Stability

Definitions

An effective length:

$$l_e = \alpha l$$

where the coefficient α depends on the static scheme case:

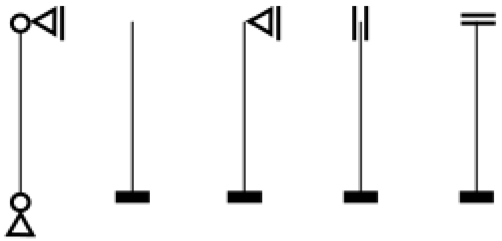

Fig. 12.1 Effective length coefficient

The slenderness ratio:

$$\lambda = \frac{l_e}{i_{\min}}$$

Euler critical force (in an elastic range):

$$P_E = \frac{\pi^2 E I_{\min}}{l_e^2} = \frac{\pi^2 E A}{\lambda^2}$$

(valid in the linear elastic range, $\lambda > \lambda_{\lim}$).

In the out-of-linearly elastic range:

Tetmeyer-Jasiński proposal

$$\sigma_{T-J} = a - b\lambda, \quad \sigma_{T-J}(0) = R_e, \quad \sigma_{T-J}(\lambda_{\lim}) = R_H \quad \rightarrow \quad a = R_e, \quad b = \frac{R_e - R_H}{\lambda_{\lim}}$$

$$P_{T-J} = A\sigma_{T-J} = AR_e - A(R_e - R_H)\frac{\lambda}{\lambda_{\lim}}$$

Johnson-Ostenfeld proposal

$$\sigma_{J-O} = C - B\lambda^2, \quad \sigma_{J-O}(0) = R_e, \quad \sigma_{J-O}(\lambda_{gr}) = R_H \quad \rightarrow \quad C = R_e, \quad B = \frac{R_e - R_H}{\lambda_{\lim}^2}$$

$$P_{J-O} = A\sigma_{J-O} = AR_e - A(R_e - R_H)\left(\frac{\lambda}{\lambda_{\lim}}\right)^2 .$$

(valid in non-linear range, $\lambda < \lambda_{\text{lim}}$)

Examples

Example 12.1

Determine the bearing capacity of the bar in Fig. 12.2. The cross-section dimensions are 20×40 cm, length *l* = 4 m, *E* = 205 GPa, $R_e = 380$ MPa, $R_H = 280$ MPa, *n* = 2.2.

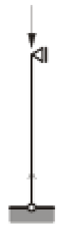

Fig. 12.2 Static scheme of a strut

Solution

cross-sectional characteristics

area: $0.2 \times 0.4 = 0.08$ m²

minimum inertia moment: $I_{\min} = \frac{0.4 \cdot 0.2^3}{12} = 2.67 \cdot 10^{-4}$ m⁴

minimum inertia radius: $i_{\min} = \sqrt{\frac{I_{\min}}{A}} = \frac{2.67 \cdot 10^{-4}}{0.08} = 0.0578$ m

effective length: $l_e = \alpha l = 1 \cdot l = l = 4$ m

slenderness ratio: $\lambda = \frac{l_e}{i_{\min}} = \frac{4}{0.0578} = 69.2$

limit slenderness ratio: $\lambda_{\text{lim}} = \pi \sqrt{\frac{E}{R_H}} = \pi \sqrt{\frac{205 \cdot 10^9}{280 \cdot 10^6}} = 85.0$

$\lambda < \lambda_{\text{lim}} \rightarrow$ non-linear range, we adopt T-J formula

$$P_{T-J} = A\sigma_{T-J} = A(a - b\lambda) = A\left(R_e - \frac{R_e - R_H}{\lambda_{\text{lim}}}\lambda\right) = 0.08 \cdot \left(380 - \frac{(380-280)}{85} 69.2\right) \cdot 10^6 = 23.9 \text{ MN}$$

acceptable value of load: $P_{acc} = \frac{P_{T-J}}{n} = \frac{23.9}{2.2} = 10.9$ MN

Example 12.2

Determine the bearing capacity of the bar in Fig. 12.3; the cross-section diameter *d* = 4 cm, *l* = 6.14 m, *E* = 205 GPa, $R_e = 450$ MPa, $R_H = 250$ MPa, *n* = 3.0.

