[\mathbf{E}^2 - I_1 \mathbf{E} + I_2 \mathbf{1} \right] \ ,}$$

Rearranging the terms gives us:

$$\begin{aligned}
\mathbf{T}_S &= \phi_3 \mathbf{E}^2 - (\phi_2 + I_1 \phi_3)\mathbf{E} + (\phi_1 + I_1 \phi_2 + I_2 \phi_3)\mathbf{1} = \\
&= k_2 \mathbf{E}^2 + k_1 \mathbf{E} + k_0 \mathbf{1}
\end{aligned} \ .$$

So **any constitutive relation for an isotropic hyperelastic material may be written down in the form of a tensorial quadratic polynomial**. The coefficients of that polynomial depend, however, on the strain state in given point too.

HOOKE's MATERIAL

Let's assume that the strain is small, close to 0. Then an approximation of the elastic potential with its power series expanded in the neighborhood of $\mathbf{E}=\mathbf{0}$ (materials natural state) may be sufficiently precise:

$$W(\mathbf{E}) \approx W(\mathbf{0}) + \frac{1}{1!}\left.\frac{\partial W}{\partial E_{ij}}\right|_{\mathbf{0}} E_{ij} + \frac{1}{2!}\left.\frac{\partial^2 W}{\partial E_{ij} E_{kl}}\right|_{\mathbf{0}} E_{ij} E_{kl} + \frac{1}{3!}\left.\frac{\partial^3 W}{\partial E_{ij} E_{kl} E_{mn}}\right|_{\mathbf{0}} E_{ij} E_{kl} E_{mn} + \ldots$$

Derivatives of elastic potential calculated in point $\mathbf{E}=\mathbf{0}$ are certain constants

$$W(\mathbf{E}) \approx W_0 + \hat{S}_{ij} E_{ij} + \frac{1}{2} \mathsf{S}_{ijkl} E_{ij} E_{kl} + \ldots$$

where:

$$W_0 = W(\mathbf{0}) \quad , \qquad \hat{S}_{ij} = \left.\frac{\partial W}{\partial E_{ij}}\right|_{\mathbf{0}} \quad , \qquad \mathsf{S}_{ijkl} = \left.\frac{\partial^2 W}{\partial E_{ij} \partial E_{kl}}\right|_{\mathbf{0}}$$

Differentiation with respect to strain components gives us:

$$S_{ij} = \frac{\partial W}{\partial E_{ij}} \approx \hat{S}_{ij} + \frac{1}{2}(\mathsf{S}_{ijkl} + \mathsf{S}_{klij}) E_{kl} + \ldots = \hat{S}_{ij} + \mathsf{S}_{ijkl} E_{kl} + \ldots$$

The value of W_0 may be chosen arbitrarily – it vanishes anyway after differentiation. Let's take it equal 0. If $\hat{S}_{ij}$ was not equal zero, then in an unstrained body there would be a non-zero stress state what (disregarding phenomena of e.g. residual stresses, which do not apply to elasticity) is against experimental evidence. We will take then $\hat{S}_{ij} = 0$. Neglecting higher order terms, the constitutive relation may be written in the form:

$$S_{ij} = \left.\frac{\partial^2 W}{\partial E_{ij} E_{kl}}\right|_{\mathbf{E}=\mathbf{0}} E_{kl}$$

A **hyperelastic material** for which an **elastic potential has a form of a second-degree function** of the strain state (no constant or linear terms) is called **Hooke's material** or **linear elastic material**.

$$W = \frac{1}{2} \mathsf{S}_{ijkl} E_{ij} E_{kl} \quad \text{where} \quad \mathsf{S}_{ijkl} = \left.\frac{\partial^2 W}{\partial E_{ij} E_{kl}}\right|_{\mathbf{E}=\mathbf{0}}$$

$$S_{ij} = \mathsf{S}_{ijkl} E_{kl}$$

A four-index matrix S_{ijkl} may be called an **elastic constants matrix**. It may be shown that it is a fourth-rank tensor. Constitutive relation for Hooke's material has a form of **generalized Hooke's Law:**

$$\mathbf{T}_S = \mathsf{S} \cdot \mathbf{E} \qquad S_{ij} = \mathsf{S}_{ijkl} E_{kl}$$

or equivalently

$$\mathbf{E} = \mathsf{C} \cdot \mathbf{T}_S \qquad E_{ij} = \mathsf{C}_{ijkl} S_{kl}$$

Where S is termed **stiffness tensor** and $\mathsf{C} = \mathsf{S}^{-1}$ is a **compliance tensor**. These tensors are characterized by a set of internal symmetries:

- due to symmetry of the stress tensor: $S_{ij} = S_{ji} \quad \Rightarrow \quad \mathsf{S}_{ijkl} = \mathsf{S}_{jikl}$
- due to symmetru of the strain tensor: $E_{kl} = E_{lk} \quad \Rightarrow \quad \mathsf{S}_{ijkl} = \mathsf{S}_{ijlk}$
- due to commutativity of differentiation: $\mathsf{S}_{ijkl} = \left.\frac{\partial}{\partial E_{ij}} \frac{\partial W}{\partial E_{kl}}\right|_0 = \left.\frac{\partial}{\partial E_{kl}} \frac{\partial W}{\partial E_{ij}}\right|_0 = \mathsf{S}_{klij}$

The same is for compliance tensor. Summing it up:

$$\begin{aligned} \mathsf{S}_{ijkl} &= \mathsf{S}_{jikl} = \mathsf{S}_{ijlk} = \mathsf{S}_{klij} \\ \mathsf{C}_{ijkl} &= \mathsf{C}_{jikl} = \mathsf{C}_{ijlk} = \mathsf{C}_{klij} \end{aligned}$$

Finally, **among 81 components of S and C only 21 of them are independent**. Writing the generalized Hooke's Law explicitly, it can be noticed that **every component of the stress tensor is a linear combination of the components of the strain tensor** with the components of stiffness tensor being coefficients of that combination:

$$\begin{aligned} S_{11} = \; & \mathsf{S}_{1111} E_{11} + \mathsf{S}_{1122} E_{22} + \mathsf{S}_{1133} E_{33} + \\ & (\mathsf{S}_{1123} + \mathsf{S}_{1132}) E_{23} + (\mathsf{S}_{1131} + \mathsf{S}_{1113}) E_{31} + (\mathsf{S}_{1112} + \mathsf{S}_{1121}) E_{12} \\ S_{12} = \; & \mathsf{S}_{1211} E_{11} + \mathsf{S}_{1222} E_{22} + \mathsf{S}_{1233} E_{33} + \\ & (\mathsf{S}_{1223} + \mathsf{S}_{1232}) E_{23} + (\mathsf{S}_{1231} + \mathsf{S}_{1213}) E_{31} + (\mathsf{S}_{1212} + \mathsf{S}_{1221}) E_{12} \\ & \dots \end{aligned}$$

It can be expressed in the matrix notation as follows:

$$\begin{bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ S_{23} \\ S_{31} \\ S_{12} \end{bmatrix} = \begin{bmatrix} \mathsf{S}_{1111} & \mathsf{S}_{1122} & \mathsf{S}_{1133} & \mathsf{S}_{1123} & \mathsf{S}_{1131} & \mathsf{S}_{1112} \\ & \mathsf{S}_{2222} & \mathsf{S}_{2233} & \mathsf{S}_{2223} & \mathsf{S}_{2231} & \mathsf{S}_{2212} \\ & & \mathsf{S}_{3333} & \mathsf{S}_{3323} & \mathsf{S}_{3331} & \mathsf{S}_{3312} \\ & & & \mathsf{S}_{2323} & \mathsf{S}_{2331} & \mathsf{S}_{2312} \\ & \mathrm{sym} & & & \mathsf{S}_{3131} & \mathsf{S}_{3112} \\ & & & & & \mathsf{S}_{1212} \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{23} \\ 2E_{31} \\ 2E_{12} \end{bmatrix}$$

$$\begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{23} \\ 2E_{31} \\ 2E_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 2C_{1123} & 2C_{1131} & 2C_{1112} \\ & C_{2222} & C_{2233} & 2C_{2223} & 2C_{2231} & 2C_{2212} \\ & & C_{3333} & 2C_{3323} & 2C_{3331} & 2C_{3312} \\ & & & 4C_{2323} & 4C_{2331} & 4C_{2312} \\ & \text{sym} & & & 4C_{3131} & 4C_{3112} \\ & & & & & 4C_{1212} \end{bmatrix} \begin{bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ S_{23} \\ S_{31} \\ S_{12} \end{bmatrix}$$

If we substitute double indices with a single number according to the rule $(11;22;33;23;31;12) \rightarrow (1;2;3;4;5;6)$, such a notation is referred to as **Voigt notation**. Much more precise notation is the one below, sometimes referred to as the **Mandel notation**:

$$\begin{bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ \sqrt{2}S_{23} \\ \sqrt{2}S_{31} \\ \sqrt{2}S_{12} \end{bmatrix} = \begin{bmatrix} S_{1111} & S_{1122} & S_{1133} & \sqrt{2}S_{1123} & \sqrt{2}S_{1131} & \sqrt{2}S_{1112} \\ & S_{2222} & S_{2233} & \sqrt{2}S_{2223} & \sqrt{2}S_{2231} & \sqrt{2}S_{2212} \\ & & S_{3333} & \sqrt{2}S_{3323} & \sqrt{2}S_{3331} & \sqrt{2}S_{3312} \\ & & & 2S_{2323} & 2S_{2331} & 2S_{2312} \\ & \text{sym} & & & 2S_{3131} & 2S_{3112} \\ & & & & & 2S_{1212} \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ \sqrt{2}E_{23} \\ \sqrt{2}E_{31} \\ \sqrt{2}E_{12} \end{bmatrix}$$

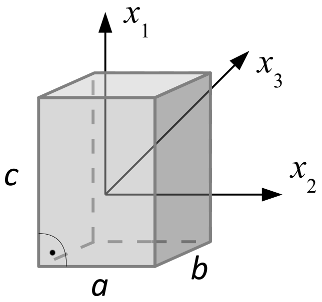
$$\begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ \sqrt{2}E_{23} \\ \sqrt{2}E_{31} \\ \sqrt{2}E_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & \sqrt{2}C_{1123} & \sqrt{2}C_{1131} & \sqrt{2}C_{1112} \\ & C_{2222} & C_{2233} & \sqrt{2}C_{2223} & \sqrt{2}C_{2231} & \sqrt{2}C_{2212} \\ & & C_{3333} & \sqrt{2}C_{3323} & \sqrt{2}C_{3331} & \sqrt{2}C_{3312} \\ & & & 2C_{2323} & 2C_{2331} & 2C_{2312} \\ & \text{sym} & & & 2C_{3131} & 2C_{3112} \\ & & & & & 2C_{1212} \end{bmatrix} \begin{bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ \sqrt{2}S_{23} \\ \sqrt{2}S_{31} \\ \sqrt{2}S_{12} \end{bmatrix}$$

ANISOTROPIC HOOKE'S MATERIAL

It can be proven that there are only 8 possible classes of anisotropy of linear elastic materials, corresponding to some extent with crystallographic systems. The higher is the symmetry, the least number of independent elastic constants is needed:

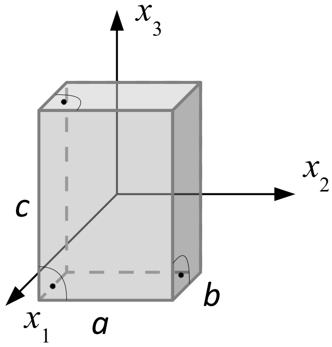
1. Anisotropy	triclinic crystal system	21 constants
2. Monoclinic symmetry	monoclinic crystal system	13 constants
3. Orthotropy	orthorhombic crystal system	9 constants
4. Trigonal symmetry	trigonal crystal system	6 constants
5. Tetragonal symmetry	tetragonal crystal system	6 constants
6. Cyllindrical symmetry	hexagonal crystal system	5 constants
7. Cubiy symmetry	cubic crystal system	3 constants
8. Isotropy		2 constants

MONOCLINIC SYMMETRY



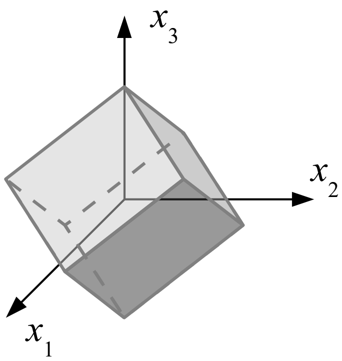
$$\mathbf{S} = \begin{bmatrix} \mathsf{S}_{1111} & \mathsf{S}_{1122} & \mathsf{S}_{1133} & \sqrt{2}\mathsf{S}_{1123} & 0 & 0 \\ & \mathsf{S}_{2222} & \mathsf{S}_{2233} & \sqrt{2}\mathsf{S}_{2223} & 0 & 0 \\ & & \mathsf{S}_{3333} & \sqrt{2}\mathsf{S}_{3323} & 0 & 0 \\ & & & 2\mathsf{S}_{2323} & 0 & 0 \\ & \text{sym} & & & 2\mathsf{S}_{3131} & 2\mathsf{S}_{3112} \\ & & & & & 2\mathsf{S}_{1212} \end{bmatrix}$$

ORTHOTROPY



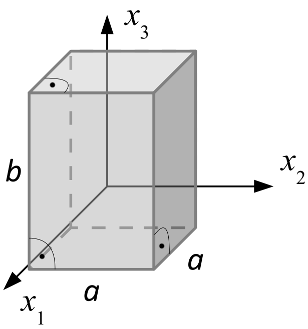
$$\mathbf{S} = \begin{bmatrix} \mathsf{S}_{1111} & \mathsf{S}_{1122} & \mathsf{S}_{1133} & 0 & 0 & 0 \\ & \mathsf{S}_{2222} & \mathsf{S}_{2233} & 0 & 0 & 0 \\ & & \mathsf{S}_{3333} & 0 & 0 & 0 \\ & & & 2\mathsf{S}_{2323} & 0 & 0 \\ & \text{sym} & & & 2\mathsf{S}_{3131} & 0 \\ & & & & & 2\mathsf{S}_{1212} \end{bmatrix}$$

TRIGONAL SYMMETRY



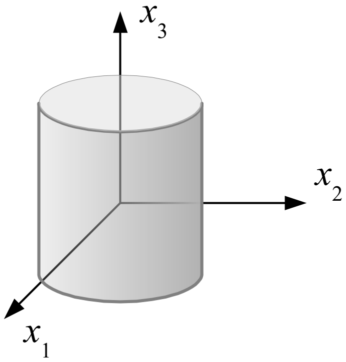
$$\mathbf{S} = \begin{bmatrix} \mathsf{S}_{1111} & \mathsf{S}_{1122} & \mathsf{S}_{1133} & \sqrt{2}\mathsf{S}_{1123} & 0 & 0 \\ & \mathsf{S}_{1111} & \mathsf{S}_{1133} & -\sqrt{2}\mathsf{S}_{1123} & 0 & 0 \\ & & \mathsf{S}_{3333} & 0 & 0 & 0 \\ & & & 2\mathsf{S}_{2323} & 0 & 0 \\ & \text{sym} & & & 2\mathsf{S}_{2323} & 2\mathsf{S}_{1123} \\ & & & & & \mathsf{S}_{1111}-\mathsf{S}_{1122} \end{bmatrix}$$

TETRAGONAL SYMMETRY



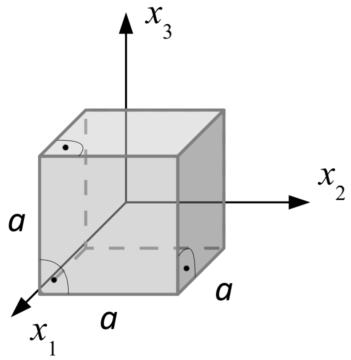
$$\mathbf{S} = \begin{bmatrix} \mathsf{S}_{1111} & \mathsf{S}_{1122} & \mathsf{S}_{1133} & 0 & 0 & 0 \\ & \mathsf{S}_{1111} & \mathsf{S}_{1133} & 0 & 0 & 0 \\ & & \mathsf{S}_{3333} & 0 & 0 & 0 \\ & & & 2\mathsf{S}_{2323} & 0 & 0 \\ & \text{sym} & & & 2\mathsf{S}_{2323} & 0 \\ & & & & & 2\mathsf{S}_{1212} \end{bmatrix}$$

CYLINDRICAL SYMMETRY (TRANSVERSAL ISOTROPY)



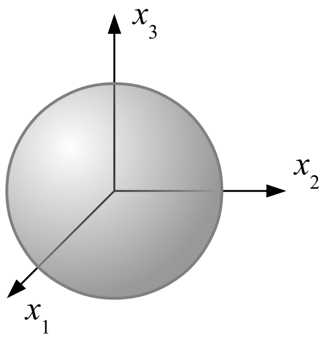
$$\mathbf{S} = \begin{bmatrix} S_{1111} & S_{1122} & S_{1133} & 0 & 0 & 0 \\ & S_{1111} & S_{1133} & 0 & 0 & 0 \\ & & S_{3333} & 0 & 0 & 0 \\ & & & 2S_{2323} & 0 & 0 \\ & \mathrm{sym} & & & 2S_{2323} & 0 \\ & & & & & S_{1111}-S_{1122} \end{bmatrix}$$

CUBIC SYMMETRY



$$\mathbf{S} = \begin{bmatrix} S_{1111} & S_{1122} & S_{1122} & 0 & 0 & 0 \\ & S_{1111} & S_{1122} & 0 & 0 & 0 \\ & & S_{1111} & 0 & 0 & 0 \\ & & & 2S_{2323} & 0 & 0 \\ & \mathrm{sym} & & & 2S_{2323} & 0 \\ & & & & & 2S_{2323} \end{bmatrix}$$

ISOTROPY



$$\mathbf{S} = \begin{bmatrix} S_{1111} & S_{1122} & S_{1122} & 0 & 0 & 0 \\ & S_{1111} & S_{1122} & 0 & 0 & 0 \\ & & S_{1111} & 0 & 0 & 0 \\ & & & S_{1111}-S_{1122} & 0 & 0 \\ & \mathrm{sym} & & & S_{1111}-S_{1122} & 0 \\ & & & & & S_{1111}-S_{1122} \end{bmatrix}$$

ISOTROPIC HOOKE'S MATERIAL

As in the elastic potential in the Hooke's material only second-degree terms may occur, not all possible combinations of strain tensor invariants may be accounted for in case of an isotropic material. As the first invariant is a linear function of tensor components **only the square of the first invariant may be used**. The **second invariant is a second-degree function itself**. The third invariant is a cubic functions and it cannot be transformed in such a way to obtain quadratic terms only – the **third invariant won't occur in the constitutive relation for isotropic Hooke's material**.

We may write then:

$$W(I_1;I_2;I_3)=W(I_1^2\ ;\ I_2)$$

The most simple proposition of such a function is a linear combination of those arguments with coefficients α i β :

$$W=\alpha(I_1)^2+\beta I_2$$

Stress tensor is then equal:

$$\mathbf{T}_S=\frac{\partial W}{\partial \mathbf{E}}=\frac{\partial W}{\partial I_1}\mathbf{1}+\frac{\partial W}{\partial I_2}(I_1\mathbf{1}-\mathbf{E})=2\alpha I_1\mathbf{1}+\beta(I_1\mathbf{1}-\mathbf{E})$$

Rearranging the terms gives us:

$$\mathbf{T}_S=(2\alpha+\beta)I_1\mathbf{1}-\beta\mathbf{E}$$

Let's introduce now:

$\lambda=(2\alpha+\beta)$ **- the first Lamé parameter**

$\mu=-\frac{1}{2}\beta$ **- the second Lamé parameter**

Linear constitutive law is as follows: $\mathbf{T}_S=2\mu\mathbf{E}+\lambda\,\mathrm{tr}(\mathbf{E})\mathbf{1} \Leftrightarrow S_{ij}=2\mu E_{ij}+\lambda E_{kk}$

or explicitly:

$$\begin{cases} S_{11}=2\mu E_{11}+\lambda(E_{11}+E_{22}+E_{33}), & S_{23}=2\mu E_{23} \\ S_{22}=2\mu E_{22}+\lambda(E_{11}+E_{22}+E_{33}), & S_{31}=2\mu E_{31} \\ S_{33}=2\mu E_{33}+\lambda(E_{11}+E_{22}+E_{33}), & S_{12}=2\mu E_{12} \end{cases}$$

The above linear system of algebraic equations is sometimes referred to as the **first form of the generalized Hooke's law**. Inversion of these relations give us the **second form**:

$$\begin{cases} E_{11}=\frac{1}{E}\left[(1+\nu)S_{11}-\nu(S_{11}+S_{22}+S_{33})\right], & E_{23}=\frac{S_{23}}{2G} \\ E_{22}=\frac{1}{E}\left[(1+\nu)S_{22}-\nu(S_{11}+S_{22}+S_{33})\right], & E_{31}=\frac{S_{31}}{2G} \\ E_{33}=\frac{1}{E}\left[(1+\nu)S_{33}-\nu(S_{11}+S_{22}+S_{33})\right], & E_{12}=\frac{S_{12}}{2G} \end{cases}$$

where:

$E=\frac{\mu(3\lambda+2\mu)}{(\lambda+\mu)}$ **- longitudinal stiffness modulus, Young modulus**

$\nu=\frac{\lambda}{2(\lambda+\mu)}$ **- transversal expansion coefficient, Poisson ratio**

$G=\mu$ **- transverse stifness modulus, Kirchhoff modulus**

There is also the third form which makes use of the decomposition of a second-rank tensor into its **isotropic** (**spherical**) **part** and its **deviator**:

$$\mathbf{T}_S = \mathbf{A}_T + \mathbf{D}_T \quad \text{where} \quad \mathbf{A}_T = \frac{1}{3}\mathrm{tr}(\mathbf{T}_S)\mathbf{1} \;\;,\;\; \mathbf{D}_T = \mathbf{T}_S - \mathbf{A}_T = \mathbf{T}_S - \frac{1}{3}\mathrm{tr}(\mathbf{T}_S)\mathbf{1}$$

$$\mathbf{E} = \mathbf{A}_E + \mathbf{D}_E \quad \text{where} \quad \mathbf{A}_E = \frac{1}{3}\mathrm{tr}(\mathbf{E})\mathbf{1} \;\;,\;\; \mathbf{D}_E = \mathbf{S} - \mathbf{A}_E = \mathbf{E} - \frac{1}{3}\mathrm{tr}(\mathbf{E})\mathbf{1}$$

Constitutive relation may be written in the following form:

$$[\mathbf{A}_T] + [\mathbf{D}_T] = \left[(2\mu + 3\lambda)\mathbf{A}_E\right] + [2\mu\,\mathbf{D}_E]$$

It can be easily checked that spherical parts and deviators are orthogonal so for two tensors their spherical parts and deviatoric parts are simultaneously equal only if the tensors are equal one to another. This results in:

$$\begin{cases} \mathbf{A}_T = 3K\,\mathbf{A}_E \\ \mathbf{D}_T = 2G\,\mathbf{D}_E \end{cases}$$

where

$K = \lambda + \frac{2}{3}\mu$ - **volumetric stiffness modulus, Helmholtz modulus**

$G = \mu$ - **transverse stiffness modulus, Kirchhoff modulus**

It can be written down in the following form:

