Limit analysis of structures

Theorems of structural limit analysis

Statically admissible stress field

A statically admissible stress field (SASF) is any stress field which is compatible with the static boundary conditions and inside the boundary the stress values don't exceed admissible values anywhere, $|\sigma_x| \le Re$.

Lower bound theorem

If a statically admissible stress field can equilibrate applied loading, the plastic load capacity is not less than the applied loading.

Kinematically admissible displacement field

A kinematically admissible displacement field (KADF) is any displacements field which is compatible with the kinematic boundary conditions.

Upper bound theorem

If, on the kinematically admissible displacement field, the virtual work of external and internal forces are equal, the plastic load capacity is not greater than the applied loading.

Examples

Example 15.1

Determine the plastic load capacity of the beam in Fig. 8.12.

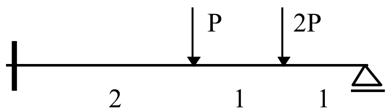


Fig. 15.1 Beam with loading

Solution

KADF

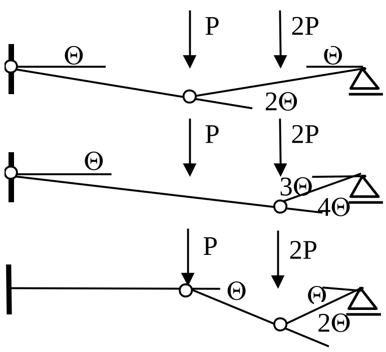


Fig. 15.2 Kinematic failure schemes

We suppose three kinematic mechanisms, Fig. 15.2, and calculate the load capacity for each scheme, respectively:

$$2\Theta P + 2P\Theta = 3\overline{\overline{M}}\Theta \quad \rightarrow \quad P_1 = 0.75\overline{\overline{M}}$$

$$2\Theta P + 2P3\Theta = 5\overline{\overline{M}}\Theta \quad \rightarrow \quad P_2 = 0.625\overline{\overline{M}}$$

$$2P\Theta = 3\overline{\overline{M}}\Theta \quad \rightarrow \quad P_3 = 1.5\overline{\overline{M}}$$

We choose the minimal value, so $\overline{\overline{P}} = 0.625\overline{\overline{M}}$.

<u>SASF</u>

We verify the above solution admitting the plastic hinges from 2nd kinematic scheme, i.e. at the fixed end and under the force $2P$, Fig. 15.3

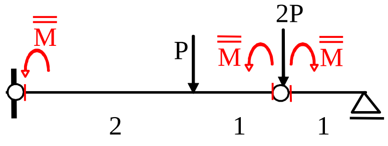


Fig. 15.3 Beam with loading and hinges

from the upper beam:

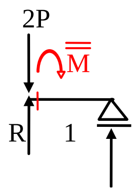


Fig. 15.4 Upper beam

we get:

$$R = 2P - \overline{\overline{M}}$$

from the lower beam equilibrium:

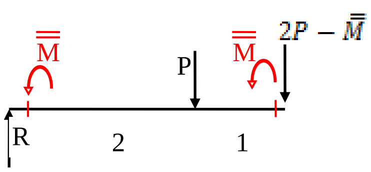


Fig. 15.5 Lower beam

we get:

$$P = \frac{5}{8}\overline{\overline{M}} = 0.625\overline{\overline{M}}$$

with the final moment distribution statically acceptable:

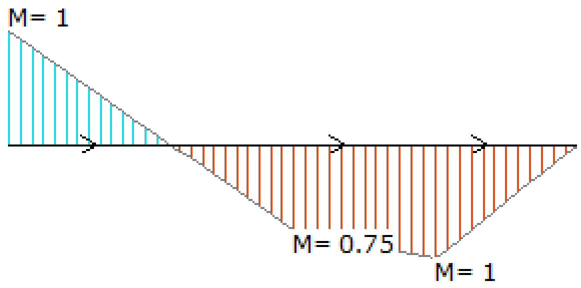


Fig. 15.6 Statically acceptable bending moment

which confirms the kinematic solution.

Example 15.2

Find the limit load capacity of the beam in Fig. 15.7 with variable cross-section capacity: $2\overline{\overline{M}}$ from the left and $\overline{\overline{M}}$ from the right.

