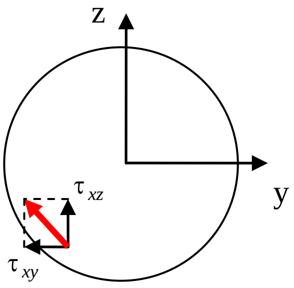
Torsion of bars with circular cross-section

Basis formulae

In the particular case of axial symmetry (a circle, a ring), the distortion function is identically equal to zero and the torsion inertia moment reduces to the polar inertia moment:

$$\phi(y,)z \equiv 0 \quad \rightarrow \quad J_s = J_0$$

It means that the plane cross-section remains a plane after the deformation, it doesn't warp.



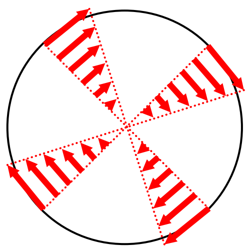


Fig. 10.1. Shear stress distribution

The resultant shear stress is:

$$\tau = \sqrt{\tau_{xy}^2 + \tau_{xz}^2} = G\theta\sqrt{z^2 + y^2} = G\theta r = \frac{M_s}{J_0} r \text{, [MPa].}$$

The shearing stress varies linearly with the distance from the axis of the shaft, Fig. 10.4.

The torsion angle is calculated from the unit torsion angle provided that the torsion moment is constant:

$$\frac{d\alpha}{dx} = \theta = \frac{M_s}{GJ_0} \quad \rightarrow \quad \alpha = \int_l \theta \mathrm{dx} = \frac{M_s l_s}{GJ_0} \text{ [1 = rad]}$$

This is the angle through which one section rotates with respect to the other. Note, that the torsion angle is dimensionless, so the angle unit in the above formula is a radian, not a degree.

When the interval is loaded by a continuous twist moment of intensity $m(x)$, the formulae become:

$$M(x) = \int_0^x m(\xi)d\xi \text{, and} \qquad \alpha(x) = \int_0^x \frac{M(\xi)}{GJ_0} d\xi$$

Design conditions

There are two main requirements:

- the requirement of the strength and
- the requirement of usability.

The first requirement means that every structure has to sustain applied load. The second one consists in several demands of durability, rigidity, resistance to severe weather conditions and so on. From the point of view of the strength of materials course two principal design conditions should be listed:

– the ultimate limit state
– the serviceability limit state, usually the stiffness of the structure.

In the particular case of the torsion, the ultimate limit state is checked by:

$$\max(\tau) \le R_t$$

where R_t is the shear strength of a given material. The suitable value can be found in the standards.

Because the maximum value of the stress is attained at the side surface for $r = R$, we introduce so-called cross-section twist factor or modulus W_0:

$$W_0 \overset{def}{=} \frac{J_0}{R},$$

so:

$$\max(\tau) = \frac{M_0}{J_0} R = \frac{M_0}{W_0}$$

The stiffness condition means that the unit torsion angle or the torsion angle doesn't exceed acceptable (permissible) values:

$$\theta \le \theta_{\text{acceptable}}, \text{ or } \alpha \le \alpha_{\text{acceptable}}$$

Transmission shafts

The principal specifications to be met in the design of a transmission shaft are the power to be transmitted and the speed of rotation of the shaft. The role of the designer is to select the material and the dimensions of the cross-section of the shaft, so that the maximum shearing stress allowable in the material will not be exceeded when the shaft is transmitting the required power at the specified speed.

The power P associated with the rotation of a rigid body subjected to a torque M_0 is:

$$P = M_0 \omega,$$

where ω is the angular velocity of the body expressed in radians per second. But, $\omega = 2\pi f$, where f is the frequency of the rotation, i.e., the number of revolutions per second. We write:

$$P = M_0 2\pi f .$$

Hollow shaft

The linear repartition of the shearing stresses in the twisted circular bar signifies that the outer fibers are the most useful and the central ones are almost not used. Therefore, the use of hollow shafts makes sense, especially in the case where the weight is more important than the price of the element.

