# Torsion of bars with noncircular cross-section

## Basis formulae

### Rectangular cross-section

In the case of rectangular cross-section the steepest slope occurs at the midpoint of the larger side of the rectangle, Fig.11.1

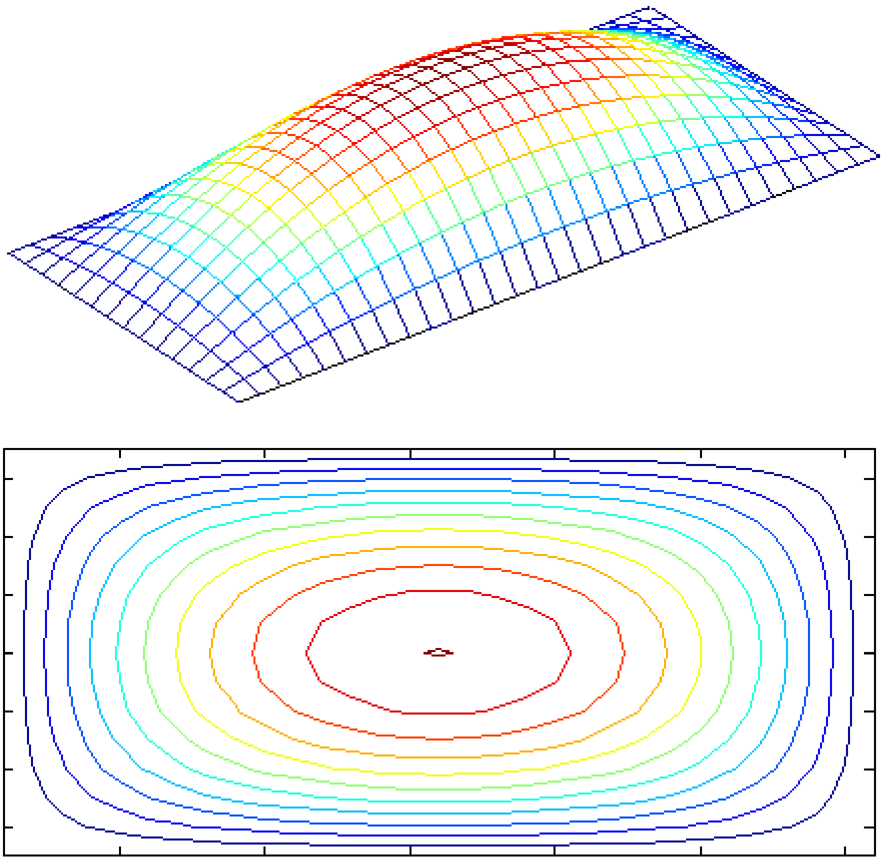

Fig. 11.1 Prandtl's function for rectangular section

Furthermore, the shearing stress is zero at the corners and at the center of the section, Fig. 11.2.

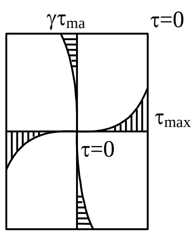


Fig. 11.2 Shearing stress in rectangular cross-section

The solution for a rectangular section may be obtained by expansion of the warping function in series and the solution can be presented as formulae:

$$\tau_{\max} = \frac{M_x}{W_x}, \quad W_x = \alpha\left(\tfrac{h}{b}\right)b^2 h, \quad \theta = \frac{M_x}{GI_x}, \quad J_x = \beta\left(\tfrac{h}{b}\right)b^3 h, \quad \tau_s = \gamma\tau_{\max},$$

where $\tau_s$ is the shearing stress at the middle of the shorter side and the coefficients $\alpha$, $\beta$, and $\gamma$ are stated in the table 11.1.

| *h/b* | 1 | 1.25 | 1.5 | 2 | 3 | 5 | 10 | ∞ |
|---|---|---|---|---|---|---|---|---|
| $\alpha$ | 0.2082 | 0.2212 | 0.2310 | 0.2459 | 0.2672 | 0.2915 | 0.3123 | 0.3333 |
| $\beta$ | 0.1406 | 0.1717 | 0.1958 | 0.2287 | 0.2633 | 0.2913 | 0.3123 | 0.3333 |
| $\gamma$ | 1.0 | 0.9159 | 0.8591 | 0.7958 | 0.7533 | 0.7429 | 0.7423 | 0.7423 |

Table 11.1 Coefficients for maximum torsion stress, torsion inertia moment and additional torsion stress

## Thin-walled cross-section

A thin-walled cross-section member is a bar the cross-section of which is made from profile, for example from bent sheet of steel. The exact solution of such twisted profiles is difficult and usually not needed in practice. There are simplified methods used instead.

