4. Cross-section forces

Definitions

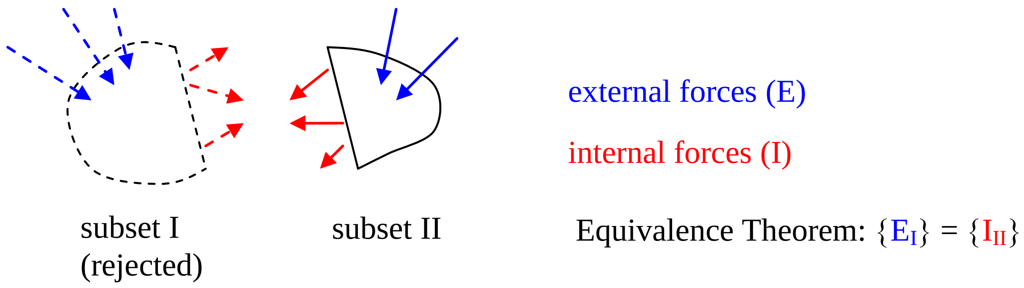


Fig. 4.1 Two separated subsets

bar – an element of a structure with length much greater (at least e.g. 6 times) than its width and height
axis of a bar – a locus of the gravity centre of all sections with minimal area
cross-section – a section perpendicular to the bar axis
prismatic bar – a bar with uniform cross-section
beam – a (usually horizontal) bar, loaded mostly perpendicular to its axis
section – a fictitious section dividing structure into (at least) two disjoint subsets; as a consequence, two equilibrated sets of internal forces arise at the place of the section
internal forces set – a forces set exerted at a section point by another (rejected) subset of the structure
external forces set – a forces set applied to the structure
proper cross-section coordinate system – a coordinate system with the origin at the cross-section gravity centre, the first axis tangent to the bar axis and other axes being principal central inertia axes of the cross-section
cross-section forces – internal forces determined in the proper cross-section coordinate system
undersides – arbitrarily distinguished side of the bar, usually bottom side (hence – the name)
characteristic interval – an interval where the cross-section forces have one equation each
characteristic point – a point which limits characteristic interval

Cross-section forces

In 2D case, there are 3 cross-section forces, cf. Fig. 4.2:

- axial force, tangent to the bar axis
- transversal or shear force, in-plane perpendicular to the bar axis
- bending moment, in-plane moment whose vector is perpendicular to the plane

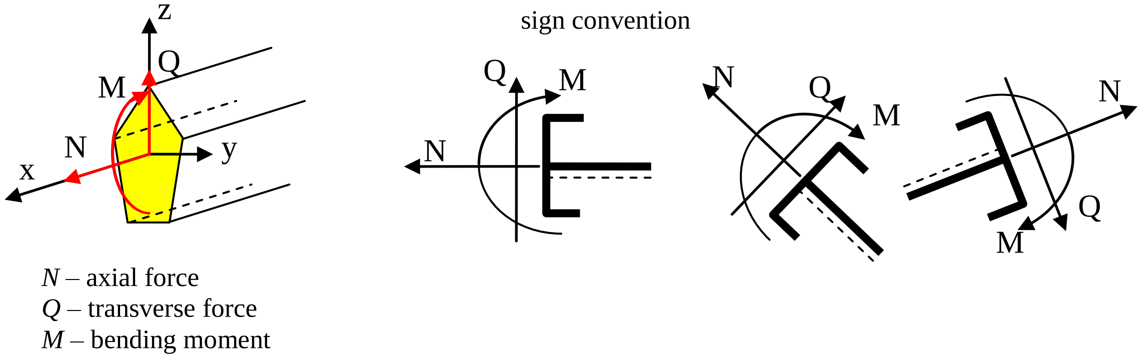


Fig. 4.2 Cross-section forces and sign convention

The aims of static calculations are the cross-section diagrams rather than their equations.

Usually, the bending moment is the most important cross-section force. So – as a rule – we always determine its extreme values and their positions.
There are two ways to determine the cross-section forces:

1. with the balance equations for the chosen subset,
2. with the theorem on equivalence of internal and external forces.

Usually, the solution of a static problem consists in construction of cross-section forces diagrams eventually.