The torsion inertia moment for the hollow shaft with outer diameter D and inner diameter d is:

$$J_0 = \frac{\pi D^4}{32} - \frac{\pi d^4}{32} = \frac{\pi}{32}\left(D^4 - d^4\right),$$

and the cross-section torsion factor is:

$$W_0 = \frac{J_0}{D} 2 = \frac{\pi}{16D}\left(D^4 - d^4\right).$$

Examples

Example 10.1

The shaft with the diameter of 4 cm and length of 2 m is fixed at one end and loaded at another end by such torque that the point A on the side surface is displaced to the point A'. The arc AA' length is 1 mm. Determine the torsion angle, the unit torsion angle, the torque, the maximum shear stress and the angular strain on the side surface. G = 80 GPa.

Solution

From the definition of the twist angle we calculate:

$$\alpha = \frac{0.1}{\pi d} 2\pi = \frac{0.2}{4} = 0.05 \text{ [rad]}.$$

For constant torsion moment we have:

$$\theta = \frac{\alpha}{l} = \frac{0.05}{2} = 0.025 \ \text{[rad/m]}.$$

Using the formula of the unit torsion angle we get:

$$M_s = GJ_s\theta = 80 \cdot 10^9 \cdot \frac{\pi 0.04^4}{32} \cdot 0.025 = 500 \ \text{[Nm]}$$

(it's quite significant value, the average value for the VW golf is about 150 Nm). The maximum stress value is:

$$\tau_{\max} = \frac{M_0}{W_0} = \frac{GJ_0\theta}{J_0} R = G\theta\frac{d}{2} = 80 \cdot 10^9 \cdot 0.025 \cdot 0.02 = 40 \cdot 10^6 \ \text{[Pa]} = 40 \ \text{[MPa]}.$$

From the Hooke's equation we determine

$$\gamma = 2\varepsilon = \frac{2\tau_{\max}}{2G} = \theta\frac{d}{2} = 0.025 \cdot 0.02 = 0.0005 \, .$$

Example 10.2

Determine the rotation angle between section A and section E of the shaft in Fig. 1.5, $d_1 = 4$ cm, $d_2 = 3$ cm, G = 80 GPa.

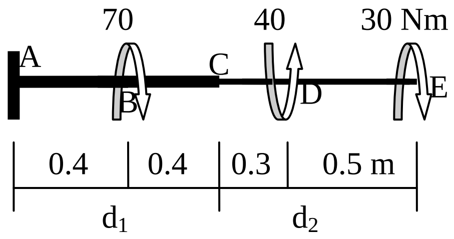


Fig. 10.2 Twisted shaft

Solution

The formula of the twist angle:

$$\alpha = \frac{M_0 l}{G J_0}$$

is valid only in the case of all parameters constant, so we have to cut the shaft into the intervals with constant cross-section inertia moment and constant torque: AB, BC, CD and DE.

We have:

AB: $l_{AB} = 0.4\,\text{m}$, $M_{AB} = 30 - 40 + 70 = 60\,\text{Nm}$, $J_{AB} = \frac{\pi d_1^4}{32} = 2.51 \cdot 10^{-7}\,\text{m}^4$,

$$\alpha_{AB} = \frac{60 \cdot 0.4}{80 \cdot 10^9 \cdot 2.51 \cdot 10^{-7}} = 0.00120\,\text{rad}$$

BC: $l_{BC} = 0.4\,\text{m}$, $M_{BC} = 30 - 40 = -10\,\text{Nm}$, $J_{BC} = \frac{\pi d_1^4}{32} = 2.51 \cdot 10^{-7}\,\text{m}^4$,

$$\alpha_{BC} = \frac{-10 \cdot 0.4}{80 \cdot 10^9 \cdot 2.51 \cdot 10^{-7}} = -0.000199\,\text{rad}$$

CD: $l_{CD} = 0.3\,\text{m}$, $M_{CD} = 30 - 40 = -10\,\text{Nm}$, $J_{CD} = \frac{\pi d_2^4}{32} = 7.95 \cdot 10^{-8}\,\text{m}^4$,

$$\alpha_{CD} = \frac{-10 \cdot 0.3}{80 \cdot 10^9 \cdot 7.95 \cdot 10^{-8}} = -0.000472\,\text{rad}$$

DE: $l_{DE} = 0.5\,\text{m}$, $M_{DE} = 30\,\text{Nm}$, $J_{DE} = \frac{\pi d_2^4}{32} = 7.95 \cdot 10^{-8}\,\text{m}^4$, $\alpha_{DE} = \frac{30 \cdot 0.5}{80 \cdot 10^9 \cdot 7.95 \cdot 10^{-8}} = 0.00236\,\text{rad}$

and the total angle of the twist is:

$$\alpha_{AE} = \alpha_{AB} + \alpha_{BC} + \alpha_{CD} + \alpha_{DE} = 0.00120 - 0.000199 - 0.000472 + 0.00236 = 0.00289\,\text{rad}.$$

Fig. 10.3 presents the diagram of the twist angle.

