Tension – part 2

Statically undetermined structures

In the situation when the statics equations are not sufficient we need reference to the complete set of equations:

- static equations (← Navier's equations + static boundary conditions)
- compatibility (geometric) equations (← Cauchy's equations + kinematic conditions)
- constitutive equations (← Hooke's equations, thermal expansion equations)

Linearized geometry

We assume that the bars lengthen or shorten but their angles do not change (the influence of their change is negligible). In this way, the length change is the segment determined by two points: the projection of actual position of the bar's end onto original bar direction and the bar end itself, Fig. 2.1.

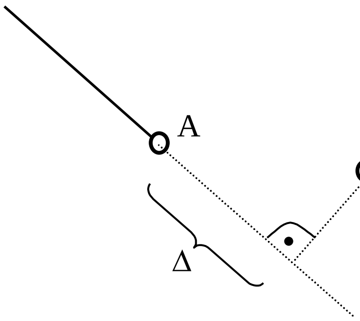

Fig. 2.1 Elongation of the bar

Please remember some concepts of displacement, Fig. 3.6:

– real displacement ("true", "actual", observed),
– possible displacement (that is possible with respect to the constraints)
– virtual displacement (parallel to the virtual velocities).

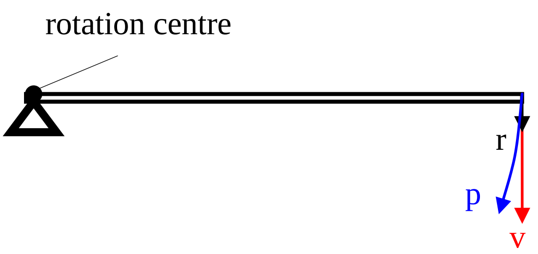

Fig. 2.2 Displacements

Examples of statically indeterminable problems

Example 1

Determine the forces in the deformable bars in Fig. 2.3.

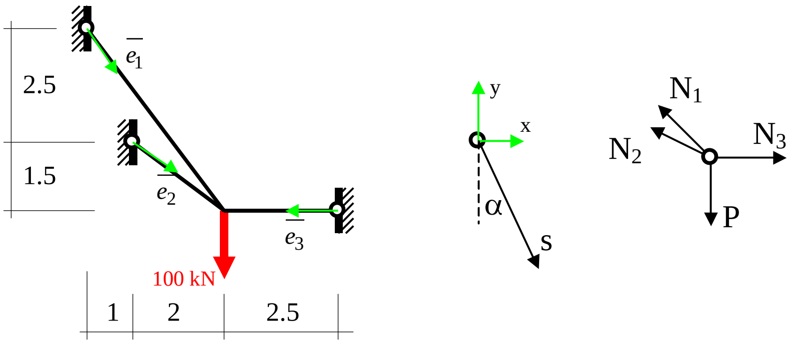

Fig. 2.3 Set of deformable bars, displacements and forces

Solution:

We have:

$$\mathbf{s} = (\sin\alpha, -\cos\alpha)$$

$$\mathbf{e_1} = (0.6, -0.8), \quad \mathbf{e_2} = (0.8, -0.6), \quad \mathbf{e_3} = (-1, 0)$$

hence:

$$\begin{aligned}
\Delta_1 &= \mathbf{e_1} \cdot \mathbf{s} = 0.6s \cdot \sin\alpha + 0.8s \cdot \cos\alpha, \\
\Delta_2 &= \mathbf{e}_2 \cdot \mathbf{s} = 0.8s \cdot \sin\alpha + 0.6s \cdot \cos\alpha, \\
\Delta_3 &= \mathbf{e}_3 \cdot \mathbf{s} = -s \cdot \sin\alpha,
\end{aligned}$$