Sign convention

The bending moment is positive when the undersides are tensioned.
The axial force is positive when its sense coincides with the positive sense of cross-section outward normal.
The transverse (shear) force is positive when its sense coincides with the positive sense of axial force turned by right angle clockwise.
The sign of bending moment depends on an arbitrary choice of the undersides. Because the sign is not an objective quantity and serves for calculations only, it may be completely arbitrary and is ignored during all verifications as irrelevant (even your mark of the ± sign, which has no sense, will be ignored). Nevertheless, the bending moment diagram should be drawn on the tensioned side of a bar[1].
The signs of axial and shear forces should be indicated in their diagrams.

Example

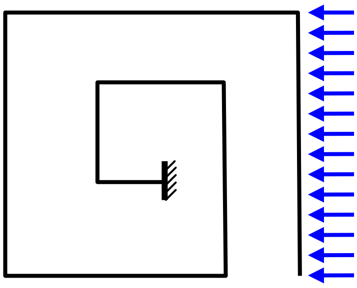


Fig. 4.3 A loop-shaped structure

Tip: Sometimes the constraints reactions are not necessary to determine the cross-section forces.

We begin from free end of the structure, Fig. 4.4.

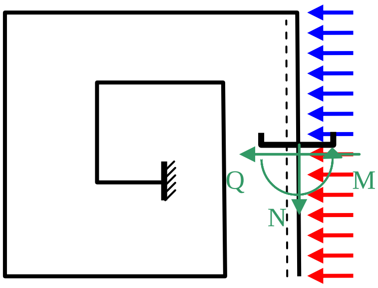


Fig. 4.4 First section of the structure

Further the section is from the free end greater is the bending moment: greater is the loading applied on rejected part of the structure and greater is the resultant's lever. The undersides are compressed, so the bending moment diagram should be drawn on the other side. The function is nonlinear and changes from

[1] It seems that civil engineers have opposite convention than mechanical ones, which draw bending moments on the compressed side.

zero at the free end to the value $q \cdot \text{length} \cdot \text{length}/2 = \frac{ql^2}{2}$ at the other end of the interval. The shear force is linear and changes from 0 to ql . The axial force is zero.
We make second section at adjacent horizontal part of the structure, Fig. 4.5.

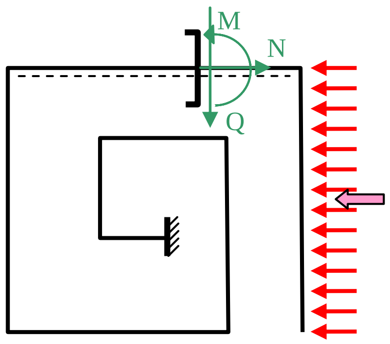


Fig. 4.5 Second section

Tip: From this interval till the last one, entire loading is applied to the rejected part of the structure. We can replace it by its resultant.

We see that undersides are compressed by constant bending moment: constant resultant and its lever. The shear force is zero and axial force is negative and constant.
Third section is made at adjacent vertical branch, Fig. 4.6.

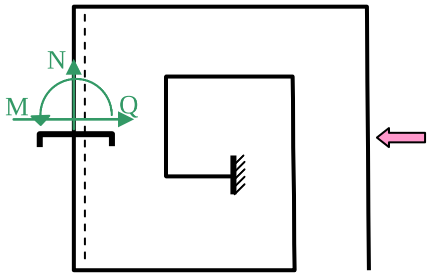


Fig. 4.6 Third section

The resultant lever changes, so the bending moment changes sign: "upper" undersides are compressed and "lower" tensioned; the change is linear. The shear force is constant and negative, the axial force is zero.
Forth section is made at ext horizontal branch, Fig. 4.7.

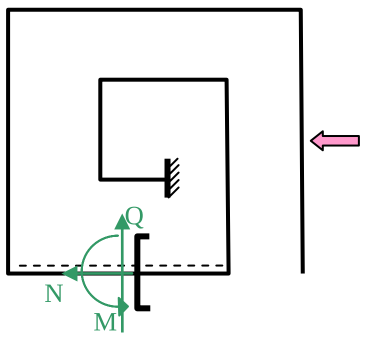


Fig. 4.7 Fourth section

Now, the undersides are tensioned by constant bending moment, the shear force is zero and the axial force is constant and positive (tension).
Repeating the reasoning we finally get the cross-section diagrams in the form as on Fig. 4.8.