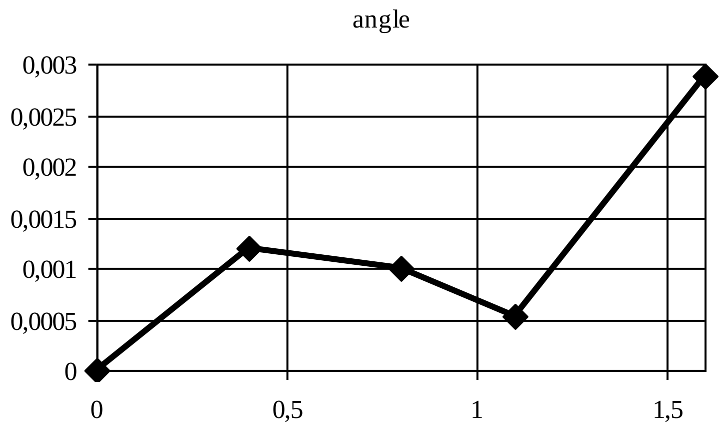


Fig. 10.3 Diagram of the twist angle

Example 10.3

Determine the diameter of the shaft in Fig. 10.4, if the acceptable values of (a) the shear stress is $R_t = 150$ MPa, (b) unit twist angle is $\theta_{acc} = 0.05$ [rad/m] and (c) the twist angle between the sections B and D is 0.02 [rad]. The Kirchhoff module value is G = 80 GPa, $d_2 = 1.2 d_1$.

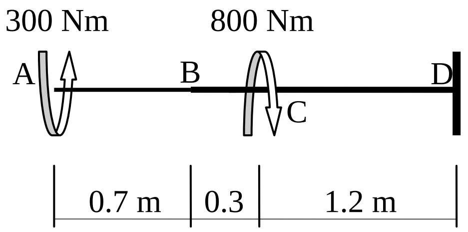


Fig. 10.4 Shaft under torsion

Solution

the 1st condition will be fulfilled when:

$$\tau_{max} = \frac{M_0}{W_0} \le R_t = 150 \text{ MPa}$$

hence the following equations:

$$\frac{M_0}{W_0} = \frac{16 M_0}{\pi d^3} \le R_t \quad \rightarrow \quad d \ge \sqrt[3]{\frac{16 M_0}{\pi R_t}}$$

and

$$d_{AB} = \sqrt[3]{\frac{16 \cdot 300}{\pi \cdot 150 \cdot 10^6}} = 0.0217 \text{ m}, \quad d_{BC} = \frac{1}{1.2} \cdot \sqrt[3]{\frac{16 \cdot 300}{\pi \cdot 150 \cdot 10^6}} = 0.0181 \text{ m}, \quad d_{CD} = \frac{1}{1.2} \cdot \sqrt[3]{\frac{16 \cdot 500}{\pi \cdot 150 \cdot 10^6}} = 0.0214 \text{ m}$$

so

$$d = \max(d_{AB}, d_{BC}, d_{CD}) = 0.0217\,\text{m}$$

the 2nd condition will be fulfilled when

$$\theta = \frac{M_0}{GJ_0} \le \theta_{\text{acc}} = 0.05$$

hence, we have:

$$\frac{32M_0}{G\pi d^4} \le 0.05 \quad \rightarrow \quad d \ge \sqrt[4]{\frac{32M_0}{80\cdot 10^9 \pi\cdot 0.05}}$$

