Strength of Materials

4. Simple beams

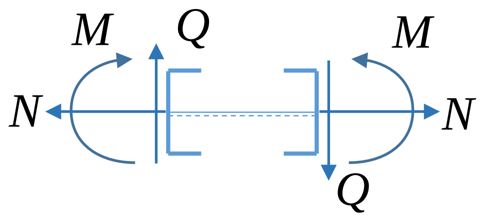
Definitions and signs convention

The cross-sectional forces – a reminder of the procedure

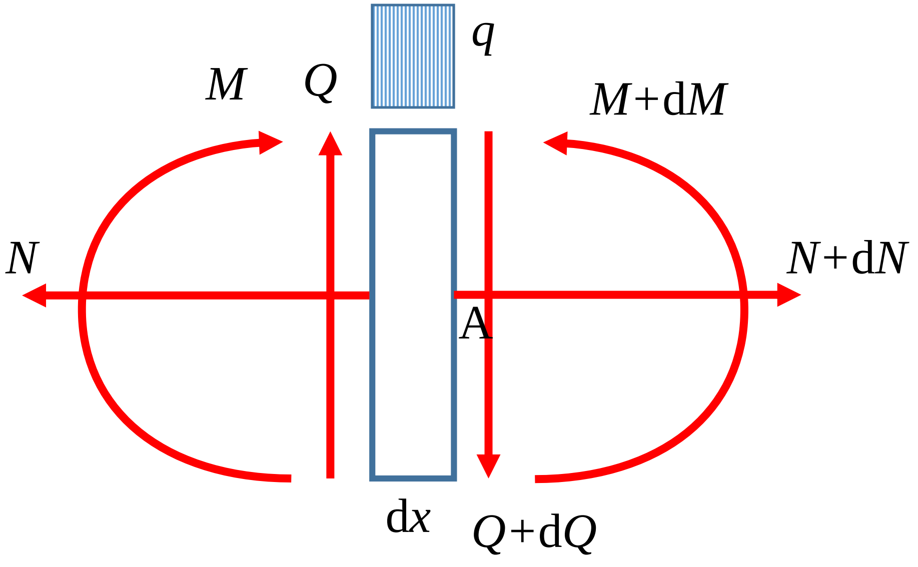
1. Verification of the structure stability
2. Determination of characteristic points and characteristic intervals
3. For each interval:
    1. A section divides the structure into two disjoint subsets
    2. The proper cross-section coordinate set is introduced
    3. The external forces, acting onto the rejected subset, are reduced at the proper cross-section coordinate set ($\{Z_I\} = \{W_{II}\}$; $\{Z_I\}$ - the external forces set of the subset *I*, $\{W_{II}\}$ - the internal forces set of the subset *II*
    4. Determination of functions of the cross-sectional forces
    5. Calculation of characteristic values of the cross-sectional forces
    6. Drawing of diagrams of the cross-sectional forces

Signs convention:

- the bending moment is drawn from tensioned side, its sign is irrelevant
- the transversal (shear) force is positive if it turns the section clockwise
- the longitudinal (axial) force is positive if acts in the sense of the outer normal



Schwedler-Zhuravski theorem



$$\Sigma X = 0 \rightarrow N - N - dN = 0 \rightarrow dN = 0 \rightarrow N = \text{const}$$

$$\Sigma Y = 0 \rightarrow Q - Q - dQ - qdx = 0 \rightarrow \frac{dQ}{dx} = -q(x)$$

$$\Sigma M_A = 0 \rightarrow M - M - dM + Qdx - qdx\frac{dx}{2} = 0 \rightarrow$$

$$\frac{dM}{dx} = Q - \frac{qdxdx}{2}$$

$\frac{qdxdx}{2} \rightarrow$ infinitesimal of higher order, $\cong 0 \rightarrow$

$$\frac{dM}{dx} = Q$$

Finally, the relationships for the cross-sectional forces are:

$$\boxed{\frac{dM}{dx} = Q(x)}$$

$$\boxed{\frac{dQ}{dx} = -q(x)}$$

$$\boxed{\frac{d^2M}{dx^2} = -q(x)}$$

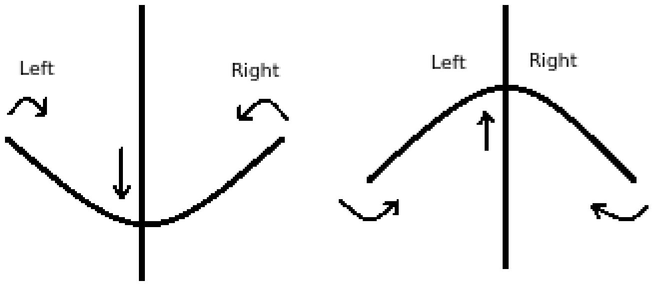
The first derivative of the bending moment with respect of variable *x* is equal to the transversal (shear) force (accurately to within the absolute value, because the sign of the bending moment can be chosen arbitrarily).
When the transversal force vanishes, the bending moment reaches its extreme value.
The bending moment function is by two order higher than the function of the continuous load intensity.

SZT – consequences

To draw diagrams of the cross-sectional forces we have to calculate their functions at all characteristic points:

- at the beginning of each characteristic interval
- at the end of each characteristic interval
- at all points of extreme values within each characteristic interval

The bending moment can be sagging (bottom side is tensioned) or hogging (upper side is in tension).



Sagging Bending Moment

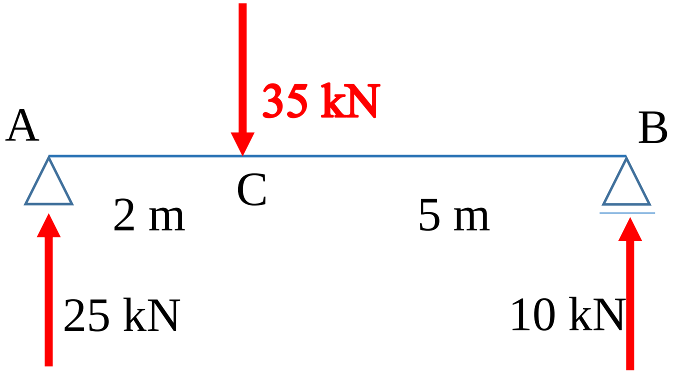
Hogging Bending Moment

The characteristic points and the characteristic intervals

The characteristic points are: the beginning and the end of the beam; the application point of the point force or the point moment; the point of beginning or end of continuous load

Attention! Technically, a hinge is not a characteristic point, however it is a special point where the bending moment vanishes.

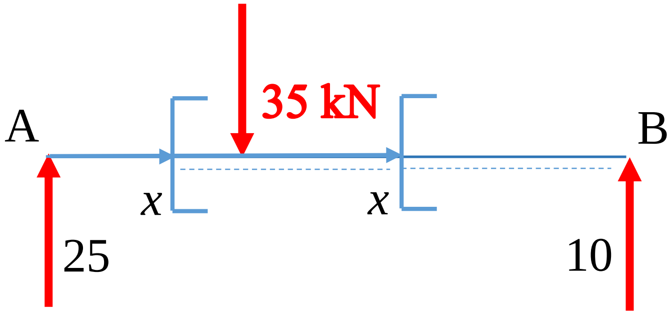
Simply supported beam with point force

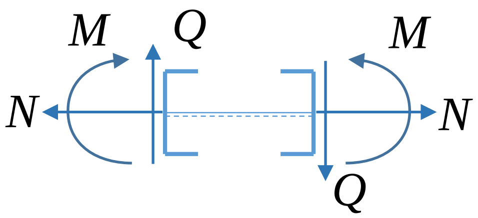


The constraints reactions are inversely proportional to the distance of the force from the supports.

$R_A = 35 \cdot 5 : 7 = 25$ kN
$R_B = 35 \cdot 2 : 7 = 10$ kN

The characteristic points are: A, B and C. The characteristic intervals are: AC and CB.





