14. Stress state

Introduction

Definitions

The stress vector p_i is internal forces' density in the plane with outer normal n_i, parallel to the axis of the coordinate system.
The stress tensor σ_{ij} is a matrix of stress vector components written in the given coordinate system:

$$p_i = \sigma_{ij} n_j$$

The components of the stress tensor in the new coordinate system may be computed by the transformation formula:
– in index notation:

$$\sigma_{ij} = a_{ik} a_{jl} \sigma_{kl}$$

– in matrix notation:

$$\mathbf{\Sigma}' = \mathbf{A} \cdot \mathbf{\Sigma} \cdot \mathbf{A}^T$$

where:
– σ_{ij} is a stress matrix with normal stress on the diagonal and shear stress elsewhere:

$$\sigma_{ij} = \underbrace{\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}}_{\text{(scientific notation)}} = \underbrace{\begin{pmatrix} \sigma_{x} & \tau_{xy} & \tau_{xz} \\ \tau_{yz} & \sigma_{y} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{z} \end{pmatrix}}_{\text{(engineering notation)}}$$

The first index specifies the direction of the outer unit normal of the section plane. The second index specifies the stress direction.
A matrix can be the stress matrix if it fulfills the partial differential equations (PDE) of internal equilibrium (Navier's equations):

$$\begin{aligned} &\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xx}}{\partial z} + P_x = 0 \\ &\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yx}}{\partial z} + P_y = 0, \quad (\sigma_{ij,j} + P_i = 0, \quad i, j = x, y, x) \\ &\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + P_z = 0 \end{aligned}$$

and the static boundary conditions:

$$\begin{aligned} q_x &= \sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z \\ q_y &= \tau_{xy} n_x + \sigma_y n_y + \tau_{yz} n_z, \quad (q_i = \sigma_{ij} n_j, \quad i, j = x, y, z) \\ q_z &= \tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z \end{aligned}$$

Stress state analysis

The main goal of the stress state analysis is to find such directions of the sections where the stresses values are extreme. This happens when the outer normal and the stress vector are parallel, and leads to the eigenvalues problem:

$$\sigma_{ij} v_j = \lambda v_i \quad \rightarrow \quad (\sigma_{ij} - \lambda \delta_{ij}) v_j = 0 .$$

The set of algebraic linear equation system has non-zero (non-trivial) solution if and only if the main determinant of the system is zero. From this condition we get a cubic equation for the principal stress values:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0\,.$$

The stress matrix in the principal directions is diagonal and the shear components are zero. From algebra we know that the principal normal stresses are extreme and the principal directions are perpendicular:

$$\underbrace{\begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yz} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix}}_{\text{in }(x,y,z)} \Rightarrow \underbrace{\begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}}_{\text{in }(1,2,3)}$$

Mohr's circles

The domain of possible results due to the transformation of a coordinate system is illustrated by Mohr's circles, Fig. 14.1 with the shaded region.

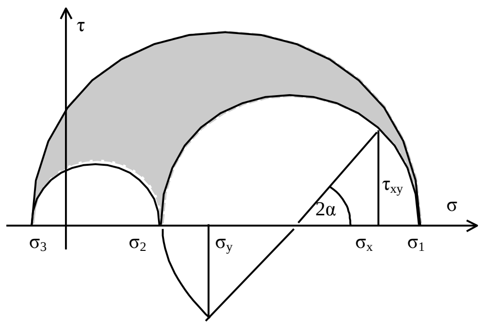


Fig. 14.1 Mohr's circles

Saint-Venant principle

Two statically equivalent load systems produce nearly the same stress in regions at a distance that is at least equal to the largest dimension in the loaded region

Examples

Three-dimensional state of stress

The stress matrix and the outer normal vector of a section are given. Determine the principal directions and stresses, the normal and tangential components of the stress vector for the section.

$$\sigma_{ij} = \begin{pmatrix} 240 & -17 & 40 \\ -17 & 85 & 60 \\ 40 & 60 & -120 \end{pmatrix},\ \mathbf{n}\left(\sqrt{7},2,-2\right)$$

Solution

We calculate the invariants:

$$I_1 = \mathrm{tr}(\sigma_{ij}) = 240 + 85 - 120 = 205$$

$$I_2 = \begin{vmatrix} 240 & -17 \\ -17 & 85 \end{vmatrix} + \begin{vmatrix} 85 & 60 \\ 60 & -120 \end{vmatrix} + \begin{vmatrix} 240 & 40 \\ 40 & -120 \end{vmatrix} = -24089$$

$$I_3 = \det(\sigma_{ij}) = 240 \cdot \begin{vmatrix} 85 & 60 \\ 60 & -120 \end{vmatrix} + 17 \cdot \begin{vmatrix} -17 & 60 \\ 40 & -120 \end{vmatrix} + 40 \cdot \begin{vmatrix} -17 & 85 \\ 40 & 60 \end{vmatrix} = 3494920$$

