Beams deflections – Macaulay's method

Introduction

A deflection is the displacement of structural element under load. In the case of the beams, we use this term for linear vertical displacement. In the technical bending theory, we make two main assumptions that:

- Bernoulli's hypothesis (about the plane cross-sections) is valid,
- the derivatives of the displacements are small.

From the course about the beam bending we know that the beam curvature is proportional to the bending moment and inversely proportional to the stiffness of bending:

$$\frac{1}{\rho(x)} = \kappa(x) = \frac{|M(x)|}{EJ_y}.$$

The mathematical formula for the curvature of a line is:

$$\kappa(x) = \frac{|w''(x)|}{\left[1+(w')^2\right]^{\frac{3}{2}}} \cong |w''(x)|,$$

hence, we get the equation of deflections with separable variables:

$$\boxed{EJ_y w''(x) = -M(x)}$$

The sign in the formula above results from admitted coordinate set of the beam axis line and the sagging bending moment, Fig. 8.1.

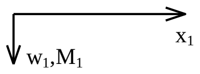

Fig. 8.1. Coordinate set

The beam deflections equation may be solved in several ways. We will limit our considerations to three of them, the most commonly used in engineering.

Analytical method

The method of integration of the curvature equation, named shortly: the analytical method, consists in direct solution by twice integration of the equation. Because of the second order of the differential equation, after the integration we have two integration constants, which should be determined from the kinematic boundary conditions. Moreover, when the bending moment is not given by one formula for the whole beam, but instead of this is given separately for each of the cross-section forces interval, the boundary value problem (the equation with its proper boundary conditions) should be formulated for each interval. The number of integration constants raises and is as twice as the number of intervals. The additional conditions are needed to determine all integral constants. These are so-called the compatibility or conformity conditions. It means that the deflection from both sides of the intermediate characteristic point should be the same. If there is no hinge at such a point the deflection derivatives from both sides should also be the same. While the first condition is obvious, the second one results from the curvature formula: a bent beam deflected axis means an infinite bending moment or zero bending stiffness. Both cases are excluded for beams.

The calculation technique will be explained in some examples.

Example 8.1

Determine the deflection of the cantilever loaded by a point force at its free end, Fig. 8.2.

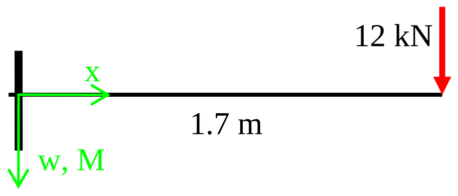

Fig. 8.2 Cantilever with load

Solution

the bending moment equation:

$$M(X) = -20.4 + 12x$$

the curvature equation:

$$EJw''(x) = 20.4 - 12x$$

after first integration:

$$EJw'(x) = 20.4x - 6x^2 + C$$

and next:

$$EJw(x) = 10.2x^2 - 2x^3 + Cx + D$$

the kinematical conditions are:

$$w(0) = 0, \quad w'(0) = 0$$

hence we find $C = D = 0$

and finally we get:

$$w(x) = \frac{19.65}{EJ}$$

Example 8.2

The simple supported beam is loaded by the point force, Fig. 8.3. Determine the maximum deflection of the beam.

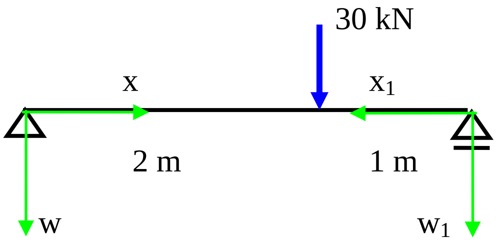

Fig. 8.3 Beam with load

Solution

We introduce the coordinate sets from both sides (cf. Fig. 8.3):

$$M(x) = 10x \qquad M(x_1) = 20x_1$$
$$EJw'' = -10x \qquad EJw_1'' = -20x_1$$
$$EJw' = -5x^2 + C \qquad EJw_1' = -10x_1^2 + C_1$$
$$EJw = -\tfrac{5}{3}x^3 + Cx + D \qquad EJw_1 = -\tfrac{10}{3}x_1^3 + C_1x_1 + D_1$$
$$w(0) = 0 \quad \to \quad D = 0 \qquad w_1(0) = 0 \quad \to \quad D_1 = 0$$

the compatibility conditions are:

$$w(2) = w_1(1)$$
$$w'(2) = -w_1'(1)$$

Note: The minus sign in the second condition results from different definition of positive derivative, see Fig. 8.4.