Law of the change of volume: $p = K\theta$

Law o the change of shape:

$$\begin{bmatrix} S_{11}-p & S_{12} & S_{13} \\ S_{21} & S_{22}-p & S_{23} \\ S_{31} & S_{32} & S_{33}-p \end{bmatrix} = 2G \begin{bmatrix} E_{11}-\frac{\theta}{3} & E_{12} & E_{13} \\ E_{21} & E_{22}-\frac{\theta}{3} & E_{23} \\ E_{31} & E_{32} & S_{33}-\frac{\theta}{3} \end{bmatrix}$$

where:

hydrostatic pressure: $p = \frac{1}{3}(S_{11} + S_{22} + S_{33})$

volumetric strain: $\theta = E_{11} + E_{22} + E_{33}$

WORK AND POWER IN DEFORMABLE SOLIDS

In the classical mechanics the notions of work and power are introduced. So we will do now. In case of mechanics of **material points** **work** W **performed by a force** $\mathbf{F}$ **along displacement** $\mathbf{x}$ is expressed as:

$$W = \mathbf{F}\cdot \mathrm{d}\,\mathbf{x}$$

Power P is defined as the **time-derivative of work**:

$$P = \frac{\mathrm{d}\,W}{\mathrm{d}\,t} = \mathbf{F}\cdot\frac{\mathrm{d}\,\mathbf{x}}{\mathrm{d}\,t} = \mathbf{F}\cdot\mathbf{v} \quad ,$$

where $\mathbf{v}$ is the velocity vector. In case of mechanics of **continua** the power will be calculated as an **integral (sum) of power of external forces acting on respective displacements**:

$$P = \iiint_V b_i v_i \mathrm{d}\,V + \iint_S q_i v_i \mathrm{d}\,S$$

V denoted the internal volume and S boundary surface of the solid. Let us express the surface tractions in terms of stress:

$$P = \iiint_V b_i v_i \mathrm{d}\,V + \iint_S t_{ji} \nu_j v_i \mathrm{d}\,S$$

Divergence theorem gives us

$$P = \iiint_V b_i v_i \mathrm{d}\,V + \iiint_V (t_{ji} v_i)_{,j} \mathrm{d}\,V = \iiint_V b_i v_i \mathrm{d}\,V + \iiint_V (t_{ji,j} v_i + t_{ji} v_{i,j}) \mathrm{d}\,V$$

Making use of the equations of motion:

$$t_{ji,j} + b_i = \rho\, a_i \qquad \Rightarrow \qquad t_{ji,j} = \rho\, a_i - b_i$$

we obtain:

$$P = \iiint_V \rho\, a_i v_i \mathrm{d}\,V + \iiint_V t_{ji} v_{i,j} \mathrm{d}\,V$$

The first integrand may be expressed as:

$$\rho\, a_i v_i = \rho\, \dot{v}_i v_i = \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{1}{2} v_i v_i\right)$$

what is the time-derivative of the **kinetic energy**:

$$\frac{\mathrm{d}}{\mathrm{d}\,t} E_k = \frac{\mathrm{d}}{\mathrm{d}\,t} \iiint_V \frac{1}{2} v_i v_i \mathrm{d}\,V$$

Concerning the second integral ,the velocity gradient $\nabla_{\mathbf{x}}\mathbf{v} = \mathbf{L}$ may be **decomposed into its symmetric part** (stretch rate tensor $\mathbf{D}$) and **antisymmetric part** (spin tensor $\mathbf{W}$) $\mathbf{L} = \mathbf{D} + \mathbf{W}$:

$$D_{ij} = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right) \qquad W_{ij} = \frac{1}{2}\left(v_{i,j} - v_{j,i}\right) \quad .$$

The second integrand may be written as:

$$t_{ji} v_{i,j} = t_{ji} D_{ij} + t_{ji} W_{ij}$$

Dot product of symmetric stress tensor and antisymmetric spin tensor is equal so **power may be expressed as a sum of time-derivative of kinetic energy and elastic strain power** - an integral of $t_{ji} D_{ij}$ which will be termed the elastic strain power density:

$$P = \underbrace{\frac{\mathrm{d}}{\mathrm{d}\,t} \iiint_V \frac{1}{2} v_i v_i \,\mathrm{d}\,V}_{E_k} + \iiint_V t_{ji} D_{ji} \,\mathrm{d}\,V$$

ENERGY CONJUGATE PAIRS OF STRESS AND STRAIN MEASURES

It can be shown that certain pairs of all introduced measures of stress and strain may be interpreted in such a way that the **dot product of the measure of stress and time derivative of measure of strain gives us the elastic strain power density**. Let us consider e.g. The linear part of the spatial strain tensor $\boldsymbol{\eta}$ [1]:

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i} - u_{k,i} u_{k,j}\right) \quad \overset{u_{i,j} \ll 1}{\longrightarrow} \quad e_{ij} \approx \eta_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

We can write:

$$\dot{\eta}_{ji} = \frac{1}{2}\left(\dot{u}_{j,i} + \dot{u}_{i,j}\right) = \frac{1}{2}\left(v_{j,i} + v_{i,j}\right) = D_{ij}$$

then

$$\boxed{P_s = \iiint_V t_{ji} D_{ji} \,\mathrm{d}\,V = \iiint_V t_{ji} \cdot \dot{\eta}_{ji} \,\mathrm{d}\,V} \, ,$$

Every such a pair of stress and strain measures such that the dot product of the measure of stress and time derivative of measure of strain gives us the elastic strain power density will be termed a pair of energy conjugates**.**

1) **NOTE**: $\boldsymbol{\eta}$ should not be confused with small strain tensor $\boldsymbol{\varepsilon}$ - definitions of both look the same, the difference is that in $\boldsymbol{\eta}$ differentiation is performed with respect to spatial coordinates, and in $\boldsymbol{\varepsilon}$ with respect to material ones.

It may be shown that the below pairs provide an elastic strain power density:

ENERGY CONJUGATE PAIRS OF MEASURE OF STRESS AND STRAIN	
STRESS MEASURE	**STRAIN MEASURE**
True stress tensor $\mathbf{T}_\sigma$	**Linear part of the spatial strain tensor** $\boldsymbol{\eta} = \frac{1}{2}\left[\nabla_x \mathbf{u} + (\nabla_x \mathbf{u})^T\right]$
PK1 stress tensor $\mathbf{T}_R = J\,\mathbf{T}_\sigma \cdot \mathbf{F}^{-T}$	**Deformation gradient** $\mathbf{F} = \nabla_X \mathbf{x}$
PK2 stress tensor $\mathbf{T}_S = J\,\mathbf{F}^{-1} \cdot \mathbf{T}_\sigma \cdot \mathbf{F}^{-T}$	**Material strain tensor** $\mathbf{E} = \frac{1}{2}\left(\mathbf{U}^2 - \mathbf{1}\right)$

ELASTIC STRAIN WORK AND ENERGY

As the work was defined as a dot product of a vector of force and of a vector of displacement, a definition of work of internal forces (stress) on corresponding displacements (strains) may be proposed in a similar way. The dimension of such a product would be

$$[S_{ij}E_{ij}] = \frac{\mathrm{N}}{\mathrm{m}^2} \cdot \frac{\mathrm{m}}{\mathrm{m}} = \frac{\mathrm{N} \cdot \mathrm{m}}{\mathrm{m}^3} = \frac{\mathrm{J}}{\mathrm{m}^3}$$

So the dot product of the stress and strain tensor could be a measure of volumetric density of elastic strain work and elastic strain energy. Assuming linear constitutive law between material strain tensor and PK2 stress tensor, total **energy** accumulated in a monotonic process may be expressed as:

$$\Phi = \frac{1}{2}\iiint_{V_R} S_{ij} E_{ij}\,\mathrm{d}V_R = \frac{1}{2}\iiint_{V_R} \mathsf{S}_{ijkl} E_{ij} E_{kl}\,\mathrm{d}V_R = \frac{1}{2}\iiint_{V_R} \mathsf{C}_{ijkl} S_{ij} S_{kl}\,\mathrm{d}V_R = \iiint_{V_R} \phi\,\mathrm{d}V_R$$

Factor ½ is due to monotonic increment of both stress and strain starting from 0 and

$$\phi = \frac{1}{2} S_{ij} E_{ij} = \frac{1}{2}\mathsf{S}_{ijkl} E_{ij} E_{kl} = \frac{1}{2}\mathsf{C}_{ijkl} S_{ij} S_{kl}$$

is the (**volumetric**) **elastic strain energy density**. We may notice that it is also the **elastic potential for Hooke's material**.

SUMMARY

OGÓLNY ZWIĄZEK KONSTYTUTYWNY DLA
MATERIAŁU SPRĘŻYSTEGO

$$\mathbf{T}_S(\mathbf{X};t) = f\left(\mathbf{E}(\mathbf{X};t)\right) \quad \text{oraz} \quad \exists f^{-1}$$

OGÓLNY ZWIĄZEK KONSTYTUTYWNY DLA
MATERIAŁU HIPERSPRĘŻYSTEGO

W - potencjał sprężysty

$$\mathbf{T}_S = \frac{\partial W}{\partial \mathbf{E}} \quad \Leftrightarrow \quad S_{ij} = \frac{\partial W}{\partial E_{ij}} \quad i, j=1,2,3$$

OGÓLNY ZWIĄZEK KONSTYTUTYWNY DLA
IZOTROPOWEGO MATERIAŁU HIPERSPRĘŻYSTEGO

$$\mathbf{T}_S = \phi_3 \mathbf{E}^2 - (\phi_2 + I_1\phi_3)\mathbf{E} + (\phi_1 + I_1\phi_2 + I_2\phi_3)\mathbf{1} =$$
$$= k_2\mathbf{E}^2 + k_1\mathbf{E} + k_0\mathbf{1}$$

gdzie: I_1, I_2, I_3 są niezmiennikami tensora odkształcenia.

$$\phi_1 = \phi_1(I_1; I_2; I_3) = \frac{\partial W}{\partial I_1}$$
$$\phi_2 = \phi_2(I_1; I_2; I_3) = \frac{\partial W}{\partial I_2}$$
$$\phi_3 = \phi_3(I_1; I_2; I_3) = \frac{\partial W}{\partial I_3}$$

Alternatywnie:

$$\mathbf{T}_\sigma = \frac{2}{J}\left[\frac{1}{J^{2/3}}\left(\frac{\partial W}{\partial \bar{I}_1} + \bar{I}_1\frac{\partial W}{\partial \bar{I}_2}\right)\mathbf{B} - \frac{1}{J^{4/3}}\frac{\partial W}{\partial \bar{I}_2}\mathbf{B}^2\right] + \left[\frac{\partial W}{\partial J} - \frac{2}{3J}\left(\bar{I}_1\frac{\partial W}{\partial \bar{I}_1} + 2\bar{I}_2\frac{\partial W}{\partial \bar{I}_2}\right)\right]\mathbf{1}$$

gdzie $\mathbf{B}$ - lewy tensor deformacji

$\bar{I}_1 = J^{-2/3}\hat{I}_1$, $\bar{I}_2 = J^{-4/3}\hat{I}_2$, $J = \sqrt{\hat{I}_3}$

$\hat{I}_1, \hat{I}_2, \hat{I}_3$ są niezmiennikami prawego i lewego tensora deformacji.

ZWIĄZEK KONSTYTUTYWNY DLA
MATERIAŁU LINIOWO-SPRĘŻYSTEGO

Potencjał sprężysty postaci: $W = \frac{1}{2} \mathsf{S}_{ijkl} E_{ij} E_{kl}$

Związek konstytutywny: $S_{ij} = \mathsf{S}_{ijkl} E_{kl}$

gdzie: $\mathsf{S}_{ijkl} = \mathsf{S}_{jikl} = \mathsf{S}_{ijlk} = \mathsf{S}_{klij}$

ZWIĄZEK KONSTYTUTYWNY DLA
IZOTROPOWEGO MATERIAŁU LINIOWO-SPRĘŻYSTEGO

1 POSTAĆ

$$\begin{cases} S_{11} = 2\mu E_{11} + \lambda(E_{11}+E_{22}+E_{33}), & S_{23} = 2G E_{23} \\ S_{22} = 2\mu E_{22} + \lambda(E_{11}+E_{22}+E_{33}), & S_{31} = 2G E_{31} \\ S_{33} = 2\mu E_{33} + \lambda(E_{11}+E_{22}+E_{33}), & S_{12} = 2G E_{12} \end{cases}$$

2 POSTAĆ

$$\begin{cases} E_{11} = \frac{1}{E}\left[(1+\nu)S_{11} - \nu(S_{11}+S_{22}+S_{33})\right], & E_{23} = \frac{S_{23}}{2G} \\ E_{22} = \frac{1}{E}\left[(1+\nu)S_{22} - \nu(S_{11}+S_{22}+S_{33})\right], & E_{31} = \frac{S_{31}}{2G} \\ E_{33} = \frac{1}{E}\left[(1+\nu)S_{33} - \nu(S_{11}+S_{22}+S_{33})\right], & E_{12} = \frac{S_{12}}{2G} \end{cases}$$

3 POSTAĆ

Prawo zmiany objętości:

$$\mathbf{A}_T = 3K\,\mathbf{A}_E \quad \text{gdzie} \quad \mathbf{A}_T = \frac{1}{3}\mathrm{tr}(\mathbf{T}_S)\mathbf{1} \;\;,\;\; \mathbf{A}_E = \frac{1}{3}\mathrm{tr}(\mathbf{E})\mathbf{1}$$

Prawo zmiany postaci:

$$\mathbf{D}_T = 2G\,\mathbf{D}_E \quad \text{gdzie} \quad \mathbf{D}_T = \mathbf{T}_S - \mathbf{A}_T = \mathbf{T}_S - \frac{1}{3}\mathrm{tr}(\mathbf{T}_S)\mathbf{1} \;\;,\;\; \mathbf{D}_E = \mathbf{S} - \mathbf{A}_E = \mathbf{E} - \frac{1}{3}\mathrm{tr}(\mathbf{E})\mathbf{1}$$

ZWIĄZKI MIĘDZY STAŁYMI SPRĘŻYSTYMI

moduł Younga E	moduł Kirchhoffa $G=\mu$	moduł Helmholtza K	współczynnik Poissona ν	pierwszy parametr Lamégo λ
E	$\frac{E}{2(1+\nu)}$	$\frac{E}{3(1-2\nu)}$	ν	$\frac{E\nu}{(1+\nu)(1-2\nu)}$
$\frac{9KG}{3K+G}$	G	K	$\frac{3K-2G}{2(3K+G)}$	$K-\frac{2}{3}G$
$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	μ	$\lambda+\frac{2}{3}\mu$	$\frac{\lambda}{2(\lambda+\mu)}$	λ

5. LINEAR THEORY OF ELASTICITY

The theory presented in the previous chapters may be summarized as being governed by the following 15 equations (for hyperelastic solids in material description):

Equations of motion: $\left[S_{kj}\left(u_{i,k}+\delta_{ik}\right)\right]_{,j}+B_i=\rho_R\ddot{u}_i \qquad i=1,2,3$

Geometric relations: $E_{ij}=\frac{1}{2}\left(u_{i,j}+u_{j,i}+u_{k,i}u_{k,j}\right), \quad i,j=1,2,3$

Constitutive relations: $S_{ij}=\frac{\partial W}{\partial E_{ij}}, \quad i,j=1,2,3$

enabling us to find 15 unknown functions: 3 components of displacement vector, 6 components of strain tensor and 6 components of stress tensor. The theory is a nonlinear one, due to

- **geometric nonlinearity** – nonlinear geometric relations:

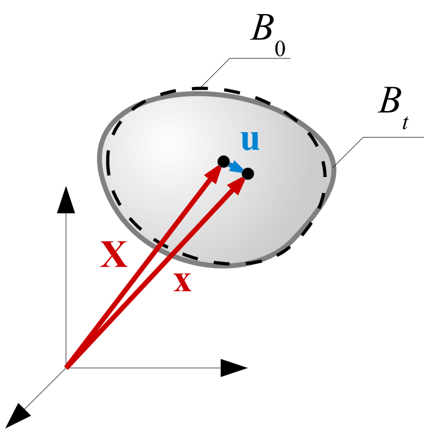
$$E_{ij}=\frac{1}{2}\left(u_{i,j}+u_{j,i}+\underbrace{u_{k,i}u_{k,j}}_{\text{nonlinear part}}\right)$$

- **physical nonlinearity** – nonlinear constitutive relations.

$S_{ij}=f\left(E_{ij}\right)$ where f may be a nonlinear functions.

Nonlinear theories are in general much more difficult to deal with than linear ones. In particular it is more often in case of non-linear problems that their solution cannot be expressed in any closed form. What's more, in non-linear theories it is often impossible to state if the solutions exists at all and even if it exists, if it is a unique one or are there many possible (or even infinite number of possible) solutions. It is common to consider – simultaneously with the non-linear original theories – their linear, simplified counterparts. In case of theory of elasticity it emerges that linearized theory is precise enough for most scientific and industrial applications.

GEOMETRIC LINEARITY – SMALL DISPLACEMENTS

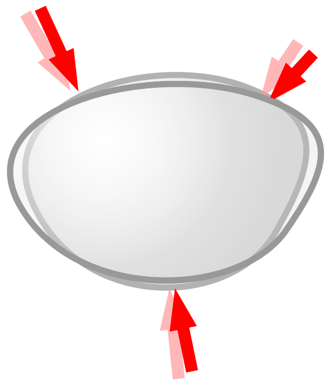


Basic assumption in geometrically linear theory of elasticity is an assumption fo **small displacements**, according to which current configuration is close to the reference one. The differenc between them are assumed to be arbitrary small, so the **distinction between spatial and material coordinates vanishes**:

$$|\mathbf{u}|=|\mathbf{x}-\mathbf{X}|\ll 1 \quad \Rightarrow \quad \mathbf{x}\approx\mathbf{X}$$

However, it does not mean that displacement is 0. We shall now denote all corrdinates with small *x*.

In particular small difference between current and reference configuration allow us to assume that position and shape of a body, points of application of external loads etc. do not change significantly after the body is deformed and the **reference configuration may be considered the domain of all functions also after deformation**.



GEOMETRIC LINEARITY – SMALL STRAINS

Another assumption is the one on **small strains**, namaly small derivatives of displacement (small variation of displacement), what results in **linear geometric relations**:

$$u_{i,j} \ll 1 \quad \Rightarrow \quad E_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i} + \underbrace{u_{k,i} u_{k,j}}_{\approx 0}\right) \approx \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) = \varepsilon_{ij}$$

It can be seen that the **Green - de Saint-Venant strain tensor may be approximated by the small strain tensor** $\mathbf{E} \approx \boldsymbol{\varepsilon}$.

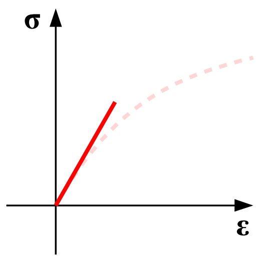
Small strain means also that surface area of any infinitely small surface element do not change significantly. Together with an assumption of small displacements – according to which current and reference configurations are approximately the same – we may conclude that also the difference between the stress measures vanish. We shall now consider **all stress measures as being equal to the Cauchy true stress tensor** $\mathbf{T}_\sigma \approx \mathbf{T}_R \approx \mathbf{T}_S$. We will denote it with $\boldsymbol{\sigma}$ and its components with σ_{ij} .

PHYSICAL LINEARITY

Assumption of **small strains** allow us also to **approximate any hyperelastic material with a linear elastic Hooke's material**, since any non-linear elastic potential may be approximated with only first terms of its power series expansion in the neighborhood of $\mathbf{E} \approx \mathbf{0}$ (small strains):

$$W(\mathbf{E}) \approx \frac{1}{2!} \underbrace{\left.\frac{\partial^2 W}{\partial E_{ij} E_{kl}}\right|_{\mathbf{0}}}_{\mathsf{S}_{ijkl}} \varepsilon_{ij} \varepsilon_{kl} + \underbrace{\frac{1}{3!} \left.\frac{\partial^3 W}{\partial E_{ij} E_{kl} E_{mn}}\right|_{\mathbf{0}} E_{ij} E_{kl} E_{mn} + \ldots}_{0}$$

$$\Rightarrow \quad S_{ij} = \frac{\partial W}{\partial E_{ij}} \approx \mathsf{S}_{ijkl} E_{kl}$$



EQUATIONS OF LINEAR THEORY OF ELASTICITY

Finally, the set of **governing equations of the linear theory of elasticity** is as follows:

Equations of motions:	$\sigma_{ij,j} + b_i = \rho \ddot{u}_i$	$i=1,2,3$
Geometric relations:	$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j}+u_{j,i}\right)$	$i,j=1,2,3$
Constitutive relations:	$\sigma_{ij} = S_{ijkl}\,\varepsilon_{kl}$	$i,j=1,2,3$

These equations must be also equipped with proper kinematic and static boundary conditions as well as with proper initial conditions.

LAMÉ DISPLACEMENT EQUATIONS

We shall focus on isotropic solids, so the constitutive relations will be of the following form:

$$\sigma_{ij} = 2G\,\varepsilon_{ij} + \lambda\delta_{ij}\varepsilon_{kk}$$

System of 15 governing equations for displacements, stresses and strains may be reduced to the system of 3 equations for displacements only as follows:

1. Introduce the constitutive relations in the equations of motion in order to express stress in terms of strain
2. Then substitute the geometrical relations in order to express strain in terms of displacement

The first step:

$$\sigma_{ij,j} + b_i = \rho \ddot{u}_i$$

$$\left[2G\varepsilon_{ij}+\lambda\delta_{ij}\varepsilon_{kk}\right]_{,j} + b_i = \rho \ddot{u}_i$$

$$2G\varepsilon_{ij,j}+\lambda\delta_{ij}\varepsilon_{kk,j} + b_i = \rho \ddot{u}_i$$

The second step:

$$2G\cdot\frac{1}{2}\left(u_{i,jj}+u_{j,ij}\right)+\lambda\delta_{ij}\,\frac{1}{2}\left(u_{k,kj}+u_{k,kj}\right) + b_i = \rho \ddot{u}_i$$

$$G\left(u_{i,jj}+u_{j,ij}\right)+\lambda\delta_{ij}u_{k,kj} + b_i = \rho \ddot{u}_i$$

$$G\left(u_{i,jj}+u_{j,ij}\right)+\lambda u_{j,ji} + b_i = \rho \ddot{u}_i$$

Finally:

$$G u_{i,jj} + (G+\lambda) u_{j,ji} + b_i = \rho \ddot{u}_i \qquad i=1,2,3$$

or written explicitly for each value of *i*:

$$\begin{cases} G\nabla^2 u_1 + (G+\lambda)(u_{1,11}+u_{2,21}+u_{3,31}) + b_1 = \rho \ddot{u}_1 \\ G\nabla^2 u_2 + (G+\lambda)(u_{1,12}+u_{2,22}+u_{3,32}) + b_2 = \rho \ddot{u}_2 \\ G\nabla^2 u_3 + (G+\lambda)(u_{1,13}+u_{2,23}+u_{3,33}) + b_3 = \rho \ddot{u}_3 \end{cases} ,$$

where ∇^2 is the Laplace operator (laplacian). The above system is inhomogeneous a system of three partial differential equations of the 2nd order for the components of displacement vector. They are sometimes referred to as the **Lamé equations** or **Cauchy-Navier equations**.

