13. Cross-section characteristics

Introduction

Definitions

cross-section area

$$A = \iint_A \mathrm{d}A, \ [\mathrm{m}^2]$$

static moments of area (first moments of area, first moments of inertia)

$$S_y = \iint_A z\mathrm{d}A, \quad S_z = \iint_A y\mathrm{d}A, \ [\mathrm{m}^3]$$

gravity centre (when the calculation concerns a geometrical shape only the term *centroid* is used)

$$C(y_c, z_c): \quad y_c = \frac{S_z}{A}, \quad z_c = \frac{S_y}{A}$$

centroidal axis, central axis – an axis passing through the centroid (the gravity centre)
(second moments of area), inertia moments (about *y* and *z* axes and product moment)

$$J_y = \iint_A z^2\mathrm{d}A, \quad J_z = \iint_A y^2\mathrm{d}A, \quad J_{yz} = \iint_A yz\mathrm{d}A, \ [\mathrm{m}^4]$$

central inertia moment – inertia moments about central axis
matrix of inertia moments

$$\begin{pmatrix} J_y & -J_{yz} \\ -J_{yz} & J_z \end{pmatrix}$$

principal central inertia moments – inertia moments resulting from the eigenvalue problem

$$J_1 = \frac{J_y + J_z}{2} + \sqrt{\left(\frac{J_y - J_z}{2}\right)^2 + J_{yz}^{\ 2}}, \quad J_2 = \frac{J_y + J_z}{2} - \sqrt{\left(\frac{J_y - J_z}{2}\right)^2 + J_{yz}^{\ 2}}, \quad \tan\alpha = \frac{J_1 - J_y}{-J_{yz}},$$

principal central axes – such axes x_1, x_2 that:

$$\begin{pmatrix} J_y & -J_{yz} \\ -J_{yz} & J_z \end{pmatrix} \rightarrow \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}$$

principal central inertia radii

$$i_1 = \sqrt{\frac{J_1}{A}}, \quad i_2 = \sqrt{\frac{J_2}{A}}, \ [\mathrm{m}]$$

ellipse of inertia – an ellipse in which half-axes are equal to principal inertia radii
polar inertia moment

$$J_O = \iint_A r^2\mathrm{d}A, \ [\mathrm{m}^4]$$

Superposition principle

Due to additivity of the double integrals, every integral may be calculated over subareas. It means that a cross-section may be divided into simpler geometric figures and the total results are the sum of the partial results.

Parallel axes theorem (Steiner's formulae)

Let's suppose the first and second inertia moments about the central axis *c* are known. Their values about any parallel axis *l*, at distance *d* from central axis *c*, read:

$$S_l = S_c + A \cdot d, \quad J_l = J_c + A \cdot d^2$$

Formulae to memorize

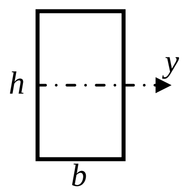
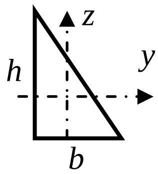
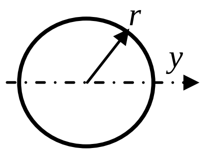
figure	drawing	area	inertia moment
rectangle		bh	$J_y = \frac{bh^3}{12}$
right triangle		$\frac{1}{2}bh$	$J_y = \frac{bh^3}{36}$ $J_{yz} = -\frac{b^2h^2}{72}$ [1]
circle		πr^2	$J_y = \frac{\pi r^4}{4}$ $J_0 = \frac{\pi r^4}{2}$

Table 13.1 Characteristics of basic figures – formulae to memorize

Tensor calculus conclusions

The principal inertia moments are extreme (maximal and minimal).
Every symmetry axis is the principal central axis. Second axis is perpendicular to the symmetry axis and passes through the gravity centre.
Tip: The characteristics calculations for symmetric cross-section are simpler because we can compute principal moments directly.
If a cross-section has more then two symmetry axes, every central axis is principal too. In that case every central moment is also a principal one.

Example

Determine the geometric characteristics for the section in Fig. 13.1.

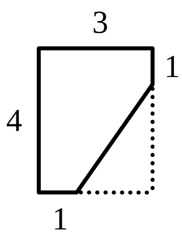


Fig. 13.1 Cross-section

Solution

The calculation is made for the rectangle "minus" the triangle.
area:

[1] the sign of the product moment changes each time a reflection is made

$$A = 3 \cdot 4 - \frac{1}{2} \cdot 2 \cdot 3$$

gravity centre:

$$y_c = \frac{12 \cdot 1.5 - 3 \cdot \left(1 + \frac{2}{3} \cdot 2\right)}{9} = 1.222 \text{ cm}$$

$$z_c = \frac{12 \cdot 2 - 3 \cdot \frac{1}{3} \cdot 3}{9} = 2.333 \text{ cm}$$

central inertia moments:

$$J_{yc} = \frac{3 \cdot 4^3}{12} + 12 \cdot (2 - 2.333)^2 - \frac{2 \cdot 3^3}{12} - 3 \cdot (1 - 2.333)^2 = 10.5 \text{ cm}^4$$

$$J_{zc} = \frac{3^3 \cdot 4}{12} + 12 \cdot (1.5 - 1.222)^2 - \frac{2^3 \cdot 3}{12} - 3 \cdot \left(1 + \frac{2}{3} \cdot 2 - 1.222\right)^2 = 5.556 \text{ cm}^4$$