The cubic equation:

$$\sigma^3 - 205 \cdot \sigma^2 - 24089 \cdot \sigma - 3494920 = 0$$

has the roots:

$$\sigma_1 = 245.1, \quad \sigma_2 = 101.0, \quad \sigma_3 = -141.1\,.$$

The principal directions are given by the transformation matrix:

$$a_{ij}=\begin{pmatrix}0.9929 & 0.0404 & -0.1123\\ 0.0689 & 0.9624 & -0.2627\\ 0.9583 & 0.2686 & 0.9583\end{pmatrix}\begin{matrix}x\\ y\\ z\end{matrix}.$$
$$\qquad\quad 1 \qquad\quad 2 \qquad\quad 3$$

We calculate the length of the normal vector:

$$|\mathbf{n}|=\sqrt{7+4+4}=3.8730$$

and we normalize the vector (to obtain unit normal vector)

$$\mathbf{n}(0.6831, 0.5164, -0.5164)$$

The stress vector is:

$$p_j=\sigma_{ij}n_i \quad\to\quad p(134.5, 1.30, 120.3)$$

its length:

$$|\mathbf{p}|=180.5$$

the normal component of the stress vector is:

$$\sigma=\mathbf{p}\cdot\mathbf{n}=p_i n_i=30.45$$

and the tangential component is:

$$|\tau|=\sqrt{p^2-\sigma^2}=177.9$$

Plane stress

For the given stress tensor, determine principal stresses and directions.

$$T_\sigma=\begin{pmatrix}40 & 16\\ 16 & -10\end{pmatrix}.$$

Solution

The principal stresses for the plane state of stress are:

$$\sigma_1=\frac{\sigma_x+\sigma_y}{2}+\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^{\;2}}=\frac{40-10}{2}+\sqrt{\left(\frac{40+10}{2}\right)^2+16^2}=44.68$$
$$\sigma_2=\frac{\sigma_x+\sigma_y}{2}-\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^{\;2}}=\frac{40-10}{2}-\sqrt{\left(\frac{40+10}{2}\right)^2+16^2}=-14.68,$$

The principal directions:

$$\tan\alpha_1=\frac{\sigma_1-\sigma_x}{\tau_{xy}}=\frac{44.68-40}{16}=0.2925 \quad\to\quad \alpha_1=0.285, \quad (16.3°)$$
$$\tan\alpha_2=\frac{\sigma_2-\sigma_x}{\tau_{xy}}=\frac{-14.68-40}{16}=-3.4175 \quad\Rightarrow\quad \alpha_2=-1.286, \quad (-73.7°)$$

The transformation matrix and the stress tensor in principal directions are, cf. Fig. 14.2:

	1	2
x	0.9598	−0.2808
y	0.2808	0.9598

$$T_\sigma=\begin{pmatrix}44.68 & 0\\ 0 & -14.68\end{pmatrix}$$

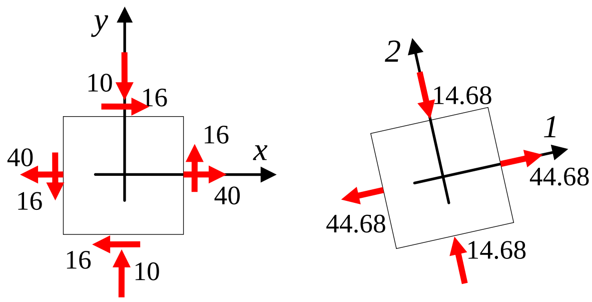


Fig. 14.2 Plane state of stress

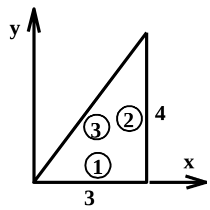
or, in 3D notation:

	1	2	3
x	-0.9598	0	0.2808
y	-0.2808	0	-0.9598
z	0	1	0

$$T_\sigma = \begin{pmatrix} 44.68 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -14.68 \end{pmatrix}$$

Static boundary conditions

Given the stress tensor, determine the mass forces and the boundary loading of the shield.