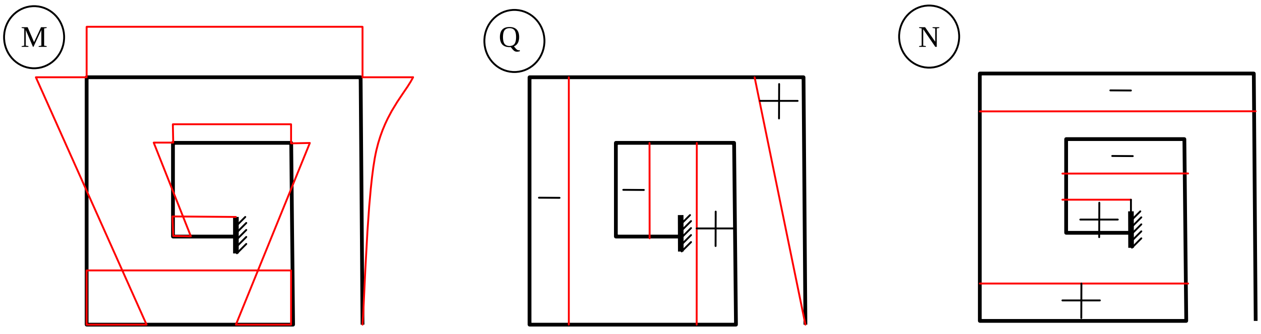


Fig. 4.8 Cross-sections diagrams

8 basic cases of simple beam (Fig. 4.9)

1. Three point bending[2] ("p-l-over-four" case)

GR[3]: The pointed tip at the bending moment diagram is always in the same sense as the point force.
GR: The jump in the shear force diagram is equal to the point force (its sense and value).
Maximal value of the bending moment is attained for the point force applied in the mid-span and is equal

$M_{\max} = \frac{Pl}{4}$ (hence its name).

2. Point moment

Constraint reactions equilibrate the applied point moment.
GR: The reactions values depend on the value of the point moment load only, not its position.
GR: At the cross-section of the point moment application there is a jump of the bending moment. The jump is equal to the value of the applied load (and not the upper/lower value of the bending moment).
GR: The jump of the bending moment results from the applied point moment. It means that, unless the point moment is applied, the bending moment is zero at the beam ends.

3. "q-l-square-over-eight" case

GR: The convexity of the bending moment diagram is in the sense of the applied loading.
GR: The maximum of the bending moment occurs at the section of the shear force zeroing.
GR: If it is not said otherwise, the extremum of the bending moment should always be determined (its position and value).

[2] Because the entire beam is bent, it means rather that the bending is caused by forces applied at 3 points
[3] General Rule

