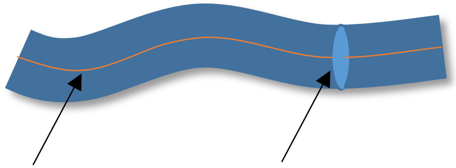
Strength of Materials

2. Cross-sectional forces

Bar structures

A bar – a structural element with length much greater than its width and height

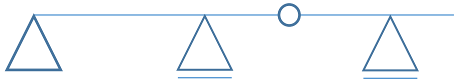
Straight bars: beams, frames, and trusses; curvilinear bars: arches



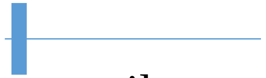
axis cross-section

simply supported beam
with a pin and a roller

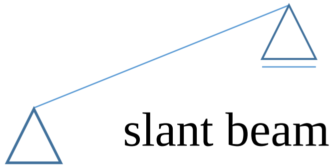
one span
overhung beam



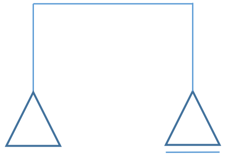
hinged (Gerber) beam



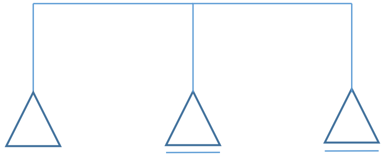
cantilever
(clamped beam)



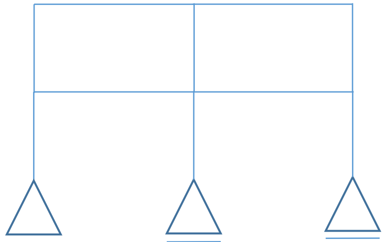
slant beam



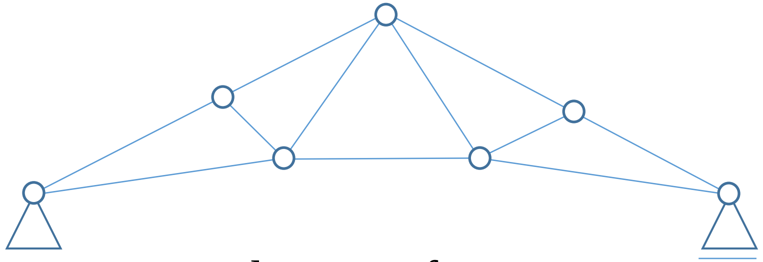
one-bay frame



two-bay frame

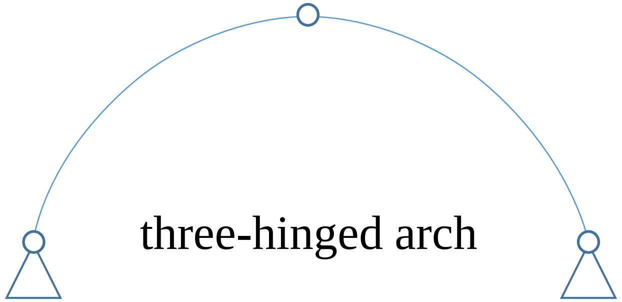


two-story two-bay frame
(two-storey two-bay frame)



planar roof truss

dead-end tower
(anchor pylon,
3D truss)



three-hinged arch

Loadings are applied to the bar axis

Kinematic stability (rigidity)

Degrees of freedom (DoF) – number of independent parameters that describe a movement (number of independent motions that are allowed to the body)

If DoF > 0 the system is a mechanism (and can be calculated dynamically, with the use of d'Alembert forces of inertia)

dynamic equilibrium: $\Sigma \boldsymbol{F} + m\ddot{\boldsymbol{x}} = \boldsymbol{0}$

If DoF = 0 the system is blocked, unmovable, stiff, still, geometrically rigid, stable, and can be analyzed by static methods; it's in static equilibrium

static equilibrium: $\Sigma \boldsymbol{F} = \boldsymbol{0}$

Make sure of stability of the system before you perform static calculations

A system can be stable/unstable as a whole (with constraints considered; external stability) and free body stable (considered without the constraints; internal stability)

constraint	DoF	movements	blocked	reactions
pin	1	1R	2T	2F
roller	2	1R, 1T	1T	1P
fixed end	0	none	1R, 2T	2F, 1M
guided support	1	1T	1R, 1T	1P, 1M

R – rotation
T – translation
F – force
P – perpendicular force
M - moment

External and internal stability

Very important question is: will the structure keep its shape under loading?
The answer YES means that the structure is stable.

The cases are:

Internal stability	External stability	Static equilibrium possible
YES	NO	NO
NO	NO	NO
YES	YES	YES
NO	YES	YES[1)]

[1)] but there will be some problems with static calculations; so, each step of analysis is important

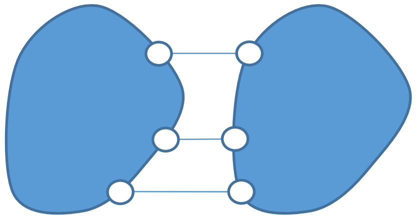
There are plenty of method to determine the structure stability. The most instructive are three of them.

