Biaxial bending

Basis formulae

There are two cases, apparently different: (a) the bending of the cross-section by the moment the direction of which is not parallel to the one of the principal central axes, and (b) the cross-section loaded by two bending moments acting in the directions of the principal central axes, Fig. 4.1.

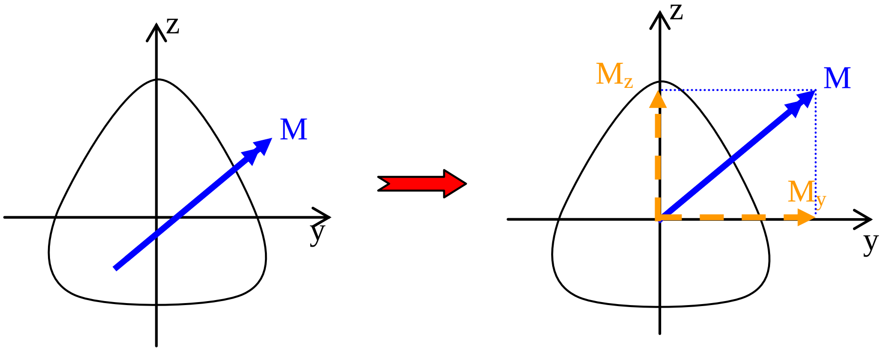


Fig. 4.1 Bending moment and its decomposition

It is obvious from the decomposition shown, that the problem is the same in both cases..

Note, that we always consider the directions of the bending moments in relation with the directions of the principal central axes.

Tip: there is no biaxial bending of such cross-sections as circle, square, and any regular polygon; there is no biaxial bending because any central axis of a cross-section considered is the principal axis, too.

The normal stress is the sum of the normal stresses of two bending components:

$$\sigma_x = \frac{M_y}{J_y} z - \frac{M_z}{J_z} y \,.$$

The distribution of the normal stress in the cross-section is linear.

The neutral axis is a straight line that passes through the cross-section centroid:

$$\sigma_x = 0 \quad \rightarrow \quad z = \frac{M_z}{M_y} \cdot \frac{J_y}{J_z} y \,.$$

The neutral axis doesn't agree with the direction of the resultant bending moment: it declines more or less slightly in the direction of the axis of the greatest flexibility (of smaller principal central inertia moment).

The maximal value of the normal stress is reached at the points the most distant from the neutral axis.

From the ultimate limit state we have the design inequality:

$$\max\left|\sigma_x\right| \le R \,,$$

where R is the material strength.

To determine the most exerted points of the cross-section we have to know the position of the neutral axis.

Examples

Example 4.1

The T-shape steel purlin, Fig. 4.3, is bent by the horizontal vector of bending moment. Knowing that the acceptable value of the normal stress is 150 MPa, determine the value of the allowable moment. The cross-section characteristics are: $s = h = 60$ mm, $e = 18.6$ mm, $I_y = 23.8$ cm^4, $I_z = 12.2$ cm^4, the angle $\alpha = 12.5°$.

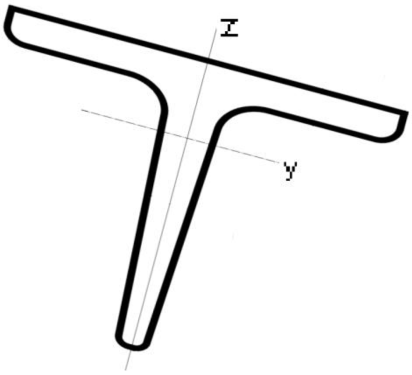


Fig. 4.2 T-shape purlin

Solution

the decomposition of the bending moment:

$$M_y = M\cos\alpha, \quad M_z = M\sin\alpha,$$

the normal stress:

$$\sigma_x = \frac{M_y}{J_y} z - \frac{M_z}{J_z} y = M\left(\frac{\cos\alpha}{J_y} z - \frac{\sin\alpha}{J_z} y\right),$$

the neutral axis equation:

$$z = \tan\alpha \cdot \frac{23.8}{12.2} y = 0.4325y$$

from the cross-section drawing, Fig. 4.3:

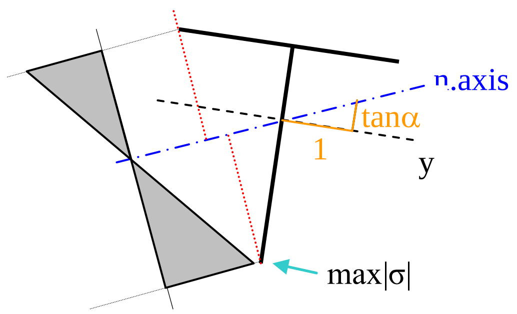


Fig. 4.3 Cross-section with neutral axis

we get the maximal stress at the web corner, near the symmetry axis, (0, -41.4) in [mm]:

$$\sigma_x = M(0.041021\cdot(-4.14))\cdot 10^6 = 0.1698\cdot 10^6 M$$

from ultimate limit state, we have:

$$|\sigma_x| \le r \quad \to \quad 0.1698\cdot 10^6 M \le 150\cdot 10^6 \quad \to \quad M \le 883 \text{ Nm}$$

Example 4.2

Determine the cross-section parameter *a*, Fig. 4.4, knowing that the acceptable stress value is *R* = 300 MPa.

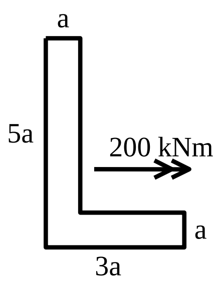


Fig. 4.4 Cross-section with load

Solution

position of the centroid: y_c = 1.17 a, z_c = 2.17 a

central inertia moments: J_y = 30.75 a^4, J_z = 10.75 a^4, J_{yz} = -10 a^4

eigenvalues: J_1 = 34.89 a^4, J_2 = 6.61 a^4, principal directions: α = 22.49°

bending moments: M_1 = Mcosα = 185 kNm, M_2 = -M sinα = -76.5 kNm

normal stress distribution: $\sigma_x = \frac{M_1}{J_1}x_2 - \frac{M_2}{J_2}x_1 = \frac{5.30x_2 + 11.57x_1}{a^4}10^3$

neutral axis equation: $x_2 = -2.19x_1$

we calculate the coordinates of the corners in principal central coordinates from the transformation formula

pt	y	z	x_1	x_2	from neutral axis.	σ [MPa]
A	-1.17 a	-2.17 a	-1.91 a	-1.56 a	2.39 a	-243
B	2.83 a	-2.17 a	1.78 a	-3.09 a	0.34 a	35
C	2.83 a	-1.17 a	2.17 a	-2.16 a	1.07 a	109
D	-0.17 a	3.83 a	1.31 a	3.60 a	2.69 a → max.	274
E	-1.17 a	3.83 a	0.38 a	3.99 a	2.01 a	205

Tab. 4.1 Calculation data for the problem 3.5.

The most distant point from the neutral axis is point D where the stress value is the greatest.

The cross-section design: $\sigma_x(1.31\ a, 3.60\ a) \le R \quad \Rightarrow \quad a \ge 0.0485\ m \approx 0.05\ m$

The normal stress for assumed value of the parameter *a* are stated in the Tab. 4.1 and the diagram of the normal stress distribution is shown in Fig. 4.5.

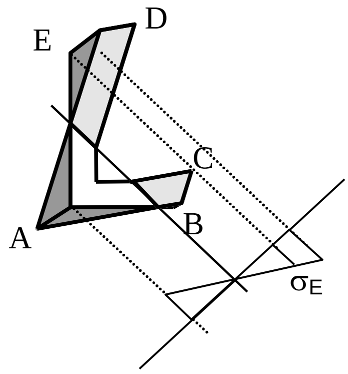


Fig. 4.5 Normal stress distribution

Example 4.3

A beam cross section is formed by nailing together two 50 mm by 150 mm boards as indicated in Figure 4.6 and loaded by a moment of 2 kNm. The plane of the load passes through the centroid of the cross section as indicated. Determine the maximum flexure stress in the cross section and the orientation of the neutral axis.