so

$$d_{AB} = \sqrt[4]{\frac{32\cdot 300}{80\cdot 10^9 \pi\cdot 0.05}} = 0.0296\,\text{m}, \qquad d_{BC} = \frac{1}{1.2}\cdot\sqrt[4]{\frac{32\cdot 300}{80\cdot 10^9 \pi\cdot 0.05}} = 0.0246\,\text{m},$$

$$d_{CD} = \frac{1}{1.2}\cdot\sqrt[4]{\frac{32\cdot 500}{80\cdot 10^9 \pi\cdot 0.05}} = 0.0280\,\text{m}$$

so

$$d = \max(d_{AB}, d_{BC}, d_{CD}) = 0.0296$$

the 3rd condition will be fulfilled when

$$\alpha_{BD} \le \alpha_{acc} = 0.02\,\text{rad}$$

hence, we have:

$$\alpha_{BD} = \alpha_{BC} + \alpha_{CD} = \frac{M_{BC} l_{BC}}{GJ_{0BC}} + \frac{M_{CD} l_{CD}}{GJ_{0CD}} = \left| \frac{300\cdot 0.3\cdot 32}{80\cdot 10^9\cdot\pi\cdot(1.2\cdot d)^4} - \frac{500\cdot 1.2\cdot 32}{80\cdot 10^9\cdot\pi\cdot(1.2\cdot d)^4} \right| \le 0.02$$

so

$$d \ge \frac{1}{1.2}\cdot\sqrt[4]{\frac{32}{\pi\cdot 80\cdot 10^9\cdot 0.02}(500\cdot 1.2 - 300\cdot 0.3)} = 0.0354\,\text{m}$$

Taking into account the results obtained we finally assume:

$$d = 0.036\text{m} = 3.6\text{cm}$$

Example 10.4

For the shaft in Fig. 1.8 determine the function of the twist moment and draw the unit torsion angle diagram. Determine the values and draw the diagram of the torsion angle. $d_1 = 3\,\text{cm}$, $d_2 = 3.1\,\text{cm}$, $G = 80$ GPa.

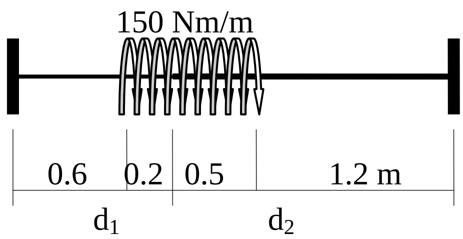


Fig. 10.5 Shaft with loading

Solution

In order to solve the redundant problem we need to go back to the BVP and the complete set of its equations. As we know, there are:

– the statics equations of the equilibrium of the forces (Navier's equations with the SBC),
– the geometric equations of compatibility (Cauchy's equations with KBC), and
– the constitutive equations describing behavior of the material.

For the particular problem of the shaft twist we have:

– one statics equation $\sum M_0 = 0$ (the resultant torque should be zero for static equilibrium)
– one geometric equation of compatibility $\alpha_{AE} = -\alpha_{EA} = 0$ (the twist angle between one end and another, both fixed)
– the constitutive equations $\alpha = \dfrac{M_0 l}{GJ_0}$

We release the shaft end from constraints:

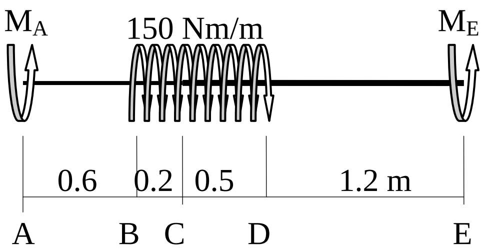


Fig. 10.6 Shaft under loading

We have two unknowns M_A, M_E and two equations, the static one and the kinematic one:

$$M_A - 150 \cdot 0.7 + M_E = 0$$

$$\alpha_{AB} + \alpha_{BC} + \alpha_{CD} + \alpha_{DE} = 0$$

The second equation can be written as:

$$\frac{M_A \cdot 0.6}{GJ_{01}} + \frac{(M_A - 150 \cdot 0.5 \cdot 0.2) \cdot 0.2}{GJ_{01}} + \frac{(M_A - 150 \cdot 0.2 - 150 \cdot 0.5 \cdot 0.5) \cdot 0.5}{GJ_{02}} + \frac{(M_A - 150 \cdot 0.7) \cdot 1.2}{GJ_{02}} = 0$$

Multiplying both sides by GJ_{01}, with $J_{01}/J_{02} = 0.8771$ we get:

$$0.6M_A + 0.2M_A - 3 + 0.8771 \cdot 0.5M_A - 29.6 + 1.05M_A - 110.5 = 0,$$

hence we obtain:

$$M_A = 62.47\,\text{Nm},\ M_E = 42.53\,\text{Nm}.$$

Fig. 10.7, 10.8 and 10.9 present the diagrams of the twist moment, unit twist angle and twist angle.