Rigid connection of two members

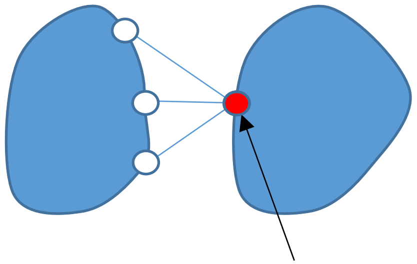
Shield (member) – a (sub)system that's been proved as geometrically stable. It has only one instantaneous rotation center (IRC)

Two shields connection theorem (2ST)
Two members connection is rigid (stable) if the members are connected by three bars, that are not parallel nor their directions intersect at one point (so, the relative movement between the members is excluded)

unstable connections



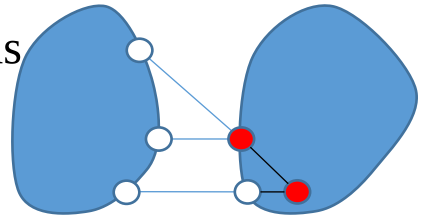
parallel connection bars (intersection point at infinity = instantaneous rotation center)



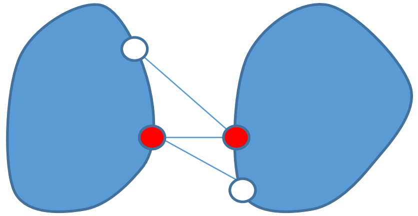
connection bars' directions intersect at one point

instantaneous rotation center (IRC)

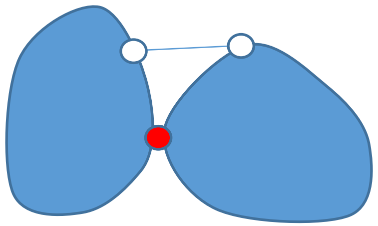
stable connections



two instantaneous rotation centers (excluded)



(as above)



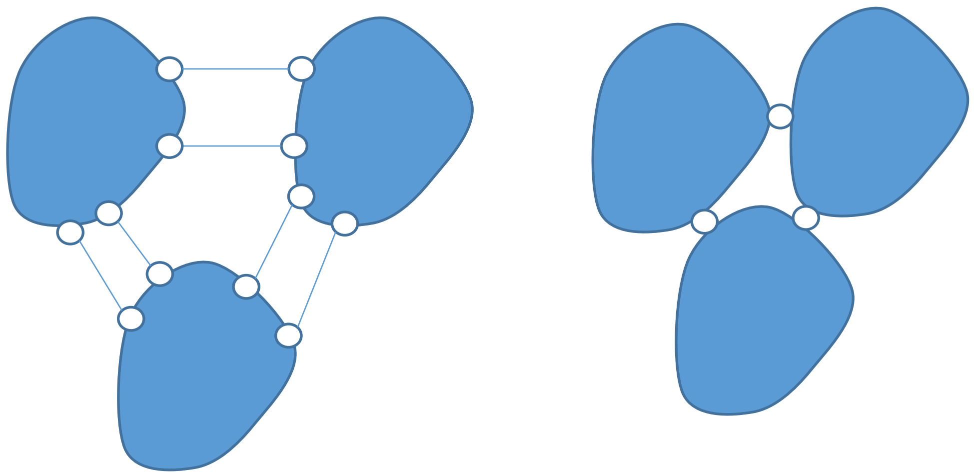
(a hinge is equivalent to two bars connecting at the hinge)

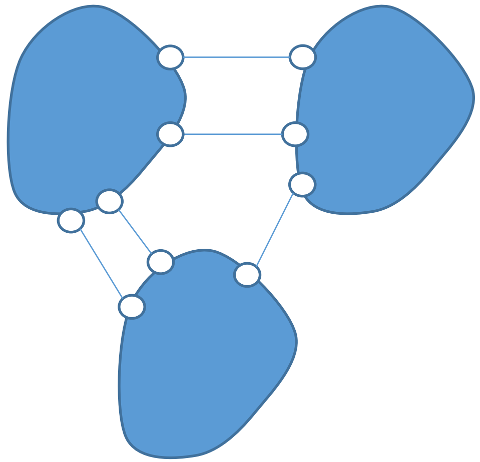
Rigid connection of three members

Three members connection is rigid if each of two members are connected by two bars (or a hinge) and the points of intersection of each pairs are not collinear. (3ST)

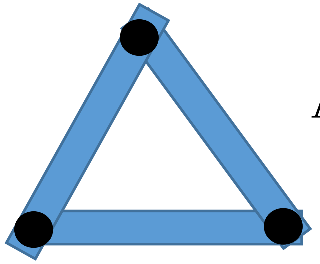
rigid connections

unstable connections

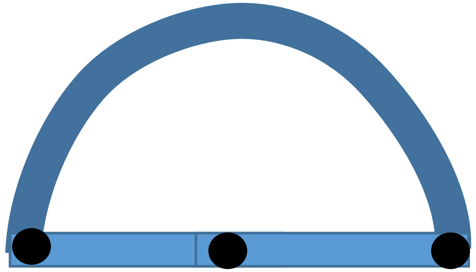




one bar missing



A connection *in triangle* is rigid

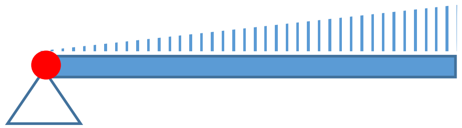


connection points are collinear

Virtual velocities

Virtual velocity – velocity consistent with the constraints

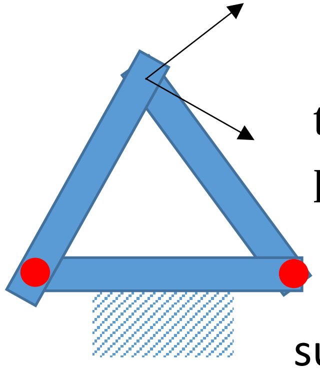
Virtual velocity theorem (VVT)
A system is geometrically stable iff (if and only if) there is no consistent field of virtual velocities.
A system is geometrically unstable iff there exists a consistent field of virtual velocities.



linear distribution of VV

IRC at pin

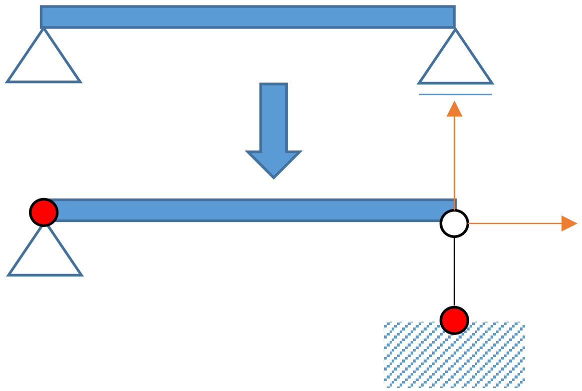
internal stability



two different velocities at one point -> contradiction

support (soil, ground, …)

external stability



two virtual velocities at one point: contradiction

Static equilibrium – balance equations

3D: 6 balance equations:

