Eccentric tension/compression

Basis formulae

$$\sigma_x = \frac{N}{A} + \frac{Nz_N}{J_y} z + \frac{Ny_N}{J_z} y = \frac{N}{A}\left(1 + \frac{z_N z}{i_y^2} + \frac{y_N y}{i_z^2}\right)$$

The relation obtained shows that the distribution of stresses across the section is linear. As before, we need to know the neutral axis position. The most distant fibers from the neutral axis are the most exerted section part, where the normal stress is maximal.

The neutral axis equation can be written in the form:

$$\sigma_x = 0 \quad \rightarrow \quad z = -\frac{i_y^2}{z_N}\left(1 + \frac{y_N y}{i_z^2}\right).$$

The neutral axis is the straight line, but this time it doesn't pass through the cross section centroid.

From the ultimate limit state we get the design condition:

$$\max|\sigma_x| \le R,$$

where the maximum value of the normal stress is attained at the most distant points from the neutral axis.

Tip: Massive members in eccentric compression can be calculated as in tension with the change of the stress sign.

Examples

Example 5.1

Determine the maximum value of the normal stress in the cross-section loaded by the force N = 150 kN, Fig. 5.1 (the dimensions in cm).

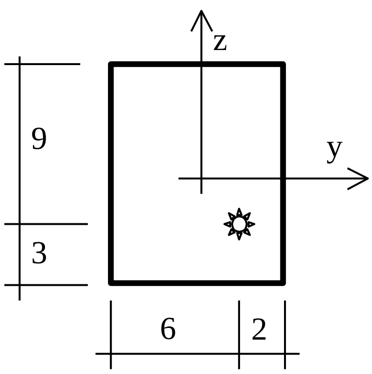

Fig. 5.1 Eccentrically loaded section

Solution

1. The cross-section forces:

$N = 150$ kN, $M_y = -150 \cdot 0.03 = -4.5$ kNm, $M_z = -150 \cdot 0.02 = -3$ kNm

2. The geometric characteristics of the cross-section: A = 96 cm^2, I_y = 1152 cm^4, I_z = 512 cm^4

3. The normal stress distribution:

$$\sigma_x = \frac{N}{A} + \frac{M_y}{I_y} z - \frac{M_z}{I_z} y = \frac{150 \cdot 10^3}{96 \cdot 10^{-4}} - \frac{4.5 \cdot 10^3}{1152 \cdot 10^{-8}} z + \frac{3 \cdot 10^3}{512 \cdot 10^{-8}} y = 15.7 - 391z + 586y \text{ MPa}$$

4. The equation of the neutral axis:

$$z = 1.5y + 0.04$$

5. The maximum stress will occur at the right lower corner:

$$\max|\sigma_x| = 15.7 - 391 \cdot (-0.06) + 586 \cdot 0.04 = 62.6 \text{ MPa}$$

Example 5.2

Determine the cross-section parameter a, Fig. 5.2, knowing that the tension force is N = 150 kN and the bending moment is M_y = 75 kNm. Assume R = 250 MPa.

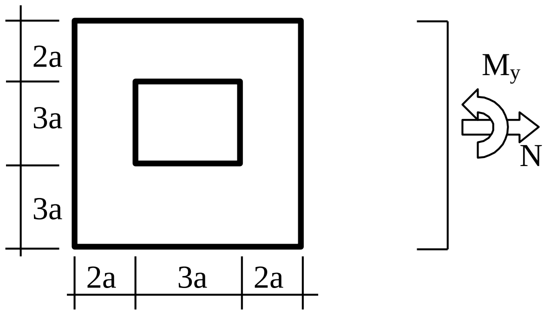

Fig. 5.2 Box section loads

Solution:

1. The cross-section characteristics: A = (56-9) a^2 = 47 a^2, z_c = 3.9 a, I_y = 289 a^4

2. The normal stress distribution:

$$\sigma_x = \frac{N}{A} + \frac{M_y}{I_y} z = \frac{150 \cdot 10^3}{47a^2} - \frac{75 \cdot 10^3}{289a^4} z$$