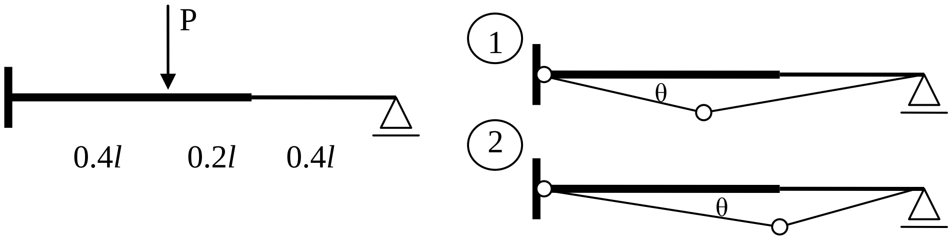


Fig. 15.7 Beam with variable stiffness and collapse schemes

For the kinematical schemes we have:

1) $\Theta_1 = \frac{2}{3}\Theta,\qquad 2\overline{\overline{M}}(\Theta + \frac{5}{3}\Theta) = P\Theta 0.4l \quad\rightarrow\quad \overline{\overline{P}} = 13.33\frac{\overline{\overline{M}}}{l}$

2) $\Theta_1 = 1{,}5\Theta,\qquad 2\overline{\overline{M}}\Theta + 2.5\overline{\overline{M}}\Theta = P\Theta 0.4l \quad\rightarrow\quad \overline{\overline{P}} = 11.25\frac{\overline{\overline{M}}}{l}$

The lowest value of the kinematic approach solution is $\overline{\overline{P}} = 11.25\frac{\overline{\overline{M}}}{l}$.

We check the solution by the static approach. With the plastic hinge at the fixed end, we get (cf. Fig. below):

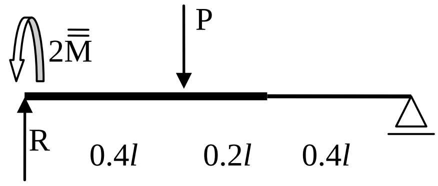


Fig. 15.8 Beam with plastic hinge

$$R = \frac{2\overline{\overline{M}}}{l} + 0.6P$$

$$M(0.6l) = 0.6Rl - 2\overline{\overline{M}} - 0.2Pl = -0.8\overline{\overline{M}} + 0.16Pl$$

$$M(0.6l) = \overline{\overline{M}} \rightarrow -0.8\overline{\overline{M}} + 0.16Pl = \overline{\overline{M}} \rightarrow \overline{\overline{P}} = 11.25\frac{\overline{\overline{M}}}{l}$$

(the same value as from kinematic approach)

Example 15.3

Find the limit load of the frame below.

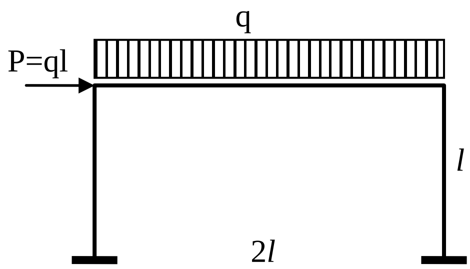


Fig. 15.9 Portal frame

1. Kinematically admissible schemes of collapse

We verify 3 schemes of collapse: beam type, frame type and mixed:

Fig. 15.10 Kinematical schemes of collapse

– beam scheme

$$2\int_0^l \theta x q \mathrm{dx} = 4\overline{\overline{M}}\theta \quad \rightarrow \quad \overline{\overline{q}} = 4\frac{\overline{\overline{M}}}{l^2}$$

– frame scheme

$$ql\theta l = 4\overline{\overline{M}}\theta \quad \rightarrow \quad \overline{\overline{q}} = 4\frac{\overline{\overline{M}}}{l^2}$$

– mixed scheme

$$ql\theta l + 2\int_{o}^{l}\theta x q \,\mathrm{dx} = \overline{\overline{M}}\theta + 2\overline{\overline{M}}\theta + 2\overline{\overline{M}}\theta + \overline{\overline{M}}\theta \quad \rightarrow \quad \overline{\overline{q}} = 3\frac{\overline{\overline{M}}}{l^{2}}$$

