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3. Pure and biaxial bending

Introduction

Bending is a major concept used in the design of many structural components, such as beams and girders. It is a case of prismatic member loading, when the direction of the bending moment vector (as a so-called free vector) is parallel to principal central axis of the cross-section.
The simplest examples of the pure bending are a barbell or an axle in a trailer. (These cases are commonly, however not correctly, named 4-point bending).
Some results of the pure bending will be also used in the analysis of other types of loading, such as eccentric axial loading and transverse loading.
The BVP solution can be found by the static approach, similarly to the case of tension.

Bernoulli's theorem

A cross-section, plane and perpendicular to the bar axis before loading, remains a plane and perpendicular to the deformed bar axis after the loading.
This does not rule out the possibility of deformations within the cross-section plane.
We will use the above theorem as the kinematic approximate assumption to other loading cases.

Strain and stress state

The longitudinal normal strain ε_x varies linearly with the distance from the neutral surface. The neutral axis (surface) intersects the cross-section and divides it into two regions: compressed and tensioned. The neutral axis agrees with the principal central axis of the applied bending moment.
Due to the Hooke's law, the repartition of normal stress, the flexural stress, is also linear:

$$\sigma_x = \frac{M_y}{J_y} z \,,$$

where the distance z is measured from the principal central axis.
The maximum absolute value of the stress is attained at the fibers which are the most distant from the neutral axis:

$$\max|\sigma_x| = \frac{|M_y|}{J_y} z_{\max} \,.$$

We denote the elastic section modulus W_y as:

$$W_y \overset{def}{=} \frac{J_y}{z_{\max}} \quad [\mathrm{m}^3]$$

so,

$$\boxed{\max|\sigma_x| = \frac{|M_y|}{W_y}}$$

Maximum stress is proportional to the bending moment and inversely proportional to the elastic section modulus.
From the ultimate limit state we have the design inequality:

$$\max|\sigma_x| \le R \quad \rightarrow \quad \frac{M_y}{W_y} \le R \,.$$

By virtue of Saint-Venant's principle, the relations obtained can be used to compute stresses as long as the section considered is not too close to the points where the couples are applied.
Three kinds of problems can arise:

- we seek for material that can sustain the loading (it happens rarely, usually the choice of material depends on other reasons – technological, economic, etc.),
- we determine the largest permissible load for given material and cross-section geometry,
- we determine the cross-section geometry needed to bear the loading.

Curvature

The curvature of the bent beam is:

$$\kappa = \frac{1}{\rho} = \frac{M_y}{EJ_y} \quad \left(= \frac{|w''|}{\sqrt{1+(w')^2}^{\,3}} \approx |w''| \right)$$

Examples

Example 4.1

There are three cross-sections with the same height and area. Which section will be the most useful for the pure bending? Assume the same value of bending moment.

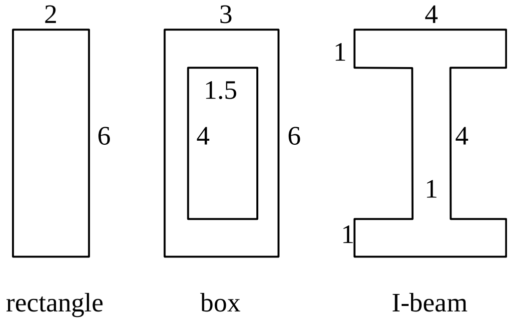

Fig. 3.1 Three cross-sections

Solution

We calculate the elastic section modulus:

- for the rectangle $W_y = 12$ cm^3 (100%)
- for the box $W_y = 15.33$ cm^3 (128%)
- for the I-beam $W_y = 18.67$ cm^3 (156%)

Assuming the maximum value of the stress for rectangle as 100%, the corresponding values for the box and I-beam are 78% and 64%, respectively.

The results show that the best cross-section geometry for the pure bending is the one with the material which is the most distant from the neutral axis; it means S-beam (standard beam) and W-beam (wide flange beam), Fig. 3.2.

