Transverse bending

Basis formulae

The normal stress formula is similar to that of the pure (circular) bending:

$$\sigma_x = \frac{M_y(x)}{J_y} z,$$

but the bending moment as well as the stress are functions of the *x* axis coordinate.

The formula for the shearing stress in the cross-section plane is stated as:

$$\tau_{xz} = \frac{Q(x) S_y^I(z)}{J_y b(z)},$$

where:

$Q(x)$ is the cross-section shear force,

$S_y^I(z)$ is the static moment of cut cross-section part,

J_y is the inertia moment of the whole cross-section,

$b(z)$ is the width of the cut line.

The sense and sign of the shear stress depends on the sense of the cross-section shear force and the assumed sense of the *z*-coordinate but the dependence is not as straightforward as it seems to be. Let's consider an element cut from a beam with positive shear force, Fig. 7.1.

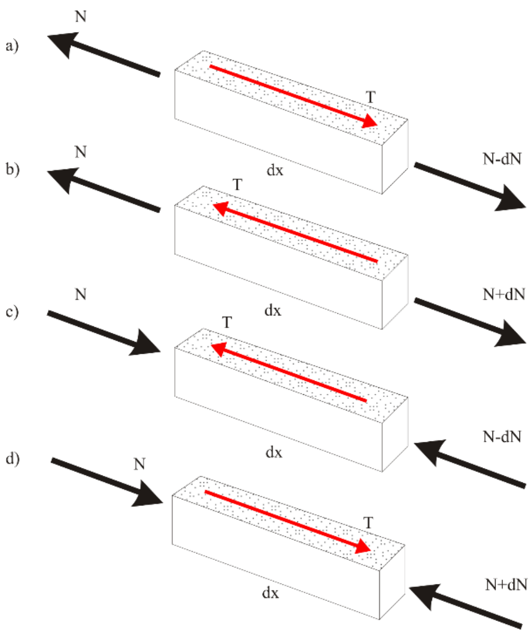

Fig. 7.1 Longitudinal shear force direction

The direction of the shearing stresses may be obtained from the direction of the longitudinal force. Let's consider the balance of the longitudinal forces acting. Positive shear force means the increasing moment with bottom fibers tensioned or the decreasing moment with bottom fibers compressed. In both cases, the element's balance is ensured by the shearing forces *T* of the same sense. These longitudinal shear forces cause in turn the shearing stress, τ_{zx}, with the same direction and sense. Knowing that the shear stresses act at the edge in pairs, $\tau_{zx} = \tau_{xz}$, we can determine the sign of the shear stress. The sign is not objective; it depends on the adopted coordinate axes.

Usually, the maximum value of the shear stress is attained at the neutral axis for the normal stress.

Design

In most cases the dominant criterion in the design of a beam for strength is the maximum value of the normal stress in the beam:

$$\max|\sigma_x| = \frac{M_y}{W_y} \leq R.$$

Next, the condition for the shear stress should be verified:

$$\max|\tau_{xz}| \leq R_t$$

where R_t is the shearing strength, a material constant.

Examples

Example 7.1

Determine the shear stress distribution for the cross-sections: a) rectangular, b) triangular (isosceles triangle *b*×*h*) and c) circular.

Solution

a)

We calculate:

$$\tau_{xz} = \frac{Q\left(\frac{h}{2}\frac{h}{4} - \frac{z^2}{2}\right)b}{\frac{bh^3}{12}b} = \frac{3Q\left[1-\left(\frac{2z}{h}\right)^2\right]}{2bh}.$$

The maximum value is in the neutral axis:

$$\max(\tau_{xz}) = \tau_{xz}(z=0) = \frac{3}{2}\frac{Q}{A}.$$

The stress distribution is parabolic with the maximum 50 % greater than the cross-section average.

b)

width and height as *z*-functions:

$$b(z) = \tfrac{2}{3}b - \tfrac{b}{h}z, \quad h(z) = \tfrac{2}{3}h - z$$

cross-section geometry term:

$$\frac{S_y(z)}{b(z)} = \tfrac{1}{2}h(z)\left(z + \tfrac{h(z)}{3}\right)$$

we search a cut line with the maximum value of the shear stress:

$$\frac{\partial \tau_{zx}}{\partial z} = 0 \quad \rightarrow \quad \frac{\partial}{\partial z}\left[\left(\tfrac{2}{3}h - z\right)\left(\tfrac{2}{9}h + \tfrac{2}{3}z\right)\right] = 0 \quad \rightarrow \quad z = \tfrac{h}{6}$$

$$\max|\tau_{xz}| = \frac{Q}{J_y}\frac{h^2}{12} = \frac{36Q}{bh^3}\frac{h^2}{12} = 3\frac{Q}{bh} = \tfrac{3}{2}\tfrac{Q}{A}$$