The first interval

$x \leq 2$ m

$M(x) = 25x,\ M(0) = 0,\ M(2) = 50$ kNm

$Q(x) = 25 = \text{const}$

$N(x) \equiv 0$

The second interval

$2 \leq x \leq 7$ m

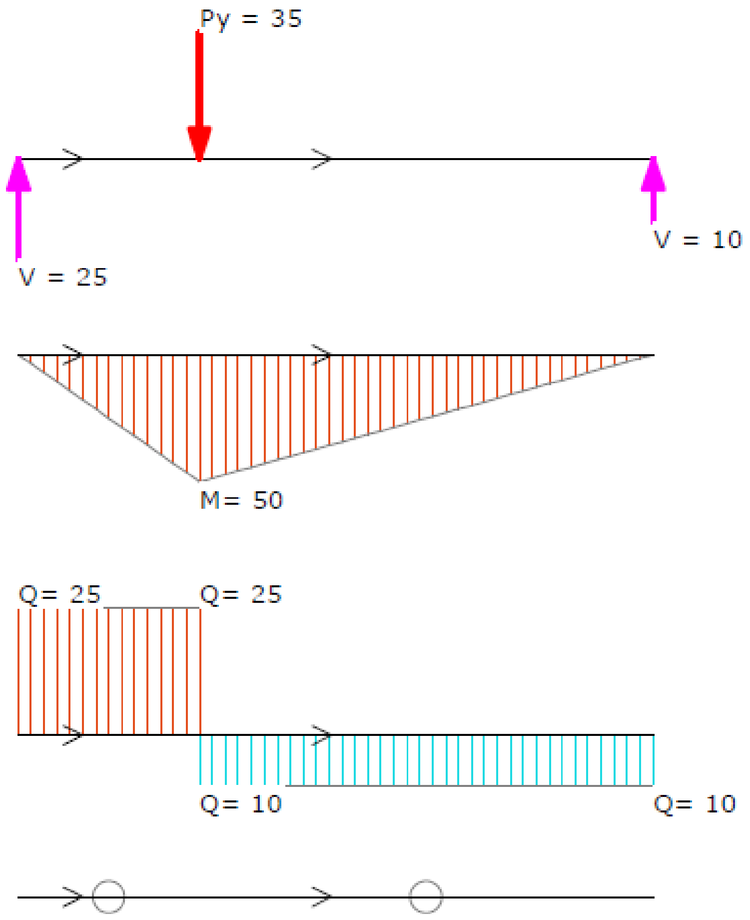
$M(x) = 25x - 35(x-2),\ M(2) = 50$ kNm, $M(7) = 25 \cdot 7 - 35 \cdot 5 = 0$

$Q(x) = 25 - 35 = -10$ kN $= \text{const}$

$N(x) \equiv 0$

Simply supported beam with point force – cont.

The diagrams of the cross-sectional forces



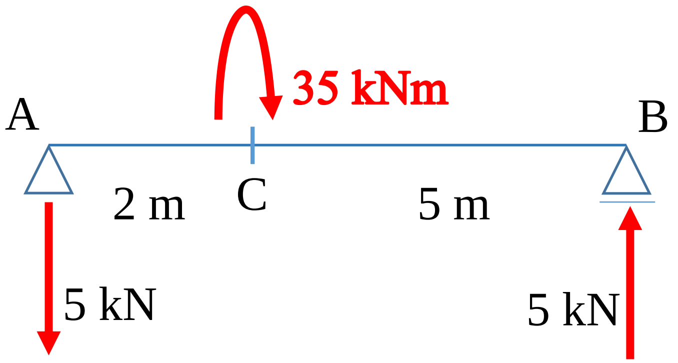
The bending moment diagram from tensioned side

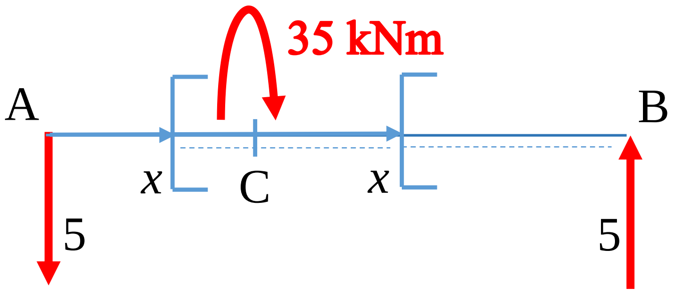
The point force down – the kink down

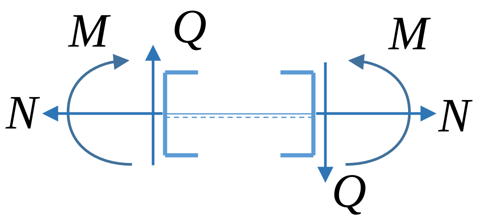
The bending moments at the beam ends are always zero (unless the point moment is applied there)

The jump of the transversal force is equal to the point force value

Simply supported beam with point moment







The constraints reactions have opposite senses and don't depend on the bending moment position.

$R_A = 35\!:7 = 5$ kN

$R_B = 35\!:7 = 5$ kN

There are three characteristic points: $A, B,$ and C and two characteristic intervals: AC and CB

The first interval

$0 \leq x \leq 2$ m

$M(x) = -5x,\ M(0) = 0, \mathrm{M}(2) = -10$ kNm

$Q(x) = -5$ kN = const

$N(x) \equiv 0$

The second interval

$2 \leq x \leq 7$

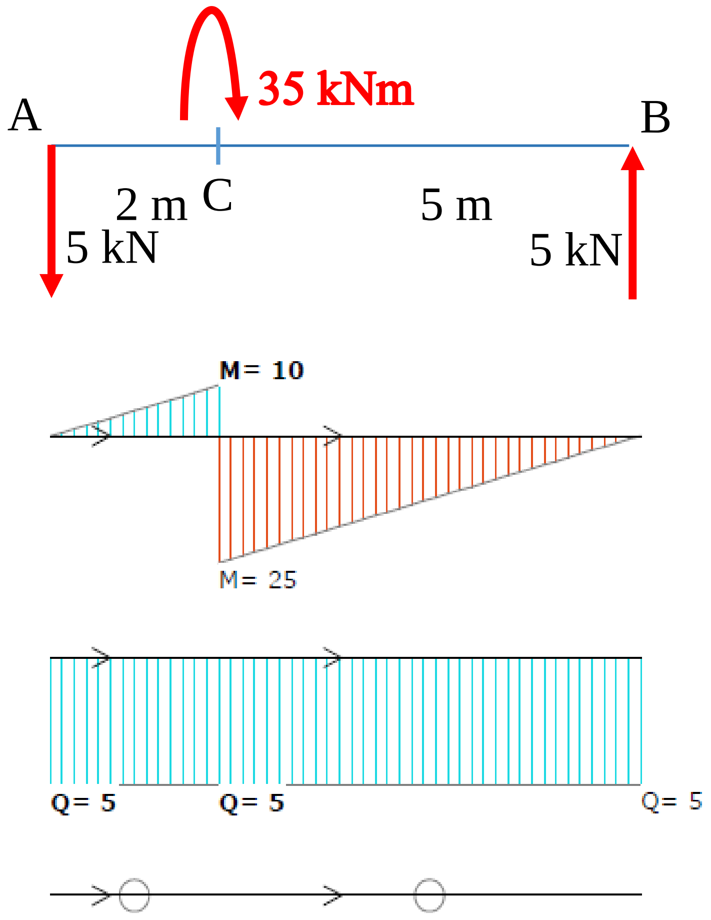
$M(x) = -5x + 35, M(2) = 35$ kNm, $M(7) = 0$

$Q(x) = -5$ kN = const

$N(x) \equiv 0$

Simply supported beam with point moment – cont.

The diagrams of the cross-sectional forces

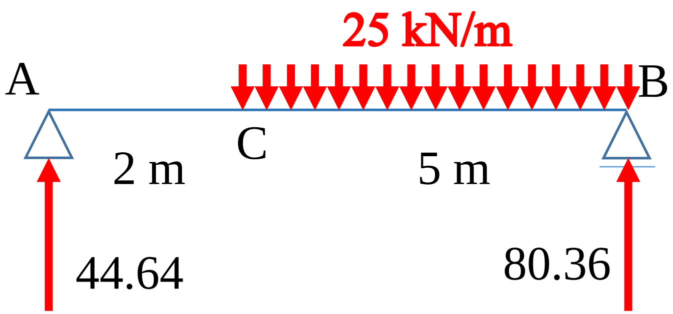


The bending moment diagram from tensioned side

The jump on bending moment diagram pops up at the point where the point moment is applied, and the jump is equal to the value of the applied moment
The lines on the bending moment diagram are parallel – the second one is translated down by the value of the applied moment

The bending moments at the beam ends are always zero (unless the point moment is applied there)

Simply supported beam with continuous load

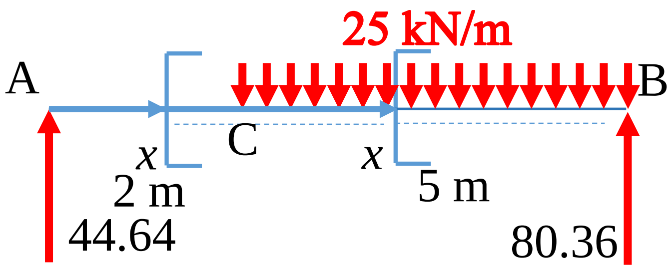


The constraints reactions:

$\Sigma M_B = 0 \rightarrow R_A = 25 \cdot 5 \cdot 2.5 : 7 = 44.64$ kN

$\Sigma M_A = 0 \rightarrow R_B = 25 \cdot 5 \cdot 4.5 : 7 = 80.36$ kN

There are 3 characteristic points: $A, B,$ and C, and two characteristic intervals: AC and CB