Eliminating:

$$s \cdot \sin\alpha = -\Delta_3, \quad s \cdot \cos\alpha = \frac{\Delta_2 + 0.8\Delta_3}{0.6}$$

we get:

$$3\Delta_1 - 4\Delta_2 - 1.4\Delta_3 = 0 \tag{1*}$$

Assuming $EA = \text{idem}$ and substituting the Hooke's law we change kinematic unknowns in (1*) into static unknowns:

$$15N_1 - 10N_2 - 3.5N_3 = 0 \tag{1}$$

Moreover, we have two static equations:

$$-0.6N_1 - 0.8N_2 + N_3 = 0 \tag{2}$$

$$0.8N_1 + 0.6N_2 - P = 0 \tag{3}$$

which form with equation (1) the set of three equations with three unknowns. The solution of the set gives us:

$$N_1 = 0.712P, \quad N_2 = 0.717P, \quad N_3 = 1.00P .$$

To check the linear geometry assumption we calculate the loaded point displacement, E = 200 GPa, A = 4 cm^2: $\Delta_1 = \ldots = 4.24$ mm, $\Delta_2 = \ldots = 2.13$ mm, $\Delta_3 = \ldots = 2.98$ mm and $\mathbf{s}(2.98,-7.52)$ mm. This proves that the displacements are sufficiently small.

Tip: Both schemes of virtual displacements and static equilibrium should be compatible. The static forces equilibrium scheme, compatible with the virtual displacements scheme, can be always selected, but not vice versa. Not for every static scheme, the compatible scheme of virtual displacements exists. Therefore, we proceed first with the displacements draft and next with the forces draft.

Example 2

In this example the direction of the loaded point's displacement is imposed (but not its sense) by the rigid link, Fig. 2.4:

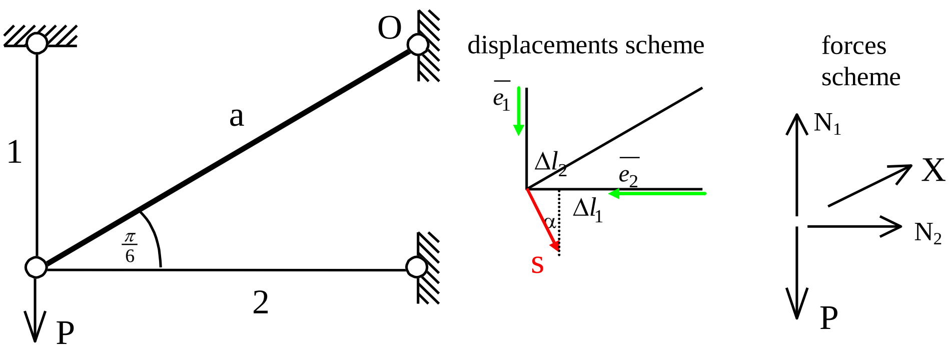

Fig. 2.4 Structure, displacements and forces schemes

Solution:

$$\bar{s} = s(\sin\alpha, -\cos\alpha), \quad \overline{e_1} = (0,-1), \quad \overline{e_2} = (-1,0)$$

hence:

$$\Delta_1 = \bar{s} \cdot \overline{e_1} = s\cos\alpha, \quad \Delta_2 = \bar{s} \cdot \overline{e_2} = -s\sin\alpha .$$

and the geometric equation has the form:

$$\Delta_1 = -\sqrt{3}\Delta_2$$

inserting the constitutive equations, we have:

$$\frac{N_1 l_1}{EF} = \sqrt{3}\frac{N_2 l_2}{EF}$$

so

$$N_1 = 3N_2 \qquad (1)$$

The static equation completes the set:

$$(P - N_1)a\cos\tfrac{\pi}{6} + N_2 a\sin\tfrac{\pi}{6} = 0. \tag{2}$$

The solution of the equations set is:

$$N_1 = \frac{3\sqrt{3}P}{3\sqrt{3}+1} = 0.8386P, \quad N_2 = -\frac{\sqrt{3}P}{3\sqrt{3}+1} = -0.2795P \text{ (compression)}.$$

Example 3

For the truss in Fig. 2.5 determine the axial forces in the bars.

