15. Strain state

Introduction

Definitions

deformation – occurs when the distance between the trajectories of two points changes and/or the angle between the two lines on the body changes
we distinguish two configurations: original (undeformed) and final (deformed)
displacement – difference of a point position in original and final configuration
material (Lagrangian) description – original material configuration is used as a reference (usually it is the case in solid mechanics, where we observe the displacement of a point at the bridge mid-span due to loading)
spatial (Eulerian) description – spatial coordinate set is used as a reference (usually it is the case in hydromechanics, where we consider the pressure or velocity at some point in a pipeline, we are not interested in the fact which particle passes through the point)
Lagrangian strain – a strain computed by using original geometry as a reference
Eulerian strain – a strain computed by using deformed geometry as a reference
normal strain – a ratio of deformation to original length
shear strain – a half-change of angle from a right angle
Note: strain is dimensionless
strain value is much smaller than unity (less than 0.001 for instance), we call it infinitesimal
for infinitesimal strains the difference between Lagrangian and Eulerian strains becomes unimportant
strain matrix - a matrix with normal strains on the diagonal and shear strains elsewhere.
The strain matrix is a tensor and transforms according t transformation formula:
– in index notation:

$$\varepsilon_{ij} = a_{ik} a_{jl} \varepsilon_{kl}$$

– in matrix notation:

$$\mathrm{E}' = \mathbf{A} \cdot \mathrm{E} \cdot \mathbf{A}^T$$

where:
– ε_{ij} is a strain matrix with normal strain on the diagonal and shear strain elsewhere:

$$\varepsilon_{ij} = \underbrace{\begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix}}_{\text{(scientific notation)}} = \underbrace{\begin{pmatrix} \varepsilon_{x} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yz} & \varepsilon_{y} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{z} \end{pmatrix}}_{\text{(engineering notation)}} = \underbrace{\begin{pmatrix} \varepsilon_{x} & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yz} & \varepsilon_{y} & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_{z} \end{pmatrix}}_{\text{("old" engineering notation)}}$$

Single index specifies the direction of the fiber, a pair of indexes specifies the directions of the fibers that determine the considered angle.
A matrix can be the strain matrix if it fulfills the compatibility equations:

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0$$

and the kinematic boundary conditions.
The relationships between the displacement vector and infinitesimal strains are expressed in partial derivatives (Cauchy's equations):

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z}, \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \quad \varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \quad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right).$$

The strain state analysis is identical to the stress state analysis.

Strain measurement in plane strain state

The strains can be measured by the strain gauges. The normal strain is measured by a gauge constructed from a single wire that is wound back and forth, Fig. 15.1, attached to the surface of the tested object.

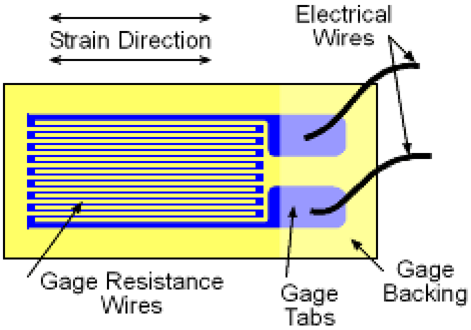

Fig. 15.1 Strain gauge

The electrical resistance changes in the direction opposite to the strain change. Most strain gauge measurement devices automatically collaborate the resistance change to the strain, so the device output is the actual strain.

Since a single gauge can only measure the strain in a single direction only, two gauges are needed to determine strain in the *x* and *y* directions. However, there is no gauge which is capable of measuring shear strain.

Because any transformed normal strain is a function of the coordinate strains, $\varepsilon_x, \varepsilon_y, \varepsilon_{xy}$, three rotated different gauges give three equations with three unknowns $\varepsilon_x, \varepsilon_y, \varepsilon_{xy}$. Any three gauges used together at one location on a stressed object are called a strain rosette, Fig. 15.2.

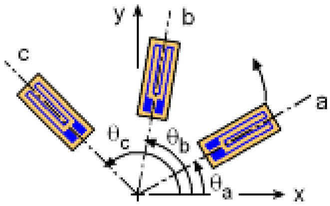

Fig. 15.2 Strain gauge rosette

Example – strain rosette 45°

The strains measured by the strain rosette 45° are: $\varepsilon_a = 0.0005$, $\varepsilon_b = 0.0002$, $\varepsilon_c = -0.0003$. Determine the strain components and compute principal strains and their directions.