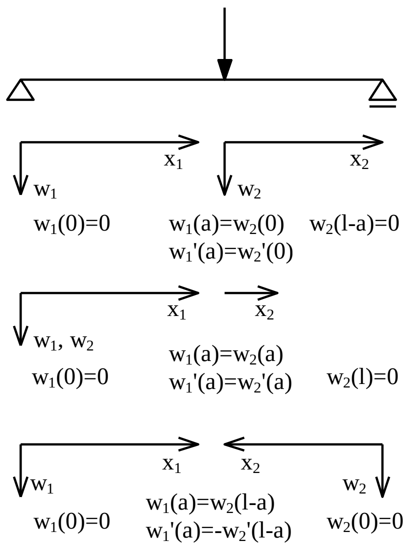

Fig. 8.4. Different coordinate sets

performing calculation, we get: $C = 13.33$, and finally:

$$w(x) = \frac{-\frac{5}{3}x^3 + 13.33x}{EJ}.$$

Calculating the position of extreme deflection, we get:

$$w'(x) = 0 \quad \rightarrow \quad -\frac{15}{3}x^2 + 13.33 = 0 \quad \rightarrow \quad x = 1{,}63 \quad \rightarrow \quad w_{\max} = \frac{14.52}{EJ}$$

Example 8.3

Determine the deflection of a simply supported beam, loaded by a point force in the middle of the span.

Solution

1st interval: $0 < x_1 < l/2$: $M(x_1) = \frac{P}{2}x_1 \quad \Rightarrow \quad EJ_y w_1(x_1) = -\frac{P}{12}x_1^3 + C_1 x_1 + D_1$,

2nd interval: $0 < x_2 < l/2$: $M(x_2) = \frac{P}{2}x_2 \quad \Rightarrow \quad EJ_y w_2(x_2) = -\frac{P}{12}x_2^3 + C_2 x_2 + D_2$,

boundary and compatibility conditions: $w_1(0) = 0, \quad w_2(0) = 0, \quad w_1(\frac{l}{2}) = w_2(\frac{l}{2}), \quad w_1'(\frac{l}{2}) = -w_2'(\frac{l}{2})$,

hence: $D_1 = D_2 = 0, \quad C_1 = C_2, -\frac{P}{16}l^2 + C_1 = \frac{P}{16}l^2 - C_2 \quad \Rightarrow \quad C_1 = \frac{P}{16}l^2$.

The final form of the equation may be written:

$$EJ_y w(x) = -\frac{P}{12}x^3 + \frac{P}{16}l^2 x.$$

The maximum deflection in the middle is:

$$f = w(\tfrac{l}{2}) = \frac{Pl^3}{48EJ_y}.$$

Macaulay's method

If there are many intervals of the cross-section forces, the analytic method is cumbersome. The formulation of the problem may be significantly simplified, when the compatibility conditions are automatically fulfilled. This can be done by writing equations in a particular way:

1. We adopt one coordinate set for all intervals: x, $w(x)$, $M(x)$.
2. The equation in i-th interval is written in such a way, that it contains the equation for the previous interval. In this way the compatibility conditions are automatically fulfilled.
3. Instead of writing each equation separately, we write one general equation marking alternative terms by the Macaulay's brackets. The integral constants are shared for all intervals.
4. The terms within the Macaulay's brackets, $< x - a_i >^n$, signify the positive difference, which means they are positive or zero.
5. To fulfill the requirements of the point (2), the method may be applied to the intervals with the same constant bending stiffness. Moreover, the continuous loadings and the point moments should be treated in a special way.

Example 8.4

Determine the deflections of the beam in Fig. 8.5.