We get upper bound estimation for the smallest value from the mixed scheme.:

$$\overline{\overline{q}} = \leq 3\frac{\overline{\overline{M}}}{l^{2}}$$

2. We check is the mixed scheme statically admissible?

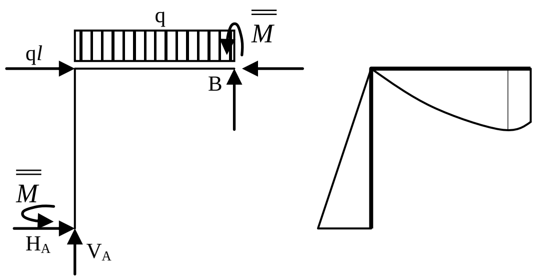
We calculate:

$$H_A = -\frac{\overline{\overline{M}}}{l}$$

$$R_A = \frac{ql}{2} + \frac{\overline{\overline{M}}}{l}$$

the shear force at spandrel beam from the left:

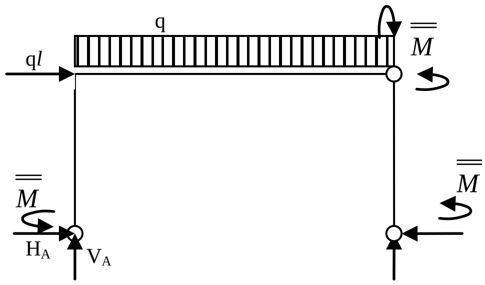
$$Q_R = R_A - ql = \frac{\overline{\overline{M}}}{l} - \frac{ql}{2} = -\frac{\overline{\overline{M}}}{2l}$$



Calculation scheme

The shear force changes the sign, the extreme value of the bending moment exceeds admissible limit value. The scheme is not admissible.

We look for the hinged section at the spandrel beam.



Calculation scheme

we calculate:

$$R_A = \frac{\overline{\overline{M}}}{l} + \frac{ql}{2}, \quad H_A = \frac{2\overline{\overline{M}}}{l} - ql$$

and the shear force in the spandrel beam is:

$$Q(x) = \frac{\overline{\overline{M}}}{l} + \frac{ql}{2} - qx = 0$$

so:

$$x = \frac{\overline{\overline{M}}}{ql} + \frac{l}{2}$$

and:

$$M(x) = R_A x - H_A l - \overline{\overline{M}} - \tfrac{1}{2} qx^2$$

and in the same time

$$M(x) = \overline{\overline{M}}$$

so, after the transformations, we have:

$$\frac{9}{4} q^2 - 7q \frac{\overline{\overline{M}}}{l^2} + \left(\frac{\overline{\overline{M}}}{l^2} \right)^2 = 0$$

and finally:

$$\overline{\overline{q}} = 2.96 \frac{\overline{\overline{M}}}{l^2}.$$

3. We verify the solution by kinematic approach, assuming the kinematic scheme of collapse with the hinge at the spandrel beam is located at *a*, to the left from the middle:

$$q\theta l^2 + \int_0^{l-a} \theta qx \mathrm{dx} + \int_0^{l+a} \theta \frac{l-a}{l+a} qx \mathrm{dx} = 4\overline{\overline{M}}\theta + 2\overline{\overline{M}}\theta \frac{l-a}{l+a}$$

after simple transformations, we have:

$$q = \frac{2\overline{\overline{M}}}{l} \frac{3l + a}{(2l - a)(l + a)}$$

We calculate the extreme:

$$\frac{\partial q}{\partial a} = 0 \quad \rightarrow \quad (2l - a)(l + a) - (3l + a)(l - 2a) = 0$$

we get the equation:

$$a^2 + 6al - l^2 = 0$$

with the core:

$$a = 0.162l$$

and finally:

$$\overline{\overline{q}} = \frac{2\overline{\overline{M}}}{l^2} \frac{3 + 0.162}{(2 - 0.162)(1 + 0.162)} = 2.96 \frac{\overline{\overline{M}}}{l^2} .$$

The result is the same as from the static approach.

Example 15.4

Determine the plastic limit load for the beam shown in the figure below, using the static and the kinematic approach, and $R_e = 330$ [MPa].