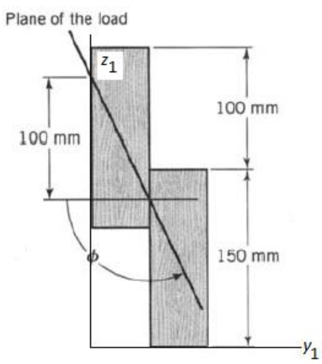


Fig. 4.6 Cross section for ex. 4.3

Solution

position of the centroid: $C(50, 125)$

central inertia moments

$$I_{y0} = \frac{50 \cdot 150^3}{12} + 50 \cdot 150 \cdot (175 - 125)^2 + \frac{50 \cdot 150^3}{12} + 50 \cdot 150 \cdot (75 - 125)^2 = 6.563 \cdot 10^7 \text{ mm}^4$$

$$I_{z0} = \frac{50^3 \cdot 150}{12} + 50 \cdot 150 \cdot 25^2 + \frac{50^3 \cdot 150}{12} + 50 \cdot 150 \cdot 25^2 = 1.25 \cdot 10^7 \text{ mm}^4$$

$$I_{y0z0} = 50 \cdot 150 \cdot (25 - 50) \cdot (175 - 125) + 50 \cdot 150 \cdot (75 - 50) \cdot (75 - 125) = -1.875 \cdot 10^7 \text{ mm}^4$$

principal central inertia moments and directions

$$I_y = \frac{6.563 \cdot 10^7 + 1.25 \cdot 10^7}{2} + \sqrt{\left(\frac{6.563 \cdot 10^7 - 1.25 \cdot 10^7}{2}\right)^2 + 1.875 \cdot 10^7} = 7.158 \cdot 10^7 \text{ mm}^4$$

$$I_z = \frac{6.563 \cdot 10^7 + 1.25 \cdot 10^7}{2} - \sqrt{\left(\frac{6.563 \cdot 10^7 - 1.25 \cdot 10^7}{2}\right)^2 + 1.875 \cdot 10^7} = 6.549 \cdot 10^6 \text{ mm}^4$$

$$\tan\alpha = \frac{7.158 \cdot 10^8 - 6.563 \cdot 10^7}{-(-1.875 \cdot 10^7)} = 0.3173 \to \alpha = 17.61°$$

bending moment decomposition

moment direction relative to horizontal axis: $90 - \tan^{-1} 2 = 90 - 63.43 = 26.57°$

moment direction relative to the principal central axis *y*: $26.57 - 17.61 = 8.955°$

$M_y = 2 \cdot \cos(8.955) = 1.976$ kNm, $M_z = 2 \cdot \sin(8.955) = 0.3113$ kNm

normal stress

$$\sigma_x = \frac{M_y}{I_y} z - \frac{M_z}{I_z} y = \frac{1976}{7.158 \cdot 10^{-5}} z - \frac{311.3}{6.549 \cdot 10^{-6}} y = 27.61z - 47.53y \text{ (in MPa)}$$

position of the neutral axis

$$\sigma_x = 0 \to z = 1.721y \to \ \beta = 59.85°$$

maximum flexure stress at left upper corner (input → central → principal central)

$$A(0, 250) \to (-50, 125) \to (-9.840, 134.3)$$

$$\sigma_x^A = 27.61 \cdot 0.1343 - 47.53 \cdot (-0.00984) = 4.175 \text{ MPa}$$

Example 4.4

A girder that supports a brick wall is built up of an s-310×47 I-beam ($A_1 = 6030$ mm^2, $I_{y1} = 90.7 \times 10^6$ mm^4, $I_{z1} = 3.90 \times 10^6$ mm^4), a C-310×31 channel ($A_2 = 3930$ mm^2, $I_{y2} = 53.7 \times 10^6$mm^4, $I_{z2} = 1.61 \times 10^6$ mm^4), and a cover plate 300 mm by 10 mm riveted together (Fig. 4.7). The girder cross section is loaded by moment $M = 90$ kNm (bottom tensioned). Determine the orientation of the neutral axis and the maximum tensile and compressive stress.