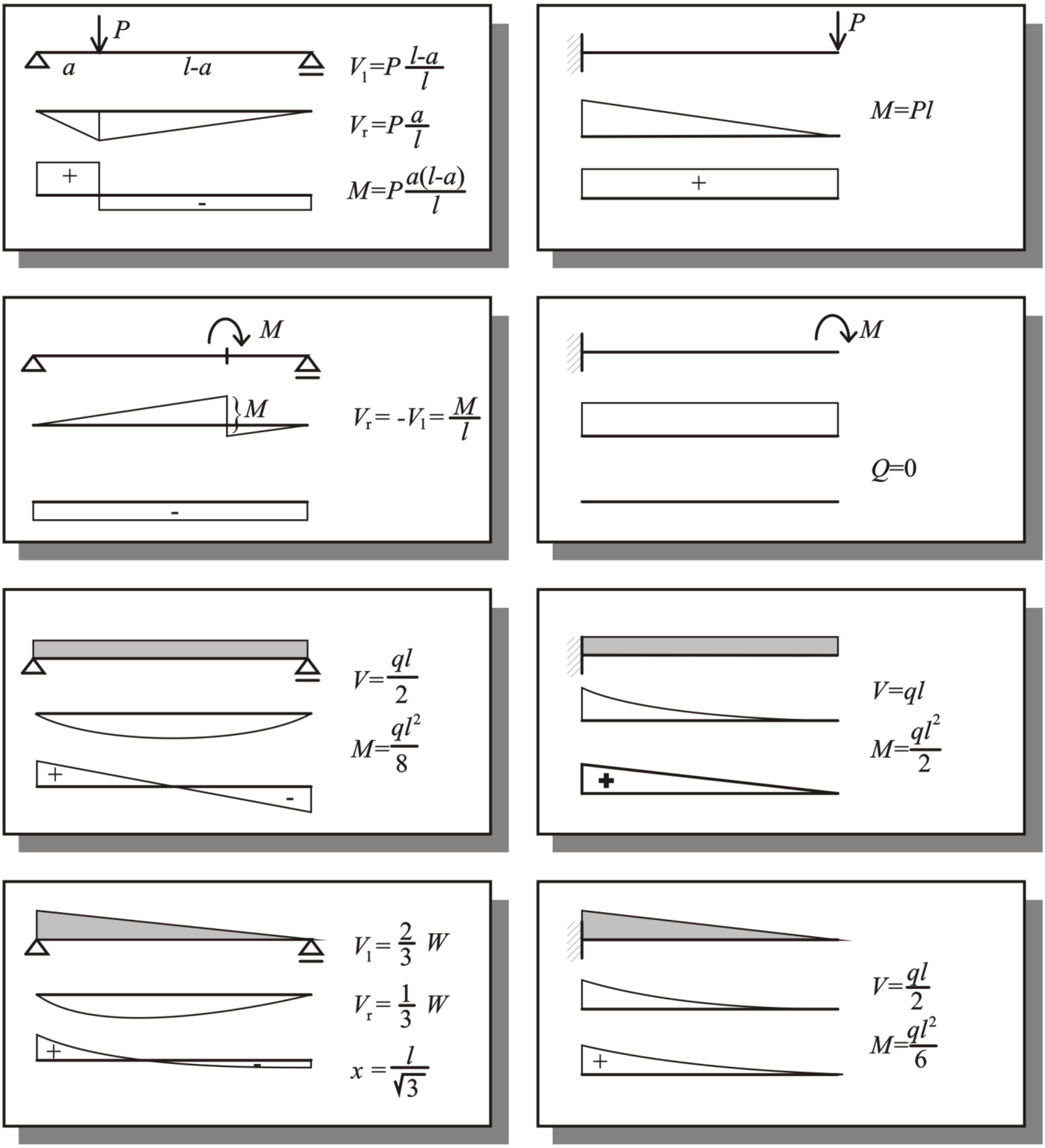


Fig. 4.9 Eight basic cases

4. Triangular loading

The resultant of the triangular loading is equilibrated by reactions which are 1/3 and 2/3 of its value.

GR: In the shear force diagrams, smaller rate coincides with smaller intensity and higher rate with higher load intensity.

5. Cantilever with the point force

In this case the upper side is tensioned and shearing force is constant.

6. Cantilever with the point moment

The load is equilibrated by the fixing moment only; the shear force zeroes.

7 and 8. Cantilever with continuous loading

The bending moment convexity follows the sense of loadings.

Use of superposition – example

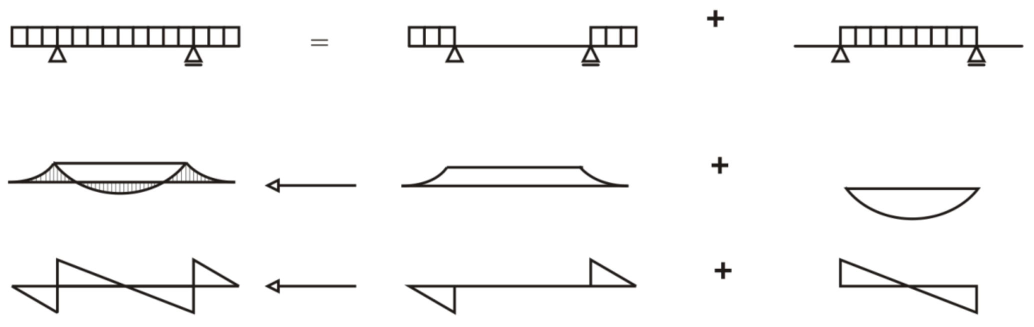
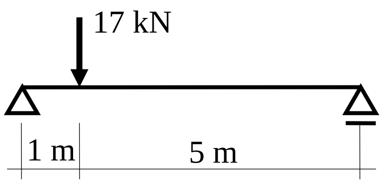


Fig. 4.10 Additivity of the results

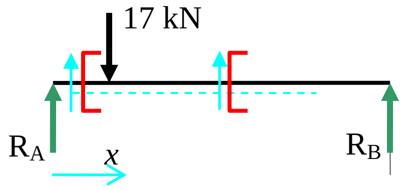
Sample problems

Problem no 1

Construct the diagrams of the cross-section forces for the beam in the figure.