3. The equation of the neutral axis: $z = 12.3a^2$

4. The distribution of normal stress shows that the greatest normal stress occurs at bottom fibers (the most distant). We write the design condition in the form:

$$\max \sigma_x = \sigma_x(z = -3.9a) = \frac{3190}{a^2} + \frac{1012}{a^3} \le R = 250 \cdot 10^6$$

The above inequality can be solved numerically or – as long as we don't need exact solution – by the trial and error method. As the first approximation, we can use separate solutions for the tension and the bending, obtaining: a = 0.0036 m and a = 0.016 m, respectively.

5. Assuming a = 0.0165 m, we get the maximum normal stress: 237 MPa < R.

Example 5.3

Determine the maximum value of the normal stress in the section loaded by the bending moment M_y = 65 Nm and tensioned by axial force N = 0.2 kN, Fig. 5.3 (the section dimensions in cm):

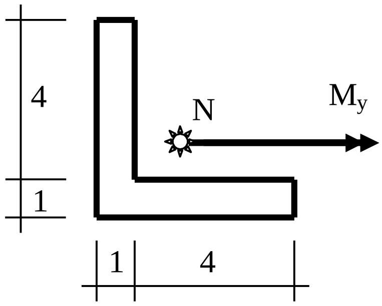

Fig. 5.3 Cross-section with loading

Solution

1. The cross-section characteristics: A = 9 cm^2, due to the oblique symmetry axis we start straight away from the principal central axes (as a difference of two squares):

   the centroid position: z_c = 1.611 cm,

   the inertia moment about the symmetry axis = $\frac{5^4}{12}-\frac{4^4}{12}=30.75\ \text{cm}^4 = I_y$

   the second inertia moment:

$$\frac{5^4}{12}+25\left(1.611\cdot\sqrt{2}-\frac{\sqrt{2}}{2}5\right)^2-\frac{4^4}{12}-16\left(1.611\cdot\sqrt{2}-\frac{\sqrt{2}}{2}4-\sqrt{2}\right)^2=8.53\ \text{cm}^4 = I_z$$

2. The cross-section forces: N = 200 N, $M_y = -M_z$ = 45.96 Nm
3. The normal stress distribution:

$$\sigma_x=\frac{N}{A}+\frac{M_y}{I_y}z-\frac{M_z}{I_z}y=\frac{200}{9\cdot 10^{-4}}+\frac{45.96}{30.75\cdot 10^{-8}}z+\frac{45.96}{8.53\cdot 10^{-8}}y$$

4. The neutral axis equation:

$$z=-0.0015-3.7y$$

5. The above equation shows, that the most distant point from the neutral axis is p. A; its coordinates in the principal central coordinate set are:

$$A\left(4\frac{\sqrt{2}}{2}+\sqrt{2}-1.611\sqrt{2}=2.38,\quad 4\frac{\sqrt{2}}{2}=2.83\right)\text{(in cm)}$$

6. The extreme normal stress at the point A is:

$$\sigma_x^A = 17.2\ \text{MPa}.$$

Example 5.4

Determine the cross-section parameter a, Fig. 5.4, knowing that the applied eccentric force is N = 45 kN and the eccentricities are proportional to the cross-section dimensions. Assume R = 250 MPa.