Initial conditions and kinematic boundary conditions are the same as in the original formulation. Static boundary conditions may be expressed in terms of displacements in the following form

$$\frac{1}{2} S_{ijkl}(u_{k,l}+u_{l,k}) \nu_j = q_i \quad \text{on} \quad S_q \quad \text{- loaded part of boundary}$$

BELTRAMI – MICHELL STRESS COMPATIBILITY EQUATIONS

Another formulation of the system of governing equations makes use of the strain compatibility equations:

$$\varepsilon_{ik,jl} - \varepsilon_{jk,il} - \varepsilon_{il,jk} + \varepsilon_{jl,ik} = 0 \qquad i,j,k,l=1,2,3$$

Introducing the constitutive relations for isotropic solids:

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sigma_{nn} \qquad i,j=1,2,3$$

gives us:

$$\frac{1+\nu}{E}[\sigma_{ik,jl} - \sigma_{jk,il} - \sigma_{il,jk} + \sigma_{jl,ik}] - \frac{\nu}{E}[\delta_{ik}\sigma_{nn,jl} - \delta_{jk}\sigma_{nn,il} - \delta_{il}\sigma_{nn,jk} + \delta_{jl}\sigma_{nn,ik}] = 0$$

Let's perform contraction (summation) of idices *i* and *k*:

$$\frac{1+\nu}{E}[\sigma_{ii,jl} - \sigma_{ji,il} - \sigma_{il,ji} + \sigma_{jl,ii}] - \frac{\nu}{E}[\delta_{ii}\sigma_{nn,jl} - \delta_{ji}\sigma_{nn,il} - \delta_{il}\sigma_{nn,ji} + \delta_{jl}\sigma_{nn,ii}] = 0$$

Let's account for properties of Kronecker deltas:

$$\frac{1+\nu}{E}[\sigma_{ii,jl} - \sigma_{ji,il} - \sigma_{il,ji} + \sigma_{jl,ii}] - \frac{\nu}{E}[3\sigma_{nn,jl} - \sigma_{nn,jl} - \sigma_{nn,jl} + \delta_{jl}\sigma_{nn,ii}] = 0$$

Let's perform some rearrangements:

$$\frac{1+\nu}{E}[\sigma_{ii,jl}-\sigma_{ji,il}-\sigma_{il,ji}+\sigma_{jl,ii}]-\frac{\nu}{E}[\sigma_{nn,jl}+\delta_{jl}\sigma_{nn,ii}]=0$$

$$[\sigma_{nn,jl}-\sigma_{ji,il}-\sigma_{il,ji}+\sigma_{jl,ii}]-\frac{\nu}{1+\nu}[\sigma_{nn,jl}+\delta_{jl}\sigma_{nn,ii}]=0$$

$$\sigma_{jl,ii}+\left[1-\frac{\nu}{1+\nu}\right]\sigma_{nn,jl}-\sigma_{ji,il}-\sigma_{li,ij}-\frac{\nu}{1+\nu}\delta_{jl}\sigma_{nn,ii}=0$$

Now, we can accound for the equilibrium equations:

$$\sigma_{ji,i}+b_j=0$$

We may differentiate it with respect to x_l and change free subscripts:

$$-\sigma_{ji,il}=b_{j,l}\quad,\quad-\sigma_{li,ij}=b_{l,j}\quad,$$

what may be then substituted into previous equations:

$$\sigma_{jl,ii}+\left[1-\frac{\nu}{1+\nu}\right]\sigma_{nn,jl}+\left(b_{j,l}+b_{l,j}\right)-\frac{\nu}{1+\nu}\delta_{jl}\sigma_{nn,ii}=0\quad.$$

We will find some further useful relations if we contract indices *j* and *l*:

$$\sigma_{kk,ii}+\left[1-\frac{\nu}{1+\nu}\right]\sigma_{nn,kk}+\left(b_{k,k}+b_{k,k}\right)-\frac{\nu}{1+\nu}\delta_{kk}\sigma_{nn,ii}=0$$

Rearrangement of dummy indices gives us:

$$\sigma_{nn,kk}+\left[1-\frac{\nu}{1+\nu}\right]\sigma_{nn,kk}+2b_{k,k}-\frac{3\nu}{1+\nu}\sigma_{nn,kk}=0$$

and after some algebra:

$$\sigma_{nn,kk}=-\frac{1+\nu}{1-\nu}b_{k,k}$$

This may be again substituted in the transformed compatibility equations resulting in:

$$\boxed{\sigma_{ij,kk}+\frac{1}{1+\nu}\sigma_{kk,ij}+\left(b_{i,j}+b_{j,i}\right)+\frac{\nu}{1-\nu}\delta_{ij}b_{k,k}=0\qquad i,j=1,2,3\quad.}$$

The above system is an inhomogeneous system of six partial differential equations of the 2nd order for stress tensor components. They are referred to as the **Beltrami – Michell stress compatibility equations**. **They may be written as follows.**

$$\nabla^2\sigma_{11} + \frac{1}{1+\nu}\left(\sigma_{11,11}+\sigma_{22,11}+\sigma_{33,11}\right) + 2b_{1,1} + \frac{\nu}{1-\nu}\left(b_{1,1}+b_{2,2}+b_{3,3}\right) = 0$$
$$\nabla^2\sigma_{22} + \frac{1}{1+\nu}\left(\sigma_{11,22}+\sigma_{22,22}+\sigma_{33,22}\right) + 2b_{2,2} + \frac{\nu}{1-\nu}\left(b_{1,1}+b_{2,2}+b_{3,3}\right) = 0$$
$$\nabla^2\sigma_{33} + \frac{1}{1+\nu}\left(\sigma_{11,33}+\sigma_{22,33}+\sigma_{33,33}\right) + 2b_{3,3} + \frac{\nu}{1-\nu}\left(b_{1,1}+b_{2,2}+b_{3,3}\right) = 0$$
$$\nabla^2\sigma_{23} + \frac{1}{1+\nu}\left(\sigma_{11,23}+\sigma_{22,23}+\sigma_{33,23}\right) + \left(b_{2,3}+b_{3,2}\right) = 0$$
$$\nabla^2\sigma_{31} + \frac{1}{1+\nu}\left(\sigma_{11,31}+\sigma_{22,31}+\sigma_{33,31}\right) + \left(b_{3,1}+b_{1,3}\right) = 0$$
$$\nabla^2\sigma_{12} + \frac{1}{1+\nu}\left(\sigma_{11,12}+\sigma_{22,12}+\sigma_{33,12}\right) + \left(b_{1,2}+b_{2,1}\right) = 0$$

If the body forces are neglected, a homogeneous system is obtained:

$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{kk,ij} = 0 \qquad i,j=1,2,3 \ .$$

after writing it out:

$$\nabla^2\sigma_{11} + \frac{1}{1+\nu}\left(\sigma_{11,11}+\sigma_{22,11}+\sigma_{33,11}\right) = 0 \qquad \nabla^2\sigma_{23} + \frac{1}{1+\nu}\left(\sigma_{11,23}+\sigma_{22,23}+\sigma_{33,23}\right) = 0$$
$$\nabla^2\sigma_{22} + \frac{1}{1+\nu}\left(\sigma_{11,22}+\sigma_{22,22}+\sigma_{33,22}\right) = 0 \qquad \nabla^2\sigma_{31} + \frac{1}{1+\nu}\left(\sigma_{11,31}+\sigma_{22,31}+\sigma_{33,31}\right) = 0$$
$$\nabla^2\sigma_{33} + \frac{1}{1+\nu}\left(\sigma_{11,33}+\sigma_{22,33}+\sigma_{33,33}\right) = 0 \qquad \nabla^2\sigma_{12} + \frac{1}{1+\nu}\left(\sigma_{11,12}+\sigma_{22,12}+\sigma_{33,12}\right) = 0$$

It is important to note that the **strain distribution obtained from the solution of the Beltrami-Michell equations satisfies the compatibility relations** (so that it may be integrated to obtain displacement field) **if and only if the stress distribution satisfies the equilibrium equations, what is not guaranteed for the solution of the Beltrami-Michell equations**. **A solution of Beltrami-Michell equations is a solution to the problem of theory of elasticity if and only if it additionally satisfies the equilibrium equations**.

Proper boundary conditions must be formulated – static boundary conditions are not a problem. There is a non-trivial difficulty in formulating kinematic boundary conditions in terms of stress tensor components.

BELTRAMI STRESS FUNCTIONS

Let us consider a stress state as follows:

$$\sigma_{ij} = \epsilon_{ikm}\epsilon_{jln}\Phi_{kl,mn} \quad ,$$

where Φ_{kl} is **any symmetric second rank tensor field which is at least four times differentiable** – it will be called the **Beltrami stress tensor** and its components will be termed **Beltrami stress functions**. Substituting them in the equilibrium equations in case of statics:

$$\sigma_{ij,j} = \epsilon_{ikm}\epsilon_{jln}\Phi_{kl,mnj} = 0$$

It may be shown that accounting for properties of the permutation symbols, symmetry of Beltrami stress tensor and symmetry of differentiation results in that **the equilibrium equations are always satisfied for any stress functions chosen**. It means that if only they satisfy also Beltrami-Michell stress compatibility equations, they provide a solution to the problem of theory of elasticity – strains calculated with the use of isotropic generalized Hooke's law satisfy the compatibility conditions and thus can be integrated in order to obtain displacement field. Let's denote:

$$\mathbf{\Phi} = \begin{bmatrix} A & B & C \\ B & D & E \\ C & E & F \end{bmatrix} \quad ,$$

where $A, B, \ldots$ are the stress functions $A(x_1, x_2, x_3), B(x_1, x_2, x_3) \ldots$. Stress state components may be calculated as follows:

$$\sigma_{11} = \frac{\partial^2 D}{\partial x_3^2} - 2\frac{\partial^2 E}{\partial x_2 \partial x_3} + \frac{\partial^2 F}{\partial x_2^2} \qquad \sigma_{23} = \frac{\partial^2 C}{\partial x_1 \partial x_2} - \frac{\partial^2 E}{\partial x_1^2} - \frac{\partial^2 A}{\partial x_2 \partial x_3} + \frac{\partial^2 B}{\partial x_3 \partial x_1}$$

$$\sigma_{22} = \frac{\partial^2 F}{\partial x_1^2} - 2\frac{\partial^2 C}{\partial x_3 \partial x_1} + \frac{\partial^2 A}{\partial x_3^2} \qquad \sigma_{31} = \frac{\partial^2 B}{\partial x_2 \partial x_3} - \frac{\partial^2 C}{\partial x_2^2} - \frac{\partial^2 D}{\partial x_3 \partial x_1} + \frac{\partial^2 E}{\partial x_1 \partial x_2}$$

$$\sigma_{33} = \frac{\partial^2 A}{\partial x_2^2} - 2\frac{\partial^2 B}{\partial x_1 \partial x_2} + \frac{\partial^2 D}{\partial x_1^2} \qquad \sigma_{12} = \frac{\partial^2 E}{\partial x_3 \partial x_1} - \frac{\partial^2 B}{\partial x_3^2} - \frac{\partial^2 F}{\partial x_1 \partial x_2} + \frac{\partial^2 C}{\partial x_2 \partial x_3}$$

Substituting it into Beltrami-Michell equations gives us a system of 6 linear partial differential equations of the 2nd order for stress functions $A, B, \ldots$. Their form is rather complex and this is the only reason for not writing them down in this place.

Specific problems of theory of elasticity may be solved with the use of the following simplified forms of Beltrami stress tensor:

MAXWELL STRESS FUNCTION

Assumed form of the Beltrami stress tensor: $\mathbf{\Phi} = \begin{bmatrix} A & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & F \end{bmatrix}$

Stress state components:

$$\sigma_{11} = \frac{\partial^2 D}{\partial x_3^2} + \frac{\partial^2 F}{\partial x_2^2} \qquad \sigma_{23} = -\frac{\partial^2 A}{\partial x_2 \partial x_3}$$

$$\sigma_{22} = \frac{\partial^2 F}{\partial x_1^2} + \frac{\partial^2 A}{\partial x_3^2} \qquad \sigma_{31} = -\frac{\partial^2 D}{\partial x_3 \partial x_1}$$

$$\sigma_{33} = \frac{\partial^2 A}{\partial x_2^2} + \frac{\partial^2 D}{\partial x_1^2} \qquad \sigma_{12} = -\frac{\partial^2 F}{\partial x_1 \partial x_2}$$

AIRY STRESS FUNCTION

Assumed form of the Beltrami stress tensor: $\mathbf{\Phi} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & F \end{bmatrix}$,

We assume that F do not depend on x_3 .

Stress state components:

$$\sigma_{11} = \frac{\partial^2 F}{\partial x_2^2} \qquad \sigma_{23} = 0$$

$$\sigma_{22} = \frac{\partial^2 F}{\partial x_1^2} \qquad \sigma_{31} = 0$$

$$\sigma_{33} = 0 \qquad \sigma_{12} = -\frac{\partial^2 F}{\partial x_1 \partial x_2}$$

MORERA STRESS FUNCTION

Assumed form of the Beltrami stress tensor: $\mathbf{\Phi} = \begin{bmatrix} 0 & B & C \\ B & 0 & E \\ C & E & 0 \end{bmatrix}$

Stress state components:

$$\sigma_{11} = -2\frac{\partial^2 E}{\partial x_2 \partial x_3} \qquad \sigma_{23} = \frac{\partial^2 C}{\partial x_1 \partial x_2} - \frac{\partial^2 E}{\partial x_1^2} + \frac{\partial^2 B}{\partial x_3 \partial x_1}$$

$$\sigma_{22} = -2\frac{\partial^2 C}{\partial x_3 \partial x_1} \qquad \sigma_{31} = \frac{\partial^2 B}{\partial x_2 \partial x_3} - \frac{\partial^2 C}{\partial x_2^2} + \frac{\partial^2 E}{\partial x_1 \partial x_2}$$

$$\sigma_{33} = -2\frac{\partial^2 B}{\partial x_1 \partial x_2} \qquad \sigma_{12} = \frac{\partial^2 E}{\partial x_3 \partial x_1} - \frac{\partial^2 B}{\partial x_3^2} + \frac{\partial^2 C}{\partial x_2 \partial x_3}$$

EQUATIONS OF LINEAR THEORY OF ELASTICITY IN CURVILINEAR COORDINATES

CYLINDIRCAL COORDINATES

Coordinate transformation:

$$\begin{cases} r = \sqrt{x_1^2 + x_2^2} \\ \phi = \operatorname{arctg} \dfrac{x_2}{x_1} \\ z = x_3 \end{cases} \Leftrightarrow \begin{cases} x_1 = r \cos\phi \\ x_2 = r \sin\phi \\ x_3 = z \end{cases}$$

Equations of motion:

$$\sigma_{rr,r} + \frac{1}{r}\sigma_{r\phi,\phi} + \sigma_{rz,z} + \frac{\sigma_{rr} - \sigma_{\phi\phi}}{r} + b_r = \rho \ddot{u}_r$$

$$\sigma_{r\phi,r} + \frac{1}{r}\sigma_{\phi\phi,\phi} + \sigma_{\phi z,z} + \frac{2}{r}\sigma_{r\phi} + b_\phi = \rho \ddot{u}_\phi$$

$$\sigma_{rz,r} + \frac{1}{r}\sigma_{\phi z,\phi} + \sigma_{zz,z} + \frac{1}{r}\sigma_{rz} + b_z = \rho \ddot{u}_z$$

Geometric relations:

$$\varepsilon_{rr} = u_{r,r} \qquad \varepsilon_{\phi z} = \frac{1}{2}\left(\frac{1}{r}u_{z,\phi} + u_{\phi,z}\right)$$

$$\varepsilon_{\phi\phi} = \frac{1}{r}u_{\phi,\phi} + \frac{u_r}{r} \qquad \varepsilon_{zr} = \frac{1}{2}\left(u_{z,r} + u_{r,z}\right)$$

$$\varepsilon_{zz} = u_{z,z} \qquad \varepsilon_{r\phi} = \frac{1}{2}\left(u_{\phi,r} - \frac{u_\phi}{r} + \frac{1}{r}u_{r,\phi}\right)$$

SPHERICAL COORDINATES

Coordinate transformation:

$$\begin{cases} r = \sqrt{x_1^2 + x_2^2 + x_3^2} \\ \psi = \arccos \dfrac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \\ \phi = \operatorname{arctg} \dfrac{x_2}{x_1} \end{cases} \Leftrightarrow \begin{cases} x_1 = r \sin\psi \cos\phi \\ x_2 = r \sin\psi \sin\phi \\ x_3 = r \cos\psi \end{cases}$$

Equations of motion:

$$\sigma_{rr,r} + \frac{1}{r}\sigma_{r\psi,\psi} + \frac{1}{r\sin\psi}\sigma_{r\phi,\phi} + \frac{1}{r}\left(2\sigma_{rr} - \sigma_{\psi\psi} - \sigma_{\phi\phi} + \sigma_{r\psi}\operatorname{ctg}\psi\right) + b_r = \rho \ddot{u}_r$$

$$\sigma_{r\psi,r} + \frac{1}{r}\sigma_{\psi\psi,\psi} + \frac{1}{r\sin\psi}\sigma_{\psi\phi,\phi} + \frac{1}{r}\left[\left(\sigma_{\psi\psi} - \sigma_{\phi\phi}\right)\operatorname{ctg}\psi + 3\sigma_{r\psi}\right] + b_\psi = \rho \ddot{u}_\psi$$

$$\sigma_{r\phi,r} + \frac{1}{r}\sigma_{\psi\phi,\psi} + \frac{1}{r\sin\psi}\sigma_{\phi\phi,\phi} + \frac{1}{r}\left(2\sigma_{\psi\phi}\operatorname{ctg}\phi + 3\sigma_{r\phi}\right) + b_\phi = \rho \ddot{u}_\phi$$

Geometric relations:

$$\varepsilon_{rr} = u_{r,r} \qquad \varepsilon_{\psi\phi} = \frac{1}{2}\left(\frac{1}{r\sin\psi}u_{\psi,\phi} + \frac{1}{r}u_{\phi,\psi} - \frac{\operatorname{ctg}\psi}{r}u_{\phi}\right)$$

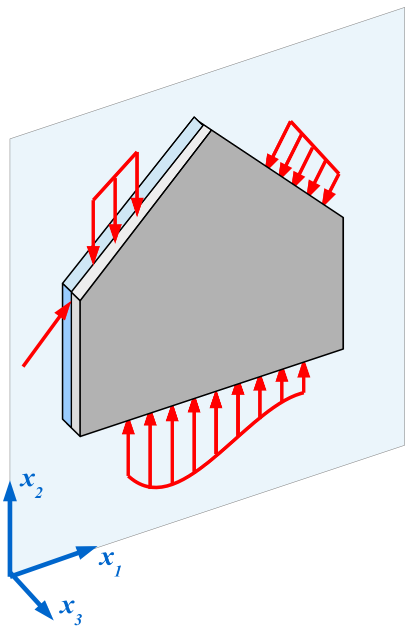
$$\varepsilon_{\psi\psi} = \frac{1}{r}u_{\psi,\psi} + \frac{u_r}{r} \qquad \varepsilon_{\phi r} = \frac{1}{2}\left(\frac{1}{r\sin\psi}u_{r,\phi} + u_{\phi,r} - \frac{u_\phi}{r}\right)$$

$$\varepsilon_{\phi\phi} = \frac{1}{r\sin\psi}u_{\phi,\phi} + \frac{u_r}{r} + \frac{\operatorname{ctg}\psi}{r}u_{\psi} \qquad \varepsilon_{r\psi} = \frac{1}{2}\left(u_{\psi,r} - \frac{u_\psi}{r} + \frac{1}{r}u_{r,\psi}\right)$$

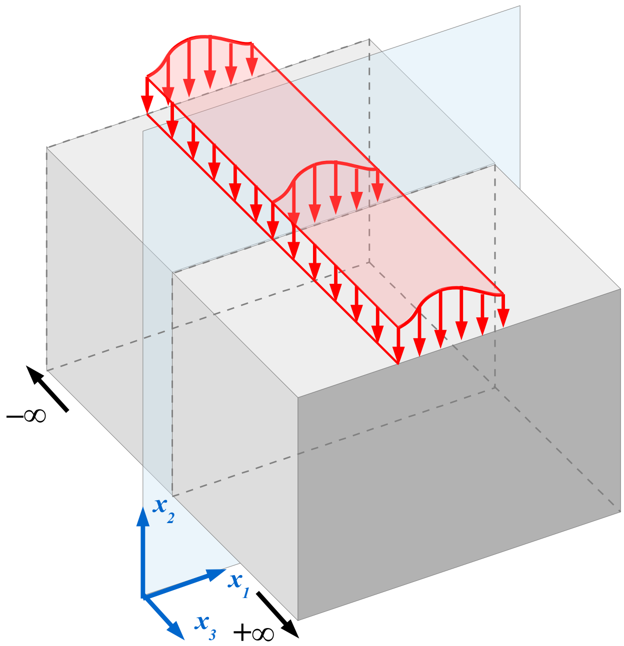
PLANE STATE PROBLEMS

It is not a general rule however it is often easier to find a solution to a partial differential equation for a problem of lower dimension. Considering 2-dimensional problems of theory of elasticity enable us to find certain closed form analytic solutions which emerge to be extremely useful. Reduction of dimension may be due to assumption of one of the below simplifications:

- **plane stress state**
- **plane strain state**



Plane stress state



Plane strain state

We shall narrow our considerations to **isotropic** solids only.

PLANE STRESS STATE

Plane stress state is described by a **plane stress tensor**:

$$\boldsymbol{\sigma}(\mathbf{x}) = \begin{bmatrix} \sigma_{11}(x_1; x_2) & \sigma_{12}(x_1; x_2) & 0 \\ \sigma_{12}(x_1; x_2) & \sigma_{22}(x_1; x_2) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hooke's law enable us to determine the strain state given by an **anti-plane strain tensor**

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \begin{bmatrix} \frac{1}{E}[\sigma_{11} - \nu\sigma_{22}] & \frac{\sigma_{12}}{2G} & 0 \\ \frac{\sigma_{12}}{2G} & \frac{1}{E}[\sigma_{22} - \nu\sigma_{11}] & 0 \\ 0 & 0 & -\frac{\nu}{E}[\sigma_{11} + \sigma_{22}] \end{bmatrix}$$

The body cannot be loaded perpendicularly to the plane of problem. Distribution of all stress, strain and displacement components depends only on $(x_1; x_2)$ and is constant along x_3 . Plane elastic solids under plane stress state are usually termed membranes.

PLANE STRAIN STATE

Plane stress state is described by a **plane strain tensor**:

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \begin{bmatrix} \varepsilon_{11}(x_1; x_2) & \varepsilon_{12}(x_1; x_2) & 0 \\ \varepsilon_{12}(x_1; x_2) & \varepsilon_{22}(x_1; x_2) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hooke's law enable us to determine the stress state given by an **anti-plane stress tensor**

$$\boldsymbol{\sigma}(\mathbf{x}) = \begin{bmatrix} (2G+\lambda)\varepsilon_{11} + \lambda\varepsilon_{22} & 2G\varepsilon_{12} & 0 \\ 2G\varepsilon_{12} & (2G+\lambda)\varepsilon_{22} + \lambda\varepsilon_{11} & 0 \\ 0 & 0 & \lambda[\varepsilon_{11} + \varepsilon_{22}] \end{bmatrix}$$

Any displacement along a direction perpendicular to the plane of the problem must be constrained. Distribution of all stress, strain and displacement components depends only on $(x_1; x_2)$ and is constant along x_3 .