- 3 projections of forces onto the coordinate set axes: $\sum P_i = 0; i = x, y, z$
- 3 moments about the axes: $\sum M_i = 0\,; i = x, y, z$

2D: 3 balance equations: three forms

1st form: $\sum X = 0;\ \sum Y = 0;\ \sum M_A = 0$

2nd form: $\sum M_A = 0;\ \sum M_B = 0;\ \sum L = 0;$ L not perpendicular to $\overline{AB}$

3rd form: $\sum M_A = 0;\ \sum M_B = 0;\ \sum M_C = 0;$ A, B, C not collinear

2D set of concurrent forces: two equations: $\sum X = 0;\ \sum Y = 0$

2D set of parallel forces: two equations: $\sum M_A = 0;\ \sum X = 0;$ X not perpendicular to the forces direction

Statically determinate structure (isostatic) – structure with the constraints reactions that can be determined from the balance equations only

Statically indeterminate (undetermined) structure (hyperstatic) – structure with the constraints reactions that cannot be determined from the balance equations only

Calculation of constraints reactions

Advice: whenever it is possible, use an uncoupled set of equations

Usually, calculations of the constraints reactions begin all static calculations. With wrong reactions everything else will be wrong also:

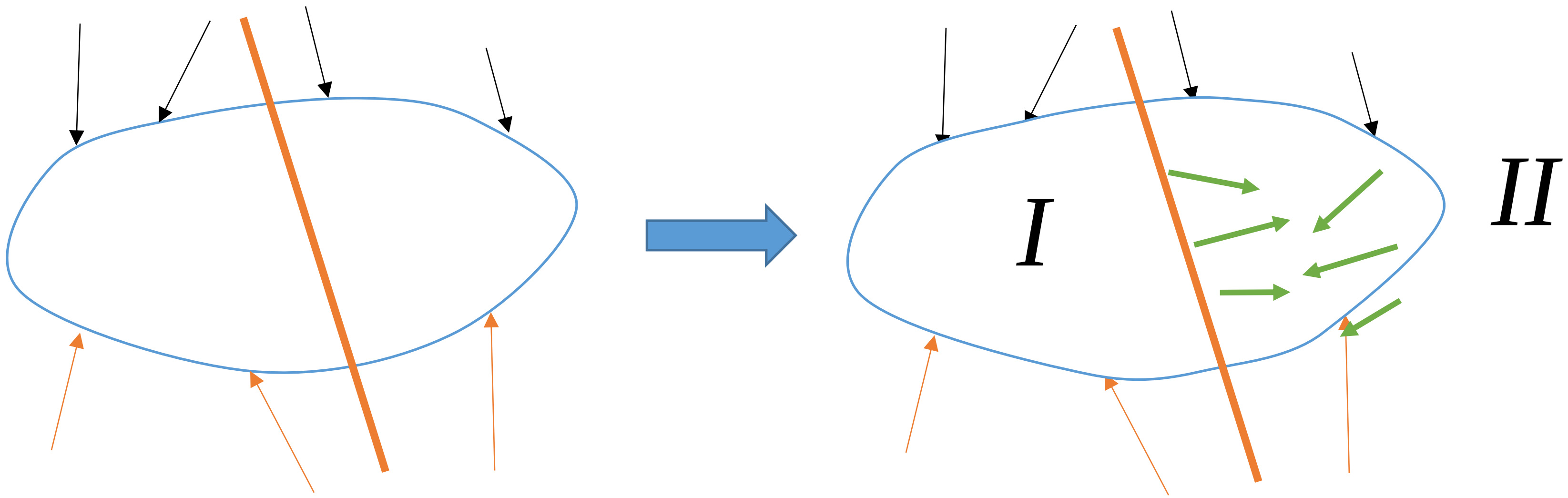
- bending moments
- shear forces
- axial forces
- stresses
- strains
- deflections

due to lack of static equilibrium.

Advice: carefully perform these calculations and always verify the results.

An ordinary calculation error is treated rather tolerantly. Other errors, like wrong procedure or wrong (inadequate) method or reactions forces are treated much more seriously.

Internal forces - definitions



balance of external forces

$$Z_I + Z_{II} = 0$$

balance of internal forces (action & reaction)

$$W_I + W_{II} = 0$$

balance of every sub-part

$$Z_I + W_I = 0; \quad Z_{II} + W_{II} = 0 \;\rightarrow\; Z_I = -W_I; \; Z_{II} = -W_{II} \;\rightarrow\; Z_I = W_{II}; \; Z_{II} = W_I$$

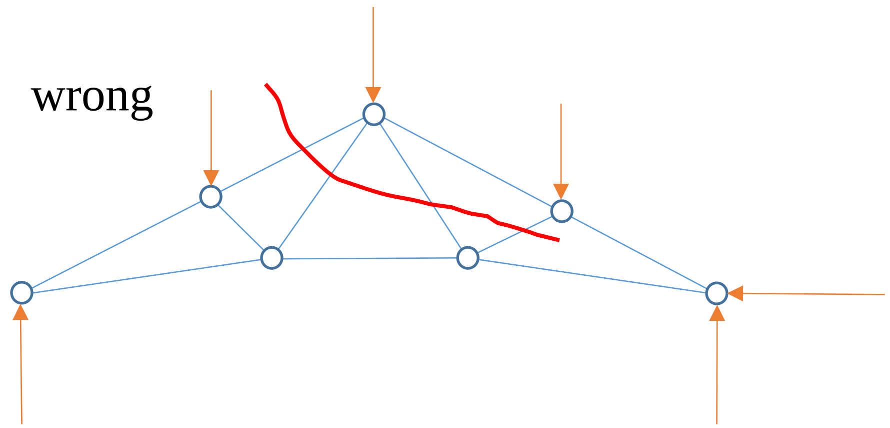
Theorem on equivalence of the external forces applied to one part of body and the internal forces applied to another part of body