$$J_{zycc} = 0 + 12 \cdot (2 - 2.333) \cdot (1.5 - 1.222) - \frac{2^2 \cdot 3^2}{72} - 3 \cdot (1 - 2.333) \cdot \left(1 + \frac{2}{3} \cdot 2 - 1.222\right) = 2.833 \text{ cm}^4$$

principal central inertia moments and direction

$$J_1 = \frac{10.5 + 5.556}{2} + \sqrt{\left(\frac{10.5 - 5.556}{2}\right)^2 + 2.833^2} = 11.79 \text{ cm}^4$$

$$J_2 = \frac{10.5 + 5.556}{2} - \sqrt{\left(\frac{10.5 - 5.556}{2}\right)^2 + 2.833^2} = 4.27 \text{ cm}^4$$

$$\tan \alpha = \frac{11.79 - 10.5}{-2.833} = -0.4553 \quad \rightarrow \quad \alpha = -0.4273 \quad (-24.48°)$$

principal central inertia radii:

$$i_1 = \sqrt{\frac{11.79}{9}} = 1.14 \text{ cm}$$

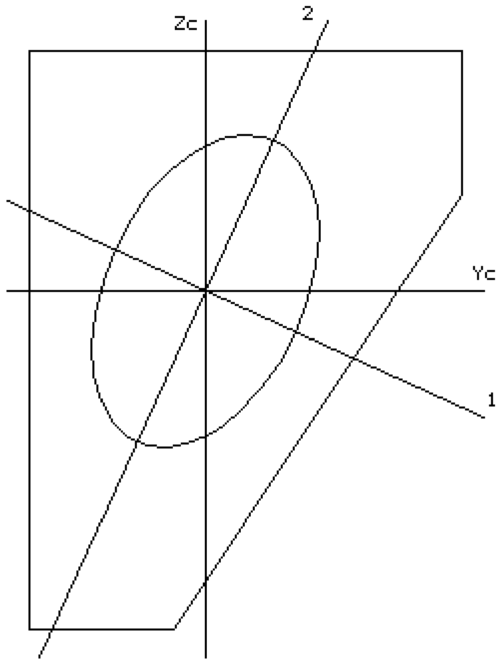


Fig. 13.2 Inertia ellipse of the cross-section

$$i_2 = \sqrt{\frac{4.27}{9}} = 0.69\,\text{cm}$$

The inertia ellipse is shown in Fig. 13.2.

Workshop theme

Adopt numerical input data for the cross-section in Fig. 13. 3 and determine principal central inertia moments. Draw the inertia ellipse.

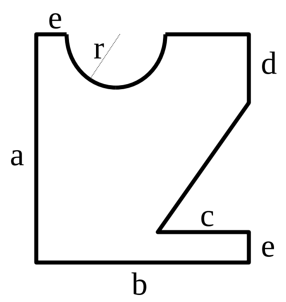


Fig. 13.3 Cross-section to compute

Review problems

Adopt numerical input data and determine principal central inertia moments.

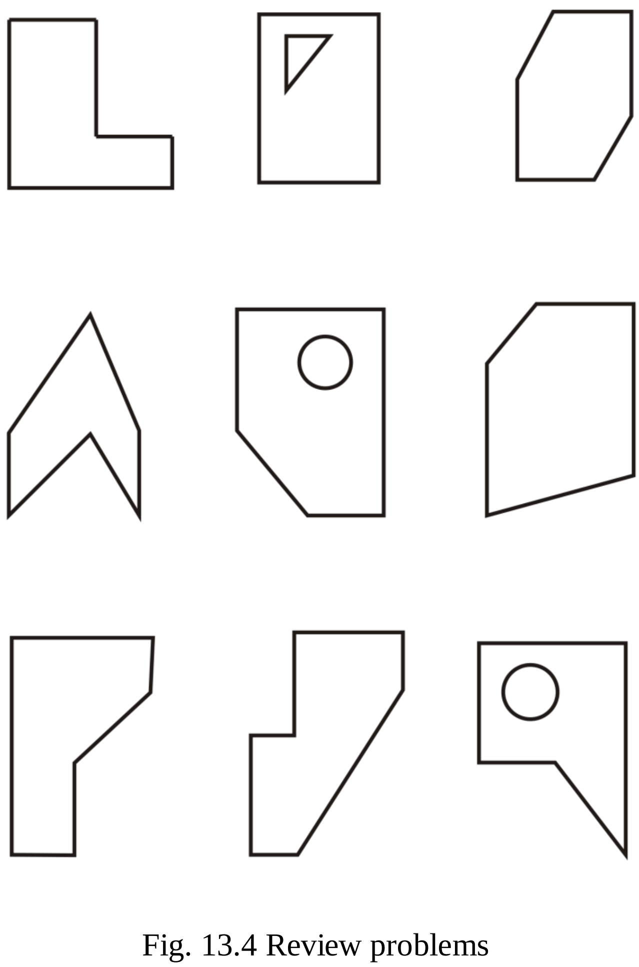


Fig. 13.4 Review problems

Addendum

Hints

Tip: Inertia moment about the central axis is minimal in reference to other parallel axes.

Tip: The gravity centre is always inside a circumscribed polygon.

Except for a few cases, all calculations in strength of materials are carried on with respect to the principal central axes.

Glossary

first moment of area, static moment – moment statyczny
second moment of area, inertia moment – moment bezwładności
centroid, gravity centre – środek ciężkości
centroidal axis, central axis – oś centralna
centroidal inertia moment, central inertia moment – centralny moment bezwładności
inertia product moment – moment dewiacji
principal central axis – oś główna centralna
principal central inertia moment – główny centralny moment bezwładności
inertia radius – promień bezwładności
ellipse of inertia – elipsa bezwładności