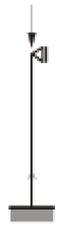

Fig. 12.3 Bar with load

Solution

cross-sectional characteristics

area: $\frac{\pi d^2}{4} = \frac{\pi \cdot 0.04^2}{4} = 1.26 \cdot 10^{-3}$ m^2

minimum inertia moment: $I_{\min} = \frac{\pi d^4}{4} = \frac{\pi \cdot 0.04^4}{4} = 2.01 \cdot 10^{-6}$ m^4

minimum inertia radius: $i_{\min} = \sqrt{\frac{I_{\min}}{A}} = \sqrt{\frac{2.01 \cdot 10^{-6}}{1.26 \cdot 10^{-3}}} = 0.0399$ m

effective length: $l_e = \alpha l = 6.14 \cdot 0.699 = 4.29$ m

slenderness ratio: $\lambda = \frac{l_e}{i_{\min}} = \frac{4.29}{0.0399} = 107.5$

limit slenderness ratio: $\lambda_{\lim} = \pi \sqrt{\frac{E}{R_H}} = \pi \sqrt{\frac{205 \cdot 10^9}{250 \cdot 10^6}} = 89.96$

$\lambda > \lambda_{\lim} \rightarrow$ linear elastic range, we adopt Euler formula

$$P_E = \frac{\pi^2 E I_{\min}}{l_e^2} = \frac{\pi^2 \cdot 205 \cdot 10^9 \cdot 2.01 \cdot 10^{-6}}{4.29^2} = 221.0 \text{ kN}$$

acceptable value of load: $P_{acc} = \frac{P_E}{n} = \frac{221.0}{3.0} = 73.65$ kN

Example 12.3

Determine the parameter a of the rectangular cross-section $a \times 2a$ of the bar in Fig. 12.4, if P = 1 MN, l = 3.5 m, E = 205 GPa, $R_e = 450$ MPa, $R_H = 250$ MPa, n = 3.0.

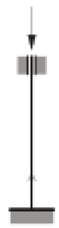

Fig. 12.4 Bar with load

Solution

the critical value of force: $P_{cr} = nP = 3 \cdot 1 \cdot 10^6 = 3$ MN the effective length: $l_e = 0.5l = 1.75$ m

limit slenderness ratio: $\lambda_{\text{lim}} = \pi\sqrt{\frac{E}{R_H}} = \pi\sqrt{\frac{205\cdot 10^9}{250\cdot 10^6}} = 89.96$

cross-sectional characteristics

area: $2a^2$

minimum inertia moment: $I_{\text{min}} = \frac{2a\cdot a^3}{12} = \frac{a^4}{6}$

minimum inertia radius: $i_{\text{min}} = \sqrt{\frac{I_{\text{min}}}{A}} = \sqrt{\frac{a^4}{6\cdot 2a^2}} = \frac{a}{\sqrt{12}}$

we try a solution in the linear elastic range:

$$P_E = \frac{\pi^2 E I_{\text{min}}}{l_e^2} \to I_{\text{min}} = \frac{P_E l_e^2}{\pi^2 E} = \frac{3\cdot 10^6\cdot 1.75^2}{\pi^2\cdot 205\cdot 10^9} = 4.54\cdot 10^{-6}\ \text{m}^4$$

the parameter a: $\frac{a^4}{6} = 4.54\cdot 10^{-6} \to a = 0.0723$ m

range checking

the slenderness ratio: $\lambda = \frac{l_e}{i_{\text{min}}} = \frac{1.75\cdot\sqrt{12}}{0.0723} = 83.84 < \lambda_{\text{lim}}$ (wrong range!)

solution in the non-linear range:

$$P_{T-J} = A\sigma_{T-J} = 2a^2\left(R_e - \frac{R_e - R_H}{\lambda_{\text{lim}}}\lambda\right) = 2a^2\left(450 - \frac{450-250}{89.96}\cdot\frac{1.75\cdot\sqrt{12}}{a}\right)\cdot 10^6 = 3\cdot 10^6$$

$$\to a = 0.0747\ \text{m}$$

Example 12.4

Chose a suitable profile INP for a bar in Fig. 12.5, if P = 60 kN, l = 4 m, E = 205 GPa, $R_e = 450$ MPa, $R_H = 250$ MPa, n = 2.8.

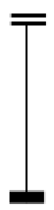

Fig. 12.5 Bar with load

Solution

the critical value of the force: $P_{cr} = nP = 168$ kN

the effective length: $l_e = 1\cdot l = 4$ m

we try a solution in the linear elastic range:

$$P_E = \frac{\pi^2 EI_{\min}}{l_e^2} \rightarrow I_{\min} = \frac{P_E l_e^2}{\pi^2 E} = \frac{168 \cdot 10^3 \cdot 4^2}{\pi^2 \cdot 205 \cdot 10^9} = 1.329 \cdot 10^{-6}\ \mathrm{m}^4$$

the least profile which satisfies condition is INP 220 with $I_{\min} = 1.62 \cdot 10^{-6}\ \mathrm{m}^4$, $i_{\min} = 0.0202$ m

limit slenderness ratio: $\lambda_{\lim} = \pi\sqrt{\frac{E}{R_H}} = \pi\sqrt{\frac{205 \cdot 10^9}{250 \cdot 10^6}} = 89.96$

range checking: $\lambda = \frac{l_e}{i_{\min}} = \frac{4}{0.0202} = 198 > \lambda_{\lim}$ (correct)

Example 12.5

Determine Euler critical force for two-storey column in Fig. 12.6., if l_1 = 4 m, l_2 = 2 m.