$$W_{II} = Z_I \qquad W_I = Z_{II}$$

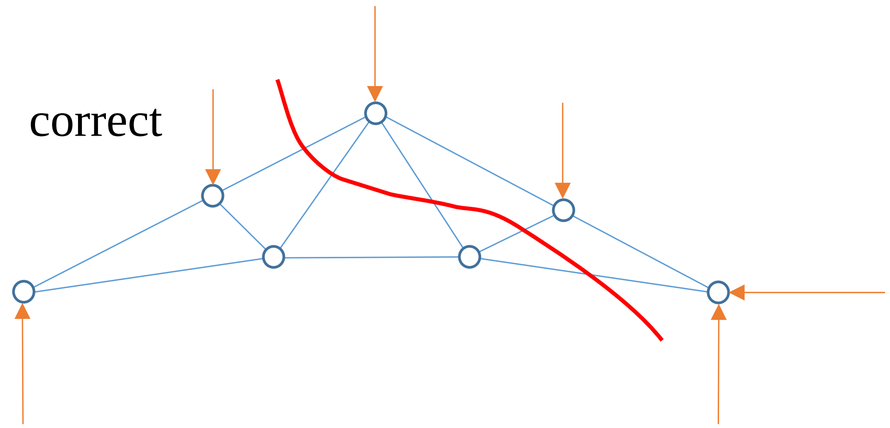
Internal forces - procedure

1. The intersection of a body into two parts
    a. the intersection should be complete, across (not just an incision)
    b. the parts should be well defined
    c. the parts should be separated
    d. it should be clear which forces are applied to each part
2. The determination of the internal forces set, making use of one of two ways possible
    a. balance of separated part of the body
    b. equivalence of the sets of internal and external forces
3. The internal forces set is not known yet; only the resultant moment and force are known

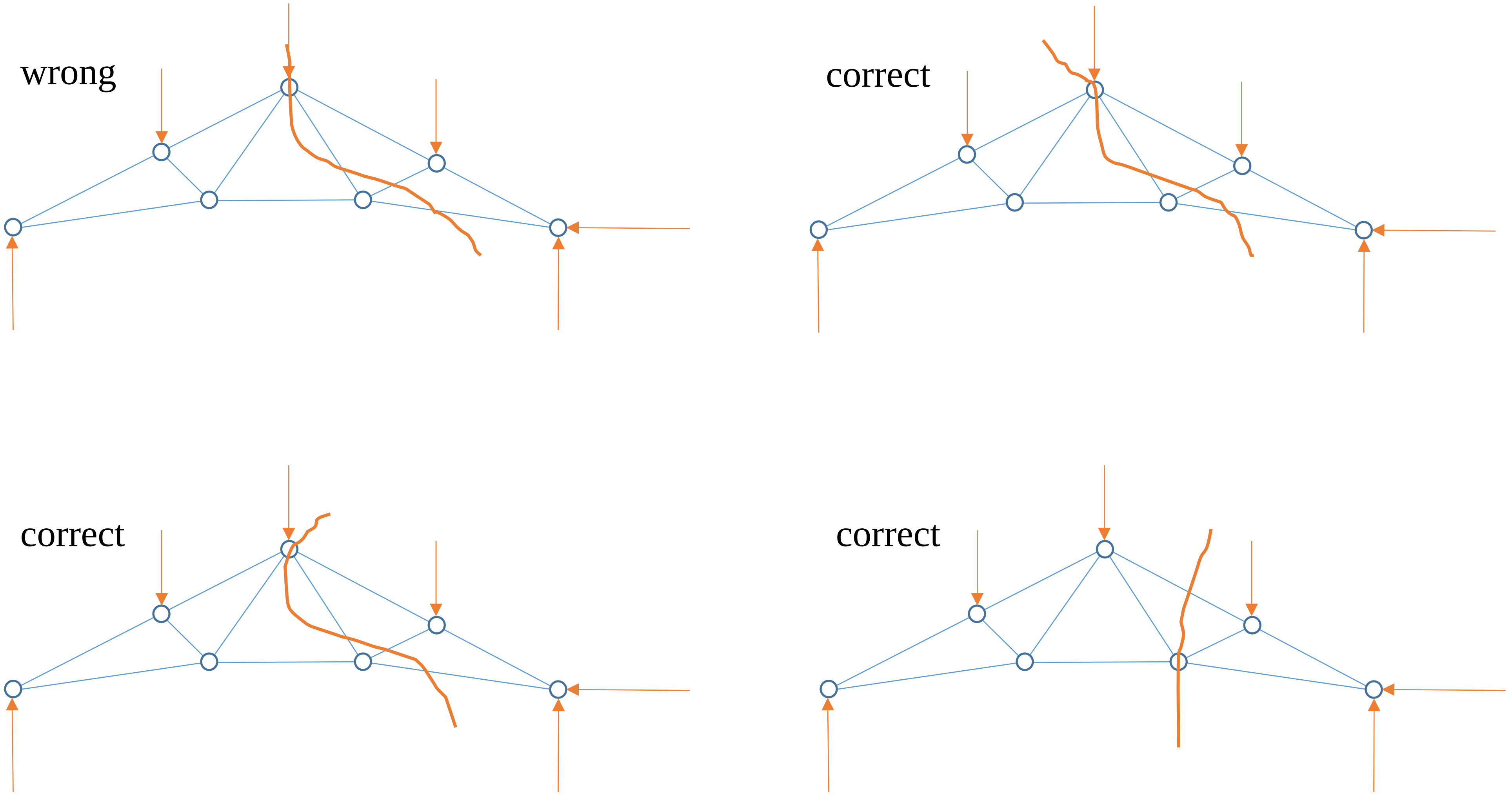
wrong



correct



Internal forces – procedure, cont.