Solution:



Constraints reactions:

$\sum M_B = 0: \quad R_A = \frac{17 \cdot 5}{6} = 14.17\,\text{kN}$, $\sum M_A = 0: \quad R_B = \frac{17 \cdot 1}{6} = 2.83\,\text{kN}$

There are two characteristic intervals

$0 < x < 1$

$M(x) = 14.17 \cdot x, \quad M(0) = 0, \quad M(1) = 14.17\,\text{kNm}$

$Q(x) = 14.17\,\text{kN}$

$N(x) \equiv 0$

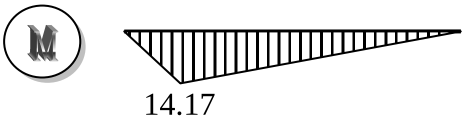
$1 < x < 6$

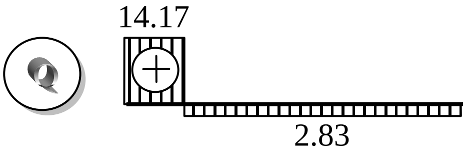
$M(x) = 14.17 \cdot x - 17 \cdot (x-1), \quad M(1) = 14.17\,\text{kNm}, \quad M(6) = 0$

$Q(x) = 14.17 - 17 = -2.83\,\text{kN}$

$N(x) \equiv 0$

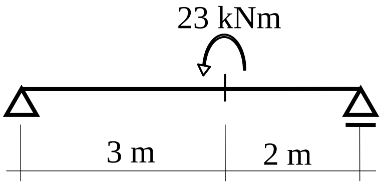
Diagrams of the cross-sections forces:



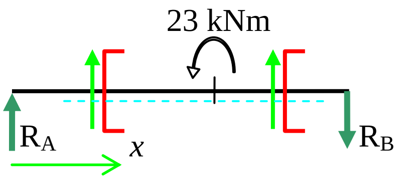


Problem no 2

Construct the diagrams of the cross-section forces for the beam in the figure.



Solution:



Constraints reactions:

$R_A = R_B = \dfrac{23}{5} = 4.6\,\text{kN}$,

There are two characteristic intervals

$0 < x < 3$

$M(x) = 4.6 \cdot x, \quad M(0) = 0, \quad M(3) = 13.8\,\text{kNm}$

$Q(x) = 4.6\,\text{kN}$

$N(x) \equiv 0$

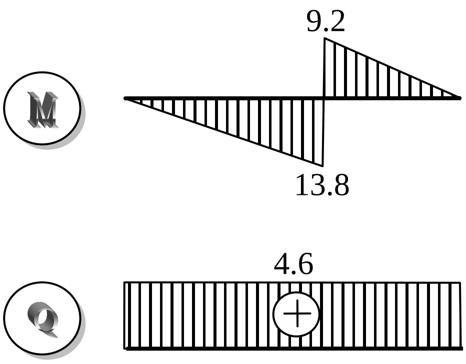
$3 < x < 5$

$M(x) = 4.6 \cdot x - 23, \quad M(3) = -9.2\,\text{kNm}, \quad M(5) = 0$

$Q(x) = 4.6\,\text{kN}$

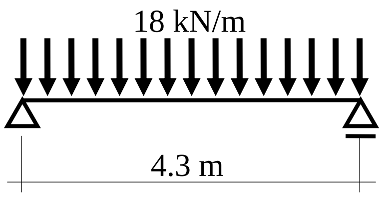
$N(x) \equiv 0$

Diagrams of the cross-sections forces:

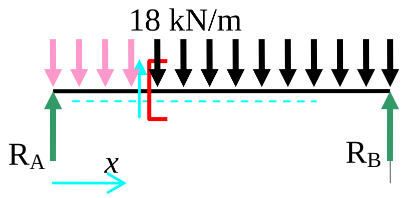


Problem no 3

Construct the diagrams of the cross-section forces for the beam in the figure.