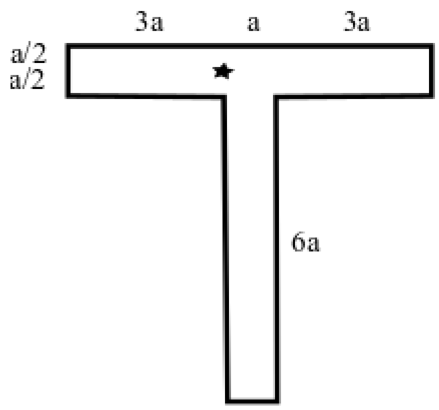

Fig. 5.4 Section loaded at variables eccentricities

Solution

1. Cross-section geometric characteristics:
   Area: $A = 7a^2 + 6a^2 = 13a^2$
   Centroid position: $z_c = \frac{7a^2 \cdot 6.5a + 6a^2 \cdot 3a}{13a^2} = 4.885a$
   Principal central inertia moments:
$$I_y = \frac{7a \cdot a^3}{12} + 7a^2 \cdot (6.5a - 4.885a)^2 + \frac{a \cdot 6^3 a^3}{12} + 6 \cdot (3a - 4.885a)^2 = 58.16a^4$$
$$I_z = \frac{a^3 \cdot 6a}{12} + \frac{a \cdot 7^3 a^3}{12} = 29.08a^4$$
2. Cross-sectional forces:
   Axial force: $N = 45$ kNm
   Bending moment about axis *y*: $M_y = 1.615a \cdot 45 \cdot 10^3 = 72675a$
   Bending moment about axis *z*: $M_z = 0.5a \cdot 45 \cdot 10^3 = 22500a$
3. Normal stress distribution:
$$\sigma_x = \frac{45 \cdot 10^3}{13a^2} + \frac{72675a}{58.16a^4} \cdot z - \frac{22500a}{29.08a^4} y = \frac{3461.5}{a^2} + \frac{1249.6}{a^3} z - \frac{773.73}{a^3} y$$
4. Neutral axis position:
   $\sigma_x = 0 \rightarrow z = 0.6192y - 2.7701a$ → left upper corner is the most distant

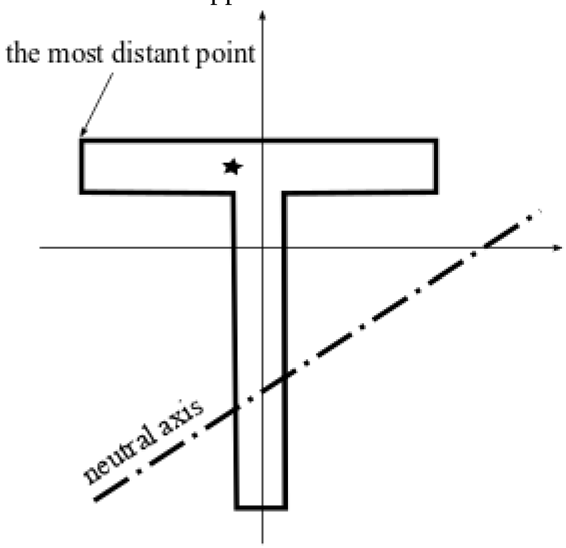

Fig. 5.5 Neutral axis and the most distant point

[We can use the analytic geometry method instead:
dividing the line general equation:

$$Ay + Bz + C = 0$$

by the norm parameter:

$$\mu = \frac{1}{\sqrt{A^2 + B^2}}$$

we get the normal form of the line (so-called Hesse's form):

$$y\cos\alpha + z\sin\alpha - p = 0$$

(where p is the distance of the line from the origin point)
and distance d from a point $1(y_1, z_1)$ to the line is given by the formula:

$$d = y_1\cos\alpha + z_1\sin\alpha - p$$

(where $d > 0$ means that the points P and of origin are on different sides of the line)
Using the above method, we get for the left upper point $d = -5.996a$ (the points on the same side of the line), and for the right lower point $d = 2.061a$ (the points on different sides of the line). The lest upper corner is the most distant from the neutral axis; at this points the normal stress is the greatest.]