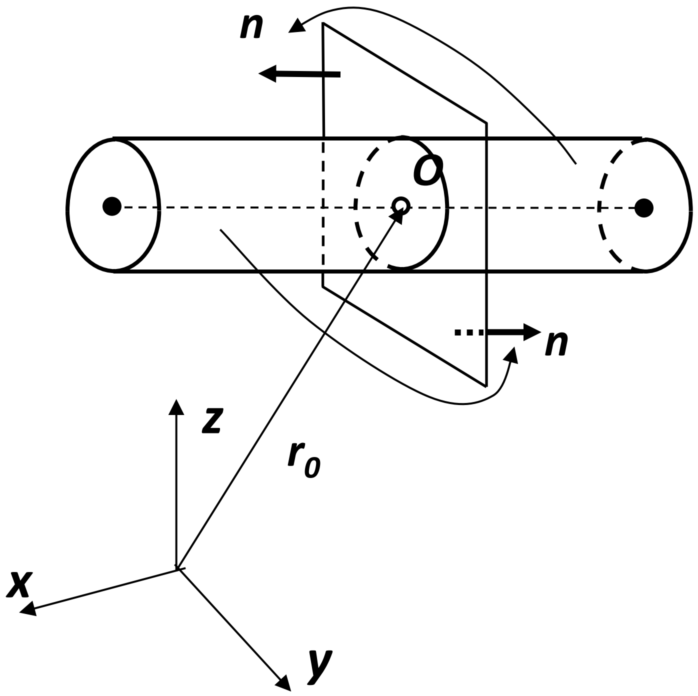
Cross sectional forces concept

proper cross-section coordinate system – a coordinate system with the origin at the cross-section gravity center, the first axis tangent to the bar axis and other axes being principal central inertia axes of the cross-section

bar axis represents the whole body and loading is applied not to the bar surface but the bar axis

Agreements

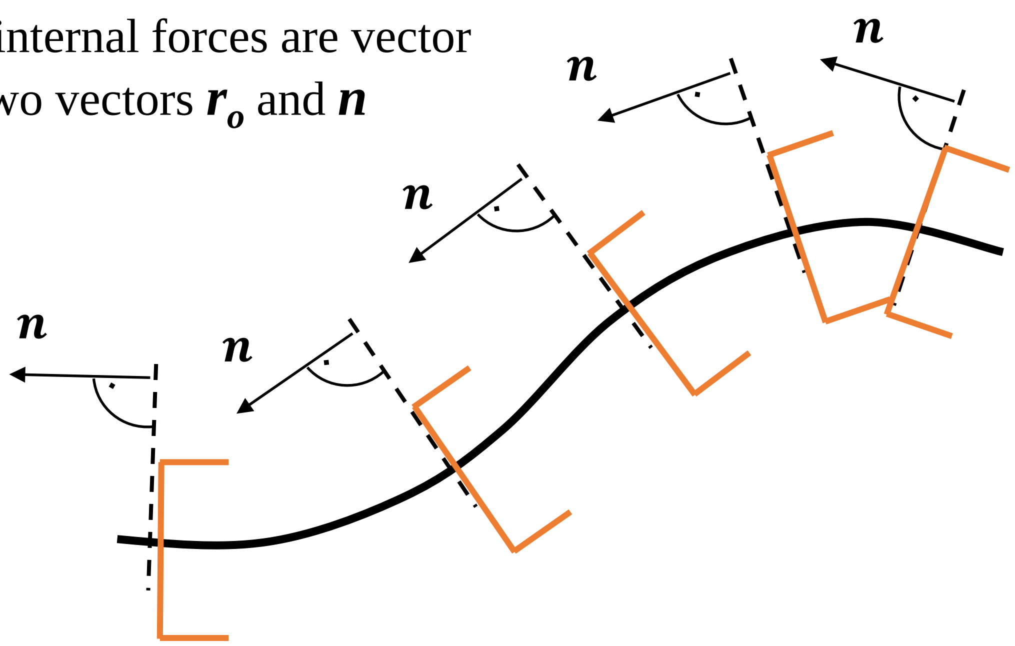
- Reduction center ***O*** is located on the bar axis by vector $\boldsymbol{r_0}$
- Internal forces are determined on the planes **perpendicular** to the bar axis (vector ***n*** is parallel to the axis)
- Vector ***n*** is an **unit outward** normal vector
- The reduction internal forces set will be reported as the cross-sectional forces



Cross sectional forces – cont.

$$S_w = S_w(r_0, n)$$
$$M_w = M_w(r_0, n)$$

Resultants of internal forces are vector functions of two vectors $\boldsymbol{r_o}$ and $\boldsymbol{n}$