Stress component σ_{33} may be also expressed by other stress components with the use of the assumption that off-plane normal strain is 0:

$$\varepsilon_{33} = \frac{1}{E}\left[\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})\right] = 0 \quad \Rightarrow \quad \sigma_{33} = \nu(\sigma_{11} + \sigma_{22})$$

Then normal strain can be expressed in the following form:

$$\varepsilon_{11}=\frac{1}{E}\left[\sigma_{11}-\nu(\sigma_{22}+\sigma_{33})\right]=\frac{1}{E}\left[\sigma_{11}-\nu\left(\sigma_{22}+\nu(\sigma_{11}+\sigma_{22})\right)\right]=$$

$$=\frac{1}{E}\left[(1-\nu^2)\sigma_{11}-(\nu+\nu^2)\sigma_{22}\right]=\frac{1-\nu^2}{E}\left[\sigma_{11}-\nu\frac{1+\nu}{1-\nu^2}\sigma_{22}\right]=\frac{1-\nu^2}{E}\left[\sigma_{11}-\frac{\nu}{1-\nu}\sigma_{22}\right]$$

Similarly for ε_{22} . If we introduce „modified elastic constants":

$$\hat{E}=\frac{E}{1-\nu^2}\,,\qquad \hat{\nu}=\frac{\nu}{1-\nu}\,,\quad G=\frac{E}{2(1+\nu)}=\frac{\hat{E}}{2(1+\hat{\nu})}$$

then the constitutive relations in any plane state (stress or strain) can be written down in the following form:

$$\begin{cases}\varepsilon_{11}=\dfrac{1}{\hat{E}}[\sigma_{11}-\hat{\nu}\sigma_{22}]\\ \varepsilon_{22}=\dfrac{1}{\hat{E}}[\sigma_{22}-\hat{\nu}\sigma_{11}]\\ \varepsilon_{12}=\dfrac{\sigma_{12}}{2G}\end{cases},\qquad \begin{cases}\hat{E}=E & \Leftrightarrow \text{ plane stress}\\ \hat{E}=\dfrac{E}{1-\nu^2} & \Leftrightarrow \text{ plane strain}\end{cases}\quad \begin{cases}\hat{\nu}=\nu & \Leftrightarrow \text{ plane stress}\\ \hat{\nu}=\dfrac{\nu}{1-\nu} & \Leftrightarrow \text{ plane strain}\end{cases}$$

what is sometimes written with the use of parameter κ :

$$\begin{cases}\hat{E}=E & \Leftrightarrow \text{ PSN}\\ \hat{E}=\dfrac{16E}{(7-\kappa)(\kappa+1)} & \Leftrightarrow \text{ PSO}\end{cases}\quad \text{where}\quad \hat{\nu}=\frac{3-\kappa}{\kappa+1}\quad \text{and}\quad \begin{cases}\kappa=\dfrac{3-\nu}{1+\nu} & \Leftrightarrow \text{ plane stress}\\ \kappa=3-4\nu & \Leftrightarrow \text{ plane strain}\end{cases}$$

Furthermore:

- plane stress: $\varepsilon_{33}=-\dfrac{\nu}{E}[\sigma_{11}+\sigma_{22}]$ $\qquad \sigma_{33}=0$,
- plane strain: $\sigma_{33}=\nu(\sigma_{11}+\sigma_{22})$ $\qquad \varepsilon_{33}=0$.

AIRY STRESS FUNCTION

Let's write down the only non-zero strain compatibility condition for plane state:

$$\varepsilon_{11,22}-2\varepsilon_{12,12}+\varepsilon_{22,11}=0$$

Expressing strain in terms of stress gives us:

$$\frac{1}{\hat{E}}[\sigma_{11}-\hat{\nu}\sigma_{22}]_{,22}-2\frac{\sigma_{12,12}}{2G}+\frac{1}{\hat{E}}[\sigma_{22}-\hat{\nu}\sigma_{11}]_{,11}=0$$

Let's rearrange the obtained result:

$$\frac{1}{\hat{E}}[\sigma_{11}-\hat{\nu}\sigma_{22}]_{,22}-\sigma_{12,12}\frac{2(1+\hat{\nu})}{\hat{E}}+\frac{1}{\hat{E}}[\sigma_{22}-\hat{\nu}\sigma_{11}]_{,11}=0$$

$$\sigma_{11,22}-\hat{\nu}\sigma_{22,22}-2(1+\hat{\nu})\sigma_{12,12}+\sigma_{22,11}-\hat{\nu}\sigma_{11,11}=0$$

$$\nabla^2\sigma_{11}-\sigma_{11,11}-\hat{\nu}\sigma_{11,11}+\Delta\sigma_{22}-\sigma_{22,22}-\hat{\nu}\sigma_{22,22}-2(1+\hat{\nu})\sigma_{12,12}+=0$$

$$\nabla^2(\sigma_{11}+\sigma_{22})-(1+\nu)[\sigma_{11,11}+\sigma_{22,22}+2\sigma_{12,12}]=0$$

$$\nabla^2(\sigma_{11}+\sigma_{22})-(1+\nu)[(\sigma_{11,1}+\sigma_{12,2})_{,1}+(\sigma_{12,1}+\sigma_{22,2})_{,2}]=0$$

Let's make use of the equilibrium equations:

$$\begin{cases}\sigma_{11,1}+\sigma_{12,2}=-b_1\\ \sigma_{12,1}+\sigma_{22,2}=-b_2\end{cases}$$

The result is:

$$\nabla^2(\sigma_{11}+\sigma_{22})+(1+\nu)(b_{1,1}+b_{2,2})=0$$

Let's introduce now the **Airy stress function** mentioned earlier:

$$F(x_1;x_2):\quad\begin{cases}F_{,11}=\sigma_{22}\\ F_{,22}=\sigma_{11}\\ F_{,12}=-\sigma_{12}-b_1x_2-b_2x_1\end{cases}$$

Equilibrium equations written down with the use of Airy stress function take form:

$$\begin{cases}F_{,221}+(-F_{,122}-b_{1,2}x_2-b_1-b_{2,2}x_1)+b_1=0\\ (-F_{,121}-b_{1,1}x_2-b_{2,1}x_1-b_2)+F_{,112}+b_2=0\end{cases}\Leftrightarrow\begin{cases}b_{1,2}x_2+b_{2,2}x_1=0\\ b_{1,1}x_2+b_{2,1}x_1=0\end{cases}$$

Assuming that the **distribution of body forces is constant in plane** $b_{i,j}=0\ (i,j=1,2)$ **the equilibrium equations are satisfied automatically**. The stress compatibility equation takes form:

$$\nabla^2(\sigma_{11}+\sigma_{22})=0$$

It means that **sum of normal stress components is a harmonic function**. When they are expressed in terms of Airy stress function, we obtain:

$$\nabla^2(F_{,22}+F_{,11})=\nabla^2(\nabla^2F)=0$$

It means that a solution to the problem of theory of elasticity is provided by a biharmonic Airy stress function:

$$\boxed{\nabla^4F=F_{,1111}+2F_{,1122}+F_{,2222}=0}$$

PLANE STATE IN POLAR COORDINATES

Many important engineering problems are characterized by axial symmetry and thus it is convenient to deal with them with the use of **polar coordinates**:

$$\begin{cases} r = \sqrt{x_1^2 + x_2^2} \\ \phi = \operatorname{arctg}\dfrac{x_2}{x_1} \end{cases} \quad \Leftrightarrow \quad \begin{cases} x_1 = r\cos\phi \\ x_2 = r\sin\phi \end{cases}$$

Relations between components of unknown functions in Cartesian and polar coordinates:

Displacement:

$$u_r = u_1\cos\phi + u_2\sin\phi$$
$$u_\phi = -u_1\sin\phi + u_2\cos\phi$$

Strain state:

$$\varepsilon_{rr} = \varepsilon_{11}\cos^2\phi + \varepsilon_{22}\sin^2\phi + \varepsilon_{12}\sin 2\phi$$
$$\varepsilon_{\phi\phi} = \varepsilon_{11}\sin^2\phi + \varepsilon_{22}\cos^2\phi - \varepsilon_{12}\sin 2\phi$$
$$\varepsilon_{r\phi} = \frac{\varepsilon_{22}-\varepsilon_{11}}{2}\sin 2\phi + \varepsilon_{12}\cos 2\phi$$

Stress state:

$$\sigma_{rr} = \sigma_{11}\cos^2\phi + \sigma_{22}\sin^2\phi + \sigma_{12}\sin 2\phi$$
$$\sigma_{\phi\phi} = \sigma_{11}\sin^2\phi + \sigma_{22}\cos^2\phi - \sigma_{12}\sin 2\phi$$
$$\sigma_{r\phi} = \frac{\sigma_{22}-\sigma_{11}}{2}\sin 2\phi + \sigma_{12}\cos 2\phi$$

Governing equations:

Equilibrium equations:

$$\sigma_{rr,r} + \frac{1}{r}\sigma_{r\phi,\phi} + \frac{\sigma_{rr}-\sigma_{\phi\phi}}{r} = 0$$
$$\sigma_{r\phi,r} + \frac{1}{r}\sigma_{\phi\phi,\phi} + \frac{2}{r}\sigma_{r\phi} = 0$$

Geometric relations:

$$\varepsilon_{rr} = u_{r,r}$$
$$\varepsilon_{\phi\phi} = \frac{1}{r}u_{\phi,\phi} + \frac{u_r}{r}$$
$$\varepsilon_{r\phi} = \frac{1}{2}\left(u_{\phi,r} - \frac{u_\phi}{r} + \frac{1}{r}u_{r,\phi}\right)$$

Constitutive relations:

$$\sigma_{rr} = (2G+\lambda)\varepsilon_{rr} + \lambda\varepsilon_{\phi\phi}$$
$$\sigma_{\phi\phi} = (2G+\lambda)\varepsilon_{\phi\phi} + \lambda\varepsilon_{rr}$$
$$\sigma_{r\phi} = 2G\varepsilon_{r\phi}$$

Strain compatibility condition:

$$\frac{1}{r^2}\frac{\partial^2\varepsilon_{rr}}{\partial\phi^2} - \frac{1}{r}\frac{\partial\varepsilon_{rr}}{\partial r} + \frac{\partial^2\varepsilon_{\phi\phi}}{\partial r^2} + \frac{2}{r}\frac{\partial\varepsilon_{\phi\phi}}{\partial r} = \frac{2}{r}\frac{\partial^2\varepsilon_{r\phi}}{\partial r\,\partial\phi} + \frac{2}{r^2}\frac{\partial\varepsilon_{r\phi}}{\partial\phi}$$

Airy stress functions formulation is still valid in polar coordinates. Biharmonic operator inpolar coordinates has the following form:

$$\nabla^4 = \Delta\Delta = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right)$$

And the **biharmonic equation for Airy stress function in polar coordinates** takes form:

$$F_{,rrrr} + \frac{2}{r^2}F_{,rr\phi\phi} + \frac{1}{r^4}F_{,\phi\phi\phi\phi} + \frac{2}{r}F_{,rrr} - \frac{2}{r^3}F_{,r\phi\phi} - \frac{1}{r^2}F_{,rr} + \frac{4}{r^4}F_{,\phi\phi} + \frac{1}{r^3}F_{,r} = 0$$

It may be shown that stress components are now expressed in terms of derivatives of Airy functions as follows:

$$\sigma_{rr} = \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \phi^2} \qquad \sigma_{\phi\phi} = \frac{\partial^2 F}{\partial r^2} \qquad \sigma_{r\phi} = \frac{1}{r^2}\frac{\partial F}{\partial \phi} - \frac{1}{r}\frac{\partial^2 F}{\partial r \partial \phi}$$

It may be shown that there exists a general solution to the biharmonic equation in polar coordinates – it is so called **Michell solution** given by the below formula:

$$\begin{aligned} F(r,\phi) &= A_{01} r^2 + A_{02} r^2 \ln r + A_{03} \ln r + A_{04}\phi + \\ &+ \left(A_{11} r^3 + A_{12} r \ln r + A_{14} r^{-1}\right)\cos\phi + A_{13} r\phi \sin\phi \\ &+ \left(B_{11} r^3 + B_{12} r \ln r + B_{14} r^{-1}\right)\sin\phi + B_{13} r\phi \cos\phi \\ &+ \sum_{n=2}^{\infty}\left[\left(A_{n1} r^{n+2} + A_{n2} r^{-n+2} + A_{n3} r^n + A_{n4} r^{-n}\right)\cos(n\phi)\right] \\ &+ \sum_{n=2}^{\infty}\left[\left(B_{n1} r^{n+2} + B_{n2} r^{-n+2} + B_{n3} r^n + B_{n4} r^{-n}\right)\sin(n\phi)\right] \end{aligned}$$

where A_{ij}, B_{ij} $(i, j = 0,1,2,\ldots)$ are constants of integration determined according to the boundary conditions.

AXIS-SYMMETRIC PROBLEMS

An important class of problems are those characterized by **axial symmetry** in which **all quantities are independent of the angle** ϕ . For axis-symmetric problems we have:

$$\frac{\partial}{\partial\phi}=0$$

It is often also assumed that:

$$u_\phi=0$$

Problems of axial symmetr for which $u_\phi\neq 0$ are sometimes referred to as quasi axis-symmetric. Assumptions of axial symmetry results in:

$$\sigma_{r\phi}=0\quad,\quad \varepsilon_{r\phi}=0\quad .$$

Governing equations:

Equilibrium equation

$$\sigma_{rr,r}+\frac{\sigma_{rr}-\sigma_{\phi\phi}}{r}=0$$

Geometric relations:

$$\varepsilon_{rr}=u_{r,r}\quad,\quad \varepsilon_{\phi\phi}=\frac{u_r}{r}\quad,\quad \varepsilon_{r\phi}=0$$

Constitutive relations:

$$\sigma_{rr}=(2G+\lambda)\varepsilon_{rr}+\lambda\varepsilon_{\phi\phi}=(2G+\lambda)u_{r,r}+\lambda\frac{u_r}{r}$$

$$\sigma_{\phi\phi}=(2G+\lambda)\varepsilon_{\phi\phi}+\lambda\varepsilon_{rr}=(2G+\lambda)\frac{u_r}{r}+\lambda u_{r,r}$$

$$\sigma_{r\phi}=0$$

Strain compatibility condition:

$$-\frac{1}{r}\frac{\partial\varepsilon_{rr}}{\partial r}+\frac{\partial^2\varepsilon_{\phi\phi}}{\partial r^2}+\frac{2}{r}\frac{\partial\varepsilon_{\phi\phi}}{\partial r}=0$$

DISPLACEMENT EQUATION FOR AXIS-SYMMETRIC PROBLEMS

An approach of finding simplified governing equation depending on displacement only, which was used in derivation of Lame equations, may be applied in plane axis-symmetric state for the only non-zero displacement component:

$$\left[(2G+\lambda)u_{r,r}+\lambda\frac{u_r}{r}\right]_{,r}+\frac{1}{r}\left[(2G+\lambda)u_{r,r}+\lambda\frac{u_r}{r}-(2G+\lambda)\frac{u_r}{r}-\lambda u_{r,r}\right]=0$$

$$\left[(2G+\lambda)u_{r,rr}+\lambda\left(\frac{u_{r,r}}{r}-\frac{u_r}{r^2}\right)\right]+\frac{2G}{r}\left(u_{r,r}-\frac{u_r}{r}\right)=0$$

$$(2G+\lambda)\left[u_{r,rr}+\frac{u_{r,r}}{r}-\frac{u_r}{r^2}\right]=0$$

Finally the **displacement equation** is of the following form:

$$u_{r,rr}+\frac{u_{r,r}}{r}-\frac{u_r}{r^2}=0$$

It is no longer a partial differential equation – it is an **ordinary differential equation of the Euler type**. Its solution is known:

$$u_r(r)=C_1 r+\frac{C_2}{r}$$

where C_1, C_2 are constants of integration determined according to the boundary conditions. The rest of the unknown components may be found easily now:

$$\varepsilon_{rr}(r)=C_1-\frac{C_2}{r^2}$$

$$\varepsilon_{\phi\phi}=C_1+\frac{C_2}{r^2}$$

$$\sigma_{rr}(r)=2C_1(G+\lambda)-\frac{2C_2G}{r^2}=2A_{01}+\frac{A_{03}}{r^2}$$

$$\sigma_{\phi\phi}(r)=2C_1(G+\lambda)+\frac{2C_2G}{r^2}=2A_{01}-\frac{A_{03}}{r^2}$$

THE USE OF AIRY STRESS FUNCTION FOR AXIS-SYMMETRIC PROBLEMS

Biharmonic operator in polar coordinates for axis-symmetric problems has the form:

$$\nabla^4 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right]\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right] =$$
$$= \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\right]\right]\right]$$

The last form is particularly useful as it enables determination of the solution by direct integration:

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial F}{\partial r}\right]\right]\right] = 0 \quad \Rightarrow \quad r\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial F}{\partial r}\right]\right] = C_1$$
$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial F}{\partial r}\right]\right] = \frac{C_1}{r} \quad \Rightarrow \quad \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial F}{\partial r}\right] = C_1\ln r + C_2$$
$$\frac{\partial}{\partial r}\left[r\frac{\partial F}{\partial r}\right] = C_1 r\ln r + C_2 r \quad \Rightarrow \quad r\frac{\partial F}{\partial r} = \frac{C_1}{4}r^2(2\ln r - 1) + \frac{C_2}{2}r^2 + C_3$$
$$\frac{\partial F}{\partial r} = \frac{C_1}{4}r(2\ln r - 1) + \frac{C_2}{2}r + \frac{C_3}{r} \quad \Rightarrow \quad F = \frac{C_1}{4}r^2(\ln r - 1) + \frac{C_2}{4}r^2 + C_3\ln r + C_4$$

Finally the general solution may be found as:

$$\boxed{F(r) = A_{00} + A_{01}r^2 + A_{02}r^2\ln r + A_{03}\ln r \quad .}$$

Stress state components:

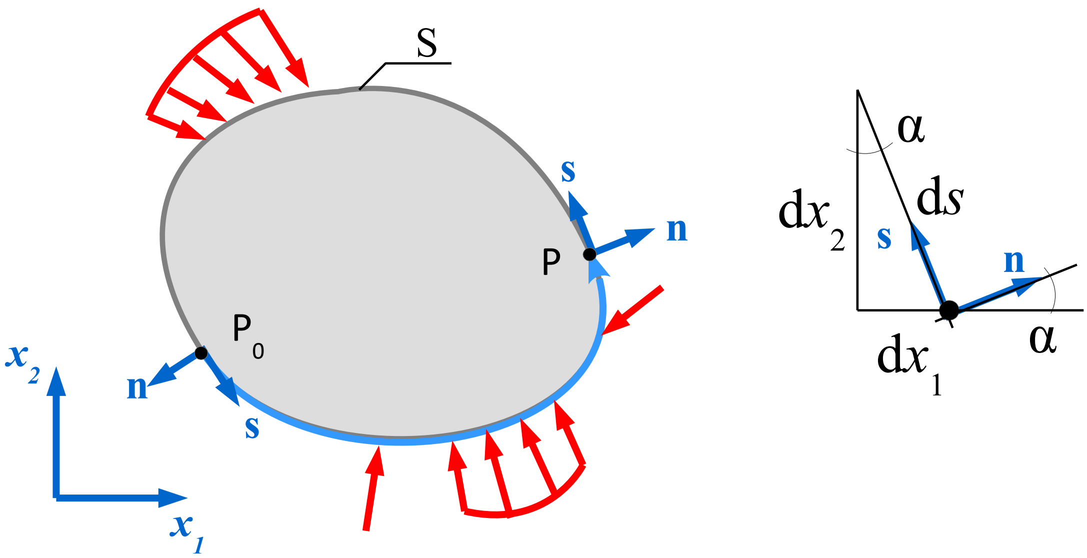
$$\sigma_{rr} = \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \phi^2} = 2A_{01} + A_{02}(2\ln r + 1) + \frac{A_{03}}{r^2}$$
$$\sigma_{\phi\phi} = \frac{\partial^2 F}{\partial r^2} = 2A_{01} + A_{02}(2\ln r + 3) - \frac{A_{03}}{r^2}$$
$$\sigma_{r\phi} = \frac{1}{r^2}\frac{\partial F}{\partial \phi} - \frac{1}{r}\frac{\partial^2 F}{\partial r\,\partial \phi} = 0$$

BOUNDARY CONDITIONS FOR AIRY STRESS FUNCTION

Biharmonic equation is fourth-order equation so it requires a number boundary conditions of proper type. Kinematic conditions are difficult to be involved – we shall narrow our considerations to the problems of purely static boundary conditions. Static boundary conditions account for stress components which are expressible in terms of the 2nd derivatives of Airy functions.

We may note that the **equation itself do not depend on elastic constants – if the boundary conditions are also independent of them**, **then** the whole solution is independent of material's mechanical properties, namely – **stress distribution in a plane state is independent of elastic constants**. This is known as the **Lévy theorem**. The stress state in such situation is the same for all materials – this fact is used in photoelasticity analysis.

We shall find now a way for finding boundary conditions for Airy stress function: Let's consider a plane membrane of thickness h in plane stress state, with boundary S.