Solution:



Constraints reactions:

$$R_A = R_B = \frac{18 \cdot 4.3}{2} = 38.7\ \text{kN}\,,$$

There is one characteristic interval

$$0 < x < 4.3$$

$$M(x) = 38.7 \cdot x - 18 \cdot \frac{x^2}{2}, \quad M(0) = 0, \quad M(4.3) = 0$$

$$Q(x) = 38.7 - 18 \cdot x, \quad Q(0) = 38.7\ \text{kN}, \quad Q(4.3) = \text{-}38.7\ \text{kN}$$

The shear force changes the sign at the point:

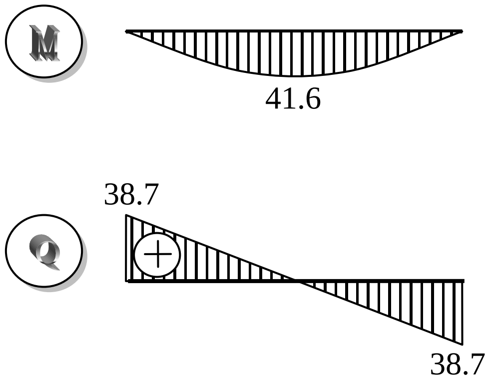
$$Q(x) = 0 \quad \rightarrow \quad x = \frac{38.7}{18} = 2.15\ \text{m}$$

The extreme value of the bending moment is:

$$M(2.15) = 38.7 \cdot 2.15 - 18 \cdot \frac{2.15^2}{2} = 41.6\ \text{kNm}$$

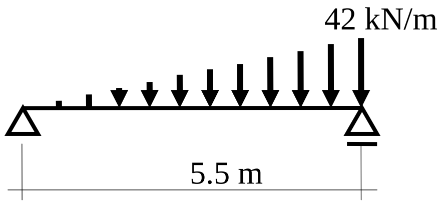
$$N(x) \equiv 0$$

Diagrams of the cross-sections forces:

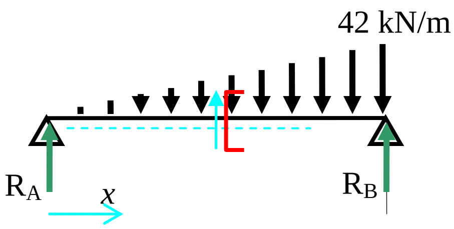


Problem no 4

Construct the diagrams of the cross-section forces for the beam in the figure.



Solution:



Constraints reactions:

$$R_A = \frac{42 \cdot 0.5 \cdot 5.5 \cdot 5.5/3}{5.5} = 38.5\,\text{kN}, \quad R_B = \frac{42 \cdot 0.5 \cdot 5.5 \cdot 5.5 \cdot 2/3}{5.5} = 77\,\text{kN}$$

There is one characteristic interval

$0 < x < 5.5$

$$M(x) = 38.5 \cdot x - \frac{42}{5.5} \cdot \frac{x^3}{6}, \quad M(0) = 0, \quad M(5.5) = 0$$

$$Q(x) = 38.5 - \frac{42}{5.5}\frac{x^2}{2}, \quad Q(0) = 38.5\,\text{kN}, \quad Q(5.5) = -77\,\text{kN}$$

The shear force changes the sign at the point:

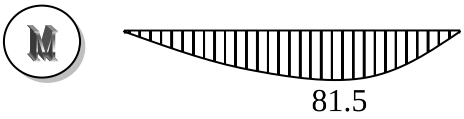
$$Q(x) = 0 \quad \rightarrow \quad 7.64\frac{x^2}{2} = 38.5 \quad \rightarrow \quad x = 3.17\,\text{m}$$