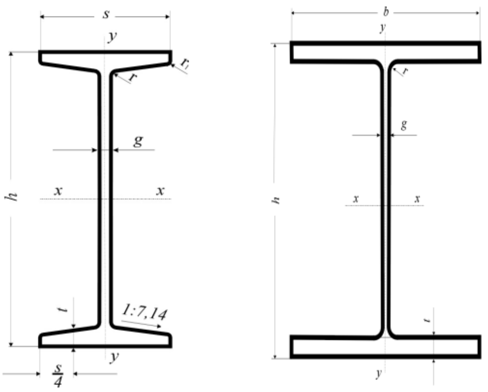

Fig. 3.2 Standard beam and wide flange beam

Example 3.2

Determine the value of the parameter a of the cross-section of the beam in Fig. 4.3 if P = 140 kN, l = 2 m, b = 0.2 m, R = 150 MPa.

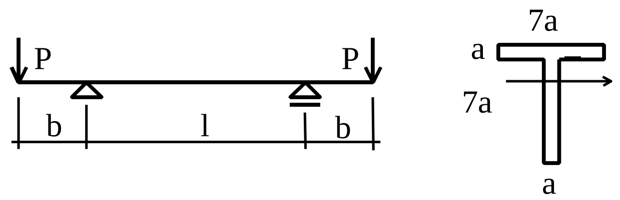

Fig.. 3.3. Beam and the cross-section

Solution

– the maximum value of the bending moment is at the span and is $M_y = Pb = 28$ kNm

– position of the cross-section centroid: $y_0 = 5.5a$

– principal central inertia moment: $J_y = 85.17a^4$

– section modulus: $W_y = \dfrac{85.17a^4}{5.5a} = 15.48a^3$

– normal stress: $\sigma_x = \dfrac{M_y}{W_y} = \dfrac{28 \cdot 10^3}{15.48a^3} \le R = 150 \cdot 10^6$

– so, the parameter: $a^3 \ge \dfrac{28 \cdot 10^3}{15.48 \cdot 150 \cdot 10^6} = 1.206 \cdot 10^{-5}, \quad \to \quad a \ge 0.0229$ m

Finally, we assume a = 2.3 cm, $J_y = 2380$ cm^4, $W_y = 188$ cm^3 and $\max|\sigma_x| = 149$ MPa.

The normal stress repartition can be presented by means of a stress solid or a stress diagram, Fig. 4.4.

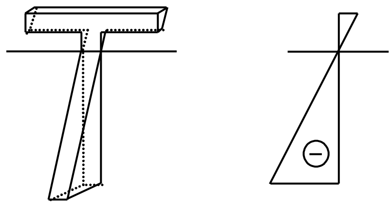

Fig. 3.4 Stress solid and stress diagram

Example 3.3

Straight rods of 6-mm diameter and 30-m length are stored by coiling the rods inside a drum of 1.25 m inside diameter. Assuming the elastic behavior of the rods, determine (a) the maximum stress in the coiled rod, (b) the corresponding bending moment in the rod. Use E = 200 GPa.

Solution

The curvature radius is known:

$$\rho = \frac{1.25 - 0.006}{2} = 0.622 \text{ m}$$

The bending moment can be calculated from the curvature formula:

$$\kappa = \frac{1}{\rho} = \frac{M_y}{EJ_y} \quad \to \quad M_y = \frac{EJ_y}{\rho} = \frac{200 \cdot 10^9 \cdot \pi \cdot 0.006^4}{64 \cdot 0.622} = 20.46 \text{ Nm}$$

The maximal stress is:

$$\max|\sigma_x| = \frac{M_y}{W_y} = \frac{20.46 \cdot 32}{\pi \cdot 0.006^3} = 964.8 \text{ MPa}$$

(much more then the steel elastic limit).

Example 3.4

Knowing that for the extruded beam shown in Fig. 3.5, the allowable stress is 84 MPa in tension and 110 MPa in compression, determine the largest moment M that can be applied. The bottom side is tensioned.

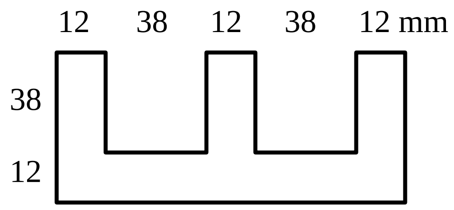

Fig. 3.5 Extruded cross-section

Solution

The cross-section characteristics:

– area: $A=(3\cdot 12+2\cdot 38)\cdot 50-2\cdot 38\cdot 38=2712\ \text{mm}^2$

– position of the centroid: $z_c=\dfrac{112\cdot 50\cdot 25-2\cdot 38\cdot 38\cdot 31}{2712}=18.61\ \text{mm}$