The maximum value is attained at the middle of the triangle height.

c)

static inertia moment:

$$S_y^I(z) = \frac{2r^2}{3}\cos^3\alpha$$

width of the cut:

$$b(z) = 2r\cos\alpha$$

so:

$$|\tau_{xz}| = \frac{Q \cdot \tfrac{2}{3}r^3\cos^3\alpha}{\frac{\pi r^4}{4} \cdot 2r\cos\alpha} = \frac{4}{3} \cdot \frac{Q}{r_2}\cos^2\alpha = \frac{4}{3} \cdot \frac{Q}{\pi r^2}\left(1 - \frac{z^2}{r^2}\right)$$

and, finally, we find the maximum value at $z = 0$:

$$\max|\tau_{xz}| = \frac{4}{3} \cdot \frac{Q}{A}.$$

Example 7.2

Determine the ratio of maximum values of the normal and shear stress in a cantilever with a rectangular cross-section ($b \times h$), loaded by a point force at its free end ($x = l$).

Solution

the maximum value of the bending moment: $M_{\max} = Pl$

the shear force: $Q = P$ (constant)

the maximum value of the normal stress: $\max\left|\sigma_x\right| = \dfrac{6Pl}{bh^2}$

the maximum value of the shear stress: $\max\left|\tau_{xz}\right| = 1.5\dfrac{P}{bh}$

the ratio: $\dfrac{\max\left|\sigma_x\right|}{\max\left|\tau_{xz}\right|} = \dfrac{6Pl}{bh^2}\cdot\dfrac{bh}{1.5P} = 4\dfrac{l}{h}$.

Usually, the normal stress is much greater than the shear stress.

Example 7.3

The welded profile IPES 600, made by the steelworks "Pokój" has the dimensions: the total height of 600 mm, the width of the flanges: 220 mm, the height of the flanges: 23 mm, the web thickness: 8 mm. Determine what part of the cross-section shear force does the web carry.

Solution

the static moment in the web:

$$S_y(z) = 1460 + 307 - 0.4z^2 = 1767 - 0.4z^2 \ \text{cm}^3$$

$$S_y(z=0) = 1767 \ \text{cm}^3$$

$$S_y(z=277) = 1460 \ \text{cm}^3$$

i.e. the diagram of the shear stress in the web is "flat"

the static moment in the flange:

$$S_y(y) = 2.3\cdot(11-y)\cdot 28.85 = 66.36\cdot(11-y) \ \text{cm}^3$$

$$S_y(y=0) = 730 \ \text{cm}^3$$

In Fig. 6.3 the diagram of the shear stress and the shear stress flow are shown.

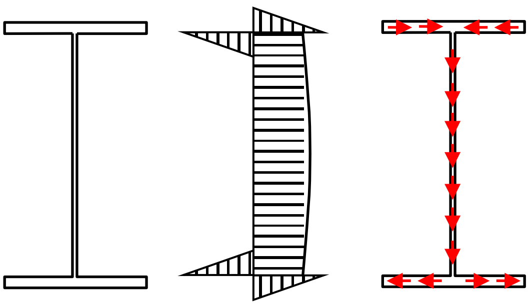

Fig. 7.2 Shear stress in IPES 600

We calculate the shear force part carried by the web:

$$Q = b \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{QS_y(z)}{J_y b} dz = \tfrac{92200}{95600} Q = 0.965Q,$$

i.e. the web carries 96.5% of the shear force.

Example 7.4.

A simply supported beam with a span length l = 4m and the rectangular cross-section $a \times 2a$ is loaded by continuous loading q = 90 kN/m. Determine the value of parameter a of the cross-section if the acceptable values of normal and shear stress are R = 300 MPa and R_t = 100 MPa, respectively. Draw the principal stresses' lines.