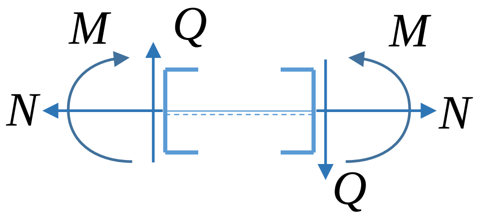
The first interval

$0 \leq x \leq 2$m

$M(x) = 44.64x; M(0) = 0; M(2) = 89.28$ kNm

$Q(x) = 44.64$ kN = const

$N(x) \equiv 0$



The second interval

$2 \leq x \leq 7$ m

$M(x) = 44.64x - 25 \cdot (x-2) \cdot \frac{x-2}{2}; M(2) = 89.28$ kNm; $M(7) = -0.02 \cong 0$

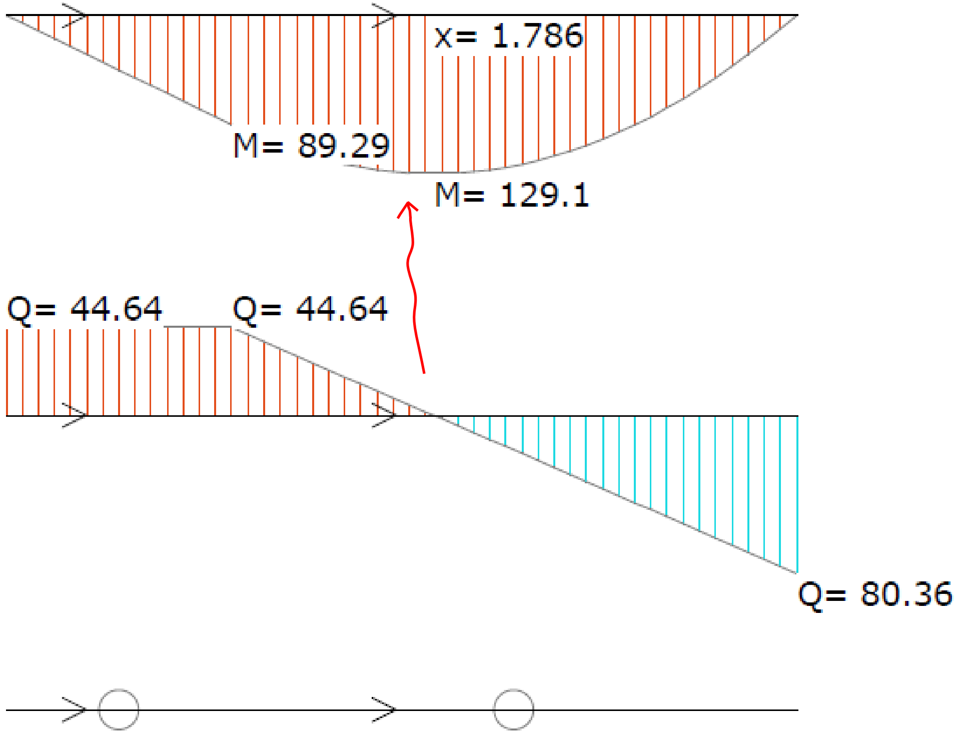
$Q(x) = 44.64 - 25 \cdot (x-2); Q(2) = 44.64$ kN; $Q(7) = -80.36$ kN (the sign changes within the interval!)

$Q(x) = 0 \rightarrow x_0 = 3.786$ m $\rightarrow M(3.786) = 44.64 \cdot 3.786 - 25 \cdot \frac{1.786^2}{2} = 129.1$ kNm (extremum)

$N(x) \equiv 0$

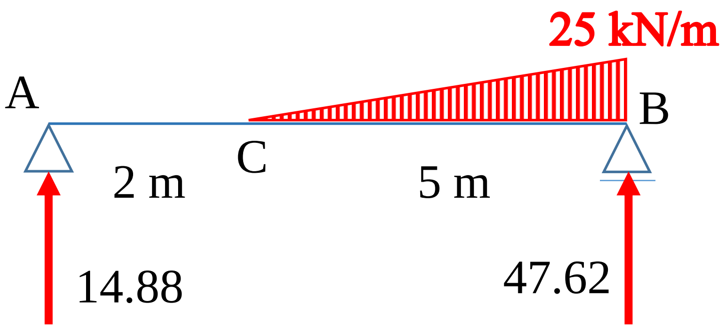
Simply supported beam with line load – cont.

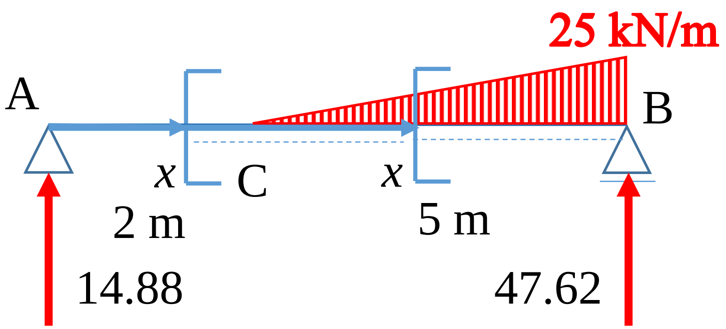
The cross-sectional forces diagrams:



In the intervals without continuous load the bending moments diagram is linear and the shear force diagram is constant.
In the intervals with constant continuous load the bending moment diagram is the second order parabola with convexity similar to the load sense (the load and convexity downwards – sagging moment).
The passage from the line to the parabola is smooth (no cusp).
The zero shear force means a maximum of the bending moment (it should be calculated).
The bending moment at the beam ends is equal to zero (unless there is a point moment applied).
There is no jumps of the bending moment if there is no point moment imposed.
The jumps of the shear force are equal to the point forces applied.

Simply supported beam with triangular load



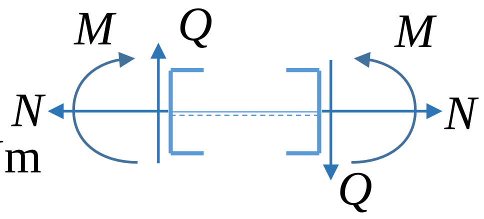


The constraints reactions:

$$\Sigma M_B = 0 \rightarrow R_A = 0.5 \cdot 5 \cdot 25 \cdot \frac{5}{3} / 8 = 14.88 \text{ kN}$$

$$\Sigma M_A = 0 \rightarrow R_B = 0.5 \cdot 5 \cdot 25 \cdot \left(\frac{2}{3} \cdot 5 + 2\right) \cdot \frac{1}{7} = 47.62 \text{ kN}$$

There are three characteristic points: A, B, and C, and two characteristic intervals: AC and CB.



The first interval: $0 \le x \le 2$ m

$M(x) = 14.88x; M(0) = 0; M(2) = 29.76$ kNm

$Q(x) = 14.88$ kN = const; $N(x) \equiv 0$

The second interval: $2 \le x \le 7$ m

$$M(x) = 14.88x - \frac{1}{2}(x-2)\frac{25}{5}(x-2)\frac{x-2}{3} = 14.88x - \frac{25}{5} \cdot \frac{(x-2)^3}{6}$$

$M(2) = 29.76$ kNm; $M(7) = 0$

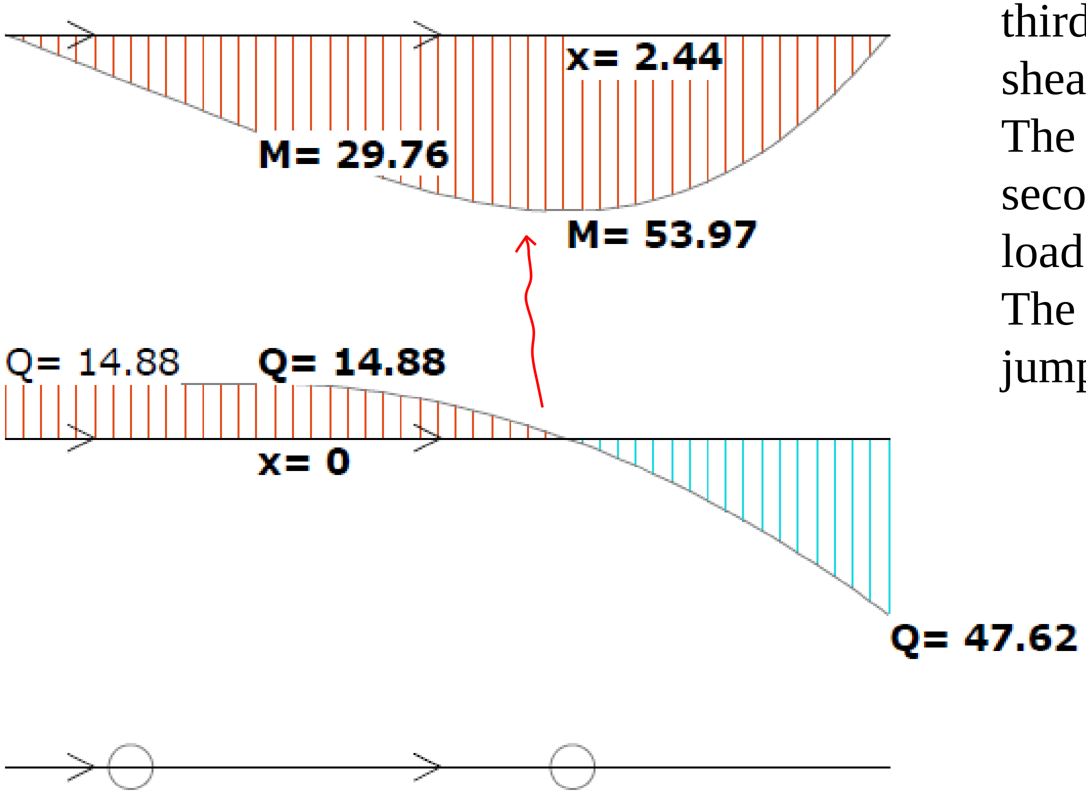
$$Q(x) = 14.88 - \frac{1}{2}(x-2)\frac{25}{5}(x-2) = 14.88 - \frac{25}{5} \cdot \frac{(x-2)^2}{2}; Q(2) = 14.88 \text{ kN}; Q(7) = -47.62 \text{ kN (changes sign)}$$