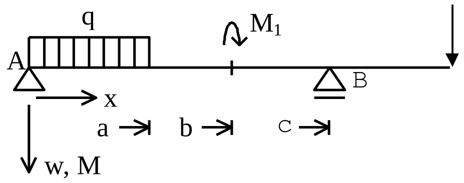

Fig. 8.5 Loaded beam

Solution

We write the equations:

$$M(x) = R_A x - \tfrac{1}{2} q x^2 + \tfrac{1}{2} q < x - a >^2 + M_1 < x - b >^0 + R_B < x - c >$$
$$EJ_y w''(x) = -R_A x + \tfrac{1}{2} q x^2 - \tfrac{1}{2} q < x - a >^2 - M_1 < x - b >^0 - R_B < x - c >$$
$$EJ_y w'(x) = C - \tfrac{1}{2} R_A x^2 + \tfrac{1}{6} q x^3 - \tfrac{1}{6} q < x - a >^3 - M_1 < x - b > - \tfrac{1}{2} R_B < x - c >^2$$
$$EJ_y w(x) = Cx + D - \tfrac{1}{6} R_A x^3 + \tfrac{1}{24} q x^4 - \tfrac{1}{24} q < x - a >^4 - \tfrac{1}{2} M_1 < x - b >^2 - \tfrac{1}{6} R_B < x - c >^3$$

The integral constants are determined from the boundary conditions: $w(0) = w(c) = 0$, so

$$D = 0, \quad Cc - \tfrac{1}{6} R_A c^3 + \tfrac{1}{24} q c^4 - \tfrac{1}{24} q (c - a)^4 - \tfrac{1}{2} M_1 (c - b)^2 = 0 \quad \Rightarrow \quad C = \ldots$$

Note, that the integral constants are written without the Macaulay's brackets.

Example 8.5

Determine the deflections of the beam in Fig. 8.6.

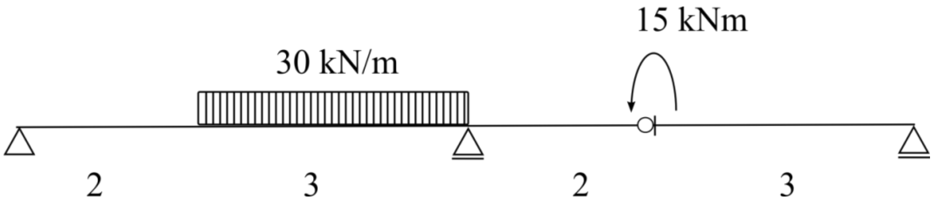

Fig. 8.6 Beam with load

Solution

statics: decomposition into the simple beams & constraints reactions

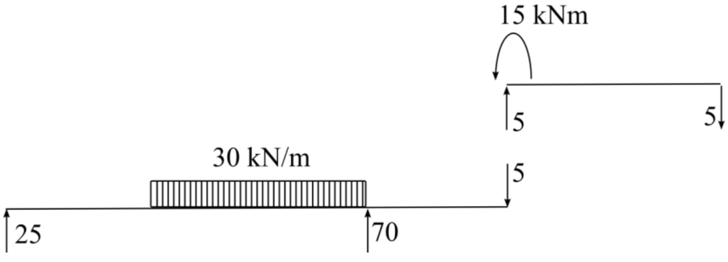

Fig. 8.7 Beam decomposition

diagram of bending moments:

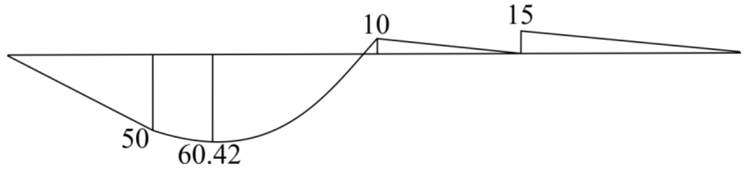

Fig. 8.8 Diagram of bending moments

diagram of shear forces:

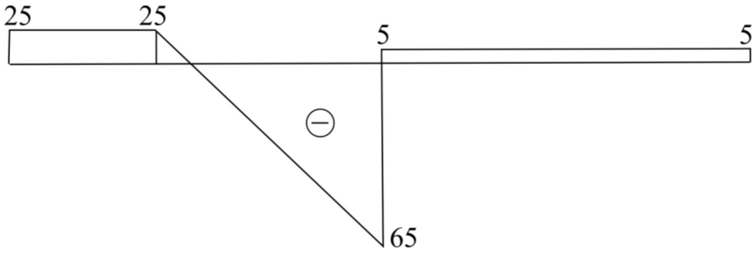

Fig. 8.9 Diagram of shear forces

$Q(x) = 25 - 30(x-2) = 0 \quad \rightarrow \quad x = 2.833$ m, $M(2.833) = 60.42$ kNm

Macauley's method

first beam:

$$EIw_1''(x_1) = -25x_1 + 15 < x_1 - 2 >^2 - 15 < x_1 - 5 >^2 - 70 < x_1 - 5 >$$

$$EIw_1'(x_1) = C_1 - 12.5x_1^2 + 5 < x_1 - 2 >^3 - 5 < +x_1 - 5 >^3 - 35 < x_1 - 5 >^2$$

$$EIw_1(x_1) = C_1x_1 + D_1 - 4.167x_1^3 + 1.25 < x_1 - 2 >^4 - 1.25 < x_1 - 5 >^4 - 11.67 < x_1 - 5 >^3$$

boundary conditions:

$$w_1(0) = 0 \rightarrow D_1 = 0$$

$$w_1(5) = 0 \rightarrow 0 = 5C_1 - 4.167 \cdot 5^3 + 1.25 \cdot 3^4 = 5C_1 - 419.6 \rightarrow C_1 = 83.93$$

additional results:

$$w_1(2) = \frac{(83.93 \cdot 2 - 4.167 \cdot 2^3)}{(EI)} = \frac{134.5}{EI}$$

$$w_1(7) = \frac{(83.93 \cdot 7 - 4.167 \cdot 7^3 + 1.25 \cdot 5^4 - 1.25 \cdot 2^4 - 11.67 \cdot 2^3)}{EI} = -\frac{173.8}{EI}$$

$$w_1'(0) = \frac{83.92}{EI}$$

$$w_1'(5) = -\frac{93.57}{EI}$$

$$w_1'(7) = -\frac{83.57}{EI}$$

second beam:

$$2EIw_2''(x_2) = -5x_2 + 15x_2^0$$

$$2EIw_2'(x_2) = C_2 - 2.5x_2^2 + 15x_2$$

$$2EIw_2(x_2) = C_2x_2 + D_2 - 0.8333x_2^3 + 7.5x_2^2$$

boundary conditions:

$$w_2(0) = -\frac{173.8}{EI} = \frac{D_2}{2EI} \to D_2 = -347.6$$

$$w_2(3) = 0 \to 0 = 3C_2 - 347.6 - 0.8333 \cdot 3^3 + 7.5 \cdot 3^2 = 3C_2 - 302.6 \to \ C_2 = 100.9$$

$$w_2'(0) = \frac{50.4}{EI}$$

$$w_2'(3) = \frac{61.7}{EI}$$

deflections diagram:

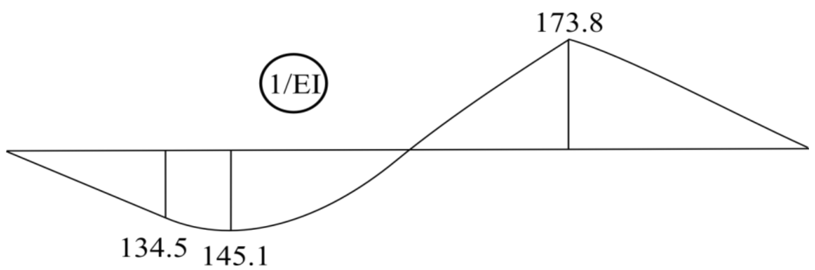

Fig. 8.10 Diagram of deflections

Example 8.6

Using the superposition principle determine the extreme value of the deflection of the beam in Fig. 8.11.