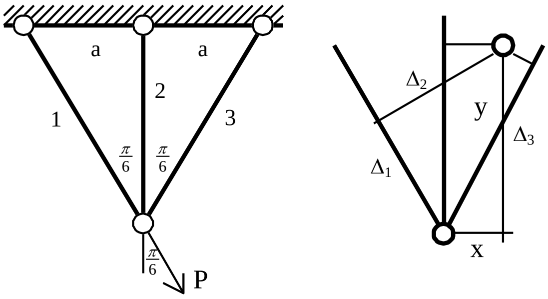

Fig. 2.5 Truss and displacement scheme

Solution:

The loaded point has two degrees of freedom. Introducing two translation parameters we express the change of the bars' length:

$$\begin{cases} \Delta_1 = \overline{e_1} \cdot \overline{AA_1} = -x\sin\tfrac{\pi}{6} + y\cos\tfrac{\pi}{6} \\ \Delta_2 = \overline{e_2} \cdot \overline{AA_1} = y \\ \Delta_3 = \overline{e_3} \cdot \overline{AA_1} = x\sin\tfrac{\pi}{6} + y\cos\tfrac{\pi}{6} \end{cases}.$$

Eliminating the auxiliary parameters, we get the geometric equation in the form:

$$\Delta_3 + \Delta_1 = \sqrt{3}\Delta_2 .$$

Expressing the elongations through the bar forces, the equation with the equations of static equilibrium gives us the set to determine the bars' forces.

Review problems

Problem 1

Determine the forces in the truss bars in Fig. 2.6 developed by the temperature growth by 30 degrees.
Assume $A_1 = A_2 = 3$ cm^2, $A_3 = 5$ cm^2, $a = 3$ m, $E = 205$ GPa, thermal expansion coefficient $\alpha = 14.5 \cdot 10^{-6}$.
Ans.: $N_1 = N_3 = -3.186$ kN, $N_2 = 5.518$ kN.

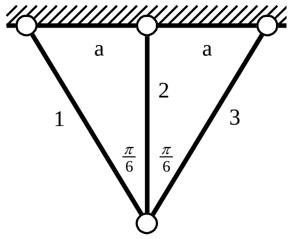

Fig. 2.6 Thermal stress in truss

Problem 2

Determine the forces in the truss bars in Fig. 2.7 developed during assembly, when the bar no 2 was shorter by $\delta = 2$ mm. Assume $A_1 = A_2 = 3$ cm^2, $A_3 = 5$ cm^2, a = 3 m, E = 205 GPa.

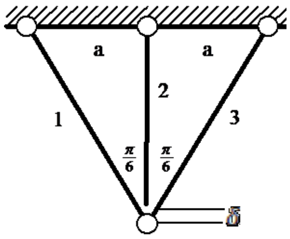

Fig. 2.7 Assembly gap

Ans.: $N_1 = N_3 = -8.46$ kN, $N_2 = 14.65$ kN.

Problem 3

Determine the forces in the deformable bars, Fig. 2.8. Assume $A_1 = 5$ cm^2, $A_2 = 4$ cm^2, $E = 205$ GPa.

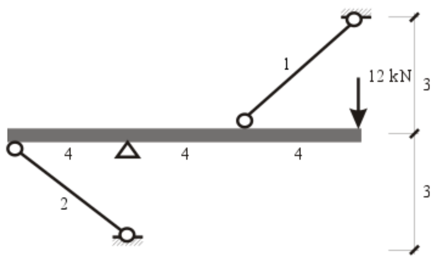

Fig. 2.8 Rigid connector and deformable bars

Ans.: $N_1 = 22.24$ kN, $N_2 = 17.76$ kN.

Problem 4

Determine the axial force diagram in the bar in Fig. 2.9. Assume E = 205 GPa, P = 20 kN, a = 3.5 cm, b = 4 cm, c = 5 cm, A = 7 cm^2.

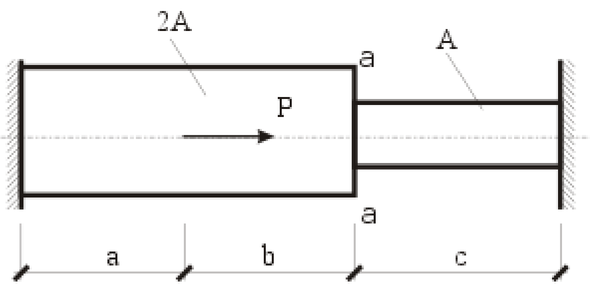

Fig. 2.9 Constrained tension

Ans.: 16 kN (a) and -4 kN (b+c)