$$Q(x) = 0 \rightarrow x_0 = \sqrt{5.952} + 2 = 4.440 \text{ m}, \rightarrow M(4.440) = 14.88 \cdot 4.44 - \frac{5}{6} \cdot 2.44^3 = 53.61 \text{ kNm (extremum)}$$

$N(x) \equiv 0$

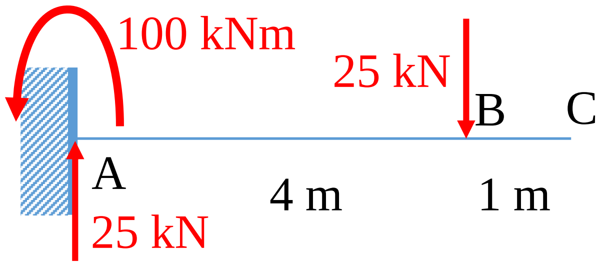
Simply supported beam with triangular load

The cross-sectional forces diagrams:



The bending moments: linear part, smooth passage into the third order parabola with extremum at the point where the shear force vanishes.
The shear force: constant part, smooth passage into the second order parabola with gradient proportional to the line load intensity.
The bending moments are zero at the beam ends and the jumps of the shear force are equal to point forces applied.

Cantilever with point force

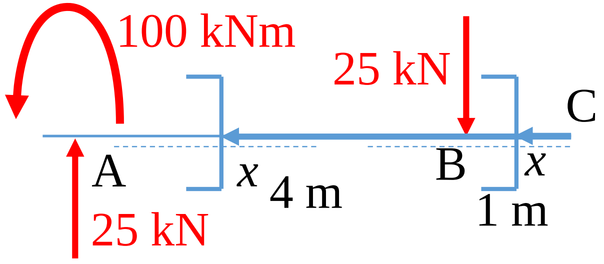


The constraints reactions:

$\Sigma Y - 0 \rightarrow R_A - 25 = 0 \rightarrow R_A = 25$ kN

$\Sigma M_A = 0 \rightarrow M_A - 25 \cdot 4 = 0 \rightarrow M_A = 100$ kNm

There are three characteristic points: $A, B,$ and C, and two characteristic intervals: AB and BC.



The first interval: $0 \leq x \leq 1$ m

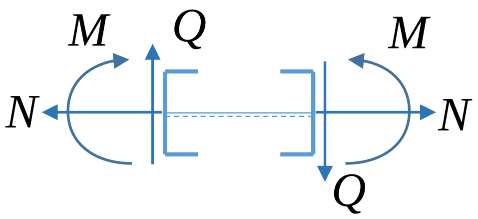
$M(x) \equiv 0; Q(x) \equiv 0; N(x) \equiv 0$

The second interval: $1 \leq x \leq 5$ m

$M(x) = -25(x-1); M(1) = 0; M(5) = -100$ kNm

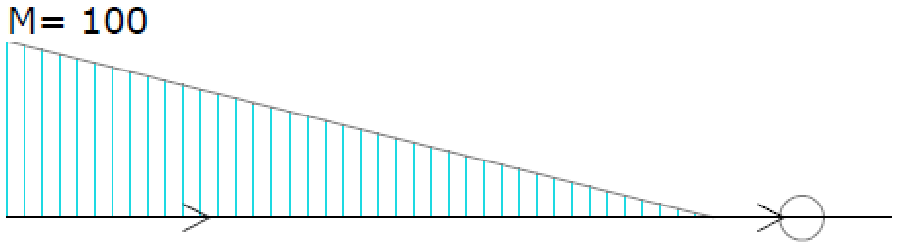
$Q(x) = 25$ kN = const

$N(x) \equiv 0$

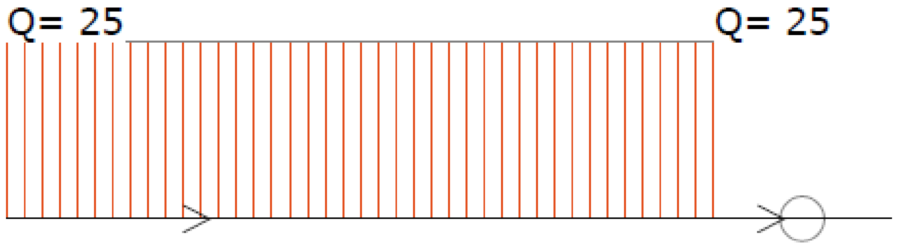


Cantilever with point force – cont.

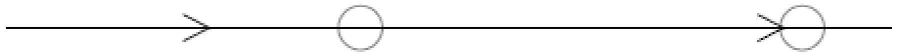
The diagrams of the cross-sectional forces



The bending moment diagram from the tensioned side.
The jump is equal to the point moment applied.
There is no line load, so the diagram is linear.

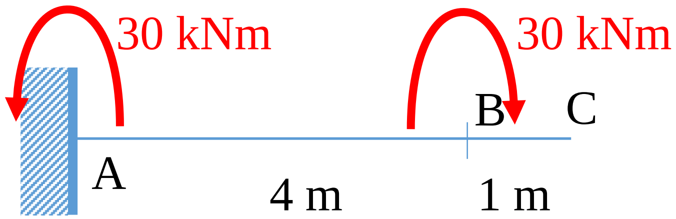


The shear force is positive and constant. The jumps are equal to the applied point forces.



The longitudinal (axial) force is identically equal zero. No need to draw the diagram; an inscription $N \equiv 0$ instead will do.

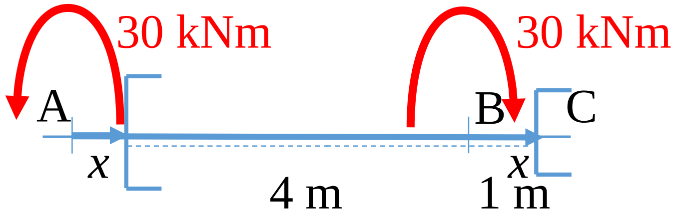
Cantilever with point moment

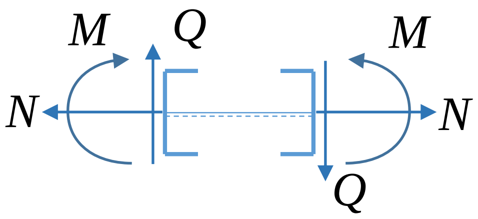


The constraints reactions

$\Sigma M_A = 0 \rightarrow M_A = 30$ kNm; $V_A = H_A = 0$

There are three characteristic points: A, B, and C, and two characteristic intervals: AB and BC.





The first interval:

$0 \leq x \leq 4$ m

$M(x) = -30$ kNm = const

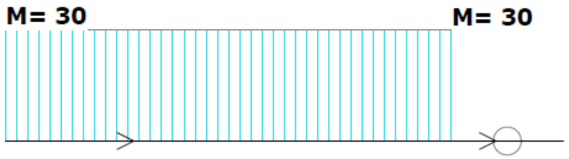
$Q(x) = N(x) \equiv 0$

The second interval:

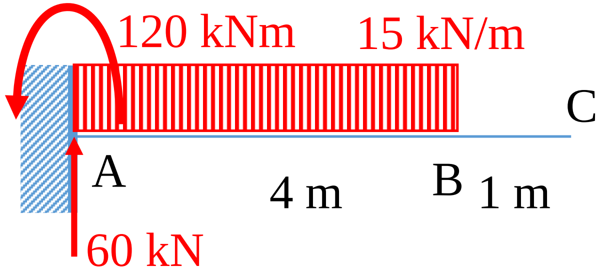
$4 \leq x \leq 5$ m

$M(x) = Q(x) = N(x) \equiv 0$

The cross-sectional forces diagrams – bending moment

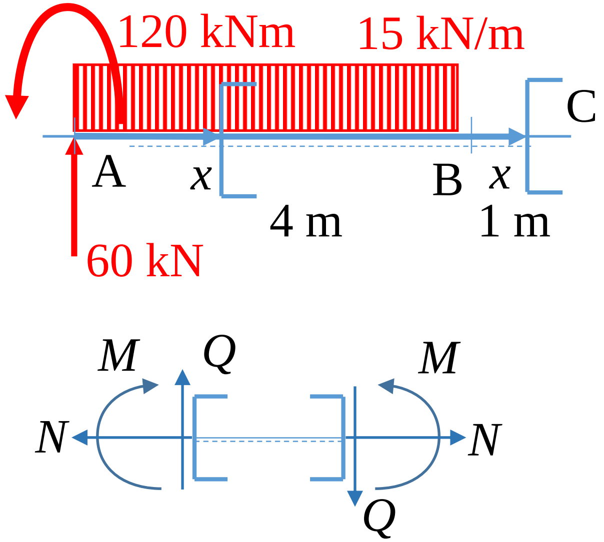


Cantilever with constant continuous load



The constraints reactions:
$\Sigma M_A = 0 \rightarrow 15 \cdot 4 \cdot 2 - M_A = 0 \rightarrow M_A = 120 \text{ kNm}$
$\Sigma Y = 0 \rightarrow V_A - 15 \cdot 4 = 0 \rightarrow V_A = 60 \text{ kN}$
$\Sigma X = 0 \rightarrow H_A = 0$

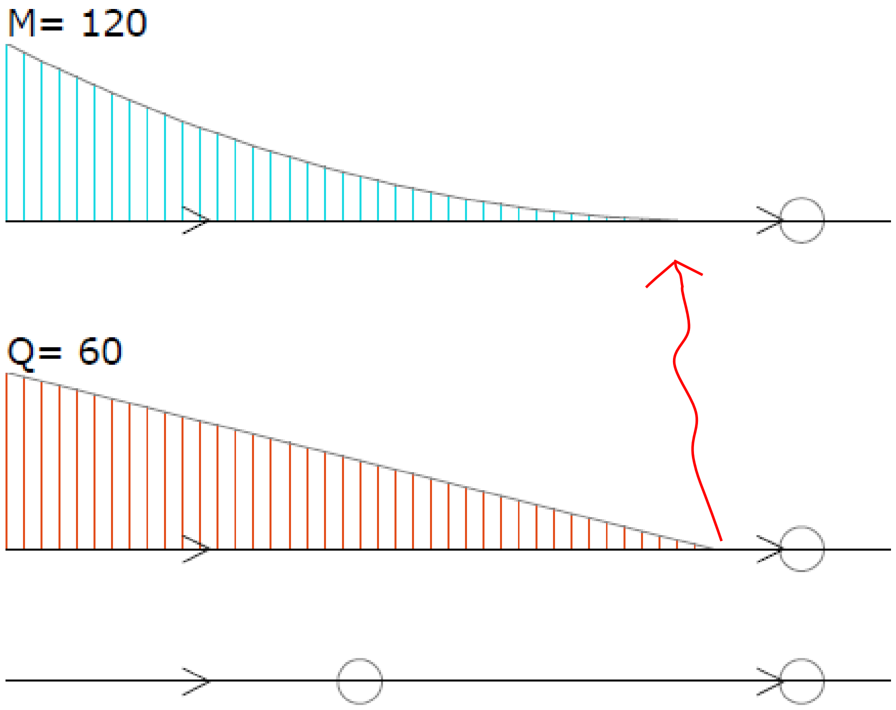
There are three characteristic points: A, B, and C, and two characteristic intervals: AB and BC



The first interval: $0 \leq x \leq 4$ m
$M(x) = -120 + 60x - qx \cdot \frac{x}{2}; M(0) = -120 \text{ kNm}$
$$M(4) = -120 + 60 \cdot 4 - 15 \cdot 8 = 0$$
$Q(x) = 60 - 15x; Q(0) = 60; Q(4) = 0$
$N(x) \equiv 0$
The second interval: $4 \leq x \leq 5$ m
$M(x) = Q(x) = N(x) \equiv 0$

Cantilever with constant continuous load – cont.

The cross-sectional forces diagrams

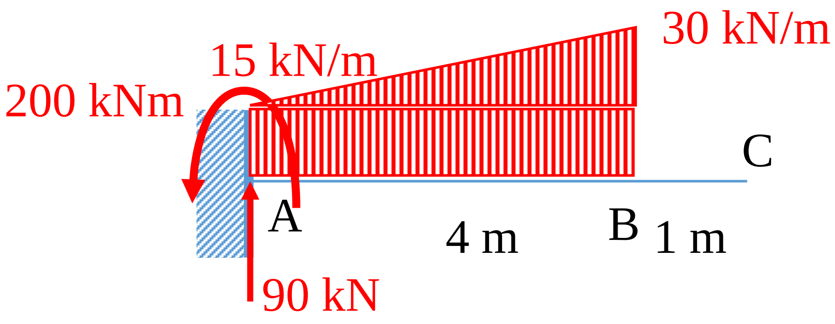


The bending moment bends in the sense of acting line load:

- load acts down -> the bending moment bends down
- load acts up -> the bending moment bends up

The shear force is linear with a jump at the point where the point force is applied.
Zero value of the shear force means an extremum of the bending moment.

Cantilever with trapezoidal load



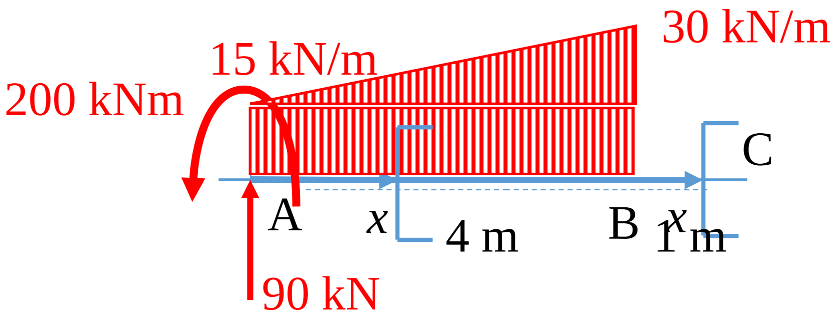
The constraints reactions:

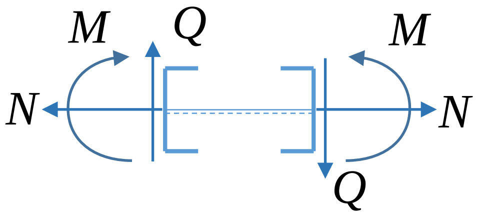
$$\Sigma M_A = 0 \rightarrow 15 \cdot 4 \cdot 2 + 15 \cdot \frac{4}{2} \cdot \frac{2}{3} \cdot 4 - \mathrm{M_A} = 0 \rightarrow \mathrm{M_A} = 200 \text{ kNm}$$

$$\Sigma Y = 0 \rightarrow V_A - 15 \cdot 4 - \frac{1}{2} \cdot 15 \cdot 4 = 0 \rightarrow V_A = 90 \text{ kN}$$

$$\Sigma X = 0 \rightarrow H_A = 0$$

There are three characteristic points: $A, B,$ and C, and two characteristic intervals: AB and BC.





The first interval: $0 \leq x \leq 4$ m

$$M(x) = 90x - 200 - \frac{15x^2}{2} - \frac{15}{4} \cdot \frac{x^3}{6}; M(0) = -200 \text{ kNm}$$

$$M(4) = 360 - 200 - 120 - 40 = 0$$

$$Q(x) = 90 - 15x - \frac{15}{4} \cdot \frac{x^2}{2}; Q(0) = 90 \text{ kN};$$

$$Q(4) = 90 - 60 - 30 = 0$$

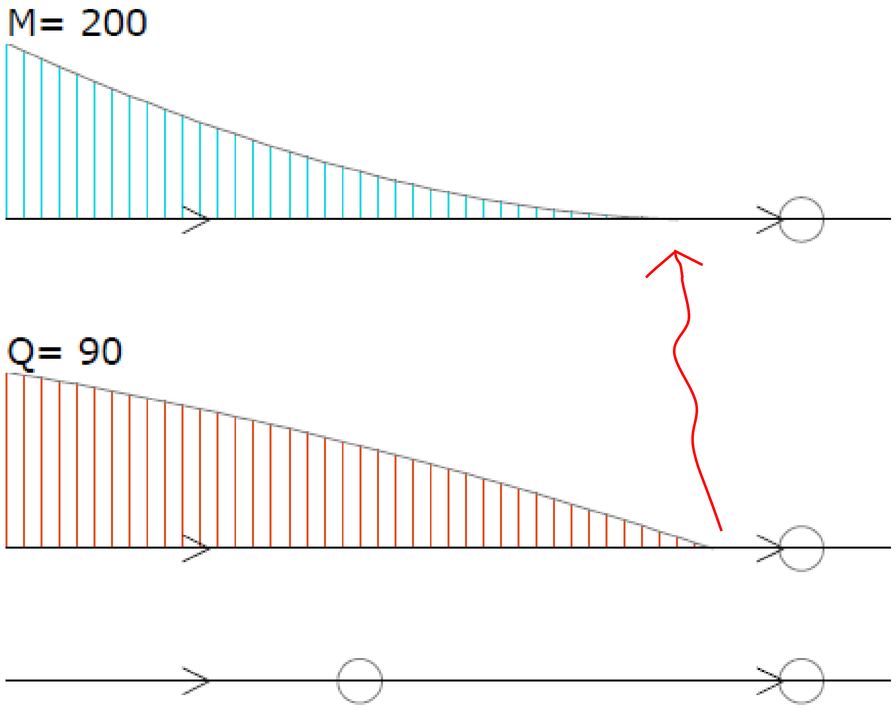
$$N(x) \equiv 0$$

The second interval: $4 \leq x \leq 5$ m

$$M(x) = Q(x) = N(x) \equiv 0$$

Cantilever with trapezoidal load – cont.