5. Design of the parameter a:

   $\max|\sigma_x| \le R \rightarrow \frac{3461.5}{a^2} + \frac{1249.6}{a^3}2.115a - \frac{773.73}{a^3}(-3.5a) \le 250 \cdot 10^6 \rightarrow \frac{8812.5}{a^2} \le 250 \cdot 10^6$

   $a \ge 0.00594$ m, we assume $a = 6$ mm
6. Checking

   $A = 468$ mm^2, $z_c = 29.31$ mm, $I_y = 75375$ mm^4, $I_z = 37688$ mm^4

   $N = 45$ kN, $M_y = 436.1$ Nm, $M_z = 135$ Nm

   $\max\ |\sigma_x| = \frac{45000}{468\cdot10^{-6}} + \frac{436.1}{75375\cdot10^{-12}}\cdot 0.01269 + \frac{135}{37688\cdot10^{-12}}\cdot 0.021 = 96.15 + 73.42 + 75.22 =$
   244.8 MPa < 250 MPa, OK

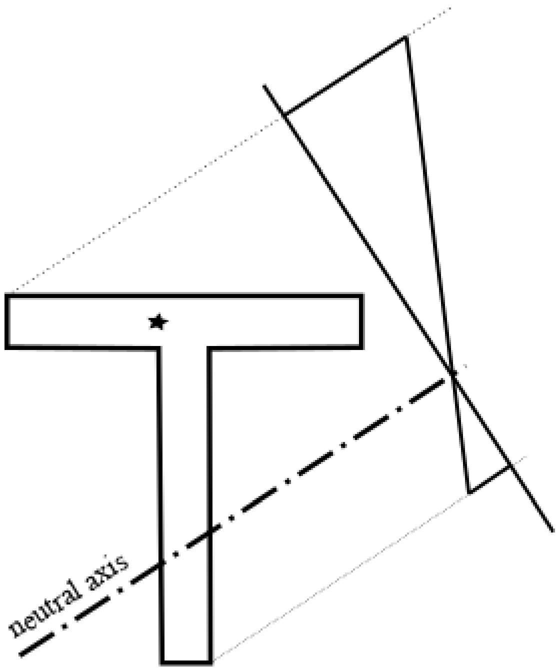

Fig. 5.6 Stress distribution

Example 5.5

Determine the cross-section parameter *a*, Fig. 5.7, knowing that the applied eccentric force is *N* = 45 kN and the eccentricities are $e_y = -0.3$ cm and $e_z = 1$ cm. Assume *R* = 250 MPa.

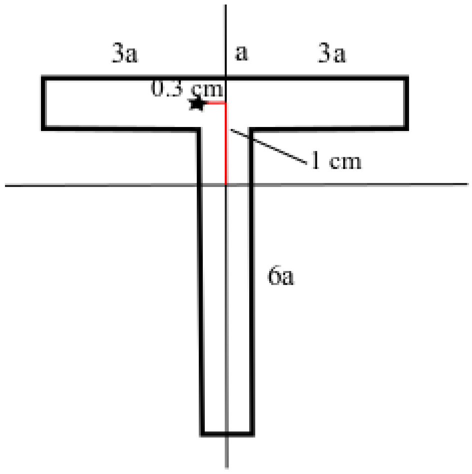

Fig. 5.7 Section loaded at constant eccentricities

Solution

1. Cross-section geometric characteristics:
   Area: $A = 7a^2 + 6a^2 = 13a^2$
   Centroid position: $z_c = \frac{7a^2 \cdot 6.5a + 6a^2 \cdot 3a}{13a^2} = 4.885a$
   Principal central inertia moments:
$$I_y = \frac{7a \cdot a^3}{12} + 7a^2 \cdot (6.5a - 4.885a)^2 + \frac{a \cdot 6^3 a^3}{12} + 6 \cdot (3a - 4.885a)^2 = 58.16a^4$$
$$I_z = \frac{a^3 \cdot 6a}{12} + \frac{a \cdot 7^3 a^3}{12} = 29.08a^4$$
2. Cross-sectional forces:
   Axial force: $N = 45$ kNm
   Bending moment about axis y: $M_y = 0.01 \cdot 45 \cdot 10^3 = 450$ Nm
   Bending moment about axis z: $M_z = 0.003 \cdot 45 \cdot 10^3 = 135$ Nm
3. Normal stress distribution:
$$\sigma_x = \frac{45 \cdot 10^3}{13a^2} + \frac{450}{58.16a^4} \cdot z - \frac{135}{29.08a^4} y = \frac{3461.5}{a^2} + \frac{7.7373}{a^4} z - \frac{4.6424}{a^4} y$$
4. Neutral axis position:
   $\sigma_x = 0 \rightarrow z = 0.6y - 447.38a^2$ → left upper corner is the most distant
5. Design of the parameter a:
$$\max|\sigma_x| \leq R \rightarrow \frac{3461.5}{a^2} + \frac{7.7373}{a^4} 2.115a - \frac{4.6424}{a^4}(-3.5a) \leq 250 \cdot 10^6$$
   from the first term: $a \geq 3.72$ mm
   from the second term: $a \geq 4.03$ mm
   from the third term: $a \geq 4.02$ mm
   trial and error method → for 6 mm max $|\sigma_x| = 247.1$ MPa.
   Answer: $a = 6$ mm.