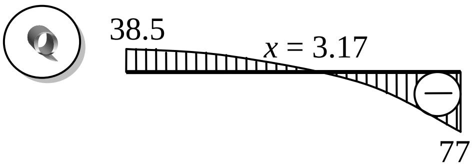
The extreme value of the bending moment is:

$$M(3.17) = 38.5 \cdot 3.17 - 7.64 \cdot \frac{3.17^3}{6} = 81.5\,\text{kNm}$$

$N(x) \equiv 0$

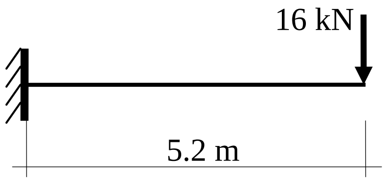
Diagrams of the cross-sections forces:



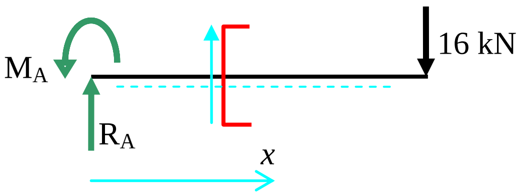


Problem no 5

Construct the diagrams of the cross-section forces for the beam in the figure.



Solution:



Constraints reactions:

$R_A = 16\,\text{kN}, \quad M_A = 16 \cdot 5.2 = 83.2\,\text{kNm}$

There is one characteristic interval

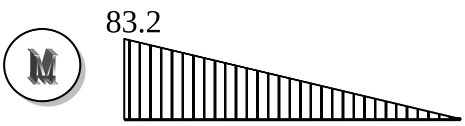
$0 < x < 5.2$

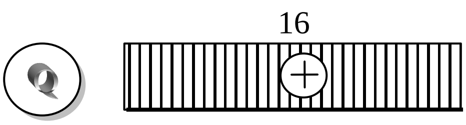
$M(x) = 16 \cdot x - 83.2, \quad M(0) = -83.2\,\text{kNm}, \quad M(5.2) = 0$

$Q(x) = 16\,\text{kN}$

$N(x) \equiv 0$

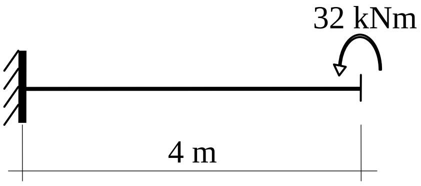
Diagrams of the cross-sections forces:



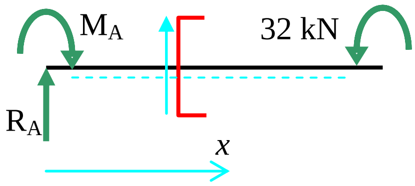


Problem no 6

Construct the diagrams of the cross-section forces for the beam in the figure.



Solution:



Constraints reactions:

$R_A = 0, \quad M_A = 32\,\text{kNm}$,

There is one characteristic interval

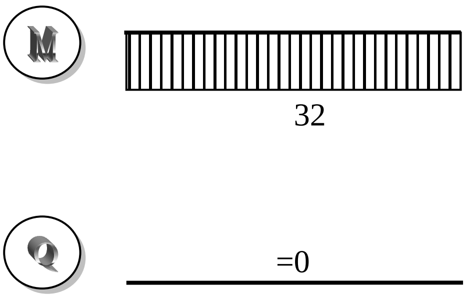
$0 < x < 4$

$M(x) = 32\,\text{kNm}$

$Q(x) \equiv 0$

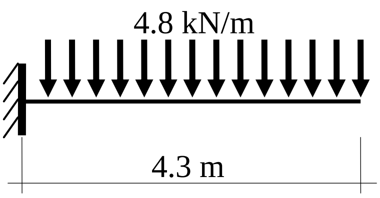
$N(x) \equiv 0$

Diagrams of the cross-sections forces:

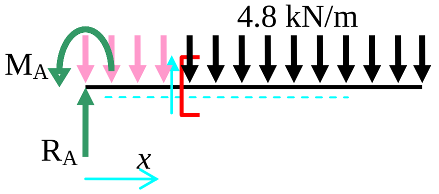


Problem no 7

Construct the diagrams of the cross-section forces for the beam in the figure.