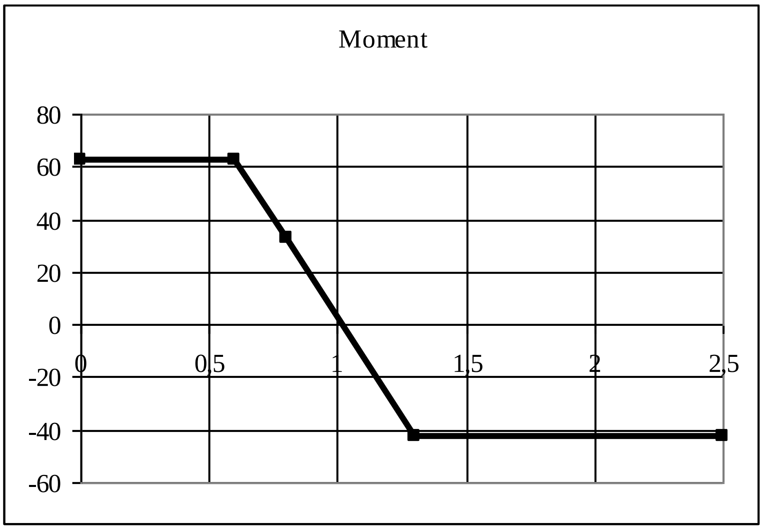


Fig. 10.7 Twist moment diagram

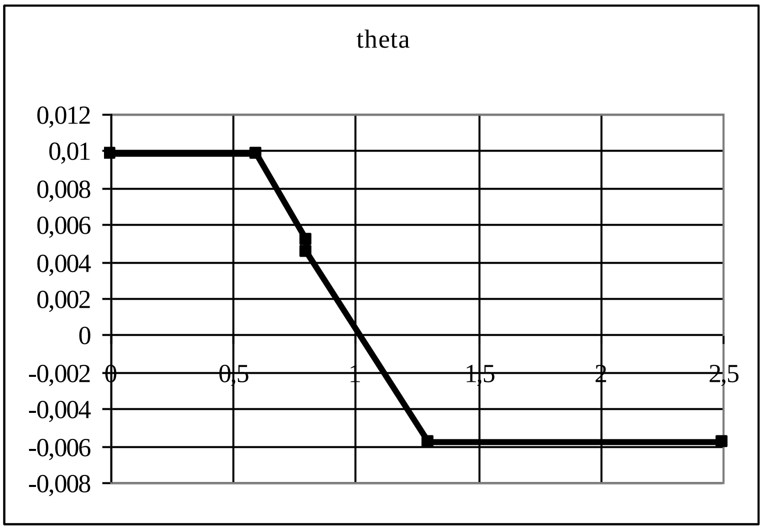


Fig. 10.8 Unit twist angle diagram

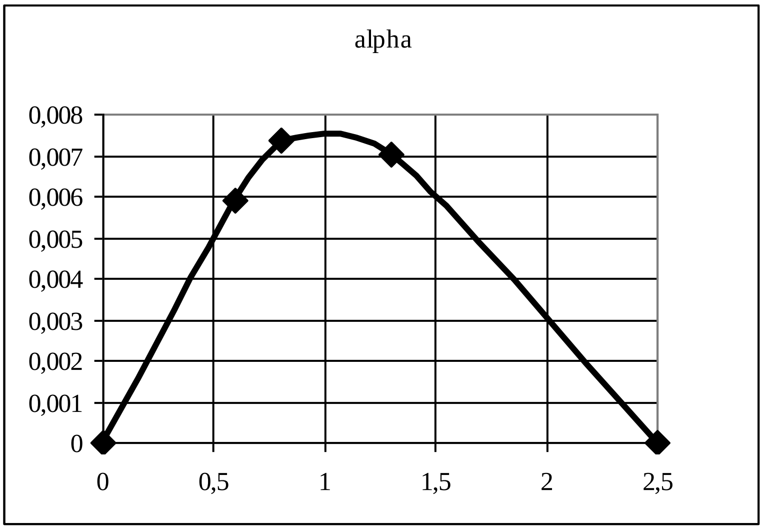


Fig. 10.9 Twist angle diagram

Review problems

Problem 10.1

An engine is connected to a generator by a large hollow shaft with inner and outer diameters of 100 mm and 150 mm, respectively. Knowing that the allowable shearing stress is 85 MPa, determine the maximum torque that can be transmitted: (a) by the shaft as designed, (b) by a solid shaft of the same weight, (c) by a hollow shaft of the same weight and of 200 mm outer diameter.

Ans.: a) 45.2 kNm, b) 23.3 kNm, c) 70.5 kNm

Problem 10.2

Knowing that each of the shafts *AB*, *BC* and *CD* consists of solid circular rods, Fig. 10.10, determine the shaft in which the maximum shearing stress occurs and the magnitude of that stress.

$d_1 = 20$ mm, $d_2 = 25$ mm, $d_3 = 30$ mm.

Ans.: 58.7 MPa in the 2nd shaft.

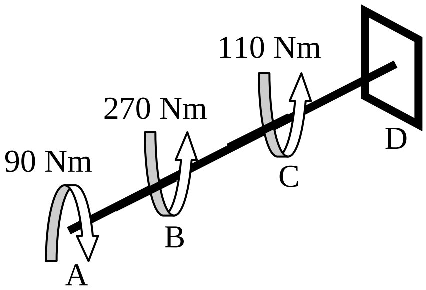


Fig. 10.10 Shafts with loads

Problem 10.3

A gear, Fig. 10.11, has two cogs: a greater one and a smaller one, with the diameters of 20 mm and 8 mm, respectively. Knowing that the allowable shearing stress is 85 MPa, determine the diameters of the shafts if the torque $M_1 = 220$ Nm.