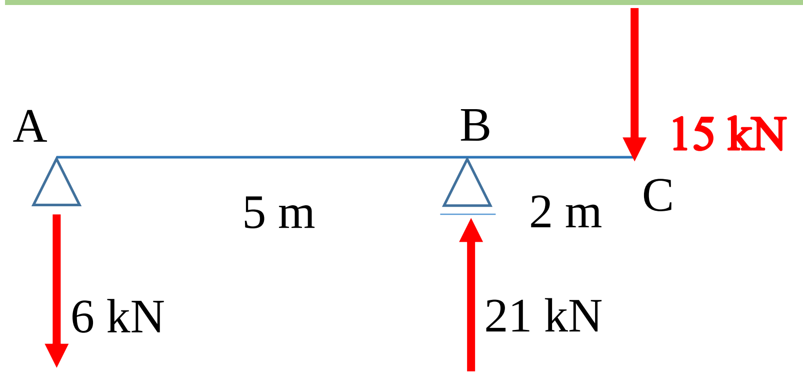
The cross-sectional forces diagrams:



The bending moments diagram is the third order parabola. The convexity sense is the same as the line load sense.

The shear forces diagram is the second order parabola. The gradient of the parabola depends on the line load intensity.

Semi-cantilever with point force

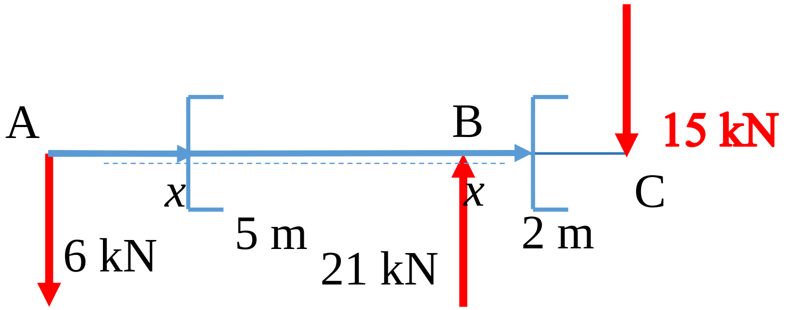


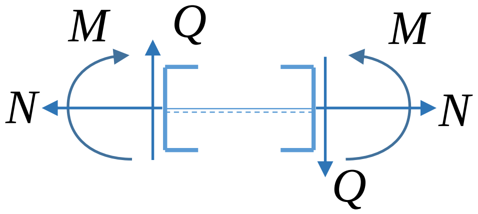
The constraints reactions

$\Sigma M_A = 0 \rightarrow R_B = 15 \cdot \frac{7}{5} = 21 \text{ kN}$

$\Sigma M_B = 0 \rightarrow R_A = 15 \cdot \frac{2}{5} = 6 \text{ kN}$

There are three characteristic points: A, B, and C, and two characteristic intervals: AB and BC.





The first interval: $0 \leq x \leq 5$ m

$M(x) = -6x; M(0) = 0; M(5) = -30 \text{ kNm}$

$Q(x) = -6 \text{ kN} = \text{const}$

$N(x) \equiv 0$

The second interval: $5 \leq x \leq 7$ m

$M(x) = -6x + 21(x-5); M(5) = -30 \text{ kNm};$

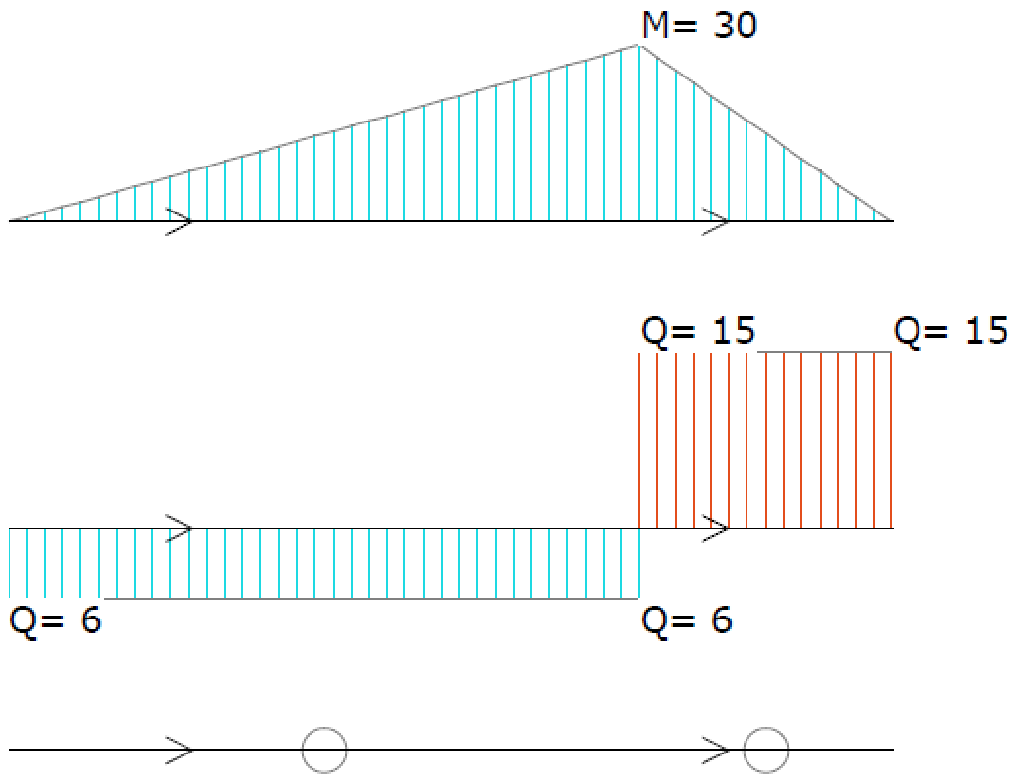
$M(7) = -42 + 42 = 0$

$Q(x) = -6 + 21 = 15 \text{ kN}$

$N(x) \equiv 0$

Semi-cantilever with point force – cont.

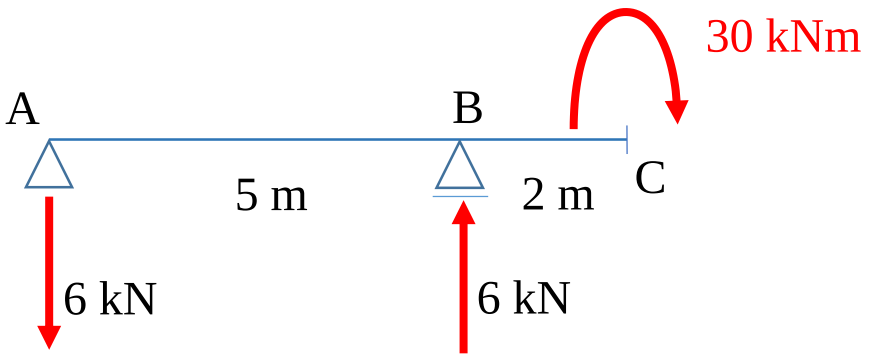
The cross-sectional forces diagrams

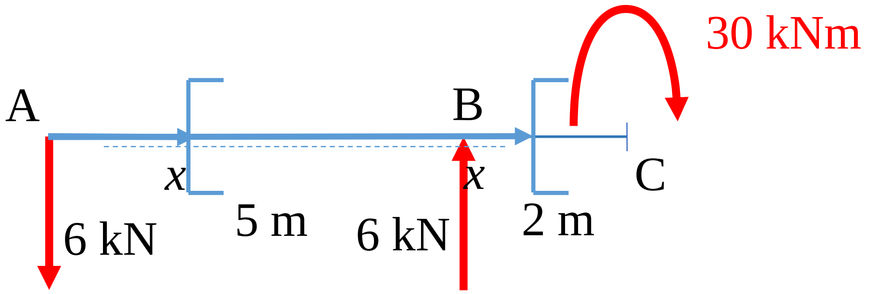


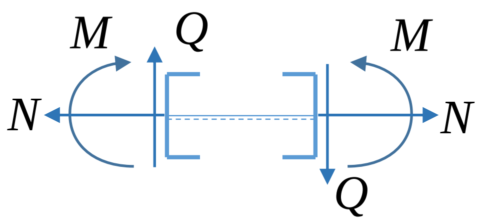
The tip on the bending moment diagram has the same sense as the applied force.
Moment values at both ends are zero.