External unit normal at the boundary is equal:

$$\mathbf{n} = \begin{cases} n_1 = \cos\alpha \\ n_2 = \sin\alpha \end{cases} ,$$

where α is an angle between direction of the unit normal and axis x_1 . Trigonometric functions of this angle may be expressed also in terms of differential relations between increment of boundary edge and increments of coordinates:

$$\mathbf{n} = \begin{cases} n_1 = \cos\alpha = \dfrac{\mathrm{d}\,x_2}{\mathrm{d}\,s} \\ n_2 = \sin\alpha = -\dfrac{\mathrm{d}\,x_1}{\mathrm{d}\,s} \end{cases} ,$$

Making the use of relation between stress components and Airy stress function (disregarding the body forces), static boundary conditions may be written down as follows:

$$\sigma_{ij} n_j = q_i \qquad \Rightarrow \qquad \begin{cases} \dfrac{\partial^2 F}{\partial x_2^2}\dfrac{\mathrm{d}\,x_2}{\mathrm{d}\,s} + \left(-\dfrac{\partial^2 F}{\partial x_1 \partial x_2}\right)\left(-\dfrac{\mathrm{d}\,x_1}{\mathrm{d}\,s}\right) = q_1 \\ \left(-\dfrac{\partial^2 F}{\partial x_1 \partial x_2}\right)\dfrac{\mathrm{d}\,x_2}{\mathrm{d}\,s} + \dfrac{\partial^2 F}{\partial x_1^2}\left(-\dfrac{\mathrm{d}\,x_1}{\mathrm{d}\,s}\right) = q_2 \end{cases}$$

The chain rule gives us:

$$\begin{cases} q_1 = \dfrac{\partial^2 F}{\partial x_2^2}\dfrac{\mathrm{d}x_2}{\mathrm{d}s} + \dfrac{\partial^2 F}{\partial x_1 \partial x_2}\dfrac{\mathrm{d}\,x_1}{\mathrm{d}\,s} = \dfrac{\mathrm{d}}{\mathrm{d}s}\left(\dfrac{\partial F}{\partial x_2}\right) \\ q_2 = -\left[\dfrac{\partial^2 F}{\partial x_1 \partial x_2}\dfrac{\mathrm{d}\,x_2}{\mathrm{d}\,s} + \dfrac{\partial^2 F}{\partial x_1^2}\dfrac{\mathrm{d}\,x_1}{\mathrm{d}\,s}\right] = -\dfrac{\mathrm{d}}{\mathrm{d}s}\left(\dfrac{\partial F}{\partial x_1}\right) \end{cases}$$

Let's sum up all the load along boundary starting from certain fixed point P_0 and ending in an arbitrary point P – it will be an integral along boundary:

$$\begin{cases} Q_1 = \iint\limits_A q_1 \,\mathrm{d}\,A = h\int\limits_{P_0}^{P} q_1 \,\mathrm{d}\,s = h\int\limits_{P_0}^{P} \dfrac{\mathrm{d}}{\mathrm{d}s}\left(\dfrac{\partial F}{\partial x_2}\right)\mathrm{d}\,s = h\dfrac{\partial F}{\partial x_2}\Bigg|_{P_0}^{P} \\ Q_2 = \iint\limits_A q_2 \,\mathrm{d}\,A = h\int\limits_{P_0}^{P} q_2 \,\mathrm{d}\,s = -h\int\limits_{P_0}^{P} \dfrac{\mathrm{d}}{\mathrm{d}s}\left(\dfrac{\partial F}{\partial x_1}\right)\mathrm{d}\,s = -h\dfrac{\partial F}{\partial x_1}\Bigg|_{P_0}^{P} \end{cases}$$

Let's find also the sum of moment of the external load about point P_0 from the load applied to the boundary between $P_0 = (x_1^{P_0}; x_2^{P_0})$ and $P = (x_1^P; x_2^P)$:

$$M = h\int\limits_{P_0}^{P}\left[q_1(x_2^P - x_2) - q_2(x_1^P - x_1)\right]\mathrm{d}\,s = h\left[x_2^P\int\limits_{P_0}^{P} q_1 \,\mathrm{d}\,s - x_1^P\int\limits_{P_0}^{P} q_2 \,\mathrm{d}\,s + \int\limits_{P_0}^{P}(q_2 x_1 - q_1 x_2)\,\mathrm{d}\,s\right]$$

Let's introduce the relation between Airy function and surface tractions:

$$M = h\left[x_2^P \dfrac{\partial F}{\partial x_2}\Bigg|_{P_0}^{P} + x_1^P \dfrac{\partial F}{\partial x_1}\Bigg|_{P_0}^{P} - \int\limits_{P_0}^{P}\dfrac{\mathrm{d}}{\mathrm{d}s}\left(\dfrac{\partial F}{\partial x_1}\right)x_1 \,\mathrm{d}\,s - \int\limits_{P_0}^{P}\dfrac{\mathrm{d}}{\mathrm{d}s}\left(\dfrac{\partial F}{\partial x_2}\right)x_2 \,\mathrm{d}\,s\right]$$

Both integrals may be calculated by integration by parts:

$$\int\limits_{P_0}^{P}\dfrac{\mathrm{d}}{\mathrm{d}s}\left(\dfrac{\partial F}{\partial x_1}\right)x_1 \,\mathrm{d}\,s = \left[x_1\dfrac{\partial F}{\partial x_1}\right]_{P_0}^{P} - \int\limits_{P_0}^{P}\dfrac{\partial F}{\partial x_1}\dfrac{\mathrm{d}\,x_1}{\mathrm{d}\,s}\mathrm{d}\,s$$

Finally:

$$M = h\left[x_2^{P_0} \frac{\partial F}{\partial x_2}\bigg|_{P_0} + x_1^{P_0} \frac{\partial F}{\partial x_1}\bigg|_{P_0} + \int_{P_0}^{P} \left(\frac{\partial F}{\partial x_1}\frac{\mathrm{d}\, x_1}{\mathrm{d}\, s} + \frac{\partial F}{\partial x_2}\frac{\mathrm{d}\, x_2}{\mathrm{d}\, s} \right) \mathrm{d}\, s \right]$$

An integrand is an total derivative with respect to natural parameter s:

$$\frac{\partial F}{\partial x_1}\frac{\mathrm{d}\, x_1}{\mathrm{d}\, s} + \frac{\partial F}{\partial x_2}\frac{\mathrm{d}\, x_2}{\mathrm{d}\, s} = \frac{\mathrm{d}\, F}{\mathrm{d}\, s}$$

so:

$$M = h\left[x_2^{P_0} \frac{\partial F}{\partial x_2}\bigg|_{P_0} + x_1^{P_0} \frac{\partial F}{\partial x_1}\bigg|_{P_0} + F\big|_{P_0}^{P} \right]$$

An infinite number of Airy functions may be found respective for given stress state – they all may differ with any combination of constant parameter and linear functions, namely the functions of the form $a_0 + a_1 x_1 + a_2 x_2$. We may chose this function in such a way that:

$$\frac{\partial F}{\partial x_1}\bigg|_{P_0} = 0 \,, \qquad \frac{\partial F}{\partial x_2}\bigg|_{P_0} = 0 \,, \qquad F\big|_{P_0} = 0$$

If we assume that position of $P = (x_1^P ; x_2^P)$ may be arbitrary chosen $P = (x_1 ; x_2)$ then we obtain the following result:

$$F\big|_P = \frac{M|_P}{h} \,, \qquad \frac{\partial F}{\partial x_1}\bigg|_P = -\frac{Q_2|_P}{h} \,, \qquad \frac{\partial F}{\partial x_2}\bigg|_P = \frac{Q_1|_P}{h}$$

Boundary conditions for Airy stress function and its first derivatives may be found by proper integrals of surface tractions – components of sum of forces and sum of moments. In particular, if the coordinate system is chosen in such a way that its first axis coincide with directions of external unit normal, the boundary conditions may be expressed in terms of normal and tangential forces at the boundary:

$$F\big|_P = \frac{M|_P}{h} \,, \qquad \frac{\partial F}{\partial n}\bigg|_P = -\frac{Q_s|_P}{h} \,, \qquad \frac{\partial F}{\partial s}\bigg|_P = \frac{Q_n|_P}{h}$$

SUMMARY

GOVERNING EQUATIONS OF LINEAR THEORY OF ELASTICITY

Equations of motion: $\sigma_{ij,j} + b_i = \rho \ddot{u}_i$

$$\begin{cases} \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} + \dfrac{\partial \sigma_{13}}{\partial x_3} + b_1 = \rho \dfrac{\partial^2 u_1}{\partial t^2} \\ \dfrac{\partial \sigma_{21}}{\partial x_1} + \dfrac{\partial \sigma_{22}}{\partial x_2} + \dfrac{\partial \sigma_{23}}{\partial x_3} + b_2 = \rho \dfrac{\partial^2 u_2}{\partial t^2} \\ \dfrac{\partial \sigma_{31}}{\partial x_1} + \dfrac{\partial \sigma_{32}}{\partial x_2} + \dfrac{\partial \sigma_{33}}{\partial x_3} + b_3 = \rho \dfrac{\partial^2 u_3}{\partial t^2} \end{cases}$$

Geometric relations: $\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} \qquad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2} \qquad \varepsilon_{33} = \frac{\partial u_3}{\partial x_3}$$

$$\varepsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \qquad \varepsilon_{31} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) \qquad \varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

Constitutive law: $\sigma_{ij} = \mathsf{S}_{ijkl}\varepsilon_{kl} \quad \Leftrightarrow \quad \varepsilon_{ij} = \mathsf{C}_{ijkl}\sigma_{kl}$

Constitutive relations for **isotropy**:

1st form: $\sigma_{ij} = 2G\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}$

$$\begin{aligned} \sigma_{11} &= 2G\varepsilon_{11} + \lambda(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}), & \sigma_{23} &= 2G\varepsilon_{23} \\ \sigma_{22} &= 2G\varepsilon_{22} + \lambda(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}), & \sigma_{31} &= 2G\varepsilon_{31} \\ \sigma_{33} &= 2G\varepsilon_{33} + \lambda(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}), & \sigma_{12} &= 2G\varepsilon_{12} \end{aligned}$$

2nd form: $\varepsilon_{ij} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\right]$

$$\begin{aligned} \varepsilon_{11} &= \frac{1}{E}\left[(1+\nu)\sigma_{11} - \nu(\sigma_{11}+\sigma_{22}+\sigma_{33})\right], & \varepsilon_{23} &= \frac{\sigma_{23}}{2G} \\ \varepsilon_{22} &= \frac{1}{E}\left[(1+\nu)\sigma_{22} - \nu(\sigma_{11}+\sigma_{22}+\sigma_{33})\right], & \varepsilon_{31} &= \frac{\sigma_{31}}{2G} \\ \varepsilon_{33} &= \frac{1}{E}\left[(1+\nu)\sigma_{33} - \nu(\sigma_{11}+\sigma_{22}+\sigma_{33})\right], & \varepsilon_{12} &= \frac{\sigma_{12}}{2G} \end{aligned}$$

LAMÉ DISPLACEMENT EQUATIONS

$$G\Delta\mathbf{u} + (G+\lambda)\nabla\cdot(\mathbf{u}\otimes\nabla) + \mathbf{b} = \rho\ddot{\mathbf{u}}$$

$$\begin{cases} G\Delta u_1 + (G+\lambda)(u_{1,11}+u_{2,21}+u_{3,31}) + b_1 = \rho\ddot{u}_1 \\ G\Delta u_2 + (G+\lambda)(u_{1,12}+u_{2,22}+u_{3,32}) + b_2 = \rho\ddot{u}_2 \\ G\Delta u_3 + (G+\lambda)(u_{1,13}+u_{2,23}+u_{3,33}) + b_3 = \rho\ddot{u}_3 \end{cases}$$

BELTRAMI-MICHELL STRESS COMPATIBILITY EQUATIONS

$$\Delta\boldsymbol{\sigma} + \frac{1}{1+\nu}\left[(\mathbf{1}\cdot\boldsymbol{\sigma})\otimes\nabla\otimes\nabla\right] + (\mathbf{b}\otimes\nabla + \nabla\otimes\mathbf{b}) + \frac{\nu}{1-\nu}\left[\mathbf{1}\cdot(\mathbf{b}\otimes\nabla)\right]\mathbf{1} = \mathbf{0}$$

$$\Delta\sigma_{11} + \frac{1}{1+\nu}(\sigma_{11,11}+\sigma_{22,11}+\sigma_{33,11}) + 2b_{1,1} + \frac{\nu}{1-\nu}(b_{1,1}+b_{2,2}+b_{3,3}) = 0$$

$$\Delta\sigma_{22} + \frac{1}{1+\nu}(\sigma_{11,22}+\sigma_{22,22}+\sigma_{33,22}) + 2b_{2,2} + \frac{\nu}{1-\nu}(b_{1,1}+b_{2,2}+b_{3,3}) = 0$$

$$\Delta\sigma_{33} + \frac{1}{1+\nu}(\sigma_{11,33}+\sigma_{22,33}+\sigma_{33,33}) + 2b_{3,3} + \frac{\nu}{1-\nu}(b_{1,1}+b_{2,2}+b_{3,3}) = 0$$

$$\Delta\sigma_{23} + \frac{1}{1+\nu}(\sigma_{11,23}+\sigma_{22,23}+\sigma_{33,23}) + (b_{2,3} + b_{3,2}) = 0$$

$$\Delta\sigma_{31} + \frac{1}{1+\nu}(\sigma_{11,31}+\sigma_{22,31}+\sigma_{33,31}) + (b_{3,1} + b_{1,3}) = 0$$

$$\Delta\sigma_{12} + \frac{1}{1+\nu}(\sigma_{11,12}+\sigma_{22,12}+\sigma_{33,12}) + (b_{1,2} + b_{2,1}) = 0$$

7. ENERGY-BASED VARIATIONAL THEOREMS

In all below considerations we will account only for purely mechanical processes – we will neglect thermal exchange between the body and its environment. In such a situation increment in energy is only due to mechanical work performed by external forces:

$$\mathrm{d}U = \mathrm{d}L$$

This work is transformed into:

- E_k **kinetic energy** accounting for **motion of particles**
- Φ **internal energy of elasticity** accounting for **reversible elastic deformation** of body

We can write then:

$$\mathrm{d}L = \mathrm{d}E_k + \mathrm{d}\Phi$$

We shall narrow our considerations to static or quasistatic processes so the kinetic energy will be neglected. We shall introduce following energy quantities:

- **Work of external forces**: $L = \iiint_V b_i u_i \,\mathrm{d}V + \iint_S q_i u_i \,\mathrm{d}S$
- **Internal energy of elastic deformation**: $\Phi = \frac{1}{2}\iiint_V \sigma_{ij}\varepsilon_{ij}\,\mathrm{d}V$
- **Total potential energu**: $\Pi = \Phi - L$

Remember that **elastic strain energy is simultaneously an elastic potential** for linear elastic solids: $W = \Phi$.

PRINCIPLE OF VIRTUAL DISPLACEMENTS

Let there be an elastic solid of volume V and boundary S . The body is loaded with external forces $\mathbf{q}$ at boundary S_q and on the rest of the boundary, S_u ,the displacements are given by function $\mathbf{g}$. Let's consider a set of **kinematically admissible displacement fields**, namely those **satisfying kinematic boundary conditions**:

$$X_u = \left\{\breve{\mathbf{u}}:\ \ \breve{\mathbf{u}}(\mathbf{x}_0) = \mathbf{g}(\mathbf{x}_0),\ \mathbf{x}_0 \in S_u\right\}$$

Among them there is one which is the true displacement – we will denote it with $\hat{\mathbf{u}}$. It is the solution of problem of theory of elasticity. All other displacement fields may be written i the form

$$\breve{\mathbf{u}} = \hat{\mathbf{u}} + \alpha\boldsymbol{\upsilon} \ ,$$

where $\boldsymbol{\upsilon}$ is a certain vector field α is a scalar. Due to fact that both $\hat{\mathbf{u}}$ and $\breve{\mathbf{u}}$ must satisfy kinematic boundary conditions at S_u so $\boldsymbol{\upsilon}$ must be equal to 0 at S_u

Let's define the **virtual displacement** $\delta\mathbf{u}$, which is an increment in displacement due to increment of α :

$$\delta\mathbf{u} = \frac{\partial \breve{\mathbf{u}}}{\partial \alpha}\mathrm{d}\alpha = \boldsymbol{\upsilon}\,\mathrm{d}\alpha$$

Let's assume that we know true displacement $\mathbf{u} = \hat{\mathbf{u}}$ - we may differentiate it in order to calculate strains $\boldsymbol{\varepsilon}$ and then stresses $\boldsymbol{\sigma}$. The equilibrium **equations** must be satisfied by those quantities:

$$\sigma_{ij,j} + b_i = 0$$

Let's maultiply this expression by virtual displacement (dot product) and integrate it over the whole volume of the body:

$$\iiint_V \left[\sigma_{ij,j}\delta u_i + b_i \delta u_i\right]\mathrm{d}V = 0$$

The first term may be transformed according to the formula for derivative of a product of functions:

$$\iiint_V \left[\left(\sigma_{ij}\delta u_i\right)_{,j} - \sigma_{ij}\delta u_{i,j} + b_i \delta u_i\right]\mathrm{d}V = 0$$

Let's divide an integral of sum into a sum of integrals. Divergence will be applied to the first of them and the second one will be put on the right hand side of equation:

$$\iint_S \sigma_{ij} n_j \delta u_i \,\mathrm{d}S + \iiint_V b_i \delta u_i \,\mathrm{d}V = \iiint_V \sigma_{ij}\delta u_{i,j}\,\mathrm{d}V$$

Surface integral may be rewritten as a sum over S_u – where $\boldsymbol{\upsilon} = \mathbf{0}$ and as a consequence $\delta\mathbf{u} = 0$ – and over S_q , where static boundary conditions of the form $\mathbf{q} = \boldsymbol{\sigma}\cdot\mathbf{n}$ hold:

$$\iint_S \sigma_{ij} n_j \delta u_i \,\mathrm{d}S = \iint_{S_u} \sigma_{ij} n_j \delta u_i \,\mathrm{d}S + \iint_{S_q} \sigma_{ij} n_j \delta u_i \,\mathrm{d}S = 0 + \iint_{S_q} q_i \delta u_i \,\mathrm{d}S \quad ,$$

as a result we obtain

$$\iint_{S_q} q_i \delta u_i \,\mathrm{d}S + \iiint_V b_i \delta u_i \,\mathrm{d}V = \iiint_V \sigma_{ij}\delta u_{i,j}\,\mathrm{d}V$$

Left hand side of this equation in a **work of external forces on virtual displacements** δL . Considering the right hand side, due to symmetry of stress tensor, we may rewrite it as follows:

$$\sigma_{ij} = \frac{1}{2}\left(\sigma_{ij} + \sigma_{ji}\right)$$

Dummy indices may be changed in the following way:

$$\sigma_{ij}\delta u_{i,j} = \frac{1}{2}\left(\sigma_{ij}\delta u_{i,j} + \sigma_{ji}\delta u_{i,j}\right) = \frac{1}{2}\left(\sigma_{ij}\delta u_{i,j} + \sigma_{ij}\delta u_{j,i}\right) = \sigma_{ij}\frac{\delta u_{i,j} + \delta u_{j,i}}{2}$$

Virtual strain corresponding to virtual displacement $\delta \mathbf{u}=0$ may be defined as

$$\delta \varepsilon_{ij} = \frac{\delta u_{i,j}+\delta u_{j,i}}{2} \quad ,$$

So finally we obtain:

$$\iint\limits_{S_q} q_i \delta u_i \mathrm{d} S + \iiint\limits_V b_i \delta u_i \mathrm{d} V = \iiint\limits_V \sigma_{ij} \delta \varepsilon_{ij} \mathrm{d} V \qquad \Leftrightarrow \qquad \delta L = \delta \Phi$$

and **the above relation must hold for any virtual displacement** $\delta \mathbf{u}$ **satisfying kinematic boundary conditions**: $\delta \mathbf{u}=0$ on S_u . This is a relation which must be satisfied by any true displacement field – the solution we are looking form. It is then a **necessary condition** **for a displacement to be a true one**. It may be shown that it is also a **sufficient condition**. Let's assume that there is a displacement field that satisfy the kinematic boundary conditions, but not necessarily guarantees satisfying the equilibrium equations. Let's start with the final result:

$$\iint\limits_{S_q} q_i \delta u_i \mathrm{d} S + \iiint\limits_V b_i \delta u_i \mathrm{d} V = \iiint\limits_V \sigma_{ij} \delta u_{i,j} \mathrm{d} V$$

An integral on the right hand side may be transformed according to the formula for derivative of product of functions and then divergence theorem may be applied:

$$\iint\limits_{S_q} q_i \delta u_i \mathrm{d} S + \iiint\limits_V b_i \delta u_i \mathrm{d} V = \iiint\limits_V (\sigma_{ij} \delta u_i)_{,j} - \sigma_{ij,j} \delta u_i \mathrm{d} V$$
$$\iint\limits_{S_q} q_i \delta u_i \mathrm{d} S + \iiint\limits_V b_i \delta u_i \mathrm{d} V = \iiint\limits_S \sigma_{ij} n_j \delta u_i \mathrm{d} V - \iiint\limits_V \sigma_{ij,j} \delta u_i \mathrm{d} V$$

Since $\mathbf{u} \in X_u$, then $\delta \mathbf{u}=0$ on S_u and surface integral over S_u is zero:

$$\iint\limits_{S_q} (q_i - \sigma_{ij} n_j) \delta u_i \mathrm{d} S + \iiint\limits_V (\sigma_{ij,j} + b_i) \delta u_i \mathrm{d} V = 0$$

Virtual displacement may be of any form. If we assume that **założymy, że** $\delta \mathbf{u}$ **takes on** S_q **values differen than 0**, then the above equation will hold (what we assume to be true) for any $\delta \mathbf{u}$ if and only if both integrands are equal to 0, what is equivalent to satisfying:

- **equilibrium equations**: $\sigma_{ij,j} + b_i = 0$
- **static boundary condition**: $\sigma_{ij} n_j = q_i$

We can see that is the assumption of the equality $\delta L = \delta \Phi$ for a certain kinematically admissible displacement leads strain and stress distribution which satisfy also equilibrium equations as well as static boundary conditions – this means that it is a solution to the problem of theory of elasticity.

To sum it up:

PRINCIPLE OF VIRTUAL DISPLACEMENTS

A necessary and sufficient condition *for a* ***kinematically admissible diesplacement field*** *to be a* ***true displacement field*** *is that* ***a work of external forces on virtual displacement was equal work of internal forces on virtual strains for any virtual displacement*** *and corresponding virtual strain*:

$$\iint_{S_q} q_i \delta u_i \,\mathrm{d}S + \iiint_V b_i \delta u_i \,\mathrm{d}V = \iiint_V \sigma_{ij} \delta \varepsilon_{ij} \,\mathrm{d}V \qquad \forall \delta \mathbf{u}$$

$$\delta L = \delta \Phi$$

It is important to note that this **principle is independent of the form of constitutive relations**. It is only assumed that strains and displacements are small.

FINITE ELEMENT METHOD

The above result is the basis for the most commonly used numerical method for solving problems of elasticity, namely the finite element method. Let's assume that displacement field is approximated as a linear combinations of a finite number of some functions $\phi_{iK}(\mathbf{x})$, termed **shape functions**:

$$u_i(\mathbf{x}) = \sum_{k=1}^{n} \alpha_{ik} \phi_{ik}(\mathbf{x}) \qquad i=1,2,3$$

in matrix form:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \phi_{11} & \cdots & \phi_{1n} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \phi_{21} & \cdots & \phi_{2n} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \phi_{31} & \cdots & \phi_{3n} \end{bmatrix}}_{N_{iK}} \underbrace{\begin{bmatrix} d_{11} \\ d_{12} \\ \vdots \\ d_{3n} \end{bmatrix}}_{d_K}$$

$$u_i = \sum_{A=1}^{3n} N_{iA} d_A \,, \qquad i=1,2,3$$

Let's determine stress and strain distribution according to that approximation

- true strains: $\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) = \frac{1}{2} \sum_{A=1}^{3N} \left(N_{iA,j} + N_{jA,i}\right) d_A$
- true stresses: $\sigma_{ij} = \mathsf{S}_{ijpq} \varepsilon_{pq} = = \frac{1}{2} \mathsf{S}_{ijpq} \sum_{A=1}^{3N} \left(N_{pA,q} + N_{qA,p}\right) d_A$
- virtual displacement: $\delta u_i = \sum_{B=1}^{3N} N_{iB} \delta d_B$
- virtual strain: $\delta \varepsilon_{ij} = \frac{1}{2}\left(\delta u_{i,j} + \delta u_{j,i}\right) = \frac{1}{2} \sum_{B=1}^{3N} \left(N_{iB,j} + N_{jB,i}\right) \delta d_B$

Principle of virtual displacements:

$$\iint_{S_q} q_i \delta u_i \, \mathrm{d}S + \iiint_V b_i \delta u_i \, \mathrm{d}V = \iiint_V \sigma_{ij} \delta \varepsilon_{ij} \, \mathrm{d}V$$

$$\iint_{S_q} \left[q_i \sum_{B=1}^{3N} N_{iB} \delta d_B \right] \mathrm{d}S + \iiint_V \left[b_i \sum_{B=1}^{3N} N_{iB} \delta d_B \right] \mathrm{d}V =$$
$$= \iiint_V \frac{1}{2} S_{ijpq} \sum_{A=1}^{3N} \left(N_{pA,q} + N_{A,p} \right) d_A \cdot \frac{1}{2} \sum_{B=1}^{3N} \left(N_{iB,j} + N_{jB,i} \right) \delta d_B \, \mathrm{d}V$$

$$\sum_{B=1}^{3N} \left[\left(\iint_{S_q} q_i N_{iB} \, \mathrm{d}S + \iiint_V b_i N_{iB} \, \mathrm{d}V \right) \delta d_B \right] =$$
$$= \sum_{A=1}^{3N} \sum_{B=1}^{3N} \left[\left(\iiint_V \frac{1}{2} S_{ijpq} \left(N_{pA,q} + N_{qA,p} \right) \left(N_{iB,j} + N_{jB,i} \right) \mathrm{d}V \right) d_A \delta d_B \right]$$

Let's denote:

- **load vector:**

$$f_B = \sum_{B=1}^{3N} \left[\left(\iint_{S_q} q_i N_{iB} \, \mathrm{d}S + \iiint_V b_i N_{iB} \, \mathrm{d}V \right) \right]$$

- **stiffness matrix:**

$$K_{BA} = \sum_{B=1}^{3N} \left[\left(\iiint_V \frac{1}{2} S_{ijpq} \left(N_{pA,q} + N_{qA,p} \right) \left(N_{iB,j} + N_{jB,i} \right) \mathrm{d}V \right) \right]$$

The principle may be written as:

$$\sum_{A=1}^{3N} K_{BA} \cdot d_A \cdot \delta d_B = f_B \cdot \delta d_B \qquad \forall \delta d_B$$

As it must hold for any δq_B, then:

$$\sum_{A=1}^{3N} K_{BA} \cdot d_A = f_B \qquad B = 1, \ldots, 3N$$

in matrix form:

$$\mathbf{K} \cdot \mathbf{d} = \mathbf{f}$$

where:

$\mathbf{K}$ – stiffness matrix
$\mathbf{d}$ – displacement vector
$\mathbf{f}$ – load vector

Whis means that we have to solve a system of linear algebraic equations for q_A, with coefficients K_{BA} and right hand sides f_B to obtain an approximate solution to the problem of theory of elasticity.