Review problems

Problem 5.1

Determine the parameter a, Fig. 5.8, assuming P = 1 MN and R = 220 MPa. (Ans.: a = 2.10 cm)

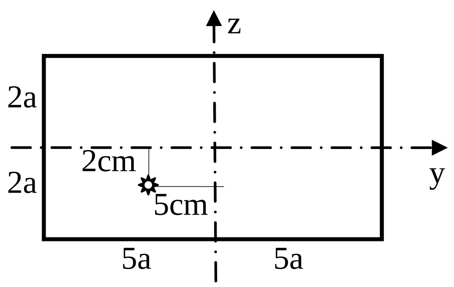

Fig. 5.8 Section loaded at constant eccentricities

Problem 5.2

Determine the parameter *a*, Fig. 5.9, assuming *P* = 1 MN and *R* = 220 MPa. (Ans.: *a* = 2.05 cm)

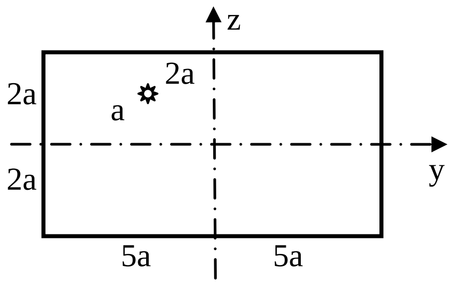

Fig. 5.9 Section loaded at variable eccentricities

Problem 5.3

Determine the normal stress distribution across the section in Fig. 5.10, *a* = 2 cm, loaded by a force *N* = 150 kN. The force is applied to a rigid plate, welded at the member end section. Draw the diagram of the normal stress. (Ans.: max σ_x =59.34 MPa at lower right corner)

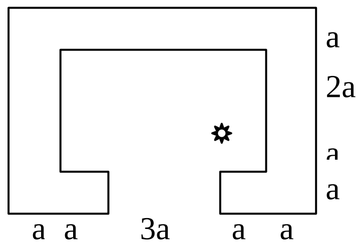

Fig. 5.10 Section with loading point

Problem 5.4

The spot footing, Fig. 5.11, is loaded by a moment *M* = 1.2 MNm and axial force *N* = 2.5 MN. Determine the *d* dimension so that the normal stress distribution at the bottom of footing would be uniform. (Ans.: *d* = 1.96 m)

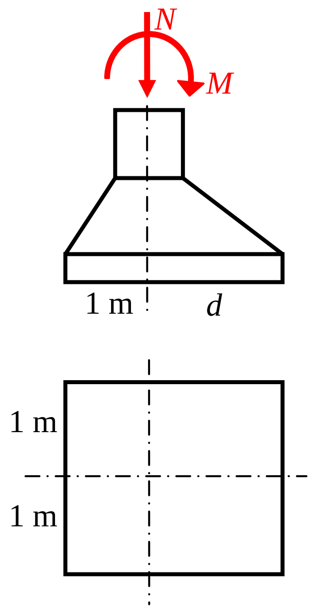

Fig 5.11 Spot footing