– principal central inertia moment: $J_y=\dfrac{112\cdot 50^3}{12}+112\cdot 50\cdot (25-18.61)^2-2\cdot\left[\dfrac{38^4}{12}+38^2\cdot(31-18.61)^2\right]=604500$ mm^4

– stress repartition:

$$\sigma_x=\frac{M_y}{J_y}z,\quad \sigma_x(z=-18.61)=\frac{-M_y}{604500\cdot 10^{-12}}(-0.01861)\le 84\cdot 10^6,\quad \sigma_x(z=31.39)=\frac{|M_y|}{604500\cdot 10^{-12}}(0.03139)\le 110\cdot 10^6$$

so, we have, respectively:

$M_y \le 2.73$ kNm and $M_y \le 2.12$ kNm,

and, finally:

$M_y \le 2.12$ kNm

Biaxial bending

At first glance there are two cases, seemingly different: (a) the bending of the cross-section by the moment the direction of which is not parallel to the one of the principal central axes, and (b) the cross-section loaded by two bending moments acting in the direction of the principal central axes, Fig. 3.6.

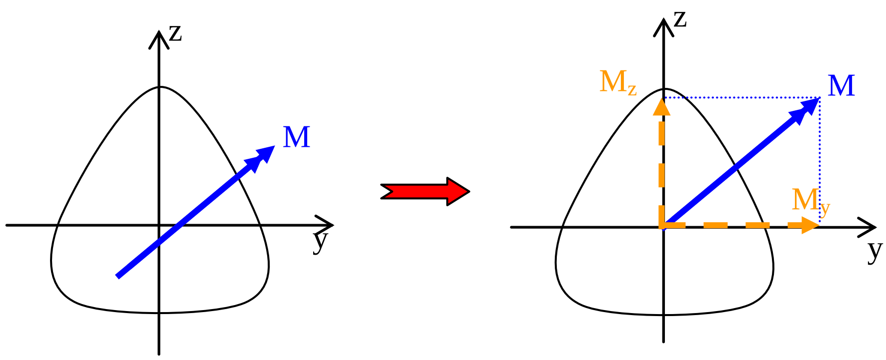

Fig. 3.6 Bending moment and its decomposition

Note, that we always consider the directions of the bending moments in relation with the directions of the principal central axes.

Tip: there is no biaxial bending of such cross-sections as circle, square, and any regular polygon; there is no biaxial bending if any central axis of a cross-section considered is the principal axis, too.
The normal stress is the sum of the normal stresses of two bending components:

$$\sigma_x = \frac{M_y}{J_y} z - \frac{M_z}{J_z} y .$$

The repartition of the normal stress in the cross-section is linear.
The neutral axis is a straight line that passes through the cross-section centroid:

$$\sigma_x = 0 \quad \rightarrow \quad z = \frac{M_z}{M_y} \cdot \frac{J_y}{J_z} y .$$

The neutral axis doesn't agree with the direction of the resultant bending moment: it declines more or less slightly in the direction of the axis of the smaller principal central inertia moment.
The maximal value of the normal stress is reached at the points the most distant from the neutral axis.
From the ultimate limit state we have the design inequality:

$$\max|\sigma_x| \le R ,$$

where R is the material strength.
To determine the most exerted points of the cross-section we have to know the position of the neutral axis.

Examples

Example 3.4

The T-shape steel purlin, Fig. 3.7, is bent by the horizontal vector of bending moment. Knowing that the acceptable value of the normal stress is 150 MPa, determine the value of the allowable moment. The cross-section characteristics are: $s = h = 60$ mm, $e = 18.6$ mm, $J_y = 23.8$ cm^4, $J_z = 12.2$ cm^4, the angle $\alpha = 12.5°$.