LAGRANGE THEOREM

Let's introduce a quatity temrs **total potential energy**:

$$\Pi = \Phi - L = \frac{1}{2}\iiint_V \sigma_{ij}\varepsilon_{ij}\,\mathrm{d}V - \left[\iiint_V b_i u_i\,\mathrm{d}V + \iint_S q_i u_i\,\mathrm{d}S\right]$$

It may be considered a functional (a function in which a variable argument in a function) of displacement field:

$$J[\mathbf{u}] = \iiint_V \frac{1}{8} C_{ijkl}(u_{k,l}+u_{l,k})(u_{i,j}+u_{j,i})\,\mathrm{d}V - \left[\iiint_V b_i u_i\,\mathrm{d}V + \iint_S q_i u_i\,\mathrm{d}S\right]$$

We will call it **Lagrange functional**. Its first variation may be calculated as follows:

$$\delta J = \frac{\mathrm{d}}{\mathrm{d}\alpha} J[\mathbf{u}+\alpha\delta\mathbf{u}]\Big|_{\alpha=0} =$$

$$= \frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{1}{2}\iiint_V \frac{1}{4} C_{ijkl}\left[(u_k+\alpha\delta u_k)_{,l}+(u_l+\alpha\delta u_l)_{,k}\right]\left[(u_i+\alpha\delta u_i)_{,j}+(u_j+\alpha\delta u_j)_{,i}\right]\mathrm{d}V\Big|_{\alpha=0} -$$
$$- \frac{\mathrm{d}}{\mathrm{d}\alpha}\left[\iiint_V b_i(u_i+\alpha\delta u_i)\,\mathrm{d}V + \iint_S q_i(u_i+\alpha\delta u_i)\,\mathrm{d}S\right]\Big|_{\alpha=0} =$$

$$= \frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{1}{2}\iiint_V \frac{1}{4} C_{ijkl}\left[(u_{k,l}+u_{l,k}) + \alpha(\delta u_{k,l}+\delta u_{l,k})\right]\left[(u_{i,j}+u_{j,i}) + \alpha(\delta u_{i,j}+\delta u_{j,i})\right]\mathrm{d}V\Big|_{\alpha=0} -$$
$$- \frac{\mathrm{d}}{\mathrm{d}\alpha}\left[\iiint_V (b_i u_i+\alpha b_i\delta u_i)\,\mathrm{d}V + \iint_S (q_i u_i+\alpha q_i\delta u_i)\,\mathrm{d}S\right]\Big|_{\alpha=0} =$$

$$= \frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{1}{2}\iiint_V C_{ijkl}\left[\varepsilon_{kl}+\alpha\delta\varepsilon_{kl}\right]\left[\varepsilon_{ij}+\alpha\delta\varepsilon_{ij}\right]\mathrm{d}V\Big|_{\alpha=0} - \left[\iiint_V b_i\delta u_i\,\mathrm{d}V + \iint_S q_i\delta u_i\,\mathrm{d}S\right] =$$

$$= \frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{1}{2}\iiint_V C_{ijkl}\left[\varepsilon_{kl}\varepsilon_{ij}+\alpha(\delta\varepsilon_{kl}\varepsilon_{ij}+\varepsilon_{kl}\delta\varepsilon_{ij})+\alpha^2\delta\varepsilon_{kl}\delta\varepsilon_{ij}\right]\mathrm{d}V\Big|_{\alpha=0} - \left[\iiint_V b_i\delta u_i\,\mathrm{d}V + \iint_S q_i\delta u_i\,\mathrm{d}S\right] =$$

$$= \frac{1}{2}\iiint_V C_{ijkl}(\delta\varepsilon_{kl}\varepsilon_{ij}+\varepsilon_{kl}\delta\varepsilon_{ij})\,\mathrm{d}V - \left[\iiint_V b_i\delta u_i\,\mathrm{d}V + \iint_S q_i\delta u_i\,\mathrm{d}S\right]$$

Due to symmetry of stiffness tensor:

$$\delta J = \iiint_V C_{ijkl}\varepsilon_{kl}\delta\varepsilon_{ij}\,\mathrm{d}V - \left[\iiint_V b_i\delta u_i\,\mathrm{d}V + \iint_S q_i\delta u_i\,\mathrm{d}S\right] =$$
$$= \iiint_V \sigma_{ij}\delta\varepsilon_{ij}\,\mathrm{d}V - \left[\iiint_V b_i\delta u_i\,\mathrm{d}V + \iint_S q_i\delta u_i\,\mathrm{d}S\right]$$

According to the principle of virtual displacements this expression is equal for any δu_i , so the first variation of Lagrange functional has a stationary point there and it may take extremal value, the largest or the smallest, depending on the value of the second variation:

$$\delta^2 J = \frac{\mathrm{d}^2}{\mathrm{d}\alpha^2} J[\mathbf{u}+\alpha\delta\mathbf{u}]\Big|_{\alpha=0} = \iiint_V \mathsf{S}_{ijkl}\,\delta\varepsilon_{ij}\,\delta\varepsilon_{kl}\,\mathrm{d}V$$

Since stiffness tensor in a positive definite tensor (strain energy must be positive)then $\delta^2 J > 0$, so **Lagrange functional has a minimal value**. It may be concluded that:

> **LAGRANGE THEOREM**
>
> *Among all* ***kinematically admissible displacement fields*** *in a linear elastic solid, the* ***true one is the only one for which total potential energy is minimal.***

It means that providing a minimum value for potential energy is equivalent to satisfying equilibrium equations and static boundary conditions. **Lagrange theorem hold only for linear elastic solids** (Hooke's material).

Similarly to the above considerations further theorems may be proved:

> **PRINCIPLE OF VIRTUAL STRESSES**
>
> ***A necessary and sufficient condition*** *for a* ***statically admissible stress field*** *to be a* ***true stress distribution*** *is that* ***a work of virtual loads on true displacements is equal to work of virtual stresses on true strains for any virtual stress*** *and corresponding virtual load*:
>
> $$\iint_S \delta q_i u_i\,\mathrm{d}S = \iiint_V \delta\sigma_{ij}\varepsilon_{ij}\,\mathrm{d}V \qquad \forall\,\delta\boldsymbol{\sigma}$$

> **CASTIGLIANO THEOREM**
>
> *Among all* ***statically admissible stress fields*** *in an linear elastic solid, the* ***true one is the only one for which total complementary energy*** *(Castigliano functional),* ***defined as***
>
> $$\Psi = \frac{1}{2}\iiint_V \sigma_{ij}\varepsilon_{ij}\,\mathrm{d}V - \iint_S q_i u_i\,\mathrm{d}S \ ,$$
>
> **is minimal**.

BETTI-MAXWELL RECIPROCAL WORK THEOREM

Let's consider an elastic body of volume V and boundary S. Let it be supported on the part of boundary denoted with S_u. Let's consider **two systems of external load**:

- **system 1:** $q_i^{(1)}$, $b_i^{(1)}$
- **system 2:** $q_i^{(2)}$, $b_i^{(2)}$

Each system yields a different solution: $u_i^{(1)}$, $\varepsilon_{ij}^{(1)}$, $\sigma_{ij}^{(1)}$ and $u_i^{(2)}$, $\varepsilon_{ij}^{(2)}$, $\sigma_{ij}^{(2)}$. Let's write down the equilibrium equations for both cases:

$$\sigma_{ij,j}^{(1)}+b_i^{(1)}=0$$
$$\sigma_{ij,j}^{(2)}+b_i^{(2)}=0$$

Let's calculate a dot product of both above expressions with displacements of the other system, and integrate it over the whole body:

$$\iiint_V \sigma_{ij,j}^{(1)} u_i^{(2)} + b_i^{(1)} u_i^{(2)} \, \mathrm{d}V = 0$$
$$\iiint_V \sigma_{ij,j}^{(2)} u_i^{(1)} + b_i^{(2)} u_i^{(1)} \, \mathrm{d}V = 0$$

Both expressions are equal to 0 since equilibrium equations hold true. Let's equate both expressions and transform according to the formula for derivative of a product of functions:

$$\iiint_V \sigma_{ij,j}^{(1)} u_i^{(2)} + b_i^{(1)} u_i^{(2)} \, \mathrm{d}V = \iiint_V \sigma_{ij,j}^{(2)} u_i^{(1)} + b_i^{(2)} u_i^{(1)} \, \mathrm{d}V$$

$$\iiint_V \left(\sigma_{ij}^{(1)} u_i^{(2)}\right)_{,j} - \sigma_{ij}^{(1)} u_{i,j}^{(2)} + b_i^{(1)} u_i^{(2)} \, \mathrm{d}V = \iiint_V \left(\sigma_{ij}^{(2)} u_i^{(1)}\right)_{,j} - \sigma_{ij}^{(2)} u_{i,j}^{(1)} + b_i^{(2)} u_i^{(1)} \, \mathrm{d}V$$

Making the use of symmetry of stress tensor and divergence theorem we may write:

$$\iint_S \sigma_{ij}^{(1)} n_j u_i^{(2)} \, \mathrm{d}S - \iiint_V \sigma_{ij}^{(1)} \frac{u_{i,j}^{(2)} + u_{j,i}^{(2)}}{2} \, \mathrm{d}V + \iiint_V b_i^{(1)} u_i^{(2)} \, \mathrm{d}V =$$
$$\iint_S \sigma_{ij}^{(2)} n_j u_i^{(1)} \, \mathrm{d}S - \iiint_V \sigma_{ij}^{(2)} \frac{u_{i,j}^{(1)} + u_{j,i}^{(1)}}{2} \, \mathrm{d}V + \iiint_V b_i^{(2)} u_i^{(1)} \, \mathrm{d}V$$

Since stress fields $\sigma_{ij}^{(1)}$, $\sigma_{ij}^{(2)}$ Are the solutions of the problem, then they must satisfy the static boundary conditions. Similarly, since $u_i^{(1)}$, $u_i^{(2)}$ are the true solutions, they must satisfy the geometric relations. We may write:

$$\iint_S q_i^{(1)} u_i^{(2)} \, \mathrm{d}S - \iiint_V \sigma_{ij}^{(1)} \varepsilon_{ij}^{(2)} \, \mathrm{d}V + \iiint_V b_i^{(1)} u_i^{(2)} \, \mathrm{d}V =$$
$$\iint_S q_i^{(2)} u_i^{(1)} \, \mathrm{d}S - \iiint_V \sigma_{ij}^{(2)} \varepsilon_{ij}^{(1)} \, \mathrm{d}V + \iiint_V b_i^{(2)} u_i^{(1)} \, \mathrm{d}V$$

It may be written in a different form:

$$\iint_S q_i^{(1)} u_i^{(2)} \,\mathrm{d}S + \iiint_V b_i^{(1)} u_i^{(2)} \,\mathrm{d}V + \Theta = \iint_S q_i^{(2)} u_i^{(1)} \,\mathrm{d}S + \iiint_V b_i^{(2)} u_i^{(1)} \,\mathrm{d}V$$

where:

$$\Theta = \iiint_V \left(\sigma_{ij}^{(2)} \varepsilon_{ij}^{(1)} - \sigma_{ij}^{(1)} \varepsilon_{ij}^{(2)} \right) \mathrm{d}V = \iiint_V \mathsf{S}_{ijkl} \left(\varepsilon_{kl}^{(2)} \varepsilon_{ij}^{(1)} - \varepsilon_{kl}^{(1)} \varepsilon_{ij}^{(2)} \right) \mathrm{d}V$$

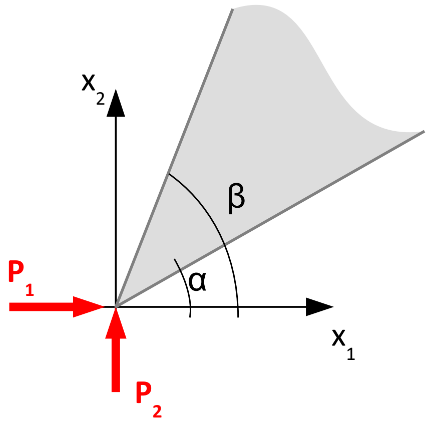
Since $\mathsf{S}_{ijkl} = \mathsf{S}_{klij}$, then $\Theta = 0$. So we may write:

BETTI-MAXWELL RECIPROCAL WORK THEOREM

*If a given **linear elastic solid** is a subject to two different systems of external loads, then **work of the first system of loads on displacements caused by the second system is equal to work of the second system of loads on displacements caused by the first system**:*

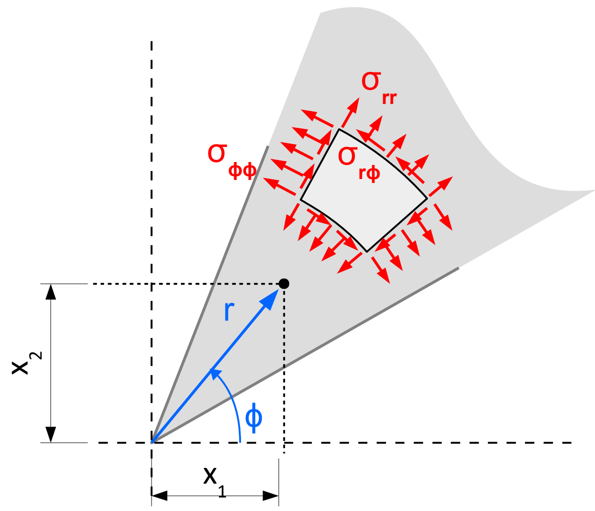
$$\iint_S q_i^{(1)} u_i^{(2)} \,\mathrm{d}S + \iiint_V b_i^{(1)} u_i^{(2)} \,\mathrm{d}V = \iint_S q_i^{(2)} u_i^{(1)} \,\mathrm{d}S + \iiint_V b_i^{(2)} u_i^{(1)} \,\mathrm{d}V$$

FLAMANT PROBLEM



The **Flamant problem** (or the **problem of an elastic wedge**) in the linear theory of elasticity is stated as follows: find the stress state, strain state and displacement field in an infinite wedge of unit thickness, made of weightless (no body forces), homogeneous, isotropic linear elastic material (Hooke's material), loaded at its tip with a point force. Let's assume that the plane of the wedge is $(x_1; x_2)$, and the origin of the coordinate system is at the wedge's tip. Point force is $\mathbf{P}=[P_1; P_2]$ $\mathrm{kN/m}$. The most convenient way of solving this problem is to use the polar coordinates:

$$\begin{cases} r=\sqrt{x_1^2+x_2^2} \\ \phi=\operatorname{arctg}\dfrac{x_2}{x_1} \end{cases} \Leftrightarrow \begin{cases} x_1=r\cos\phi \\ x_2=r\sin\phi \end{cases}$$



Governing equations are as follows

Equations of motion:

$$\sigma_{rr,r}+\frac{1}{r}\sigma_{r\phi,\phi}+\frac{\sigma_{rr}-\sigma_{\phi\phi}}{r}=0$$

$$\sigma_{r\phi,r}+\frac{1}{r}\sigma_{\phi\phi,\phi}+\frac{2}{r}\sigma_{r\phi}=0$$

Geometric relations:

$$\varepsilon_{rr}=u_{r,r}$$

$$\varepsilon_{\phi\phi}=\frac{1}{r}u_{\phi,\phi}+\frac{u_r}{r}$$

$$\varepsilon_{r\phi}=\frac{1}{2}\left(u_{\phi,r}-\frac{u_\phi}{r}+\frac{1}{r}u_{r,\phi}\right)$$

Physical relations:

$$\sigma_{rr}=(2G+\lambda)\varepsilon_{rr}+\lambda\varepsilon_{\phi\phi}$$

$$\sigma_{\phi\phi}=(2G+\lambda)\varepsilon_{\phi\phi}+\lambda\varepsilon_{rr}$$

$$\sigma_{r\phi}=2G\varepsilon_{r\phi}$$

Static boundary conditions:

- $\sigma_{\phi\phi}(r,\alpha)=0$ – no normal load at bottom boundary of the wedge
- $\sigma_{r\phi}(r,\alpha)=0$ – no tangential load at bottom boundary of the wedge
- $\sigma_{\phi\phi}(r,\beta)=0$ – no normal load at top boundary of the wedge
- $\sigma_{r\phi}(r,\beta)=0$ – no tangential load at top boundary of the wedge

We may notice that all boundary conditions are satisfied if

$$\sigma_{\phi\phi} \equiv 0 \quad \text{and} \quad \sigma_{r\phi} \equiv 0$$

Assuming that the above relations are true, equilibrium equations take form:

$$\sigma_{rr,r} + \frac{\sigma_{rr}}{r} = 0$$

The solution to the above equation is as follows:

$$\sigma_{rr}(r;\phi) = \frac{C(\phi)}{r}$$

where $C(\phi)$ is an unknown function. The stress state described as above satisfy the equilibrium equations and boundary conditions. If it also satisfies the compatibility equations (so that the strai corresponding to this stress state can be integrated to obtain displacement field) it will be a solution to the problem. Strain compatibility equations expressed in terms of stress components may be expressed as follows:

$$\Delta(\sigma_{11} + \sigma_{22}) = 0$$

Expressing it in the polar coordinates and making use of the assumptions on the form of stress state, the condition takes form:

$$\Delta \sigma_{rr} = 0$$

This is the **Laplace** (**harmonic**) **equation**, which in polar coordinates has the following form:

$$\sigma_{rr,rr} + \frac{1}{r}\sigma_{rr,r} + \frac{1}{r^2}\sigma_{rr,\phi\phi} = 0$$

Substituting the assumed form of σ_{rr} gives us

$$\frac{2}{r^3}C - \frac{1}{r^3}C + \frac{1}{r^3}\frac{\mathrm{d}^2 C}{\mathrm{d}\phi^2} = 0 \quad ,$$

what is equivalent to:

$$\frac{\mathrm{d}^2 C}{\mathrm{d}\phi^2} + C = 0$$

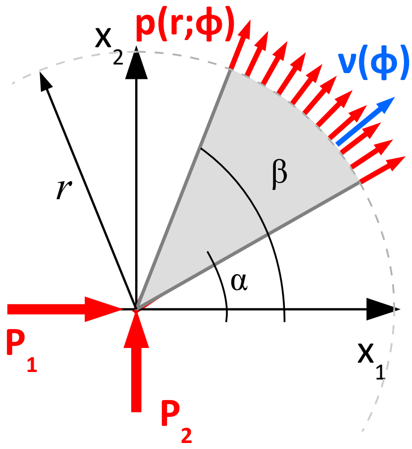
This is an ordinary differential equation for $C(\phi)$. Its solution is:

$$C(\phi) = C_1 \cos\phi + C_2 \sin\phi \quad .$$

Finally the distribution of radial normal stress is given by the following function:

$$\sigma_{rr}(r;\phi) = \frac{C_1}{r}\cos\phi + \frac{C_2}{r}\sin\phi$$

Constants of integration C_1, C_2 will be found according to the applied load. Let us consider an equilibrium of a separated part of the wedge – it is loaded at the tip with a point force, and at the imaginary circular edge of cut with a system of radial stresses. Those stresses may we expressed with the well-known relation between stress tensor and stress vector



$$\sigma_{ij}\nu_j = p_i$$

Let's assume that the external unit normal is aligned with the radial direction:

$$\mathbf{p}(r;\phi) = \sigma_{rr}(r;\phi)\cdot\boldsymbol{\nu}(\phi) \quad .$$

The stress vector may be expressed in the Cartesian coordinate as follows:

$$\mathbf{p} = [\sigma_{rr}\cos\phi \; ; \; \sigma_{rr}\sin\phi]$$

We shall now assume that this separated part is in equilibrium so the vector sum of all applied loads must be equal 0:

$$S_1 = P_1 + \int_K p_1 \, d\, s = P_1 + \int_{\phi=\alpha}^{\beta} \sigma_{rr}\cos\phi \, r \, d\phi = P_1 + \int_{\alpha}^{\beta} r\cos\phi\left[\frac{C_1}{r}\cos\phi \, d\,\phi + \frac{C_2}{r}\sin\phi\right] d\phi =$$
$$= P_1 + C_1\int_{\alpha}^{\beta}\cos^2\phi \, d\,\phi + C_2\int_{\alpha}^{\beta}\sin\phi\cos\phi \, d\,\phi =$$
$$= P_1 + \frac{C_1}{4}\left[2(\beta-\alpha) + (\sin 2\beta - \sin 2\alpha)\right] + \frac{C_2}{4}\left[\cos 2\alpha - \cos 2\beta\right] = 0$$

$$S_2 = P_2 + \int_K p_2 \, d\, s = P_2 + \int_{\phi=\alpha}^{\beta} \sigma_{rr}\sin\phi \, r \, d\phi = P_2 + \int_{\alpha}^{\beta} r\sin\phi\left[\frac{C_1}{r}\cos\phi \, d\phi + \frac{C_2}{r}\sin\phi\right] d\,\phi =$$
$$= P_2 + C_1\int_{\alpha}^{\beta}\cos\phi\sin\phi \, d\,\phi + C_2\int_{\alpha}^{\beta}\sin^2\phi \, d\,\phi =$$
$$= P_2 + \frac{C_1}{4}\left[\cos 2\alpha - \cos 2\beta\right] + \frac{C_2}{4}\left[2(\beta-\alpha) + (\sin 2\alpha - \sin 2\beta)\right] = 0$$

This gives us a system of two equations for constants C_1, C_2 .