Solution

the maximum value of the bending moment:

$$\max M_y = \frac{ql^2}{8} = \frac{90 \cdot 10^3 \cdot 4^2}{8} = 180 \,\text{kNm}$$

the section modulus:

$$W_y = \frac{a \cdot (2a)^2}{6} = \frac{2}{3} a^3$$

the design:

$$\max|\sigma_x| = \frac{\max M_y}{W_y} \le R \quad \to \quad \frac{180 \cdot 10^3 \cdot 3}{2a^3} \le 300 \cdot 10^6 \quad \to \quad a \ge 0.0965 \cong 0.1\text{m}$$

the verification of the shear stress

the maximum value of the shear force:

$$\max|Q| = 180 \text{ kN}$$

the shear stress:

$$\max|\tau_{xz}| = \frac{3}{2} \cdot \frac{180 \cdot 10^3}{0.1 \cdot 0.2} = 13.5 \,\text{MPa} < R_t$$

The principal stresses:

- in the extreme fibers there is the normal stress only, so the directions of the principal stress are vertical and horizontal,
- in the neutral axis there is the shear stress only, so the directions of principal stress are rotated by 45 degrees
- in other fibers at any z we have:

$$tg\alpha_1 = \frac{\tau}{\sigma_1}.$$

The principal stress lines are drawn in Fig. 7.3. Two families of the principal stresses' lines are perpendicular to each other. Trajectories of the tension principal stresses justify the location of reinforcement in RC beam.

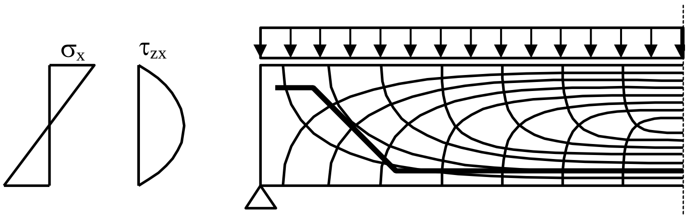

Fig. 7.3 Trajectories of the principal stress

Review problems

Problem 7.5

A timber beam with the span of 3 m and the width 90 mm is to support three concentrated forces shown in Fig. 7.4. Knowing that for the grade of timber used $\sigma_{acc} = 12$ MPa and $\tau_{acc} = 0.8$ MPa, determine the minimum required depth *d* of the beam. (Ans.: *d* = 31 cm)

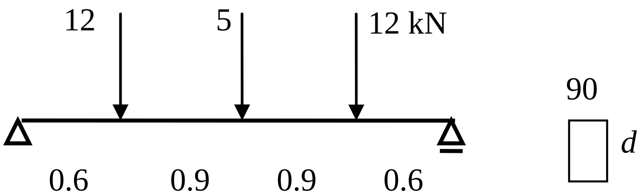

Fig. 7.4 Beam with load

Problem 7.6

For the non-uniform bending of the cross-section of the beam in Fig. 7.5, determine the normal, shear and principal stresses at a point *K* in a section $\alpha - \alpha$. Assume that the cross-section dimensions are given in [cm]. (Ans.:$\sigma_x = -279.8$ MPa, $\tau = -4.87$ MPa, $\sigma_1 = 0.1,\ \ \sigma_2 = -279.9,\ \ \alpha = -89°$)

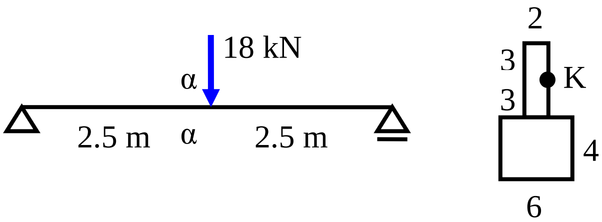

Fig. 7.5 Beam and cross-section

Problem 7.7

A timber beam of a rectangular cross-section carries a single concentrated load P in its midpoint, Fig. 7.6. Determine the depth h and the width b of the beam, knowing that R_t = 1.5 MPa, R = 12 MPa. (Ans.: $b \times h = 0.04 \times 0.5\mathrm{m}$)

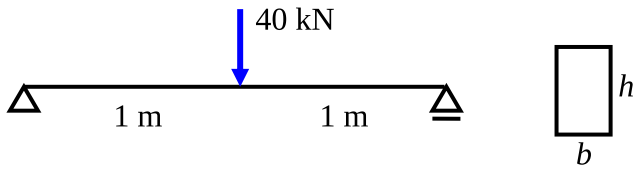

Fig. 7.6 Beam with loading