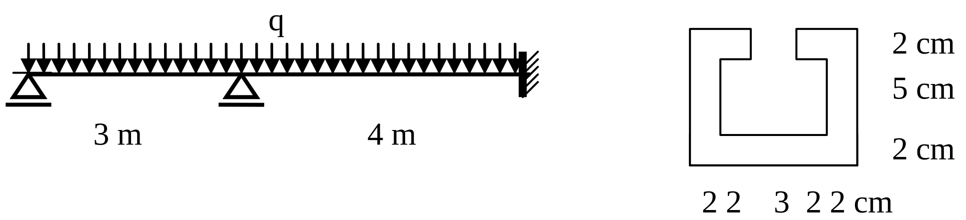


Fig. 15.11 Beam with load

Solution

1. The limit bending moment of the cross-section:

– cross-section area A = 58 cm^2
– gravity center: C(5.5, 4.14)
– principal inertia moment J_y = 512.23 cm^4
– cross-section elastic factor W_{el} = 105.35 cm^3
– position of neutral axis at limit bending moment: y_0 = 0.39 cm (downward from gravity center)
– cross-section plastic factor W_{pl} = 155.75 cm^3
– factors' relation $n = W_{pl}/W_{el}$ = 1.478
– limit bending moment $M_{pl} = 330 \cdot 10^6 \cdot 155.75 \cdot 10^{-6}$ = 51.40 kNm

2. Static approach:

left part of the beam:

– we introduce one plastic hinge which suffices to the part of the beam evolves into statically determined

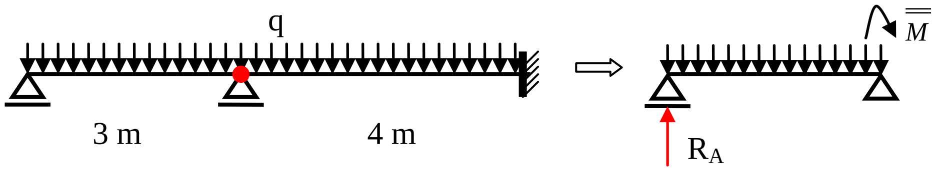


Fig. 15.12 Left part of the beam

$$R_A = \frac{4.5q - \overline{\overline{M}}}{3} = 1.5q - \tfrac{1}{3}\overline{\overline{M}}$$

– we seek the position of zero transverse force:

$$Q(x) = R_A - qx = 1.5q - \tfrac{1}{3}\overline{\overline{M}} - qx = 0 \quad \rightarrow \quad x_p = 1.5 - \frac{\overline{\overline{M}}}{3q}$$

– the bending moment at this point is:

$$M(x_p) = R_A x_p - \frac{q}{2}x_p^2 = \ldots = 1.125q - 0.5\overline{\overline{M}} + \frac{\overline{\overline{M}}^2}{18q}$$

– we assume the next plastic hinge at the point

$$M(x_p) = \overline{\overline{M}} \quad \rightarrow \quad 1.125q^2 - 1.5\overline{\overline{M}}q + \tfrac{1}{18}\overline{\overline{M}}^2 = 0 \quad \rightarrow \quad \overline{\overline{q}} = \frac{1.5 + \sqrt{2}}{2 \cdot 1.125}\overline{\overline{M}} = 1.295\overline{\overline{M}}$$

– the plastic limit load is:

$$\overline{\overline{q}} = 1.295\overline{\overline{M}} = 66.57 \text{ kN/m}$$

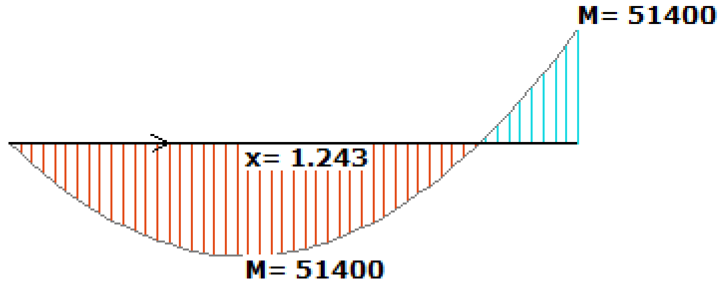


Fig. 15.13 Bending moments

right part of the beam:

– we introduce two plastic hinges which suffice to the part of the beam becomes statically determined

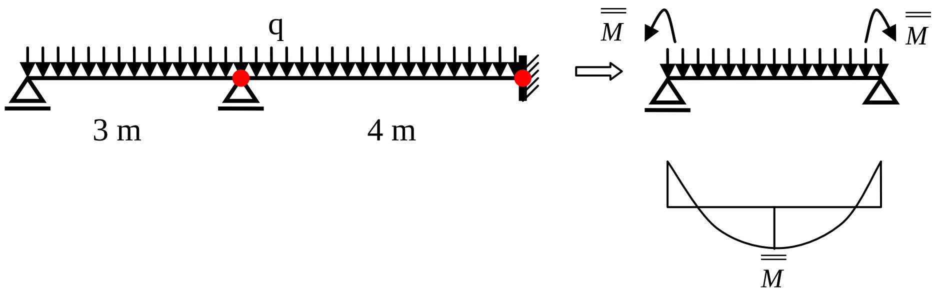


Fig. 15.14 Right part of the beam

– we assume third plastic hinge at the middle of the span:

$$\overline{\overline{M}} - \frac{ql^2}{8} = -\overline{\overline{M}} \quad \rightarrow \quad \frac{ql^2}{8} = 2\overline{\overline{M}} \quad \rightarrow \quad \overline{\overline{q}} = \frac{16}{l^2}\overline{\overline{M}} = \overline{\overline{M}} = 51.40 \text{ kN/m}$$