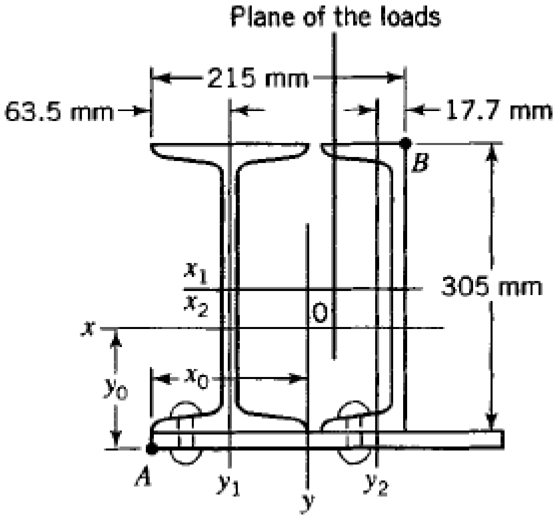


Fig. 4.7 Cross section from the problem 4.4

Solution

positions of the elements' centroids

I-beam (63.5, 162.5)

C-channel (197.3, 162.5)

plate (150, 5)

cross section area A = 6030 + 3930 + 3000=12960 mm^2

centroid of the cross section

$$y_c = \frac{6030\cdot 63.5+3930\cdot 197.3+3000\cdot 150}{12960} = 124.1 \text{ mm}$$

$$z_c = \frac{6030\cdot 162.5+3930\cdot 162.5+3000\cdot 5}{12960} = 126.0 \text{ mm}$$

central inertia moments

$$I_{yc} = 90.7\cdot 10^6 + 6030\cdot(162.5-126)^2 + 53.7\cdot 10^6 + 3930\cdot(162.5-126)^2 + 0.025\cdot 10^6 + 3000\cdot(5-126)^2 = 201.6\cdot 10^6 \text{ mm}^4$$

$$I_{zc} = 3.90\cdot 10^6 + 6030\cdot(63.5-124.1)^2 + 1.61\cdot 10^6 + 3930\cdot(197.3-124.1)^2 + 22.5\cdot 10^6 + 3000\cdot(150-124.1)^2 = 77.20\cdot 10^6 \text{ mm}^4$$

$$I_{yczc} = 0 + 6030\cdot(63.5-124.1)\cdot(162.5-126) + 0 + 3930\cdot(197.3-124.1)\cdot(182.5-126) + 0 + 3000\cdot(150-124.1)\cdot(5-126) = -12.24\cdot 10^6 \text{ mm}^4$$

principal central inertia moments

$$I_y = \frac{201.6+74.20}{2}\cdot 10^6 + \sqrt{\left(\frac{201.6-74.20}{2}\right)^2 + 12.24^2}\cdot 10^6 = 202.8\cdot 10^6 \text{ mm}^4$$

$I_z = \frac{201.6+74.20}{2} \cdot 10^6 - \sqrt{\left(\frac{201.6-74.20}{2}\right)^2 + 12.24^2} \cdot 10^6 = 73.03 \cdot 10^6$ mm^4

$$\tan\alpha = \frac{1.2}{12.24} = 0.098 \rightarrow \ \alpha = 5.60°$$

bending moment decomposition

$M_y = 90 \cdot \cos 5.6\,° = 89.57$ kNm, $M_z = 90 \cdot \sin 5.6° = 8.78$ kNm

normal stress equation

$$\sigma_x = \frac{M_y}{I_y} z - \frac{M_z}{I_z} y = \frac{89570}{202.8 \cdot 10^{-6}} z - \frac{8780}{73.03 \cdot 10^{-6}} y$$

neutral axis equation

$$\sigma_x = 0 \rightarrow z = 0.2722y$$

angle: 15.23°

this is a simple case of 4-corners cross section, maximum tensile stress is at right bottom corner B, and maximum compression stress is at left top corner A

$$A(-124.1, 189) \rightarrow (-70.09, 214.9)$$

$$B(175.9, -126) \rightarrow (136.6, -167.8)$$

so

$\sigma_x^A = \frac{89570}{202.8 \cdot 10^{-6}}(-0.0701) - \frac{8780}{73.03 \cdot 10^{-6}} 0.2149 = -56.79$ MPa

$\sigma_x^B = \frac{89570}{202.8 \cdot 10^{-6}} 0.1366 - \frac{8780}{73.03 \cdot 10^{-6}}(-0.1678) = 80.51$ MPa

(Attention! If we neglect the small angle of the principal directions we would get $\sigma_x^A = -84.4$ MPa and $\sigma_x^B = 56.3$ MPa)

Review problems

Problem 4.1

Determine the parameter a of the cross-section in Fig. 4.9. The allowable stress value is 280 MPa. Draw the stress repartition diagram. (Ans.: $a \geq 0.0056$ m)