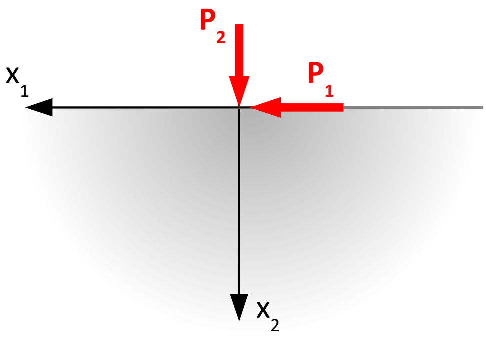
The solution is as follows:

$$C_1 = \frac{2P_1[\sin 2\beta - \sin 2\alpha - 2(\beta-\alpha)] + 2P_2[\cos 2\alpha - \cos 2\beta]}{\cos(2\beta - 2\alpha) + 2(\beta-\alpha)^2 - 1}$$

$$C_2 = \frac{2P_1[\cos 2\beta - \cos 2\alpha] + 2P_2[\sin 2\alpha - \sin 2\beta - 2(\beta-\alpha)]}{\cos(2\beta - 2\alpha) + 2(\beta-\alpha)^2 - 1}$$

HALF-PLANE LOADED WITH POINT FORCE

If we take $\alpha = 0$, $\beta = \pi$, then the wedge is transformed into a haelf-plane loaded with point force $\mathbf{P} = [P_1 ; P_2]$. The solution is as follows:



$$C_1 = -\frac{2P_1}{\pi}, \quad C_2 = -\frac{2P_2}{\pi}$$

$$\sigma_{rr}(r, \phi) = -\frac{2}{\pi r}[P_1 \cos\phi + P_2 \sin\phi]$$

Maxing the use of relations

$$\cos\phi = \frac{x_1}{r} = \frac{x_1}{\sqrt{x_1^2 + x_2^2}} \quad , \quad \sin\phi = \frac{x_2}{r} = \frac{x_2}{\sqrt{x_1^2 + x_2^2}}$$

we can express the **stress** in cartesian coordinate system according to the transformation formulae:

$$\sigma_{11} = \sigma_{rr}\cos^2\phi = = -\frac{2x_1^2}{\pi(x_1^2 + x_2^2)^2}[P_1 x_1 + P_2 x_2]$$

$$\sigma_{22} = \sigma_{rr}\sin^2\phi = = -\frac{2x_2^2}{\pi(x_1^2 + x_2^2)^2}[P_1 x_1 + P_2 x_2]$$

$$\sigma_{12} = \sigma_{rr}\sin\phi\cos\phi = = -\frac{2x_1 x_2}{\pi(x_1^2 + x_2^2)^2}[P_1 x_1 + P_2 x_2]$$

The **strain** is determined with the use of constitutive relations:

$$\varepsilon_{11} = -\frac{2(P_1 x_1 + P_2 x_2)}{\pi \hat{E}(x_1^2 + x_2^2)^2}[(1+\hat{\nu})x_1^2 - \hat{\nu}(x_1^2 + x_2^2)]$$

$$\varepsilon_{22} = -\frac{2(P_1 x_1 + P_2 x_2)}{\pi \hat{E}(x_1^2 + x_2^2)^2}[(1+\hat{\nu})x_2^2 - \hat{\nu}(x_1^2 + x_2^2)]$$

$$\varepsilon_{12} = -\frac{2(1+\hat{\nu})x_1 x_2 (P_1 x_1 + P_2 x_2)}{\pi \hat{E}(x_1^2 + x_2^2)^2}$$

where $\hat{E}, \hat{\nu}$ are generalized elastic constants:

$$\begin{cases} \hat{E} = E & \Leftrightarrow \text{Plane stress} \\ \hat{E} = \dfrac{E}{1-\nu^2} & \Leftrightarrow \text{Plane strain} \end{cases} \quad \text{and} \quad \begin{cases} \hat{\nu} = \nu & \Leftrightarrow \text{Plane stress} \\ \hat{\nu} = \dfrac{\nu}{1-\nu} & \Leftrightarrow \text{Plane strain} \end{cases}$$

Displacement can be determined by integration of strain:

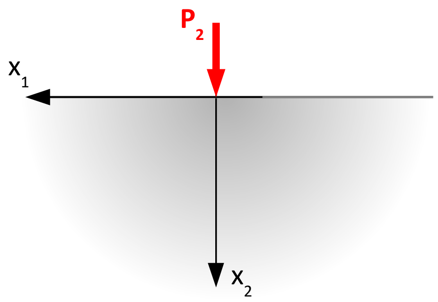
$$u_1 = \int \varepsilon_{11} \mathrm{d} x_1 + D_1(x_2) =$$
$$= -\frac{\left[\left((x_1^2+x_2^2)\ln(x_1^2+x_2^2)+(1+\hat{\nu})x_2^2\right)P_1-\left((1+\hat{\nu})x_1x_2-(1-\hat{\nu})(x_1^2+x_2^2)\operatorname{arctg}\dfrac{x_1}{x_2}\right)P_2\right]}{\pi\hat{E}(x_1^2+x_2^2)}+D_1(x_2)$$

$$u_2 = \int \varepsilon_{22} \mathrm{d} x_2 + D_2(x_1) =$$
$$= -\frac{\left[\left((x_1^2+x_2^2)\ln(x_1^2+x_2^2)+(1+\hat{\nu})x_1^2\right)P_2-\left((1+\hat{\nu})x_1x_2-(1-\hat{\nu})(x_1^2+x_2^2)\operatorname{arctg}\dfrac{x_2}{x_1}\right)P_1\right]}{\pi\hat{E}(x_1^2+x_2^2)}+D_2(x_1)$$

Functions D_1, D_2 is a general solution of homogeneous system of geometric relations and it describe the motion of the whole half-plane as a rigid body:

$$\begin{aligned} D_1(x_2) &= R x_2 + U_1 \\ D_2(x_1) &= -R x_1 + U_2 \end{aligned} \qquad R,\ U_1,\ U_2 = const.$$

HALF-PLANE LOADED WITH NORMAL FORCE



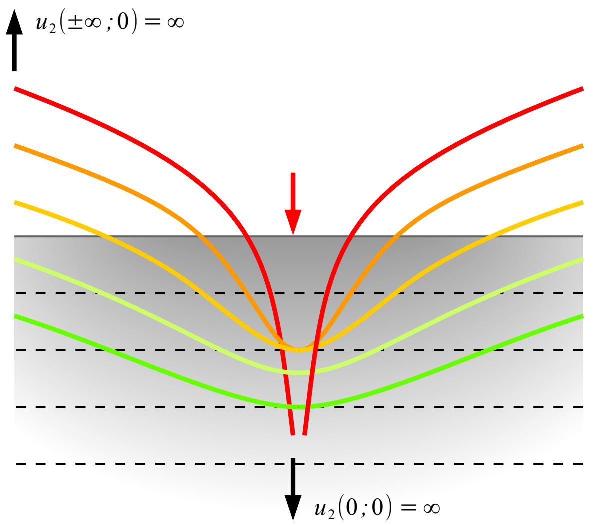
The above solution is of particular importance in case when $P_1 = 0$, namely when half-plane is loaded with a single normal force:

$$u_1 = \frac{P_2}{\pi \hat{E}} \left[(1+\hat{\nu}) \frac{x_1 x_2}{x_1^2 + x_2^2} - (1-\hat{\nu}) \operatorname{arctg} \frac{x_1}{x_2} \right] + R x_2 + U_1$$

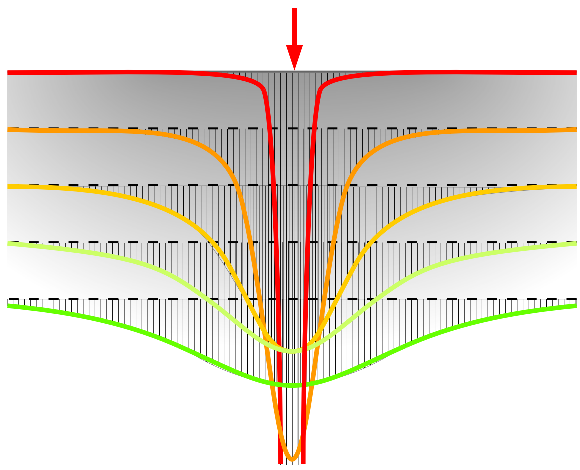
$$u_2 = -\frac{P_2}{\pi \hat{E}} \left[\ln(x_1^2 + x_2^2) + (1+\hat{\nu}) \frac{x_1^2}{x_1^2 + x_2^2} \right] - R x_1 + U_2$$

$$\sigma_{11} = -\frac{2 P_2 x_2 x_1^2}{\pi (x_1^2 + x_2^2)^2} \ , \quad \sigma_{22} = -\frac{2 P_2 x_2^3}{\pi (x_1^2 + x_2^2)^2} \ , \quad \sigma_{12} = -\frac{2 P_2 x_1 x_2^2}{\pi (x_1^2 + x_2^2)^2}$$

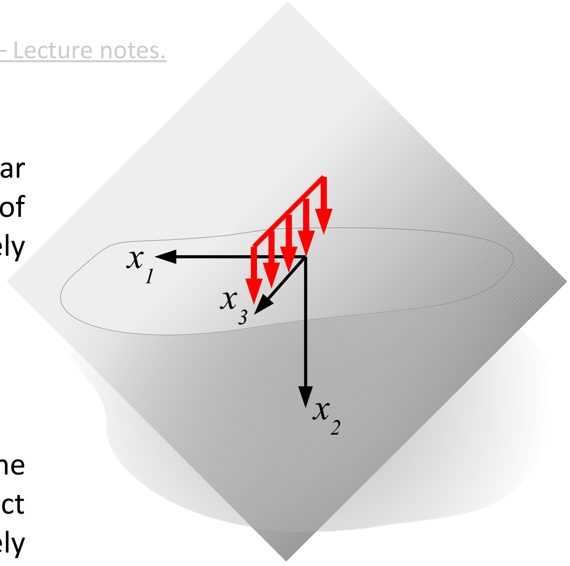
We can see that in the point of application of a force there is a singularity, namely u_2 and σ_{22} (along the direction of load) tend to infinity. The above solution may be used as a Green function (unit impulse function) to be integrated in order to obtain the solution to a problem of a half-plane loaded with a continuously distributed load. Such an integrated solution has finite values of u_2 and σ_{22} .



Vertical displacement u_2



Normal stress σ_{22}



However – unlike in case of integrals of singular solutions in 3D – independently of the choice of constants of integration the displacement of infinitely distant point is still infinite. For $x_1 \to \pm\infty$:

$$\lim_{x_1 \to \pm\infty} u_2 = -\infty$$

It may be explained in such a way, that the Flamant problem considered a plane problem is in fact a spatial problem with infinitely large load (infinitely long linear load) – what results in infinite displacement. Another feature of the discussed solution is the so called **pressure bulb**. Let us consider the radial stress

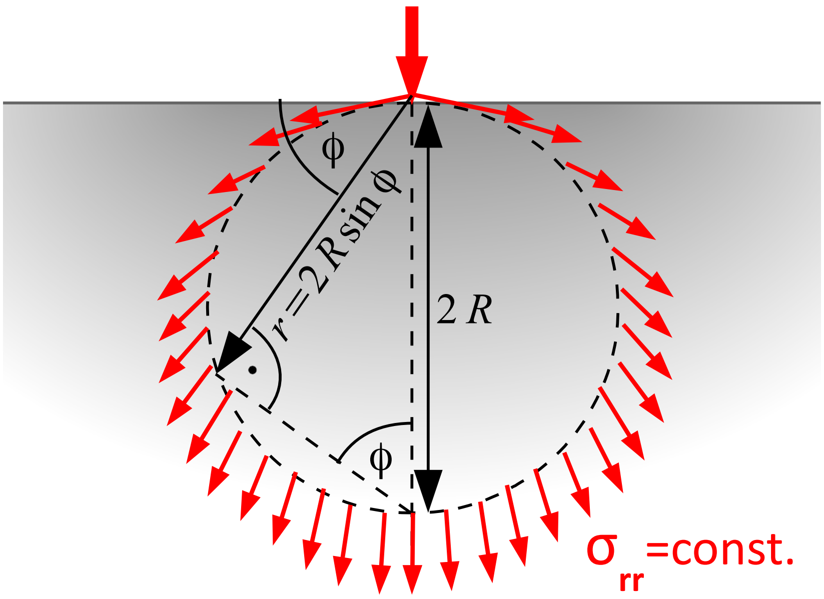
$$\sigma_{rr}(r,\phi) = -\frac{2P_2}{\pi r}\sin\phi$$

along a curve given by:

$$r = 2R\sin\phi \quad \text{where} \quad R = const.$$

It is simply a circle of radius R which is tangent to the edge of half-plane in the point of application of load. It can be easily noticed that the radial stress is constant along the circumference of such a circle (yet, orientation of this stress varies):

$$\sigma_{rr}(r,\phi) = -\frac{P_2}{\pi R}$$



HALF-PLANE LOADED WITH A UNIFORM LOAD

Flamant solution may be used as a Green function to obtain solution of more complex problems. As the considered theory is linear, the principle of superposition is valid – this means that a sum of causes results in a sum of respective results. In particular, an infinite sum may be considered an integral. The problem of a half-plane loaded with a continuous system of uniform load q distributed at interval $x_1 \in \langle -L\ ;\ L \rangle$ may be solved by integrating the Flamant solution:

Vertical displacement:

$$u_2 = -\frac{q}{\pi \hat{E}} \int_{-L}^{L} \left[\ln[(x_1-\xi)^2 + x_2^2] + \frac{(\hat{\nu}+1)(x_1-\xi)^2}{[(x_1-\xi)^2+x_2^2]} \right] d\xi =$$

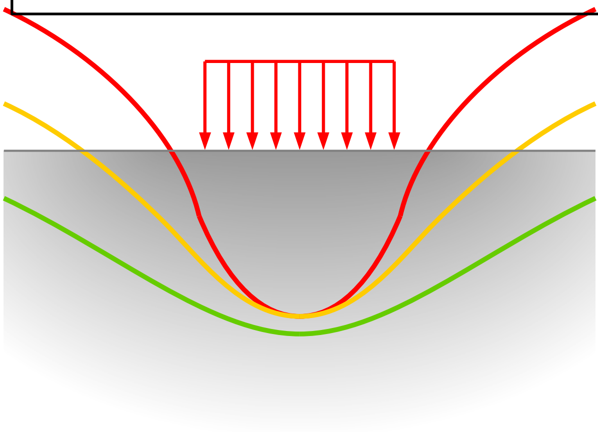
$$= \frac{qL}{\pi \hat{E}} \left\{ \left[\frac{x_1}{L} \ln \left| \frac{(x_1-L)^2+x_2^2}{(x_1+L)^2+x_2^2} \right| \right] - \ln \left| [(x_1-L)^2+x_2^2][(x_1+L)^2+x_2^2] \right| + \dots \right.$$

$$\left. \dots + (1-\hat{\nu}) \left[\frac{x_2}{L} \left(\operatorname{arctg} \frac{x_1-L}{x_2} - \operatorname{arctg} \frac{x_1+L}{x_2} \right) + 2 \right] \right\}$$

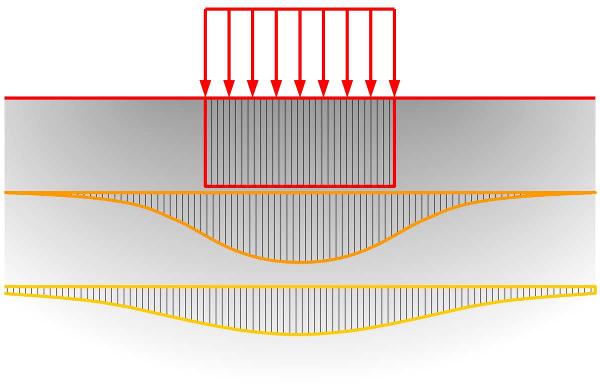
Normal stress:

$$\sigma_{22} = -\frac{2q}{\pi} \int_{-L}^{L} \frac{x_2^3}{((x_1-\xi)^2+x_2^2)^2} d\xi =$$

$$= -\frac{q}{\pi} \left[\frac{(x_1+L)x_2}{(x_1+L)^2+x_2^2} - \frac{(x_1-L)x_2}{(x_1-L)^2+x_2^2} + \operatorname{arctg} \frac{x_1+L}{x_2} - \operatorname{arctg} \frac{x_1-L}{x_2} \right]$$



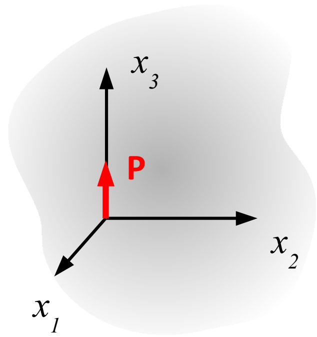
Vertical displacement u_2



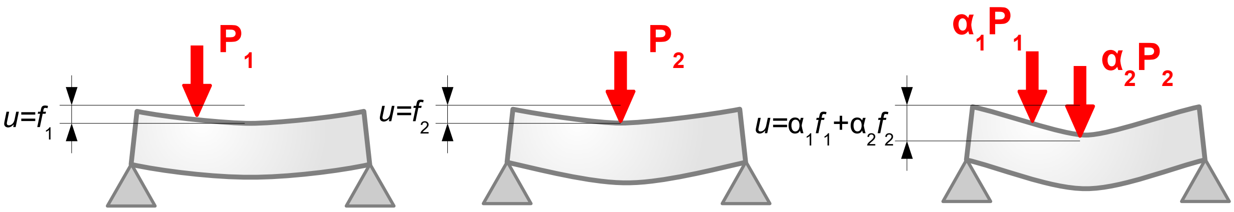
Normal stress σ_{22}

7.1 KELVIN PROBLEM

The **Kelvina problem** in the linear theory of elasticity may be stated as follows: find the stress state, strain state and displacement field in a homogeneous, isotropic linear elastic (Hooke's material) space loaded with a single point force. Let's assume that this force is equal P and the material is characterized by elastic constants λ, G . As the space is considered infinite, no surface tractions are taken into account. We assume that influence of the force on infinitely distant point is negligibly small. This solution is sometimes referred to as the **fundamental solution of the linear theory of elasticity**. As the theory is linear, the **principle of superposition** holds true, **so the result of sum of causes is the sum of results of each of the cuses determined separately**:

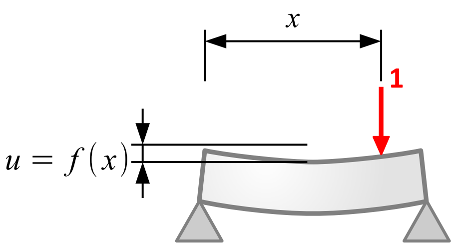


$$f(\alpha_1 \cdot \mathbf{P_1} + \alpha_2 \cdot \mathbf{P_2} + \ldots + \alpha_N \mathbf{P_n}) = \alpha_1 f(\mathbf{P_1}) + \alpha_2 f(\mathbf{P_2}) + \ldots + \alpha_N f(\mathbf{P}_N)$$

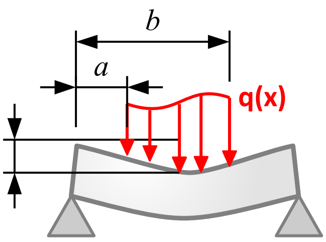


In particular, we may consider an infinite sum which under certain conditions may be calculated as a certain integral:

$$\lim_{\substack{N\to\infty \\ \Delta x_i \to 0}} f\left(\sum_{i=1}^{N} \mathbf{q}(x_i^0) \cdot \Delta x_i\right) = \lim_{\substack{N\to\infty \\ \Delta s_i \to 0}} \sum_{i=1}^{N} q(x_i^0) \cdot f\left(\mathbf{1}(x_i^0)\right) \cdot \Delta x_i = \int q(x) f(x) \mathrm{d}x$$



$$u = \int_a^b q(x) f(x) \mathrm{d}x$$



The solution due to an infinitely small impulse is called a **Green function**. Such an infinitely small impulse component of load is a point force. As it is described by a Dirac distribution, its „finite value" should be considered only in distributional sense:

$$\iiint P \delta_{\mathbf{x}_0} \mathrm{d}\mathbf{x} = P \iiint \delta_{\mathbf{x}_0} \mathrm{d}\mathbf{x} = P \quad .$$

The Kelvin problem may be posed as follows (assuming that the force is applied in the origin of a Cartesian coordinate system):

- **governing equation**: $G\nabla^2 u_i + (\lambda + G) u_{k,ik} + b_i = 0$
- **body force**: $\mathbf{b} = [0\,;0\,;P\,\delta_0]$
- **kinematic boundary conditions**: $\lim\limits_{R\to\infty} u_i = 0, \quad i=1,2,3$
  $\lim\limits_{R\to\infty} u_{i,j} = 0, \quad i,j=1,2,3$,

where $R = \sqrt{x_1^2 + x_2^2 + x_3^2}$

The problem will be solved with the use of the **Fourier integral transform**. In case of one-dimensional function $f(x)$ the transform is defined as follows:

$$\mathcal{F}\{f(x)\}(\omega) = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \mathrm{d}x \quad ,$$

where i is an imaginary unit. Function $\hat{f}(\omega)$ of variable ω is called **transform of function** $f(x)$, while $f(x)$ is called original **of function** $\hat{f}(\omega)$. **Inverse transform** is defined as follows:

$$\mathcal{F}^{-1}\{\hat{f}(\omega)\}(x) = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} \mathrm{d}x$$

It is the matter of chosen convention if „-” sign in the exponent accounted for in the definition of the transform or its inverse. It is also up to our choice how a scaling factor $(2\pi)^{-1}$ is „distributed” between those definitions. The definition of three-dimensional Fourier transform that we will use is as follows:

$$\mathcal{F}\{u_i\} = \hat{u}_i = \int_{\omega_1=-\infty}^{\infty}\left[\int_{\omega_2=-\infty}^{\infty}\left[\int_{\omega_3=-\infty}^{\infty} u_i e^{-i\omega_3 x_3} \mathrm{d}x_3\right] e^{-i\omega_2 x_2} \mathrm{d}x_2\right] e^{-i\omega_1 x_1} \mathrm{d}x_1 =$$
$$= \iiint_{-\infty}^{\infty} u_i e^{-i(x_1\omega_1 + x_2\omega_2 + x_3\omega_3)} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3$$

and its inverse is:

$$u_i = \mathcal{F}^{-1}\{\hat{u}_i\} = \frac{1}{(2\pi)^3}\iiint_{-\infty}^{\infty} \hat{u}_i e^{i(x_1\omega_1 + x_2\omega_2 + x_3\omega_3)} \mathrm{d}\omega_1 \mathrm{d}\omega_2 \mathrm{d}\omega_3$$

Integral operator $\mathscr{F}$ is linear one, so

$$\mathscr{F}\{\alpha_1 u_1+\alpha_2 u_2\} = \alpha_1 \mathscr{F}\{u_1\} + \alpha_2 \mathscr{F}\{u_2\}$$

and it may be applied to each term of displacement equations separately, putting the constants outside brackets:

$$\mathscr{F}\{G\nabla^2 u_i + (\lambda+G) u_{k,ik} + b_i\} = G\mathscr{F}\{\nabla^2 u_i\} + (\lambda+G)\mathscr{F}\{u_{k,ik}\} + \mathscr{F}\{b_i\}$$

Let's find transforms of body force components. Transform of a zero function is 0, while transform of Dirac distribution can be calculated in such a way:

$$\mathscr{F}\{b_3\} = \mathscr{F}\{P\delta_{\mathbf{0}}\} = P\iiint_{-\infty}^{\infty} \delta_{\mathbf{0}} e^{-\mathrm{i}(x_1\omega_1+x_2\omega_2+x_3\omega_3)} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 = P e^{-\mathrm{i}(x_1\omega_1+x_2\omega_2+x_3\omega_3)}\Big|_{\mathbf{x}=\mathbf{0}} = P$$

Let's find the transform of the second derivatives of displacement. Let's start with derivative with respect to x_1, x_2 :

$$\mathscr{F}\{u_{i,12}\} = \iiint_{-\infty}^{\infty} \frac{\partial^2 u_i}{\partial x_1 \partial x_2} e^{-\mathrm{i}(x_1\omega_1+x_2\omega_2+x_3\omega_3)} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3$$