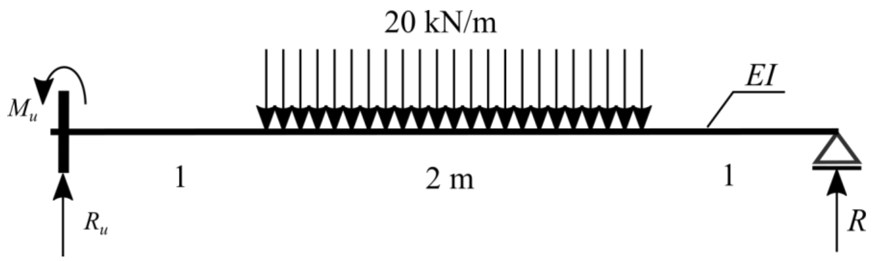

Fig. 8.11 Beam with the load

Using the superposition we split the problem into two: the active load and the hyperstatic reaction.

The reactions for the active load are: $R_u = 40$ kN, $M_u = 80$ kNm

The bending moment equation:

$$M(x) = 40 < x > -80 < x >^0 - \frac{20}{2} < x-1 >^2 + \frac{20}{2} < x-3 >^2$$

$$EIw''(x) = -40 < x > +80 < x >^0 + 10 < x-1 >^2 - 10 < x-3 >^2$$

$$EIw'(x) = C - 20 < x >^2 + 80 < x > + \frac{10}{3} < x-1 >^3 - \frac{10}{3} < x-3 >^3$$

$$EIw(x) = Cx + D - \frac{20}{3} < x >^3 + 40 < x >^2 + \frac{5}{6} < x-1 >^4 - \frac{5}{6} < x-3 >^4$$

kinematic boundary conditions: $w(0) = w'(0) = 0 \rightarrow C = D = 0$

$$w(4) = \frac{1}{EI}\left(-\frac{20}{3} \cdot 4^3 + 40 \cdot 16 + \frac{5}{6} \cdot 3^4 - \frac{5}{6} \cdot 1^4\right) = \frac{280}{EI}$$

The solution for the hyperstatic reaction: $R_u = -R$, $M_u = 4R$

The bending moment equation:

$$M(x) = -Rx + 4R < x >^0$$

$$EIw''(x) = R < x > -4R < x >^0$$

$$EIw'(x) = C + \frac{1}{2}R < x >^2 - 4R < x >$$

$$EIw(x) = Cx + D + \frac{1}{6}R < x >^3 - 2R < x >^2$$

kinematic boundary conditions: $w(0) = w'(0) = 0 \rightarrow C = D = 0$

$$w(4) = \frac{R}{EI}\left(\frac{1}{6} \cdot 4^3 - 2 \cdot 16\right) = -\frac{21.33R}{EI}$$

final deflection of the right end should be zero, so:

$\frac{280-21.33R}{RI} = 0 \rightarrow R = 13.125$ kN

Total reactions: $R_u = 26.875$ kN, $M_u = 27.5$ kNm

the bending moment final equation:

$$M(x) = -27.5 < x >^0 + 26.875 < x > \frac{20}{2} < x-1 >^2 + \frac{20}{2} < x-3 >^2$$

$$EIw''(x) = 27.5x^0 - 26.875x + 10 < x-1 >^2 - 10 < x-3 >^2$$

$$EIw'(x) = C + 27.5x - 13.44x^2 + 3.333 < x-1 >^3 - 3.333 < x-3 >^3$$

$$EIw(x) = Cx + D + 13.75x^2 - 4.48x^3 + \frac{5}{6} < x-1 >^4 - \frac{5}{6} < x-3 >^4$$

The extreme value of deflection is $w(2.28) = \frac{20.63}{EI}$

Example 8.7

To reduce the weight of a metal beam the flanges are made of steel *E* = 200 GPa and the web of aluminum *E* = 70 GPa as shown in the Figure 8.12. Determine the maximum deflection of the beam.