Solution

In this case the gauges are separated by an angle of 45°, Fig. 15.3:

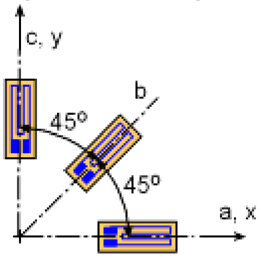

Fig. 15.3 Rosette at 45°

The direction of *a* and *x* are the same, so $\varepsilon_x = \varepsilon_a = 0.0005$. Similarly, the direction of *c* coincides with the direction of *y*, so $\varepsilon_y = \varepsilon_c = -0.0003$. For the strain in the direction of *b* we have:

$$\varepsilon_b = \varepsilon_x \cos^2(b,x) + \varepsilon_y \cos^2(b,y) + \varepsilon_{xy} \cos(b,x)\cos(b,y) + \varepsilon_{yx} \cos(by)\cos(b,x) =$$
$$= 0.0005 \cdot (\cos 45)^2 - 0.0003 \cdot (\cos 45)^2 + 2 \cdot \varepsilon_{xy} \cdot \cos 45 \cdot \cos 45 = 0.0005 \cdot 0.5 - 0.0003 \cdot 0.5 + 2 \cdot \varepsilon_{xy} \cdot 0.5 = .$$
$$= 0.00025 - -0.00015 + \varepsilon_{xy}$$

At the same time $\varepsilon_b = 0.0002$, so:

$$0.0001 + \varepsilon_{xy} = 0.0002 \quad \rightarrow \quad \varepsilon_{xy} = 0.0001$$

and

$$T_e = \begin{pmatrix} 0.0005 & 0.0001 \\ 0.0001 & -0.0003 \end{pmatrix}.$$

The principal strains are:

$$\varepsilon_1 = \frac{\varepsilon_x + \varepsilon_y}{2} + \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \varepsilon_{xy}^{\ 2}} = \frac{0.0005 - 0.0003}{2} + \sqrt{\left(\frac{0.0005 + 0.0003}{2}\right)^2 + 0.0001^2} = 0.0001 + 0.000412 = 0.000512$$

$$\varepsilon_1 = \frac{\varepsilon_x + \varepsilon_y}{2} - \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \varepsilon_{xy}^{\ 2}} = \frac{0.0005 - 0.0003}{2} - \sqrt{\left(\frac{0.0005 + 0.0003}{2}\right)^2 + 0.0001^2} = 0.0001 - 0.000412 = 0.000312$$

The principal directions are:

$$\tan\alpha = \frac{\varepsilon_1 - \varepsilon x}{\varepsilon_{xy}} = \frac{0.000512 - 0.0005}{0.0001} = 0.12 \quad \rightarrow \quad \alpha = 0.12 \quad (= 6.8°)$$

Workshop theme

The strains measured by the strain rosette 60°, Fig. 15.4, are:

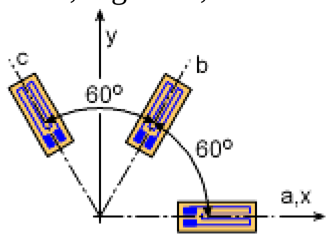

Fig. 15.4 Strain rosette 60°

$\varepsilon_a = \ldots\ldots\ldots \quad (-0.0001 \div 0.0007)$,

$\varepsilon_b = \ldots\ldots\ldots \quad (-0.0004 \div 0.0007)$,

$\varepsilon_c = \ldots\ldots\ldots \quad (-0.0002 \div 0.0002)$.

Determine the strain components and compute principal strains and their directions.

Addendum

Glossary

deformation – deformacja
original (undeformed) configuration – konfiguracja początkowa, pierwotna, nieobciążona
final (deformed) configuration – konfiguracja końcowa, aktualna, obciążona
displacement – przemieszczenie
material (Lagrangian) description – opis materialny, lagranżowski
spatial (Eulerian) description – opis przestrzenny, eulerowski
Lagrangian strain – odkształcenie we współrzędnych materialnych
Eulerian strain – odkształcenie we współrzędnych przestrzennych
normal strain – odkształcenie normalne, liniowe
shear strain – odkształcenie kątowe
compatibility equations – równania nierozdzielności

kinematic boundary conditions – kinematyczne warunki brzegowe
gauge, (AmE gage) – czujnik
strain rosette – rozeta tensometryczna