Vector $\boldsymbol{n}$ is known from the shape of bar axis

Thus, resultants of internal forces for known bar structure are function of only one vector $\boldsymbol{r_0}$

$$S_w = S_w(r_0)$$
$$M_w = M_w(r_0)$$

Cross sectional forces in 3D

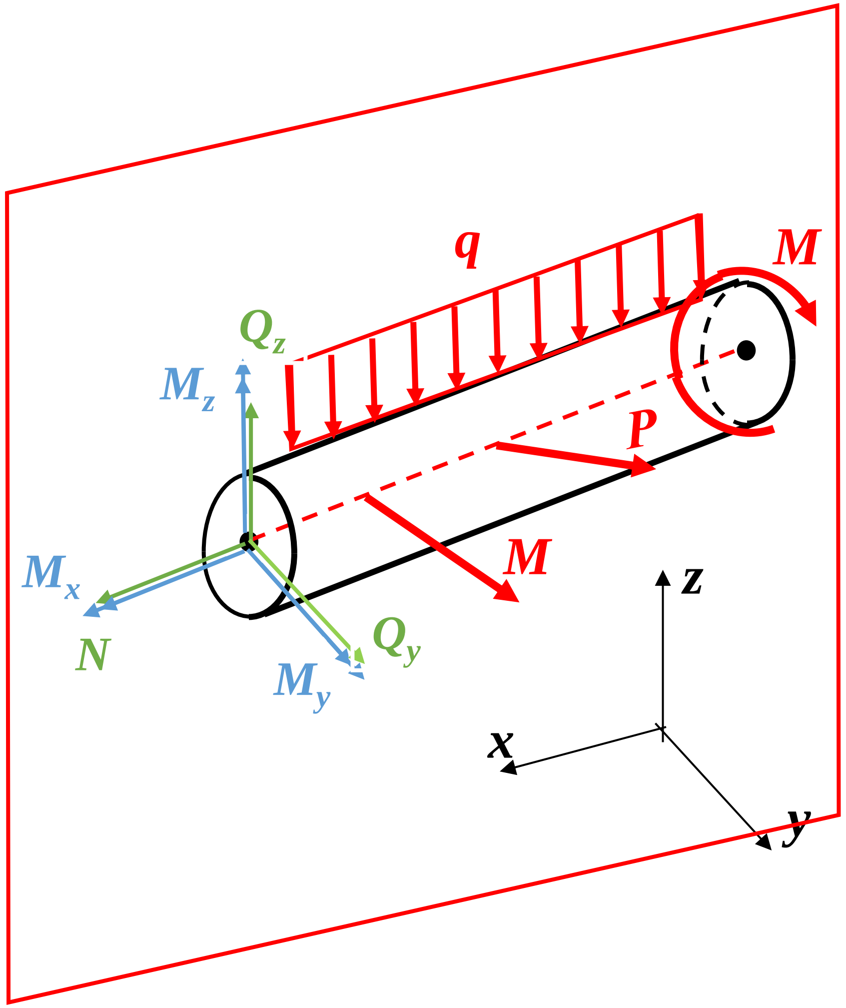
In 3D number of cross-sectional forces is six: three forces and three moments

$$S_w\{N, Q_y, Q_z\}$$

axial force, N
shear force, Q_y
shear force, Q_z

$$M_w\{M_x, M_y, M_z\}$$

torque (torsional moment), M_x
bending moment, M_y
bending moment, M_z

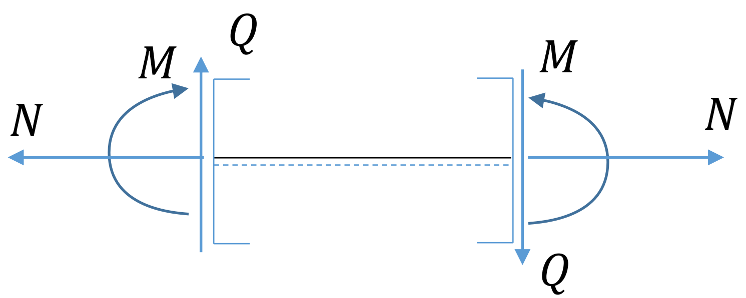


Cross sectional forces in 2D

In 2D space the cross-section forces set reduces to:

- axial force, *N*
- shear force, *Q*
- bending moment, *M*

Signs convention



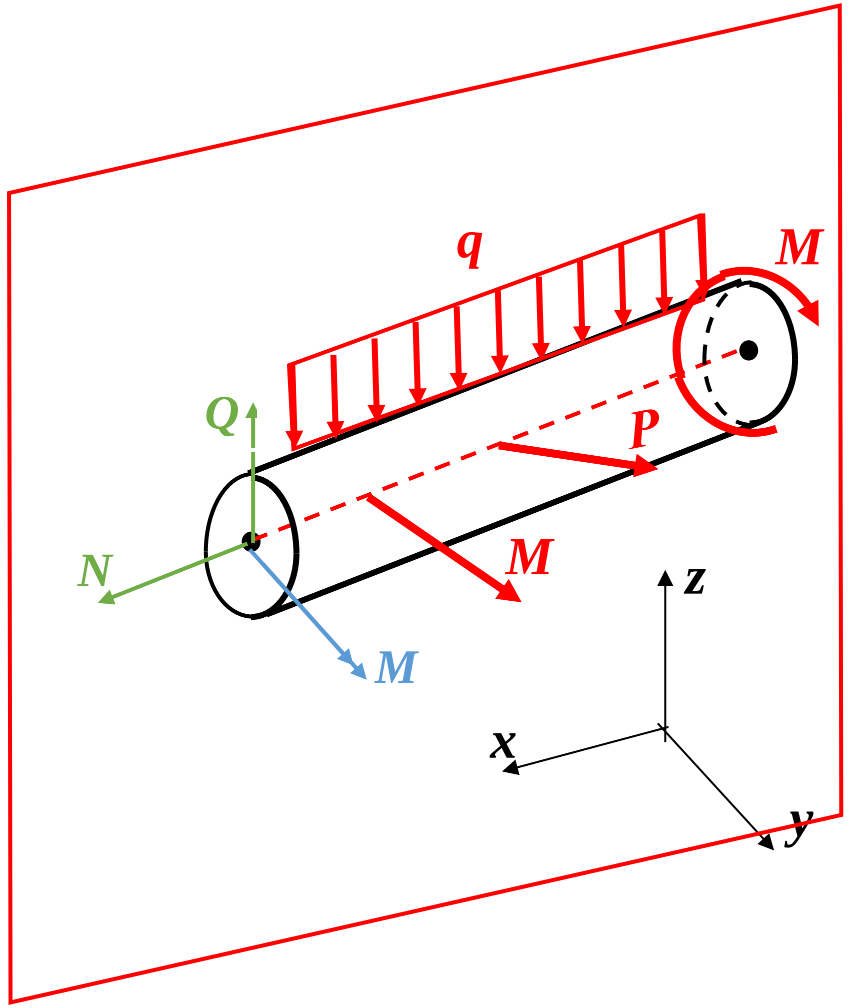
Positive bending moment bends a bar onto distinguished side