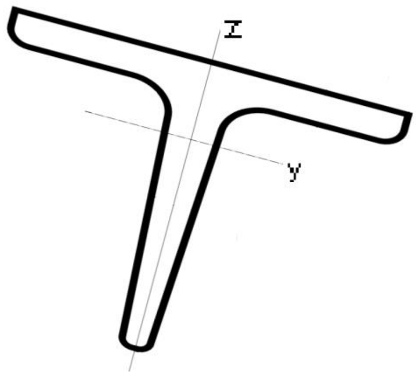

Fig. 3.7 T-shape purlin

Solution
the decomposition of the bending moment:

$$M_y = M\cos\alpha, \quad M_z = M\sin\alpha ,$$

the normal stress:

$$\sigma_x = \frac{M_y}{J_y} z - \frac{M_z}{J_z} y = M\left(\frac{\cos\alpha}{J_y} z - \frac{\sin\alpha}{J_z} y\right),$$

the neutral axis equation:

$$z = \tan\alpha \cdot \frac{23.8}{12.2} y = 0.4325 y$$

from the cross-section drawing, Fig. 3.8:

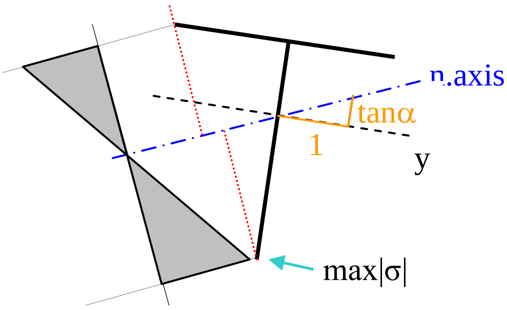

Fig. 3.8 Cross-section with neutral axis

we get the maximal stress at the web corner, near the symmetry axis, (0, -41.4) in [mm]:

$$\sigma_x = M(0.041021\cdot(-4.14))\cdot 10^6 = 0.1698\cdot 10^6 M$$

from ultimate limit state, we have:

$$|\sigma_x| \le r \quad \to \quad 0.1698\cdot 10^6 M \le 150\cdot 10^6 \quad \to \quad M \le 883 \text{ Nm}$$

Example 3.5

Determine the cross-section parameter *a*, Fig. 3.9, knowing that the acceptable stress value is *R* = 300 MPa.

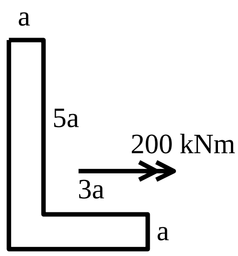

Fig. 3.9 Cross-section with loading

Solution

position of the centroid: $y_c = 1.17$ a, $z_c = 2.17$ a
central inertia moments: $J_y = 30.75$ a^4, $J_z = 10.75$ a^4, $J_{yz} = -10$ a^4
eigenvalues: $J_1 = 34.89$ a^4, $J_2 = 6.61$ a^4, principal directions: $\alpha = 22.49°$
bending moments: $M_1 = M\cos\alpha = 185$ kNm, $M_2 = -M\sin\alpha = -76.5$ kNm
normal stress distribution: $\sigma_x = \frac{M_1}{J_1}x_2 - \frac{M_2}{J_2}x_1 = \frac{5.30x_2 + 11.57x_1}{a^4}10^3$

neutral axis equation: $x_2 = -2.19x_1$

we calculate the coordinates of the corners in principal central coordinates from the transformation formula

pt	y	z	x_1	x_2	from neutral axis.	σ [MPa]
A	-1.17 a	-2.17 a	-1.91 a	-1.56 a	2.39 a	-243
B	2.83 a	-2.17 a	1.78 a	-3.09 a	0.34 a	35
C	2.83 a	-1.17 a	2.17 a	-2.16 a	1.07 a	109
D	-0.17 a	3.83 a	1.31 a	3.60 a	2.69 a → max.	274
E	-1.17 a	3.83 a	0.38 a	3.99 a	2.01 a	205

Tab. 3.1 Calculation data for the problem 3.5.

The most distant point from the neutral axis is point D where the stress value is the greatest.
The cross-section design: $\sigma_x(1.31 \text{ a}, 3.60 \text{ a}) \le R \quad \Rightarrow \quad a \ge 0.0485 \text{ m} \approx 0.05 \text{ m}$
The normal stress for assumed value of the parameter *a* are stated in the Tab. 3.1 and the diagram of the normal stress distribution is shown in Fig. 3.10.