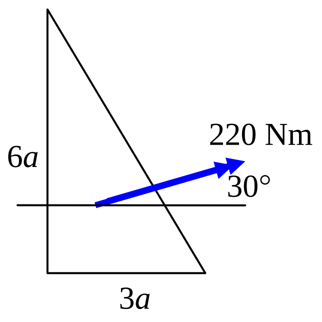


Fig. 4.9 Cross-section with loading

Problem 4.2

Determine the normal stress distribution of the cross-section in Fig. 4.10. Draw the normal stress diagram and determine the stress at p. A. (Ans.: 53.7 MPa)

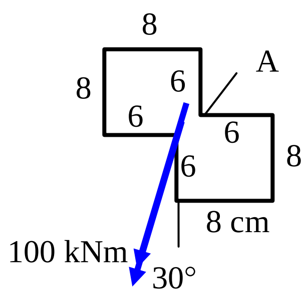


Fig. 4.10 Cross-section with loading

Problem 4.3

A C-180×15 rolled steel ($I_y = 8.87 \times 10^6$ mm^4, $I_z = 4.04 \times 10^5$ mm^4, depth = 178 mm, width = 53 mm, and $x_B = 13.7$ mm) is used as a purlin in a roof (Fig. 4.11). If the slope of the roof is ½ and the applied bending moment is 2 kNm, determine the maximum tensile and compressive stresses. (Ans.:105.2 MPa, -48.4 MPa)

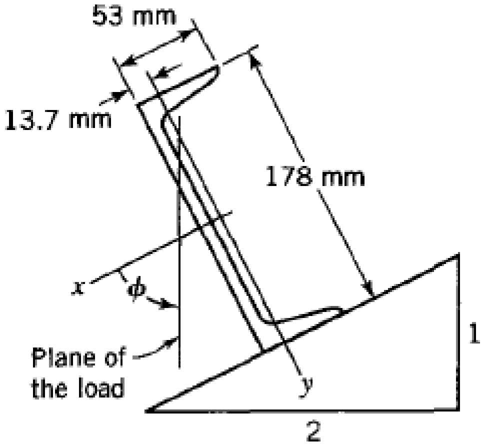


Fig. 4.11 C-channel purlin

Problem 4.4

An I-beam has the cross section shown in Fig. 4.12. The design flexural stress is limited to 120 MPa. Determine the allowable bending moment *M*. (Ans.: 13.2 kNm)

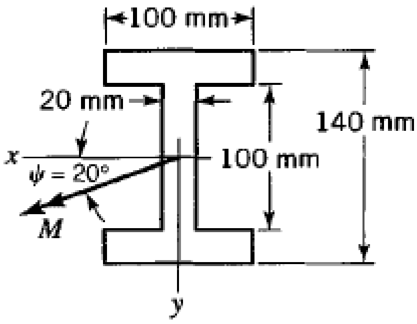
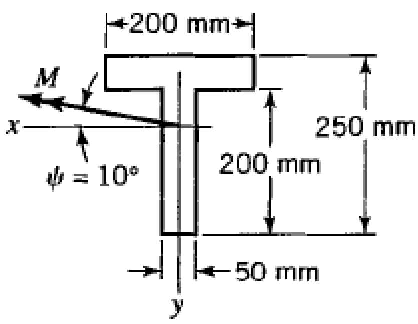


Fig. 4.12 I-beam cross section

Problem 4.5

A T-beam has the cross section shown in Fig. 4.13. The design flexural stress is limited to 150 MPa. Determine the allowable bending moment *M*. (Ans.: 97.9 kNm)



Addendum

Some useful formulae

The transformation formula for the point's coordinates:

$$x_1 = y\cos\alpha + z\sin\alpha$$
$$x_2 = -y\sin\alpha + z\cos\alpha$$

The distance from a point P to a line $y\cos\alpha + z\sin\alpha - p = 0$, ([i]):

$$d = \left| y_P \cos\alpha + z_P \sin\alpha - p \right|$$

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[i] The normal form of the line equation can be derived from the general form of the line:

$$Ay + Bz + C = 0$$

dividing by a normalizing factor:

$$\mu = -\frac{sign(C)}{\sqrt{A^2 + B^2}}$$

we get then the normal form:

$$y\cos\alpha + z\sin\alpha - p = 0$$

where the parameter p means the distance of the line from the origin of the coordinate set.