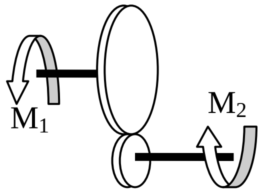


Fig. 10.11 Gear with two cogs and shafts

Ans.: $d_1 = 0.0236, d_2 = 0.0174\,\text{m}$.

Problem 10.4

Knowing that the allowable shearing stress of a brass shaft with both ends fixed, Fig. 10.12, is 55 MPa, determine the diameter of the shaft. Using the calculated diameter and knowing that the Kirchhoff's modulus is 37 GPa, determine the rotation angle of the section where the torque is applied.

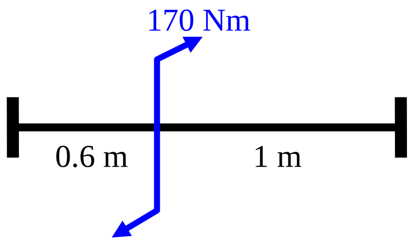


Fig. 10.12 Brass shaft with load

Ans.: $d > 0.0214\,\text{m}$, $\alpha = 0.0837\text{rad} = 4.79°$

Problem 10.5

Determine the diameter of the steel shaft, Fig. 10.13, knowing that the allowable shear stress is 80 MPa, and the maximum value of the rotation angle should not exceed the value of 2 degrees, G = 80 GPa.

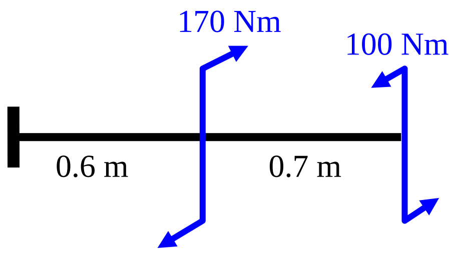


Fig. 10.13 Shaft with loading

Ans.: $d = 0.0198$ m (cf. the twist angle diagram).

Problem 10.6

Using an allowable shearing stress of 40 MPa, design a solid steel shaft of a VW engine to transmit 100 kW at the speed of 5000 rpm

Ans.: $d = 0.0290$ m