- the side is in tension
- the side is referred to as *the undersides*

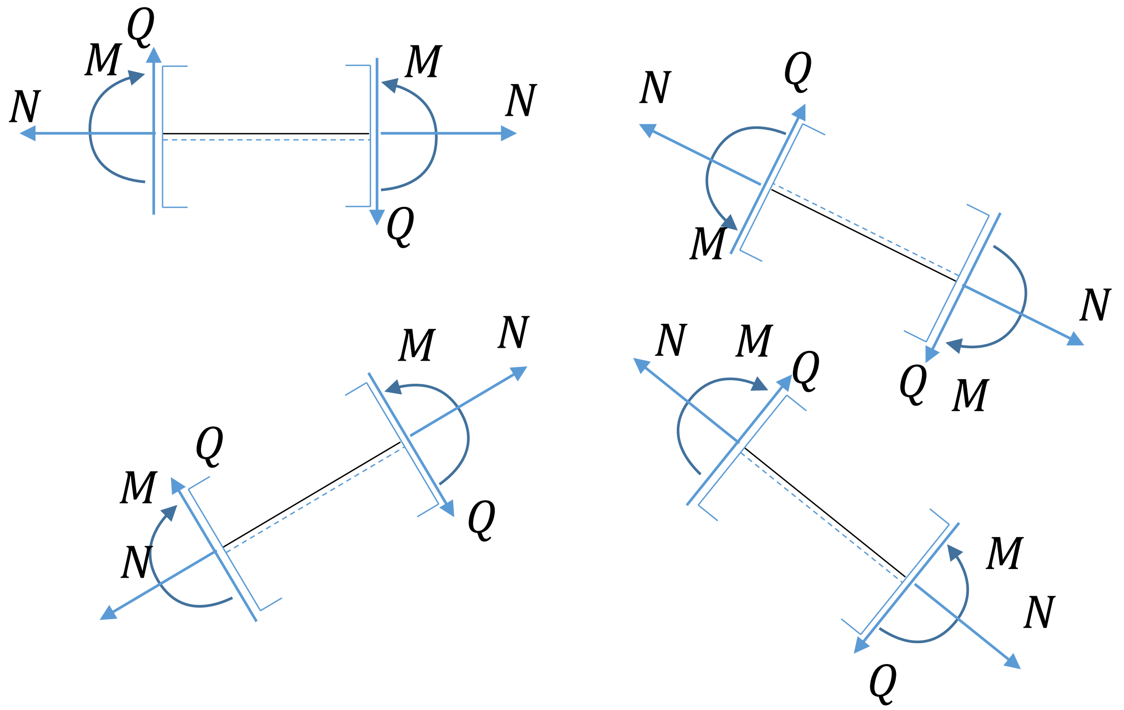
This convention is commonly used by the civil engineers; the mechanical engineers use an opposite convention

Positive axial force is like the outer normal

Positive shear force is up from the left and down from the right



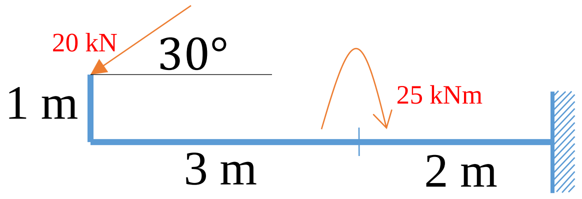
Signs convention – cont.



Bending moment is positive if marked side is tensioned.
Positive direction of axial force is outside.
Shear force positive direction turns a bar clockwise.
The sign of bending moment is meaningless, because the marked side can be chosen arbitrarily. We need the procedure just to make good attention and draw a bending moment diagram on tensioned side.

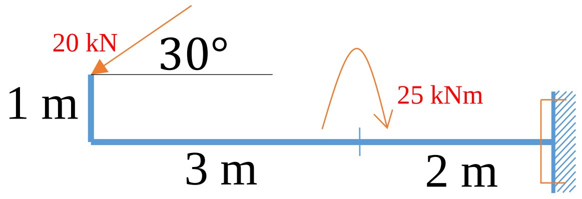
An example

Example of calculation of the cross-sectional forces at a point:
Calculate cross-sectional forces at the fixed end of the structure in Fig. aside.