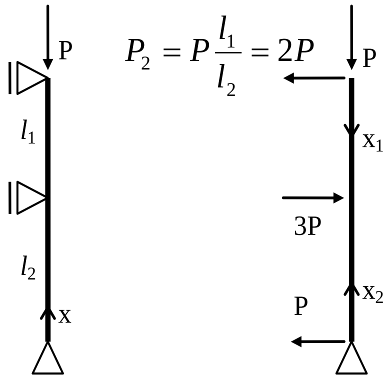

Fig. 12.6 Two-storey column

Solution

for $0 < x_1 < l_1, w_1'' + k_1^2 w_1 = -\frac{P}{EJ}x_1$, with $k^2 \equiv \frac{N}{EJ}$

we get (complementary and particular functions): $w_1(x) = A\sin k_1 x_1 + B_1 \cos k_1 x_1 - \frac{P}{N}x_1$

with the boundary conditions:

$$w_1(0) = w_1(l_1) = 0 \quad \Rightarrow \quad w_1(x_1) = \frac{P_1 l_1}{N \sin k_1 l_1}\sin k_1 x_1 - \frac{P_1}{N}x_1,$$

(similarly for $0 < x_2 < l_2$). The compatibility conditions: $w_1'(l_1) = -w_2'(l_2)$, $w_1''(l_1) = -w_2''(l_2)$.

Comparing the second order derivatives, we get $k_1 = -k_2 = k$, and from the first order derivatives we get (after rearrangement):

$$k\left(\frac{\cos kl_1}{\sin kl_1} + \frac{\cos kl_2}{\sin kl_2}\right) = \frac{l_1 + l_2}{l_1 l_2} \quad \rightarrow \quad k = 0.9642 \quad \rightarrow \quad N = k^2 EJ = 0.930EJ\,.$$

Review problems

Problem 12.1

Using allowable stress design, determine the allowable centric load for a column of 6-m effective length that is made from the following rolled-steel shape: a) HEA 220 and b) HEA 260. Use R_H = 180 MPa, R_e = 300 MPa, E = 210 GPa. Assume the safety factor equal to 2.

Ans.: 563 kN for HEA 220 and 923 kN for HEA 260.

Problem 12.2

A simple compression member of 8-m effective length is obtained by connecting two C200 steel channels with lacing bars as shown in Fig. 12.7. Knowing that the factor of safety is 1.85, determine the allowable centric load for the member. Use R_H = 180 MPa, R_e = 300 MPa, E = 210 GPa and d = 100 mm.

Ans.: 605 kN

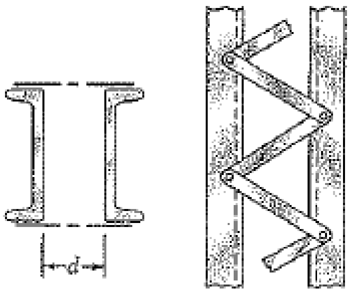

Fig. 12.7 Two channel profiles

Problem 12.3

A column of effective length L can be made by gluing together identical planks in either of the arrangements shown in Fig. 12.8. Determine the ratio of the critical load using the arrangement a) to the critical load using the arrangement b).

Ans.: 1.43

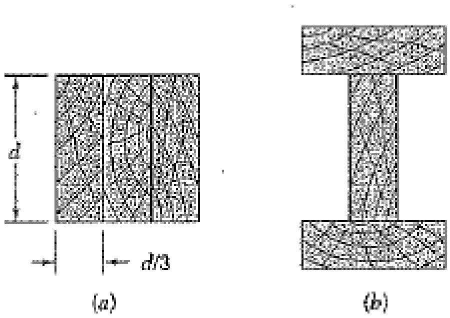

Fig. 12.8 Two arrangements of planks

Problem 12.4

Determine the critical load of an aluminum tube that is 1.5 m long and has 16-mm outer diameter and a 1.25-mm wall thickness. Use E = 70 GPa.

Ans.: 487 N

Problem 12.5

A 2-m long pin-ended column of square cross-section is to be made of wood. Assuming E = 13 GPa, R_H = 12 MPa, and using a factor of safety of 2.5 in computing Euler's critical load, determine the size of the cross-section if the column is to support (a) a 100-kN load, (b) a 200-kN load safely.

Ans.: a) 9.83 cm, b) 11.7 cm