From the point of view of practical calculation there are three types of these cross-sections, Fig. 11.3:

(a) the open thin-walled profiles developable (which can be uncurled and so-called middle-line doesn't fork),

(b) the open profiles undevelopable (which cannot be uncurled and the middle-line branches),

(c) the closed profiles (the middle-line is a closed line).

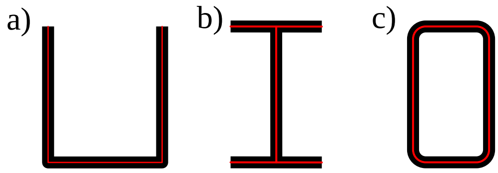


Fig. 11.3 Thin-walled profiles with middle-line (a), (b) and (c)

### Profiles with middle-line developable

Some of the thin-walled cross-sections may be substituted by one rectangle with the same area and the longer side length is equal to the middle-line of the profile.

For example, a C profile with the web 300x10 mm and the flanges 90x16 mm may be substituted by a rectangle with middle-line length $l$ = 2x90 + 300 = 480 mm and average width $b$ = 12.25 mm.

### Hydrodynamic analogy

There is an analogy between the distribution of the shearing stresses in the transverse section of a shaft and the distribution of flow lines and velocities in water flowing through a closed channel of unit depth and variable width, Fig. 11.4. In the last case, the water flow as well as the shear flow are constant, $\tau\delta = \text{const}$ .

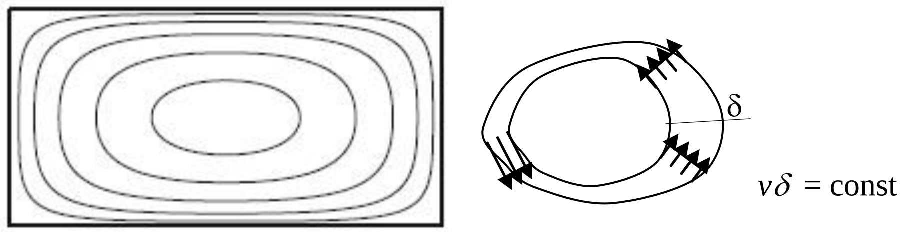


Fig. 11.4 Hydrodynamic analogy: flow lines and flow velocities in hollow shaft

### Profiles with middle-line undevelopable

The thin-walled cross-sections with the branched middle-line can be solved applying the hydrodynamic analogy. The section is cut into rectangles and it is assumed that:

– each rectangle works independently, transmitting a part of the applied torque, $M_{xi}$,

– the unit torsion angle of each rectangle is the same, $\theta_i = \text{idem}$ .

With the static equilibrium equation, we get a set of *n* equations for *n* component rectangles:

$$M_x = \sum_i M_{xi}, \quad \frac{M_{xi}}{GJ_{xi}} = \frac{M_{x(i+1)}}{GJ_{x(i+1)}}, \quad i = 1, \ldots, n-1,$$

hence we find $M_{xi}$ and the unit torsion angle $\Theta$ of the whole cross-section.

The shorter cut lines are, the better the approximation is.

### Hollow profiles

The shearing stress in closed thin-walled cross-sections may be determined by means of the hydrodynamic analogy. The flow of the stress through every section is constant: $\tau_1\delta_1 = \tau_2\delta_2 = \text{const}$ . The shearing stress, running around the cross-section and acting on a different lever, constitutes the torque as a sum of the shearing stress actions. Assuming that the stress repartition is constant in each section of the closed channel, the torque may be written as:

$$M_x = \oint_c \tau\delta ds \rho(s) = \tau\delta \oint_c \rho(s) ds = \tau\delta 2A,$$

where *F* is the area inside the middle-line of the profile, Fig. 11.5.

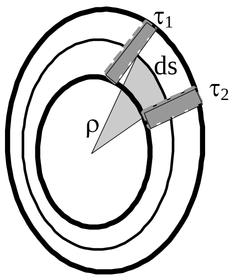


Fig. 11.5 Shearing stress circulation in closed profile

Hence, the extreme shearing stress will be attained in the section of the channel with the smallest width:

$$\tau_{\max} = \frac{M_x}{2A\delta_{\min}}.$$

The above formula is known as Bredt's first formula.
To determine the deformation caused by the torsion we use the formula:

$$\theta = \frac{M_x}{GJ},$$

where $GJ$ is the torsional stiffness, and the inertia moment is (Bredt's second formula):

$$J = \frac{4A^2}{\oint_c \frac{ds}{\delta}}, \quad \delta = \text{const} \quad \rightarrow \quad J = \frac{4A^2\delta}{c}.$$

Bredt's theory has been developed on the basis of a simplifying assumption for the stress distribution which gets closer to the actual distribution when the wall becomes thinner.

# Examples

## Example 11.1

Determine the maximum shearing stress and the unit twist angle of the cross-section in Fig. 11.6. Compare the solution for the profile with not branched middle-line with the Bredt's solution.