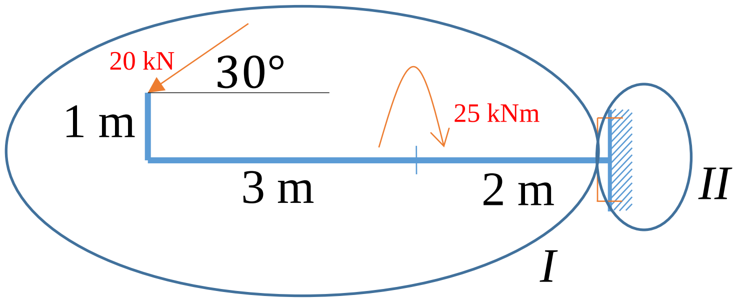


1st way of solution

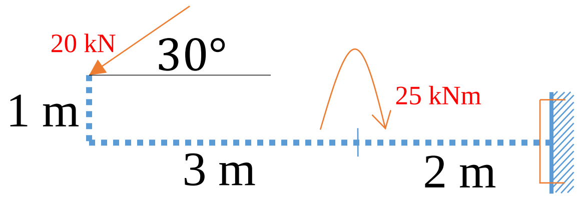
1. a cut



2. two sub-parts: *I* and *II*

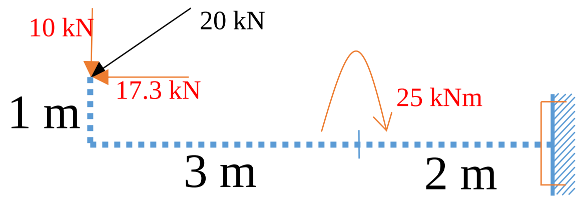


3. $\{Z_I\} = \{W_{II}\}$



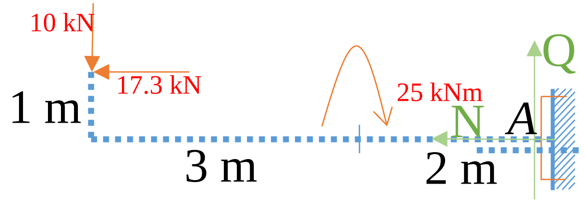
Example – cont.

4. decomposition of force vector



$20\cos 30° = 20 \cdot 0.866 = 17.3\ \text{kN}$
$20\sin 30° = 20 \cdot 0.5 = 10\ \text{kN}$

5. set of reduction



6. moment about the point *A*: $\text{M}_\text{A} = -10 \cdot 5 - 17.3 \cdot 1 + 25 = -42.3\ \text{kNm}$ (marked side is compressed)

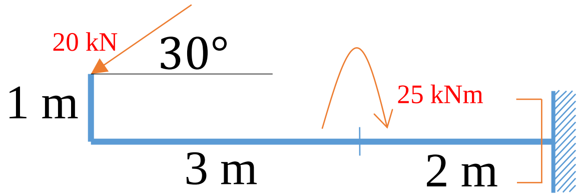
7. shear force: $Q_A = -10\ \text{kN}$

8. axial force: $N_A = 17.3\ \text{kN}$

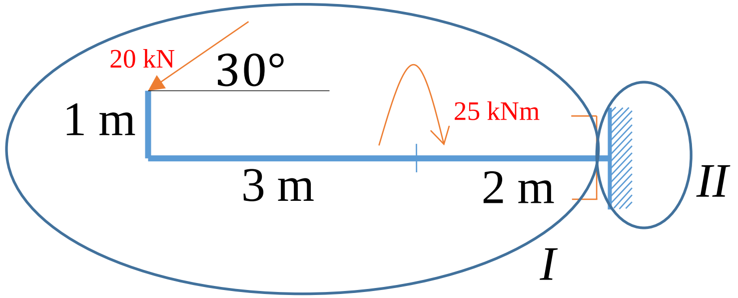
Example – cont.

2nd way of solution

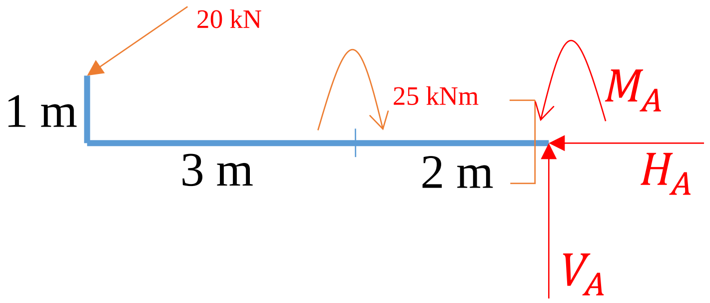
1. a cut



2. two sub-parts: *I* and *II*

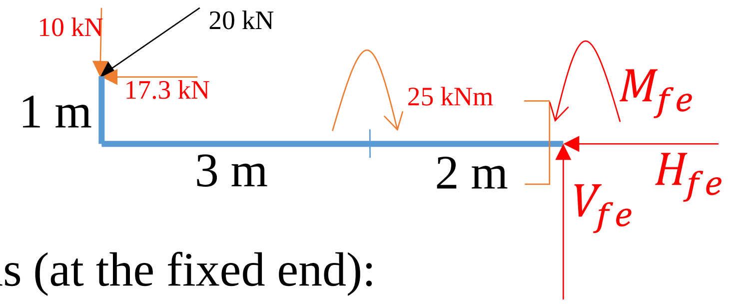


3. $\{W_I\} = \{Z_{II}\}$



Example – cont.

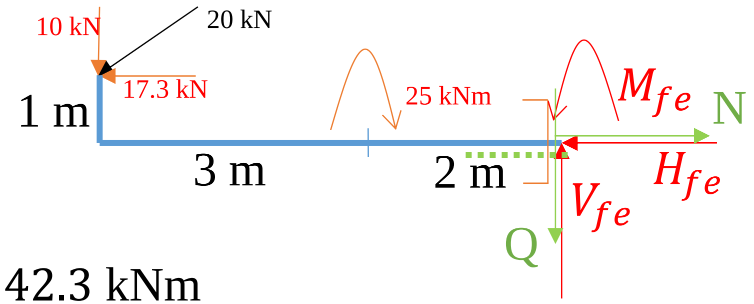
4. decomposition of force vector



5. calculation of constraints reactions (at the fixed end):

$$\sum M_A = 0 \rightarrow -10 \cdot 5 - 17.3 \cdot 1 + 25 - M_{fe} = 0 \rightarrow M_{fe} = -42.3 \text{ kNm}$$
$$\sum X = 0 \rightarrow -17.3 - H_{fe} = 0 \rightarrow H_{fe} = -17.3 \text{ kN}$$
$$\sum Y = 0 \rightarrow -10 + V_{fe} = 0 \rightarrow V_{fe} = 10 \text{ kN}$$

6. set of reduction



7. bending moment: $M_A = M_{fe} = -42.3$ kNm
8. shear force: $Q_A = -V_{fe} = -10$ kN
9. axial force: $N_A = -H_{fe} = 17.3$ kN

Thank you for your attention!