– the value for the second scheme is less then prior value, so it is evident that the first scheme is statically inacceptable (for this limit load the bending moment at right span will exceed admissible value), and the answer is:

$$\overline{\overline{q}} = 51.40 \text{ kN/m}$$

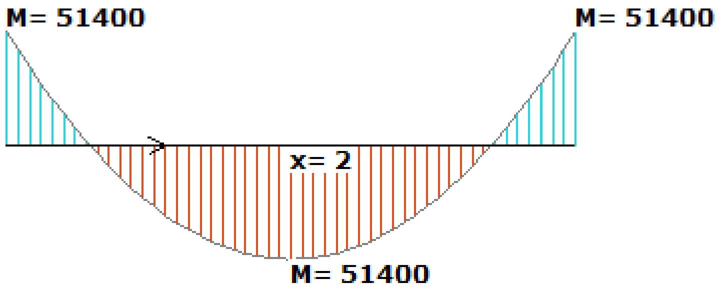


Fig. 15.15 Bending moments

3. Kinematic approach

– we introduce one degree of freedom mechanism at left and right parts of the beam:

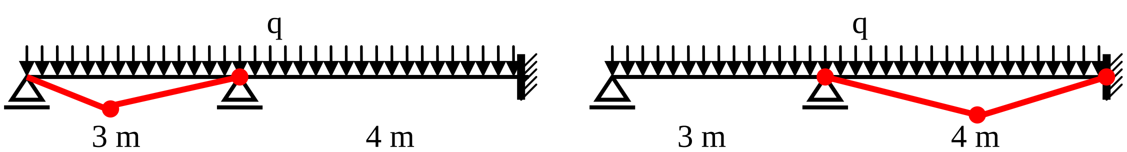


Fig. 15.16 Kinematic schemes of collapse

left part of the beam:

– from geometrical relations (the plastic hinge at *a* from the roller), we have:

$$\theta a = \theta_1 (3-a) \quad \rightarrow \quad \theta_1 = \theta \frac{a}{3-a}$$

– the work of internal forces (always positive)

$$L_{in} = \overline{\overline{M}}(\theta + \theta_1) + \overline{\overline{M}}\theta_1 = \ldots = \overline{\overline{M}}\theta \frac{3+a}{3-a}$$

– the work of external forces

$$L_{ex} = \int_0^a q\theta x dx + \int_0^{3-a} q\theta_1 x dx = \ldots = 1.5qa$$

– the bearing capacity of the scheme

$$L_{\text{int}} = L_{ext} \quad \rightarrow \quad q = \frac{2}{3} \frac{3+a}{(3-a)a} \overline{\overline{M}}$$

– we seek the extreme value of the load

$$\frac{\partial q}{\partial a} = 0 \quad \rightarrow \quad 3a - a^2 - (3+a)(3-2a) = 0 \quad \rightarrow \quad a = 3(\sqrt{2} - 1) = 1.243 \text{ m}$$

– finally, the plastic limit load is:

$$\overset{=}{q} = \frac{2}{3} \frac{3 + 3(\sqrt{2} - 1)}{(3 - 3\sqrt{2} + 3)3(\sqrt{2} - 1)} \overline{\overline{M}} = \ldots = \frac{2\sqrt{2}}{9(3\sqrt{2} - 4)} \overline{\overline{M}} = 1.295\overline{\overline{M}} = 66.56 \text{ kN/m}$$

right part of the beam:

– from the external work compared with the internal one, we have:

$$4\overline{\overline{M}}\theta = 2\int_0^2 q\theta x dx \quad \rightarrow \quad \overset{=}{q} = \overline{\overline{M}} = 51.40 \text{ kN/m}$$

– this value is less then the value in the first scheme, and because we take minimum from both cases, the answer is

$$\overset{=}{q} = 51.40 \text{ kN/m}$$