$$T_\sigma = \begin{pmatrix} 5 & -2y \\ -2y & 8 \end{pmatrix}$$

Fig. 14.3 Shield and stress state

Solution

The mass forces from Navier's equations are: **P** (2, 0)

$$\begin{cases} q_x = \sigma_x n_x + \tau_{xy} n_y \\ q_y = \tau_{xy} n_x + \sigma_y n_y \end{cases}$$

the boundary (1): $\mathbf{n}(0,-1) \quad \to \quad q_x = 2y = 0, \quad q_y = -8$

the boundary (2): $\mathbf{n}(1,0) \quad \to \quad q_x = 5, \quad q_y = -2y$

the boundary (3): $\mathbf{n}(-0.8,0.6) \quad \to \quad q_x = -4 - 1.2y, \quad q_y = 1.6y + 4.8$

The loading is drawn in Fig. 14.4.

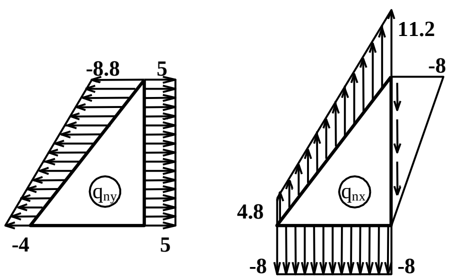


Fig. 14.4 The loading of the shield

The checking of the solution:

$\sum X = -\frac{1}{2}\cdot(8.8+4)\cdot 5 + 5\cdot 4 + 2\cdot\frac{1}{2}\cdot 3\cdot 4 = 0$,OK, similarly: $\sum Y = \ldots = 0, \quad \sum M_o = \ldots = 0$

Workshop theme

1) The stress matrix is given:

$$T_\sigma = \begin{pmatrix} 30 & 20 & 0 \\ 20 & 20 & 40 \\ 0 & 40 & -10 \end{pmatrix} \text{ MPa.}$$

Determine: a) normal stress on the section plane with the normal $\mathbf{v}\left(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}\right)$, b) shear stress in the plane parallel to the vectors: $\mathbf{v}_1\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ and $\mathbf{v}_2\left(-\frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}\right)$. Interpret the obtained results graphically.

2) Write the boundary conditions on the side surface of floating a wooden column. Assume the draught greater than the radius of the cylinder, Fig. 14.5

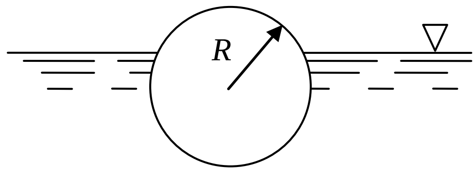


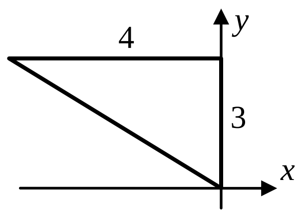
Fig. 14.5 Floating wooden column

Review problems

1) Given the plane stress matrix, find principal stresses and their directions. Illustrate the stress state before the transformation and after this.

$$T_\sigma = \begin{pmatrix} 0 & 150 \\ 150 & 0 \end{pmatrix} \text{ [MPa]}$$

2) Given the stress state and the shield shape with dimensions, Fig. 14.6, determine the mass forces and the loading on the boundary.



$$T_\sigma = \begin{pmatrix} 2xy & -7x \\ -7x & 5y \end{pmatrix}$$

Fig.

14.6 Plane stress state

3) There is uniaxial stress state in the cross-section plane of a bar, $\sigma_x = 150$ [MPa]. Determine the stress components in the plane turned by 45 degrees along *z* axis, Fig. 14.7.

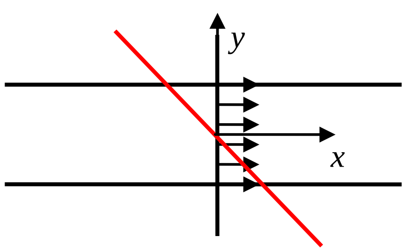


Fig. 14.7 Uniaxial stress state

Addendum

Glossary

outer (exterior) unit normal – wersor normalnej zewnętrznej
stress vector – wektor naprężenia
load vector – wektor obciążenia
stress matrix – macierz naprężenia
stress tensor – tensor naprężenia
normal stress – naprężenia normalne
shear stress – naprężenia styczne
internal balance equations – równania równowagi wewnętrznej
static boundary conditions – statyczne warunki brzegowe
stress state analysis – analiza stanu naprężenia
Mohr's circles – koła Mohra