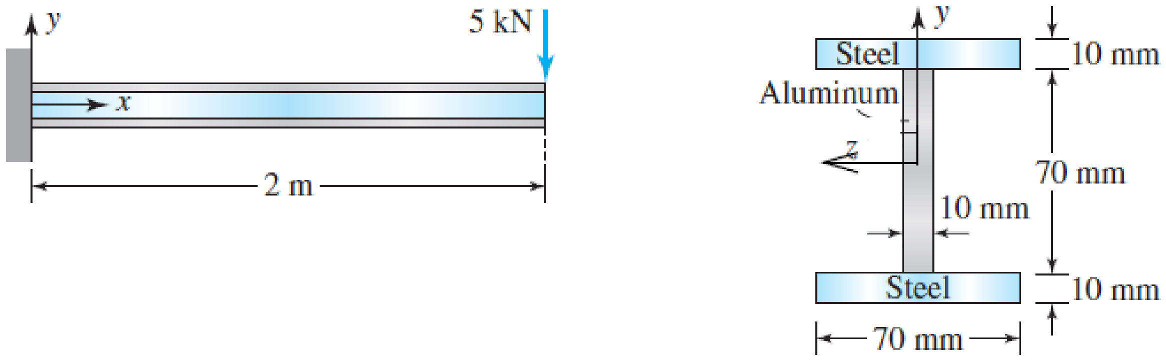

Fig. 8.12 Composite beam with the load.

Solution

the bending moment equation:

$$M(x) = 5x - 10x^0$$

$$EIw''(x) = -5x + 10x^0$$

$$EIw'(x) = C - 2.5x^2 + 10x$$

$$EIw(x) = Cx + D = -0.8333x^3 + 5x^2$$

from KBC: $w(0) = w'(0) = 0 \rightarrow C = D = 0$

$$w(2) = \frac{1}{EI}(-0.8333 \cdot 2^3 + 5 \cdot 2^2) = \frac{13.33}{EI}$$

the flexural stiffness will be calculated for homogeneous cross-section made from aluminum:

$$n = \frac{E_{st}}{E_{al}} = 2.857$$

the weighted inertia moment:

$$I_y = 2 \cdot 2.857 \cdot \left(0.07 \cdot \frac{0.01^3}{12} + 0.07 \cdot 0.01 \cdot 0.055^2\right) + 0.01 \cdot \frac{0.07^3}{12} = 1.242 \cdot 10^{-5}\ \mathrm{m}^4$$

the flexural stiffness:

$$EI = E_{al} I_y = 70 \cdot 10^9 \cdot 1.242 \cdot 10^{-5} = 869400\ \mathrm{Nm}^2$$

the maximum deflection:

$w = \frac{13.33 \cdot 10^3}{869400} = 0.0153$ m = 1.53 cm

Review problems

Determine the deflection at the point *K* of the beams in Fig. 8.17, using the Macaulay's method. (Use the statykawin program for verification of solutions).

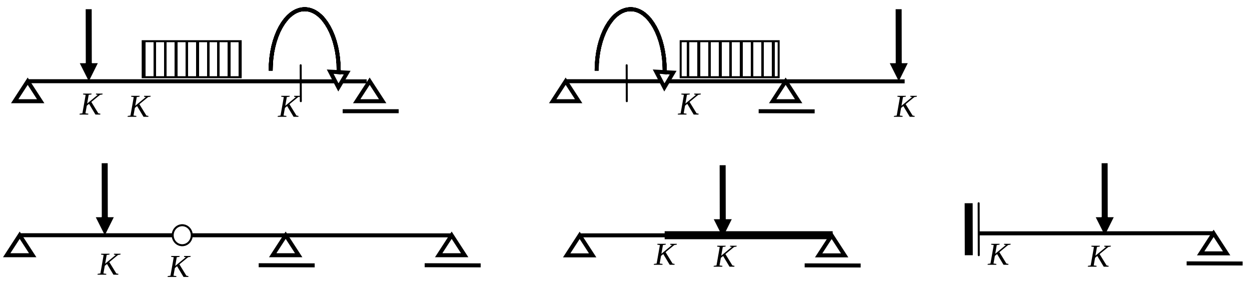

Fig. 8.13 Review problems