The jumps of the shear force follow actions of the applied transversal forces.

Semi-cantilever with point moment







The constraints reactions

$\Sigma M_A = 0 \rightarrow 5R_B - 30 = 0 \rightarrow R_B = 6$ kN

$\Sigma M_B = 0 \rightarrow 5R_A - 30 = 0 \rightarrow R_A = 6$ kN

$\Sigma X = 0 \rightarrow H_A = 0$

There are three characteristic points: A, B, and C, and two characteristic intervals: AB and BC

The first interval: $0 \leq x \leq 5$ m

$M(x) = -6x; M(0) = 0; M(5) = -30$ kNm

$Q(x) = -6$ kN = const

$N(x) \equiv 0$

The second interval: $5 \leq x \leq 7$ m

$M(x) = -6x + 6(x - 5); M(5) = -30$ kNm;

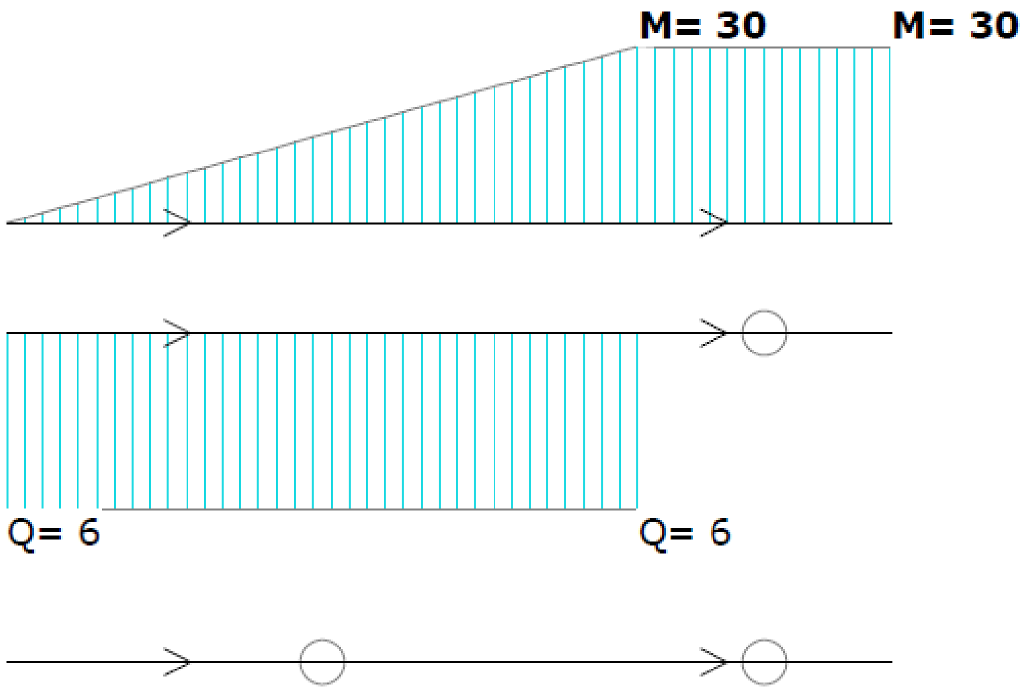
$M(7) = -42 + 12 = -30$ kNm

$Q(x) = -6 + 6 = 0$ = const

$N(x) \equiv 0$

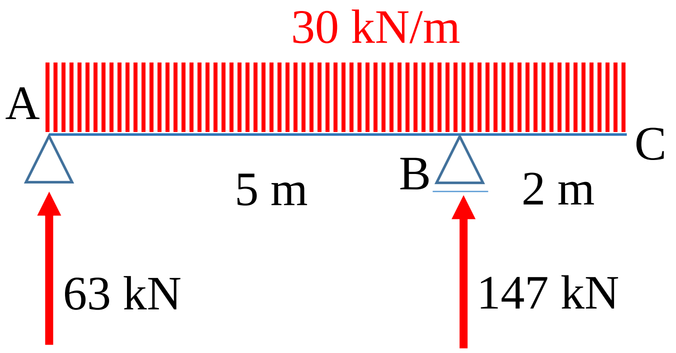
Semi-cantilever with point moment – cont.

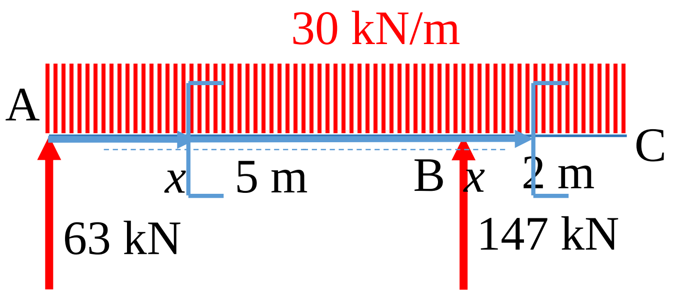
The cross-sectional forces diagrams

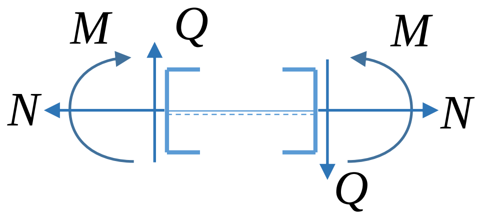


The bending moment at the beam end is equal to the point moment applied at the end.
The tip sense follow the applied force sense.

Semi-cantilever with continuous load







The constraints reactions

$\Sigma M_A = 0 \rightarrow 5R_B - 30 \cdot 7 \cdot 3.5 = 0 \rightarrow R_B = 147 \text{ kN}$

$\Sigma M_B = 0 \rightarrow 5R_A - 30 \cdot 7 \cdot 1.5 = 0 \rightarrow R_A = 63 \text{ kN}$

There are three characteristic points: A, B, and C, and two characteristic intervals: AB and BC.