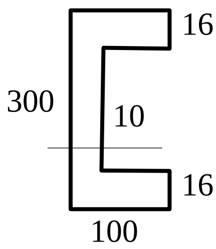


Fig. 11.6 Twisted cross-section

**Solution**

a) We consider the cross-section as a rectangle with the length equal to the length of the middle-line:

$$l = 300 - 16 + 2(100 - 5) = 474\,\text{mm}$$

To obtain the same area we take:

$$A = 16 \cdot 100 \cdot 2 + 10 \cdot (300 - 32) = 5880 \quad \rightarrow \quad \delta = {}^{5880}\!\!/_{474} = 12.41\,\text{mm}$$

We have:

$$\frac{l}{\delta} = \frac{474}{12.41} = 38.2 \quad \rightarrow \quad \alpha = \beta = \frac{1}{3}$$

The maximum shearing stress is:

$$\tau_{\max} = \frac{M_x}{W_x} = \frac{M_x}{\frac{1}{3} \cdot 12.41^2 \cdot 474} = 4.11 \cdot 10^{-5} M_x$$

$$\theta = \frac{M_x}{GJ_x} = \frac{M_x}{G} \frac{1}{\frac{1}{3} \cdot 12.41^3 \cdot 474} = 3.31 \cdot 10^{-6} \frac{M_x}{G}$$

b) We divide the cross-section into 3 rectangles, two flanges $100 \times 16$, $\frac{l}{\delta} = 6.25$ and one web $268 \times 10$, $\frac{l}{\delta} = 26.8$. The parameters $\alpha$ and $\beta$ for the web are $\alpha = \beta = \frac{1}{3}$ and for the flanges (from linear interpolation) are $\alpha = 0.2976$ and $\beta = 0.2977$. Using the Bredt's assumptions, we have:

– the static equation

$$2M_{\text{fl}} + M_{\text{w}} = M_s$$

– the compatibility equation

$$\theta_{\text{fl}} = \theta_{\text{w}} \quad \rightarrow \quad \frac{M_{\text{fl}}}{GJ_{\text{fl}}} = \frac{M_{\text{w}}}{GJ_{\text{w}}} \quad \rightarrow \quad \frac{M_{\text{fl}}}{0.2977 \cdot 16^3 \cdot 100} = \frac{M_{\text{w}}}{0.3333 \cdot 10^3 \cdot 268} \quad \rightarrow \quad M_{\text{w}} = 0.7326 M_{\text{fl}}$$

and, finally:

$$\tau_{\text{w}} = \frac{0.2681 M_s}{\frac{1}{3} \cdot 10^2 \cdot 268} = 3.00 \cdot 10^{-5} M_s, \quad \tau_{\text{fl}} = \frac{0.366 M_s}{0.2976 \cdot 16^2 \cdot 100} = 4.80 \cdot 10^{-5} M_s, \quad \max \tau = 4.80 \cdot 10^{-5} M_s$$

$$\theta = \frac{M_s}{G} \frac{0.366}{0.2977 \cdot 16^3 \cdot 100} = 3.00 \cdot 10^{-6} \frac{M_s}{G}$$

Conclusions:

Both methods give different answers, the relative errors are: for maximum shearing stress 17% and for the unit twist angle about 10%.

## Example 11.2

Determine the shearing stress in the T-beam (the web 30x5 cm and the flange 6x20 cm), twisted by the torque 20 kNm,. while l/δ = 6: α = 0.299, β = 0.299, h/b = 3.33: α = 0.272, β = 0.269.

$$\theta_1 = \theta_2 \quad \Rightarrow \quad \frac{M_1}{G \cdot 0.299 \cdot 30 \cdot 125 \cdot 10^{-8}} = \frac{M_2}{G \cdot 0.269 \cdot 20 \cdot 216 \cdot 10^{-8}}$$

$$M_1 + M_2 = 20 \cdot 10^3 \quad \rightarrow \quad M_1 = 9.82 \text{ kNm}, \quad M_2 = 10.18 \text{ kNm},$$

$$\tau_w = \frac{9820}{0.299.3 \cdot 0.0025} = 43.8 \text{ MPa}, \quad \tau_{fl} = \frac{10180}{0.2720.2 \cdot 0.0036} = 52.0 \text{ MPa}$$

## Example 11.3

To get an idea on applicability of Bredt's formulae let's analyze a particular case of twisted ring of the outer diameter $R$ and inner diameter $r$, where both exact and approximate solutions can be found. For the sake of analysis we will use a parameter $\alpha$ which is the ratio of the cross-section thickness to the middle-line radius: $\alpha = \delta/\rho$.