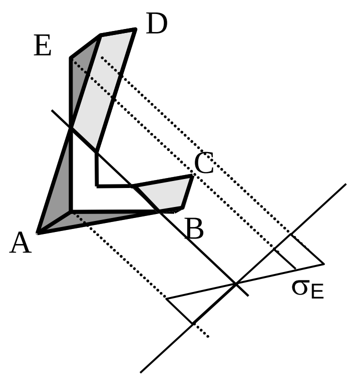

Fig. 3.10 Normal stress distribution

Review problems

Problem 3.1

Determine the cross-section parameter *a*, Fig. 3.11, if the acceptable value of the normal stress is *R* = 240 MPa. (Ans.: $a \geq 2.95$ [mm])

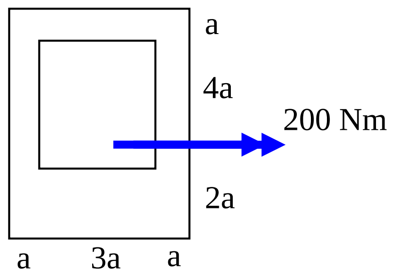

Fig. 3.11 Cross-section

Problem 3.2

Determine the maximum normal stress in the axle of the coal car, Fig. 3.12. Total weight is 72 tonnes, there are 4 axles of 120 mm diameter each, the wheel track 1435 cm, the span 1.3 m. (Ans.: 35.1 MPa)

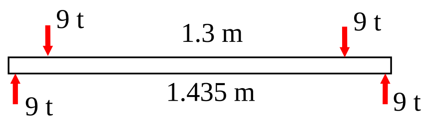

Fig. 3.12 Coal car and axle scheme

Problem 3.3

Determine the acceptable value of the bending moment, applied to the section in Fig. 3.13, knowing that the allowable stress is 80 MPa in tension and 105 MPa in compression. Draw the diagram of the stress repartition. (Ans.: M≤370 kNm, compression determines)

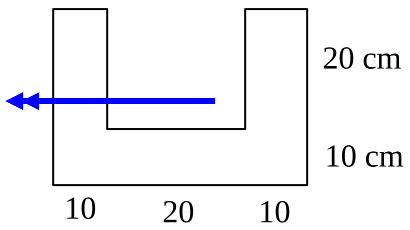

Fig. 3.13 Cross-section in bending

Problem 3.4

From the table of I-section profiles, chose an appropriate profile knowing that the acceptable value of the normal stress is 150 MPa and that the bending moment M = 12 kNm is rotated by 20° with respect to the first principal central axis, counterclockwise. (Ans.: for IPN 120 max $|\sigma_x| = 277.3$ MPa)

Problem 3.5

Determine the parameter a of the cross-section in Fig. 3.14. The allowable stress value is 280 MPa. Draw the stress repartition diagram. (Ans.: $a \geq 0.0056$ m)

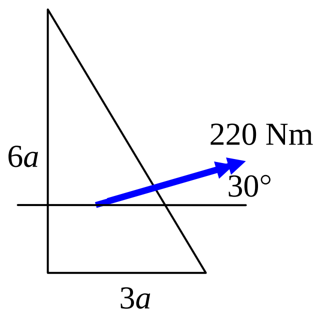

Fig. 3.14 Cross-section with loading

Problem 3.6

Determine the normal stress distribution of the cross-section in Fig. 3.15. Draw the normal stress diagram and determine the stress at p. A. (Ans.: 53.7 MPa)

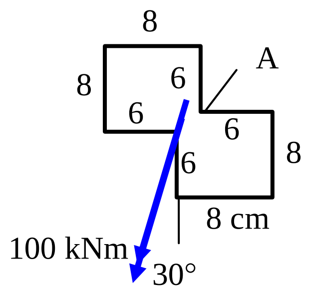

Fig. 3.15 Cross-section with loading

Addendum

Some useful formulae

The transformation formula for the point's coordinates:

$$x_1 = y\cos\alpha + z\sin\alpha$$

$$x_2 = -y\sin\alpha + z\cos\alpha$$

The distance from a point P to a line $y\cos\alpha + z\sin\alpha - p = 0$:

$$d = \left| y_P \cos\alpha + z_P \sin\alpha - p \right|$$

Glossary

free vector – wektor swobodny
barbell – sztanga
neutral axis – oś obojętna
flexural stress – naprężenia zginania
elastic section modulus – (sprężysty) wskaźnik wytrzymałości na zginanie
coil – zwój; zwijać
drum – bęben, beczka
wheel track – rozstaw kół
biaxial bending – zginanie ukośne
purlin – płatew dachowa