The first interval: $0 \le x \le 5$ m

$M(x) = 63x - 30\frac{x^2}{2}; M(0) = 0; M(5) = -60 \text{ kNm}$

$Q(x) = 63 - 30x; Q(0) = 63 \text{ kN}; Q(5) = 63 - 150 = -87 \text{ kN}$

the shear force changes sign

$Q(x_0) = 0 \rightarrow x_0 = 2.1 \text{ m}; M(2.1) = 66.15 \text{ kNm}$

$N(x) \equiv 0$

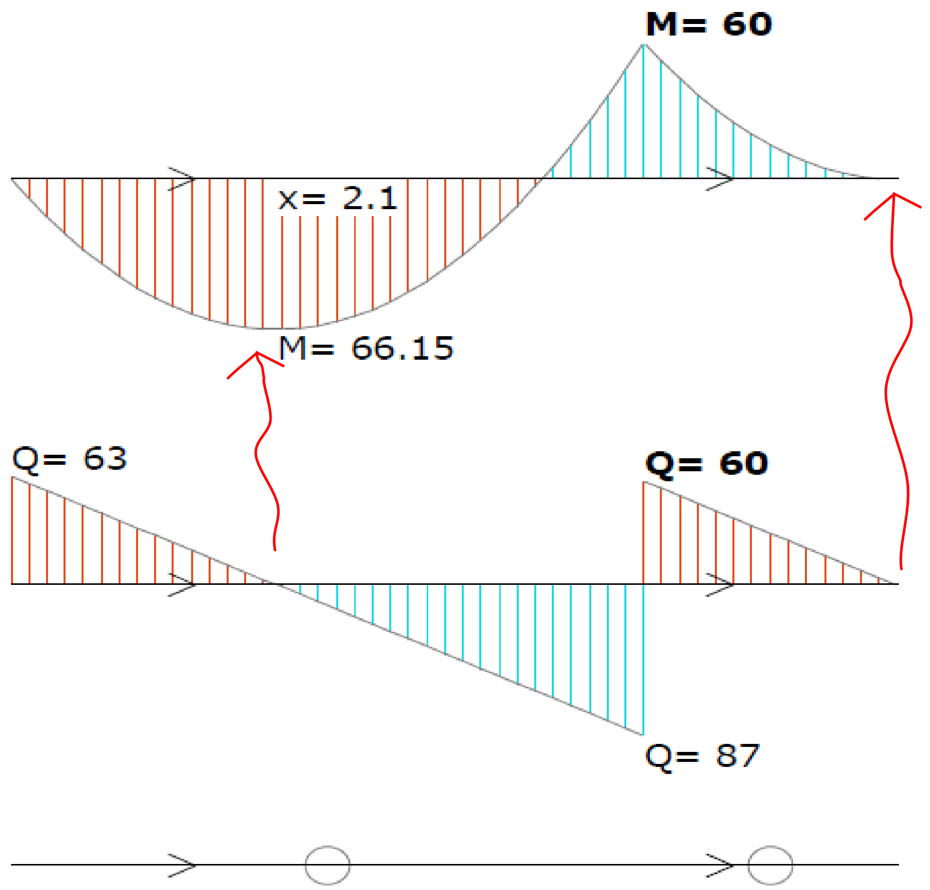
The second interval: $5 \le x \le 7$ m

$M(x) = 63x - 30\frac{x^2}{2} + 147(x-5); M(5) = -60 \text{ kNm}; M(7) = 0$

$Q(x) = 63 - 30x + 147 = 210 - 30x; Q(5) = 60 \text{ kN}; Q(7) = 0$

$N(x) \equiv 0$

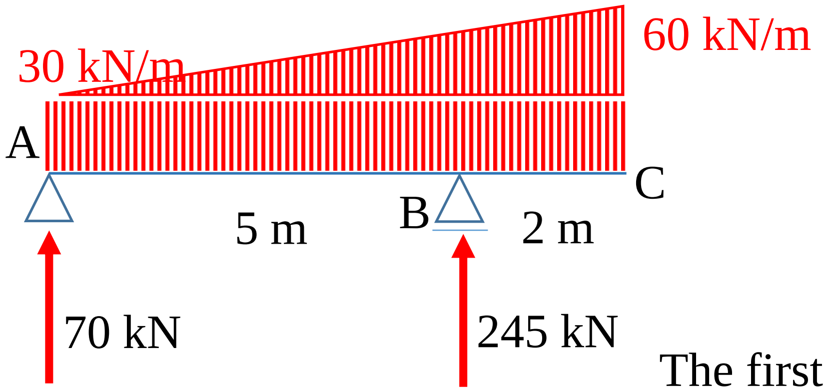
Semi-cantilever with continuous load – cont.



The convexity of the bending moment diagram follows action of the line load.

Zero value of the shear force means an extremum of the bending moment at that point.

Semi-cantilever with trapezoidal load

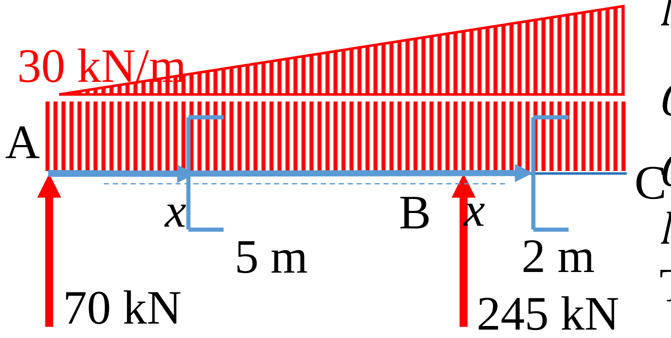


The constraints reactions

$$\Sigma M_A = 0 \rightarrow 5R_B - 30 \cdot 7 \cdot 3.5 - \frac{1}{2} 30 \cdot 7 \cdot \frac{2}{3} \cdot 7 = 0 \rightarrow R_B = 245 \text{ kN}$$

$$\Sigma M_B = 0 \rightarrow 5R_A - 30 \cdot 7 \cdot 1.5 - \frac{1}{2} 30 \cdot 7 \cdot \frac{1}{3} = 0 \rightarrow R_A = 70 \text{ kN}$$

There are three characteristic points: A, B, and C, and two characteristic intervals: AB and BC.



The first interval: $0 \leq x \leq 5$ m

$$M(x) = 70x - 30\frac{x^2}{2} - \frac{30}{7}\frac{x^3}{6}; M(0) = 0; M(5) = -114.3 \text{ kNm}$$

$$Q(x) = 70 - 30x - \frac{30}{7}\frac{x^2}{2}; Q(0) = 70 \text{ kN}; Q(5) = -133.6 \text{ kN (change of sign)}$$

$$Q(x_0) = 0 \rightarrow x_0 = 2.037 \text{ m}, M(2.037) = 74.31 \text{ kNm}$$

$$N(x) \equiv 0$$

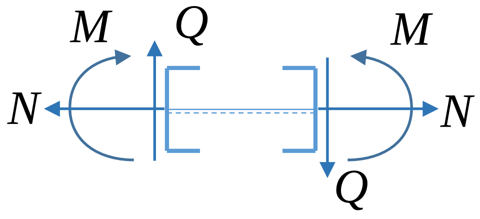
The second interval; $5 \leq x \leq 7$ m

$$M(x) = 70x - 30\frac{x^2}{2} - \frac{30}{7}\frac{x^3}{6} + 245(x-5); M(5) = -114.3\text{kNm}; M(7) = 0$$

$$Q(x) = 70 - 30x - \frac{30}{7}\frac{x^2}{2} + 245 = 315 - 30x - \frac{30}{7}\frac{x^2}{2}; Q(5) = 111.4 \text{ kN};$$

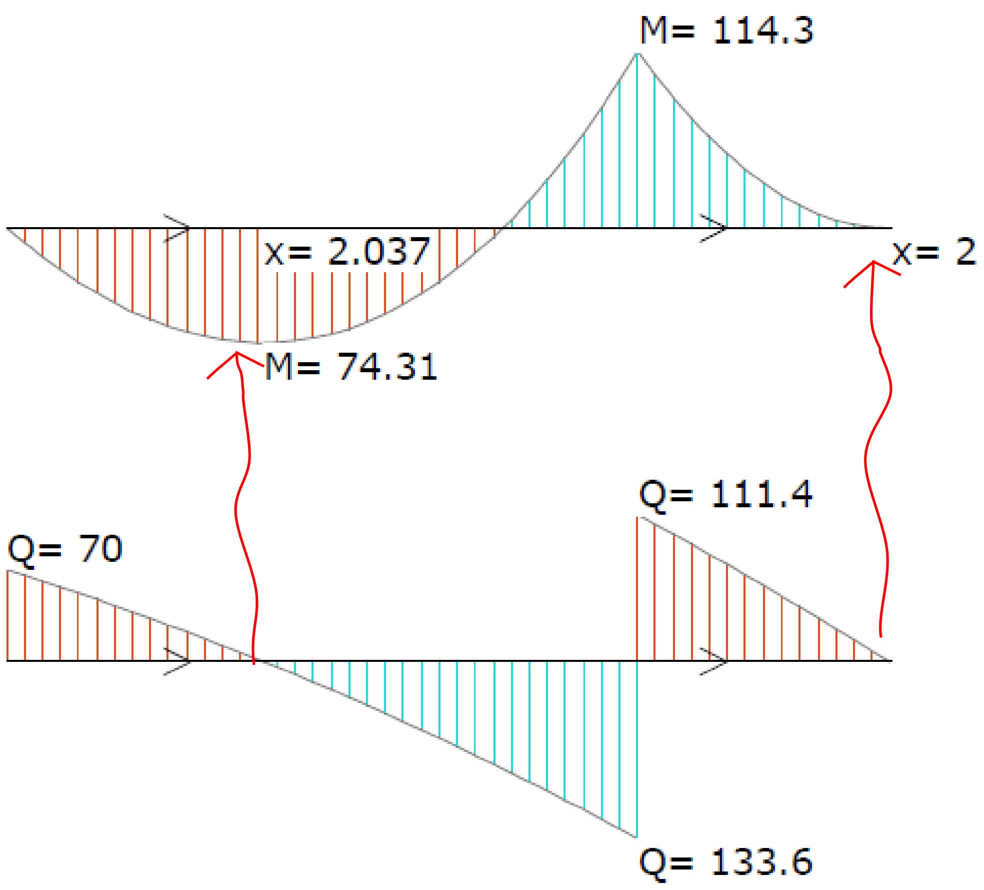
$$Q(7) = 0$$

$$N(x) \equiv 0$$



Semi-cantilever with trapezoidal load – cont.

The cross-sectional forces diagrams



The bending moments diagram is the third order parabola. The convexity sense is the same as the line load sense.

The convexity of the bending moment diagram follows action of the line load.

The shear forces diagram is the second order parabola. The gradient of the parabola depends on the line load intensity.
Zero value of the shear force means an extremum of the bending moment at that point.

Conclusions relating to cross-sectional forces

The point force (transversal = perpendicular to the beam axis):

- on the bending moment diagram there is a tip in the sense of the applied force
- on the shear force diagram there is a jump by the value of the applied force

The point moment:

- on the bending moment diagram there is a jump by the value of the applied moment
- on the shear force diagram there is no visual effects (the whole diagram is vertically displaced)

The continuous load (the line load):

- on the bending moment diagram there is convexity in the sense of the applied load
- on the shear force diagram there is a gradient (a slope) proportional to the applied load intensity (if the intensity is constant, the diagram segment is linear)

The hinge:

- it is not a characteristic point for the cross-sectional forces functions (diagrams)
- the bending moment should be zero at the hinge

The beam edge (end):

- there is zero value of the bending moment unless a point moment load is applied at the edge
- there is zero value of the shear force unless a point force is applied at the edge

If the shear force changes a sign within an interval, the position of its zero value as well as the bending moment extremum at the point should be calculated.

Thank you for your attention!