a) Bredt's solution

$$\tau_{\max}^{B} = \frac{M_x}{2A\delta} = \ldots = \frac{M_x}{\pi} \frac{1}{2\rho^2\delta}$$

$$\theta^{B} = \frac{M_x 2\pi\rho}{2GA^2\delta} = \ldots = \frac{M_x}{G\pi} \frac{1}{2\rho^3\delta}$$

b) exact solution

$$\tau_{\max}^{\text{exact}} = \frac{M_x}{W_x} = \ldots = \frac{M_x}{\pi} \frac{2\rho + \delta}{\rho\delta\left(4\rho^2 + \delta^2\right)}$$

$$\theta^{\text{exact}} = \frac{M_x}{GJ_x} = \ldots = \frac{M_x}{G\pi} \frac{2}{\rho\delta\left(4\rho^2 + \delta^2\right)}$$

The ratio of Bredt' solution to the exact solution is:

$$\frac{\tau_{\text{Bredt}}}{\tau_{\text{exact}}} = \frac{4 + \alpha^2}{4 + 2\alpha}, \quad \frac{\theta_{\text{Bredt}}}{\theta_{\text{exact}}} = \frac{4 + \alpha^2}{4}.$$

The above formulae as functions of parameter $\alpha$ are presented in Fig. 11.7.

The maximum value of shearing stress is underestimated, and quickly starts to be significant (from $\alpha \approx 0.1$); an error associated to the unit twist angle remains small up to the larger values of the parameter $\alpha$ (up to 0.5).

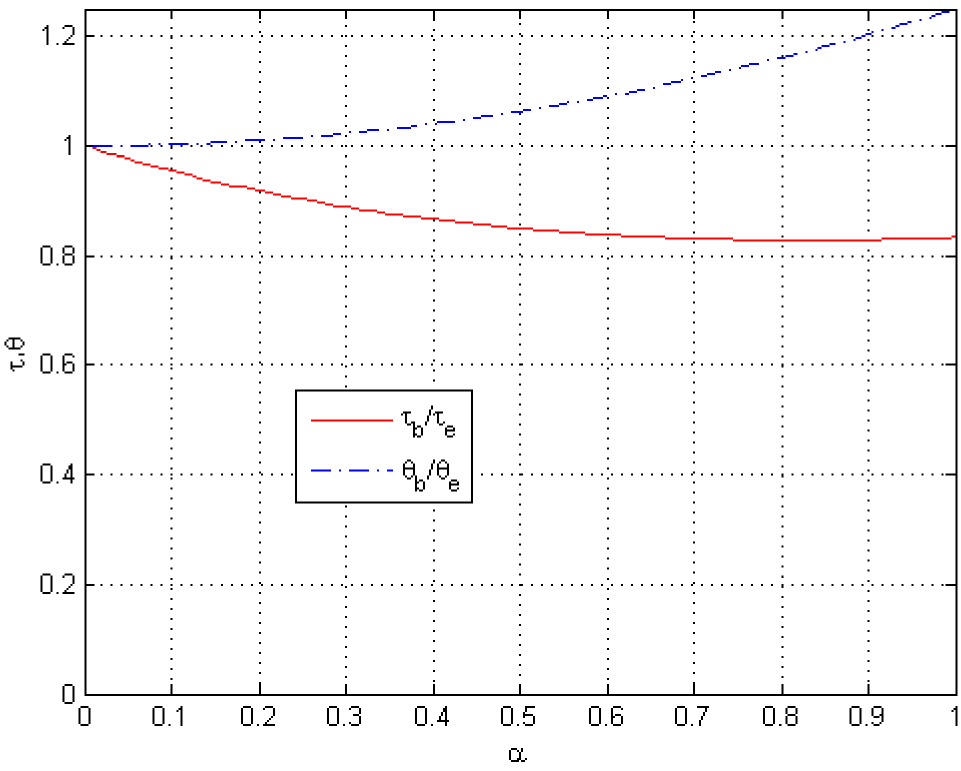


Fig. 11.7 Comparison of Bredt's solution with exact solution

## Review problems

Determine the maximum shearing stress and the unit twist angle for the cross-sections stated in Fig. 11.8.

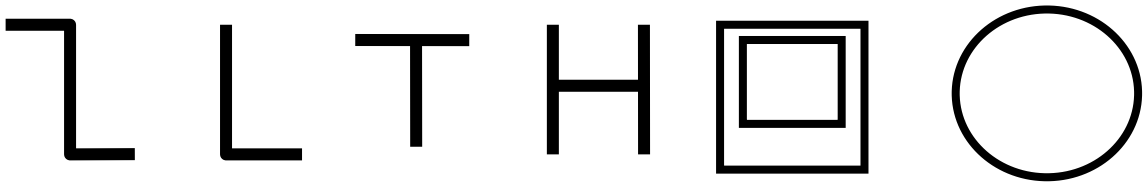

Fig. 11.8 Different profiles