Let's integrate it by parts with respect to x_1 :

$$\mathscr{F}\{u_{i,12}\} = \iiint_{-\infty}^{\infty} \frac{\partial u_i}{\partial x_1 \partial x_2} e^{-\mathrm{i}(x_1\omega_1+x_2\omega_2+x_3\omega_3)} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 =$$
$$= \iint_{-\infty}^{\infty} \left[\left[\frac{\partial u_i}{\partial x_2} e^{-\mathrm{i}(x_1\omega_1+x_2\omega_2+x_3\omega_3)} \right]_{x_1=-\infty}^{\infty} - \int_{x_1=-\infty}^{\infty} \frac{\partial u_i}{\partial x_2} \frac{\partial}{\partial x_1} e^{-\mathrm{i}(x_1\omega_1+x_2\omega_2+x_3\omega_3)} \mathrm{d}x_1 \right] \mathrm{d}x_2 \mathrm{d}x_3$$

In the first expression we shall notice that according to the Euler formula

$$e^{\mathrm{i}\phi} = \cos\phi + \mathrm{i}\sin\phi \quad \text{, where} \quad \phi = -(x_1\omega_1 + x_2\omega_2 + x_3\omega_3) \quad .$$

Due to fact that trigonometric function may take values from interval $\langle -1 ; 1 \rangle$, function $e^{\mathrm{i}\phi}$ is bounded so it has finite value also for $\phi \to \pm\infty$. Simultaneously we've assumed that $\lim_{R\to\infty} u_i = 0$ and $\lim_{R\to\infty} u_{i,j} = 0$, so a product of a finite-valued function and a function tending to 0 will also tend to 0 for $\phi \to \pm\infty$. Let's perform the differentiation

$$\mathscr{F}\{u_{i,12}\} = \iint_{-\infty}^{\infty} \left[-(-\mathrm{i}\omega_1) \int_{x_1=-\infty}^{\infty} \frac{\partial u_i}{\partial x_2} e^{-\mathrm{i}(x_1\omega_1+x_2\omega_2+x_3\omega_3)} \mathrm{d}x_1 \right] \mathrm{d}x_2 \mathrm{d}x_3 =$$
$$= \mathrm{i}\omega_1 \iiint_{-\infty}^{\infty} \frac{\partial u_i}{\partial x_2} e^{-\mathrm{i}(x_1\omega_1+x_2\omega_2+x_3\omega_3)} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3$$

and integrate by parts with respect to x_2 as previously:

$$\mathscr{F}\{u_{i,12}\} = \mathrm{i}\omega_1 \iint_{-\infty}^{\infty} \left[\left[u_i e^{-\mathrm{i}(x_1\omega_1 + x_2\omega_2 + x_3\omega_3)} \right]_{x_2=-\infty}^{\infty} - \int_{x_2=-\infty}^{\infty} u_i \frac{\partial}{\partial x_2} e^{-\mathrm{i}(x_1\omega_1 + x_2\omega_2 + x_3\omega_3)} \mathrm{d}\, x_2 \right] \mathrm{d}\, x_1 \,\mathrm{d}\, x_3 =$$
$$= (\mathrm{i}\omega_1)(\mathrm{i}\,\omega_2) \iiint_{-\infty}^{\infty} u_i e^{-\mathrm{i}(x_1\omega_1 + x_2\omega_2 + x_3\omega_3)} \mathrm{d}\, x_1 \mathrm{d}\, x_2 \mathrm{d}\, x_3 = \mathrm{i}^2 \omega_1 \omega_2 \hat{u}_i = -\omega_1 \omega_2 \hat{u}_i$$

We've made the use of definition of the Fourier transform. It emerges that transform of a derivative may be expressed by the transform of the function itself, multiplied by a certain factor, e.g.:

$$\mathscr{F}\{u_{1,11}\} = (-\mathrm{i}\,\omega_1)^2 \hat{u}_1 = -\omega_1^2 \hat{u}_1$$
$$\mathscr{F}\{\nabla^2 u_2\} = \mathscr{F}\{u_{2,11} + u_{2,22} + u_{2,33}\} = -(\omega_1^2 + \omega_2^2 + \omega_3^2)\hat{u}_2$$

Displacement equations:

$$\begin{cases} G(u_{1,11} + u_{1,22} + u_{1,33}) + (\lambda + G)(u_{1,11} + u_{2,21} + u_{3,31}) = 0 \\ G(u_{2,11} + u_{2,22} + u_{2,33}) + (\lambda + G)(u_{1,12} + u_{2,22} + u_{3,32}) = 0 \\ G(u_{3,11} + u_{3,22} + u_{3,33}) + (\lambda + G)(u_{1,13} + u_{2,23} + u_{3,33}) + P\delta_{\mathbf{0}} = 0 \end{cases}$$

are expressed in the space of transforms in the form of a **linear system of algebraic equations**:

$$\begin{cases} -G(\omega_1^2 + \omega_2^2 + \omega_3^2)\hat{u}_1 - \omega_1(\lambda + G)(\omega_1 \hat{u}_1 + \omega_2 \hat{u}_2 + \omega_3 \hat{u}_3) = 0 \\ -G(\omega_1^2 + \omega_2^2 + \omega_3^2)\hat{u}_2 - \omega_2(\lambda + G)(\omega_1 \hat{u}_1 + \omega_2 \hat{u}_2 + \omega_3 \hat{u}_3) = 0 \\ -G(\omega_1^2 + \omega_2^2 + \omega_3^2)\hat{u}_3 - \omega_3(\lambda + G)(\omega_1 \hat{u}_1 + \omega_2 \hat{u}_2 + \omega_3 \hat{u}_3) = -P \end{cases}$$

In matrix form:

$$\begin{bmatrix} G(\omega_1^2+\omega_2^2+\omega_3^2) + (\lambda+G)\omega_1^2 & (\lambda+G)\omega_1\omega_2 & (\lambda+G)\omega_3\omega_1 \\ (\lambda+G)\omega_1\omega_2 & G(\omega_1^2+\omega_2^2+\omega_3^2) + (\lambda+G)\omega_2^2 & (\lambda+G)\omega_2\omega_3 \\ (\lambda+G)\omega_3\omega_1 & (\lambda+G)\omega_2\omega_3 & G(\omega_1^2+\omega_2^2+\omega_3^2) + (\lambda+G)\omega_3^2 \end{bmatrix} \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ P \end{bmatrix}$$

It is important conclusion – instead of solving a system of differential equation, we have to solve a system of algebraic equations. What's more – it is a linear system, so the solution can be found easily:

$$\hat{u}_1 = -\frac{P(\lambda+G)}{G(\lambda+2G)} \frac{\omega_1\omega_3}{(\omega_1^2+\omega_2^2+\omega_3^2)^2}$$
$$\hat{u}_2 = -\frac{P(\lambda+G)}{G(\lambda+2G)} \frac{\omega_2\omega_3}{(\omega_1^2+\omega_2^2+\omega_3^2)^2}$$
$$\hat{u}_3 = \frac{P}{G} \frac{1}{\omega_1^2+\omega_2^2+\omega_3^2} - \frac{P(\lambda+G)}{G(\lambda+2G)} \frac{\omega_3^2}{(\omega_1^2+\omega_2^2+\omega_3^2)^2}$$

We have to find only two originals now:

$$f_1 = \mathscr{F}^{-1}\left\{\frac{1}{\omega_1^2+\omega_2^2+\omega_3^2}\right\} = ? \quad \text{oraz} \quad f_2 = \mathscr{F}^{-1}\left\{\frac{1}{(\omega_1^2+\omega_2^2+\omega_3^2)^2}\right\} = ?$$

If f_2 is found, then other required originals can be found as derivatives of this one:

$$f_{2,13} = \mathscr{F}^{-1}\left\{-\frac{\omega_1\omega_3}{(\omega_1^2+\omega_2^2+\omega_3^2)^2}\right\},\quad f_{2,23} = \mathscr{F}^{-1}\left\{-\frac{\omega_2\omega_3}{(\omega_1^2+\omega_2^2+\omega_3^2)^2}\right\},\quad f_{2,33} = \mathscr{F}^{-1}\left\{-\frac{\omega_3^2}{(\omega_1^2+\omega_2^2+\omega_3^2)^2}\right\} .$$

We have to calculate following integrals:

$$f_1 = \frac{1}{(2\pi)^3}\iiint_{-\infty}^{\infty}\frac{e^{i(\omega_1 x_1+\omega_2 x_2+\omega_3 x_3)}}{\omega_1^2+\omega_2^2+\omega_3^2}\,d\omega_1\,d\omega_2\,d\omega_3$$

$$f_2 = \frac{1}{(2\pi)^3}\iiint_{-\infty}^{\infty}\frac{e^{i(\omega_1 x_1+\omega_2 x_2+\omega_3 x_3)}}{(\omega_1^2+\omega_2^2+\omega_3^2)^2}\,d\omega_1\,d\omega_2\,d\omega_3$$

It is not an elementary task. Fortunately, we may make use of ready charts involving originals and transforms of certain functions:

$$\mathscr{F}\left\{\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\right\} = 4\pi\frac{1}{\omega_1^2+\omega_2^2+\omega_3^2} \quad\Rightarrow\quad f_1 = \frac{1}{4\pi}\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}$$

$$\mathscr{F}\left\{\sqrt{x_1^2+x_2^2+x_3^2}\right\} = -8\pi\frac{1}{(\omega_1^2+\omega_2^2+\omega_3^2)^2} \quad\Rightarrow\quad f_2 = -\frac{1}{8\pi}\sqrt{x_1^2+x_2^2+x_3^2}$$

So:

$$u_1 = \frac{P(\lambda+G)}{G(\lambda+2G)}f_{2,13}\ ,\quad u_2 = \frac{P(\lambda+G)}{G(\lambda+2G)}f_{2,23}\ ,\quad u_3 = \frac{P}{G}f_1 + \frac{P(\lambda+G)}{G(\lambda+2G)}f_{2,33}$$

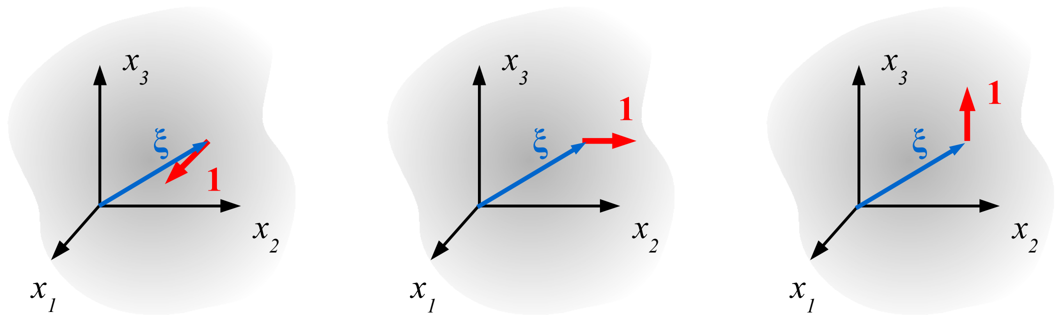
Performing proper differentiations we can write down the **fundamental solution**:

$$u_1 = \frac{P(\lambda+G)}{8\pi G(\lambda+2G)}\frac{x_1 x_3}{\left[x_1^2+x_2^2+x_3^2\right]^{3/2}}$$

$$u_2 = \frac{P(\lambda+G)}{8\pi G(\lambda+2G)}\frac{x_2 x_3}{\left[x_1^2+x_2^2+x_3^2\right]^{3/2}}$$

$$u_3 = \frac{P}{4\pi G}\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}} + \frac{P(\lambda+G)}{8\pi G(\lambda+2G)}\frac{x_3^2}{\left[x_1^2+x_2^2+x_3^2\right]^{3/2}}$$

We may now consider a system of fundamental solutions corresponding to three cases of a unit point force parallel to the one of the axes of coordinate system a applied to the same point $\boldsymbol{\xi}$ (not necessarily the origin of coordinate system):



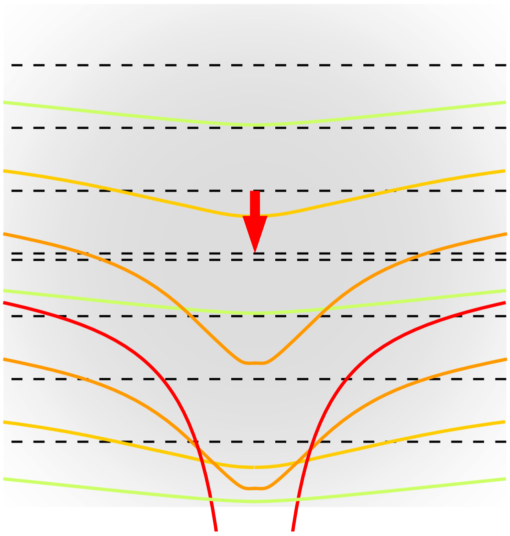
$\mathbf{u}^{(1)}, \boldsymbol{\varepsilon}^{(1)}, \boldsymbol{\sigma}^{(1)}$ $\mathbf{u}^{(2)}, \boldsymbol{\varepsilon}^{(2)}, \boldsymbol{\sigma}^{(2)}$ $\mathbf{u}^{(3)}, \boldsymbol{\varepsilon}^{(3)}, \boldsymbol{\sigma}^{(3)}$

$$u_i^{(k)} = \frac{1}{4\pi G}\left[\frac{1}{\left[(x_n-\xi_n)(x_n-\xi_n)\right]^{1/2}}\delta_{ik} + \frac{1}{4(1-\nu)}\frac{(x_i-\xi_i)(x_k-\xi_k)}{\left[(x_n-\xi_n)(x_n-\xi_n)\right]^{3/2}}\right]$$

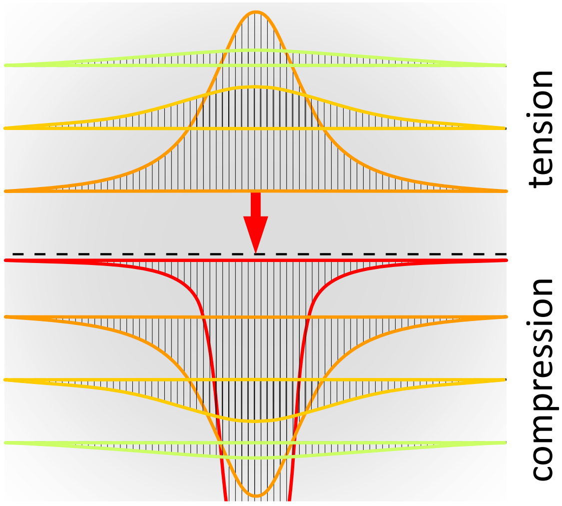
$$\varepsilon_{ij}^{(k)} = -\frac{1}{16\pi G(1-\nu)}\left[\frac{(3-4\nu)}{2}\frac{(x_j-\xi_j)\delta_{ik}+(x_i-\xi_i)\delta_{jk}}{\left[(x_n-\xi_n)(x_n-\xi_n)\right]^{3/2}} - \right.$$
$$\left. - \frac{(x_k-\xi_k)\delta_{ij}}{\left[(x_n-\xi_n)(x_n-\xi_n)\right]^{3/2}} + \frac{3(x_i-\xi_i)(x_j-\xi_j)(x_k-\xi_k)}{\left[(x_n-\xi_n)(x_n-\xi_n)\right]^{5/2}}\right]$$

$$\sigma_{ij}^{(k)} = \frac{\nu(4\nu-3)}{8\pi(1-\nu)(1-2\nu)}\frac{x_k}{\left[(x_n-\xi_n)(x_n-\xi_n)\right]^{3/2}} -$$
$$- \frac{1}{8\pi(1-\nu)}\left[\frac{(3-4\nu)}{2}\frac{(x_j-\xi_j)\delta_{ik}+(x_i-\xi_i)\delta_{jk}}{\left[(x_n-\xi_n)(x_n-\xi_n)\right]^{3/2}} - \right.$$
$$\left. - \frac{(x_k-\xi_k)\delta_{ij}}{\left[(x_n-\xi_n)(x_n-\xi_n)\right]^{3/2}} + \frac{3(x_i-\xi_i)(x_j-\xi_j)(x_k-\xi_k)}{\left[(x_n-\xi_n)(x_n-\xi_n)\right]^{5/2}}\right]$$

Such a system of solution may be used as a Green function to be integrated in order to obtain solutions of more complex problems. A characteristic feature of such unit impulse solution is that in the point of application of the point force there is a singularity – stresses, strains and displacement diverge to infinity. It has no physical meaning, as in fact all load are not point forces but distributed one, however sometimes highly concentrated, and it emerges that an integral of such a singular Green function is no longer singular.



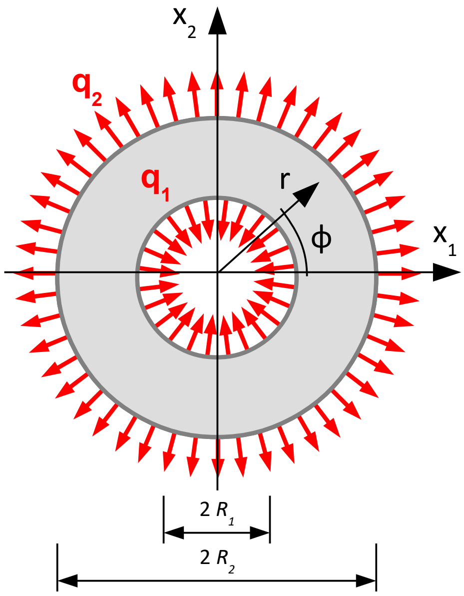
Displacement $u_3^{(3)}$



Stress $\sigma_{33}^{(3)}$

LAMÉ PROBLEM

The **Lamé problem** in linear theory of elasticity stated as follows: find the stress state, strain state and displacement field in a plane circular membrane of radius R_2 with a central opening of radius R_1 . The membrane is weightless (no body forces), made of a homogeneous isotropic linear elastic material (Hooke's material) and it is loaded with a normal load q_1 on external boundary and with a normal load q_2 on internal boundary. Let's assume that it lies in plane $(x_1; x_2)$ and the origin of the coordinate system is in the center of the opening. The most convenient way of solving the problem is to use the polar coordinates and to make use of the axial symmetry as we can notice that all unknown quantities do not vary with the change of angle ϕ and furthermore geometry and character of load indicate that also displacement $u_\phi \equiv 0$. Governing equations are as follows:



Equations of motion: $$\sigma_{rr,r} + \frac{\sigma_{rr} - \sigma_{\phi\phi}}{r} = 0$$

Geometric relations:
$$\varepsilon_{rr} = u_{r,r}$$
$$\varepsilon_{\phi\phi} = \frac{u_r}{r}$$
$$\varepsilon_{r\phi} = 0$$

Physical relations:
$$\sigma_{rr} = (2G+\lambda)\varepsilon_{rr} + \lambda\varepsilon_{\phi\phi}$$
$$\sigma_{\phi\phi} = (2G+\lambda)\varepsilon_{\phi\phi} + \lambda\varepsilon_{rr}$$
$$\sigma_{r\phi} = 0$$

Statical boundary conditions::

- **Internal boundary**: $\sigma_{rr}(R_1) = q_1$
- **External boundary**: $\sigma_{rr}(R_2) = q_2$

Due to axial symmetry we may make the use of displacement equations which for axial symmetric problems in polar coordinates take form:

$$u_{r,rr} + \frac{u_{r,r}}{r} - \frac{u_r}{r^2} = 0$$

This is so called **Euler differential equation** and its general solution is known to be of the following form:

$$u_r(r) = C_1 r + \frac{C_2}{r}$$

Strain and stress state may be thus found:

$$\varepsilon_{rr}(r) = C_1 - \frac{C_2}{r^2} \quad , \quad \varepsilon_{\phi\phi} = C_1 + \frac{C_2}{r^2}$$

$$\sigma_{rr}(r) = 2C_1(G+\lambda) - \frac{2C_2 G}{r^2} \quad , \quad \sigma_{\phi\phi}(r) = 2C_1(G+\lambda) + \frac{2C_2 G}{r^2}$$

Constants of integration are found with the use of **boundary conditions**:

$$\begin{cases} \sigma_{rr}(R_1) = 2C_1(G+\lambda) - \dfrac{2C_2 G}{R_1^2} = q_1 \\ \sigma_{rr}(R_2) = 2C_1(G+\lambda) - \dfrac{2C_2 G}{R_2^2} = q_2 \end{cases} \Rightarrow \begin{cases} C_1 = \dfrac{q_2 R_2^2 - q_1 R_1^2}{(2G+\lambda)(R_2^2 - R_1^2)} \\ C_2 = \dfrac{R_1^2 R_2^2 (q_2 - q_1)}{2G(R_2^2 - R_1^2)} \end{cases}$$

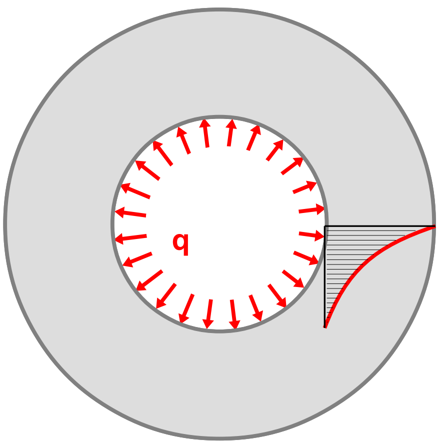
As a result we obtain the **solution of the Lamé problem**:

$$u_r(r) = \frac{q_2 R_2^2 - q_1 R_1^2}{(2G+\lambda)(R_2^2 - R_1^2)} \cdot r + \frac{R_1^2 R_2^2 (q_2 - q_1)}{2G(R_2^2 - R_1^2)} \cdot \frac{1}{r}$$

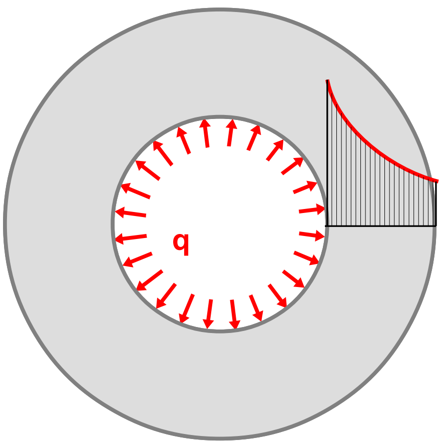
$$\varepsilon_{rr}(r) = \frac{q_2 R_2^2 - q_1 R_1^2}{(2G+\lambda)(R_2^2 - R_1^2)} - \frac{R_1^2 R_2^2 (q_2 - q_1)}{2G(R_2^2 - R_1^2)} \cdot \frac{1}{r^2} \qquad \varepsilon_{\phi\phi}(r) = \frac{q_2 R_2^2 - q_1 R_1^2}{(2G+\lambda)(R_2^2 - R_1^2)} + \frac{R_1^2 R_2^2 (q_2 - q_1)}{2G(R_2^2 - R_1^2)} \cdot \frac{1}{r^2}$$

$$\sigma_{rr}(r) = \frac{q_2 R_2^2 - q_1 R_1^2}{R_2^2 - R_1^2} - \frac{(q_2 - q_1) R_1^2 R_2^2}{R_2^2 - R_1^2} \cdot \frac{1}{r^2} \qquad \sigma_{\phi\phi}(r) = \frac{q_2 R_2^2 - q_1 R_1^2}{R_2^2 - R_1^2} + \frac{(q_2 - q_1) R_1^2 R_2^2}{R_2^2 - R_1^2} \cdot \frac{1}{r^2}$$

A general character of distribution of stress in the case of zero external load (e.g. Pipe with internal pressure) is depicted below:



radial stress σ_{rr}



circumferential stress $\sigma_{\phi\phi}$