Solution:



Constraints reactions:

$R_A = 4.8 \cdot 4.3 = 20.64\,\text{kN}, \quad M_A = 20.64 \cdot 2.15 = 44.38\,\text{kNm}$,

There is one characteristic interval

$0 < x < 4.3$

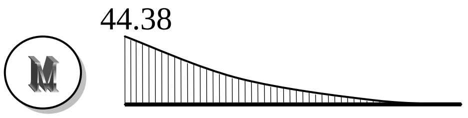
$M(x) = 20.64 \cdot x - 44.38 - 4.8 \cdot \frac{x^2}{2}, \quad M(0) = -44.38\,\text{kNm}, \quad M(4.3) = 0$

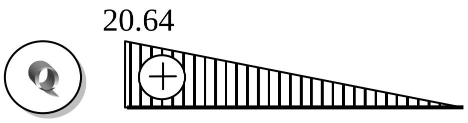
$Q(x) = 20.64 - 4.8 \cdot x, \quad Q(0) = 20.64\,\text{kN}, \quad Q(4.3) = 0$

The shear force is equal to zero at the free end, so there is minimal value of the bending moment:

$N(x) \equiv 0$

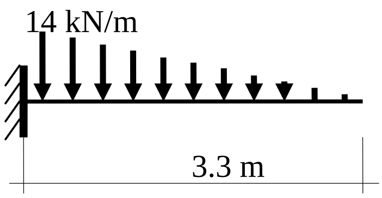
Diagrams of the cross-sections forces:



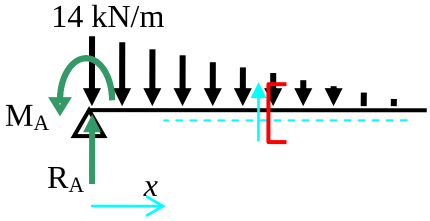


Problem no 8

Construct the diagrams of the cross-section forces for the beam in the figure.



Solution:



Constraints reactions:

$R_A = 14 \cdot 3.3 / 2 = 23.1\,\text{kN}, \quad M_A = 14 \cdot 3.3 \cdot 0.5 \cdot 3.3 / 3 = 25.41\,\text{kNm}$

There is one characteristic interval

$0 < x < 3.3$

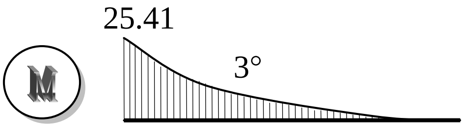
$$M(x) = -25.41 + 23..1 \cdot x - 14 \cdot \frac{x^2}{2} + \frac{14}{3.3} \cdot \frac{x^3}{6}, \quad M(0) = -25.41\,\text{kNm}, \quad M(3.3) = 0$$

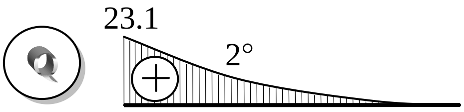
$$Q(x) = 23.1 - 14 \cdot x + \frac{14}{3.3} \cdot \frac{x^2}{2}, \quad Q(0) = 23.1\,\text{kN}, \quad Q(5.5) = 0$$

The shear force is equal to zero at the free end, so there is minimal value of the bending moment:

$N(x) \equiv 0$

Diagrams of the cross-sections forces:





Workshop theme

Write down the cross-section forces functions for 8 basic cases of simple beams.

Review problems

Practice the cross-section forces diagram for 8 basic cases at arbitrary position of the axes and loading signs. Exercise the solutions of these cases so you can draw solution for particular case within 2-3 minutes.

Addendum

Hints

Tip: There are two typical students' errors:

- an incomplete section; section does not determine the subsets properly, in effect the subsets are not disjoint
- a section without suitable internal forces or cross-section forces

Glossary

locus – m.g.p., miejsce geometryczne punktów (zbiór wszystkich takich punktów, że...)
section – przekrój
subset – podukład
disjoined subsets – układy rozłączne
cross-section – przekrój poprzeczny
prismatic bar – pręt pryzmatyczny (o stałym przekroju)
internal forces – siły wewnętrzne
cross-section forces – siły przekrojowe
cross-section proper coordinate system – układ własny przekroju poprzecznego
undersides – spody
axial force – siła podłużna, siła osiowa

transverse or transversal or shear force – siła poprzeczna, tnąca albo ścinająca
bending moment – moment zginający (gnący)
characteristic point – punkt charakterystyczny
characteristic interval – przedział charakterystyczny
simple supported beam ≈ belka jednoprzęsłowa
span – przęsło
cantilever – wspornik
overhanging beam – belka z przewieszeniem