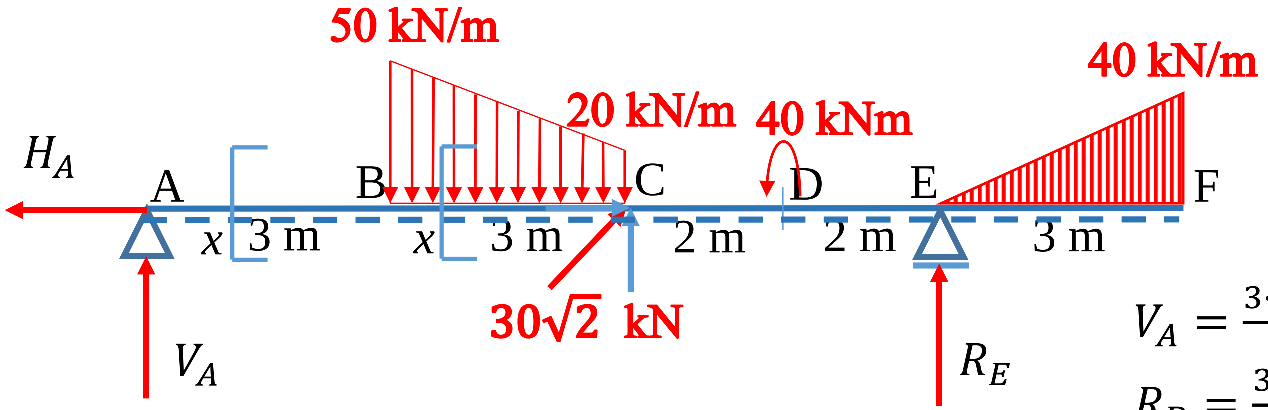
# Strength of Materials

## 5. Beams - continued

# Beam – an example



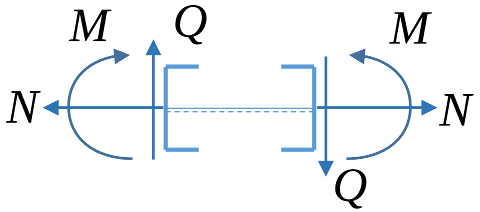
The constraints reactions:

$$H_A = 30\text{ kN}$$

$$V_A = \frac{3\cdot 20\cdot 5.5+0.5\cdot 30\cdot 3\cdot(4+2)-30\cdot 4+40-0.5\cdot 40\cdot 3\cdot 2}{10} = 40\text{ kN}$$

$$R_B = \frac{3\cdot 20\cdot 4.5+0.5\cdot 30\cdot 3\cdot 4-30\cdot 6-40+0.5\cdot 40\cdot 3\cdot 12}{10} = 95\text{ kN}$$

There are six characteristic points: $A, B, C, D, E$, and $F$, and five characteristic intervals: AB, BC, CD, DE, and EF.



The first interval: $0 \le x \le 3$ m

$M(x) = V_A x = 40x; M(0) = 0; M(3) = 120\text{ kNm}$

$Q(x) = V_A = 40\text{ kN}$

$N(x) = H_A = 30\text{ kN}$

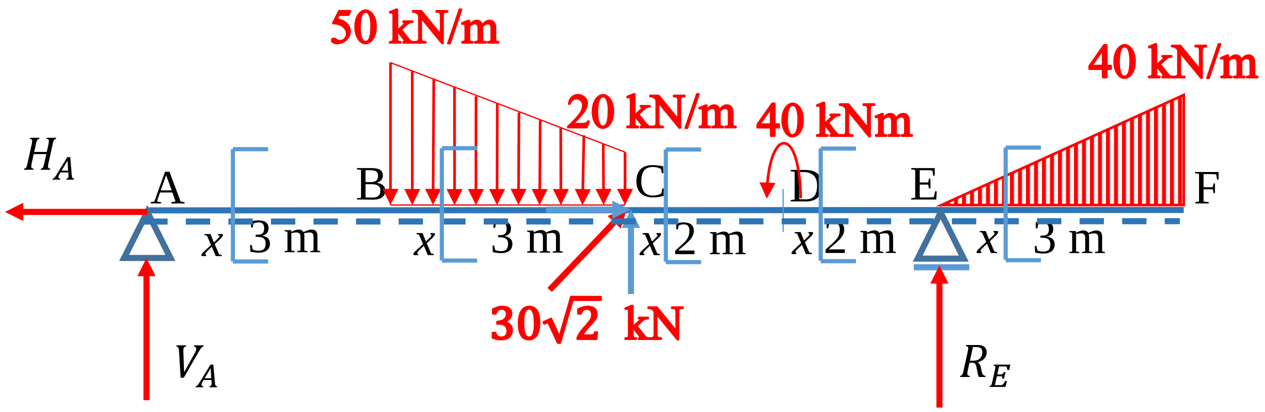
The second interval: $3 \le x \le 6$ m

$M(x) = 40x - 50\frac{(x-3)^2}{2} + \frac{30}{3}\frac{(x-3)^3}{6}; M(3) = 120\text{ kNm}; M(6) = 60\text{ kNm}$

$Q(x) = 40 - 50(x-3) + \frac{30}{3}\frac{(x-3)^2}{2}; Q(3) = 40\text{ kN}; Q(6) = -65\text{ kN}$ (shift in sign!)

$Q(x_0) = 0 \rightarrow 40 - 50x_0 + 150 + 5x_0^2 - 30x_0 + 45 = 0 \rightarrow 5x_0^2 - 80x_0 + 235 = 0$

# Example – cont.

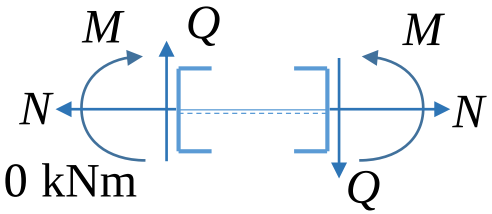


$x_1 = 12.12$ m, $x_0 = 3.8769$ m

$M(3.8769) = 40 \cdot 3.8769 - 25 \cdot 0.8769^2 + \frac{5}{3} \cdot 0.8769^3 = 137.0$ kNm

$N(x) = 30$ kN

The third interval: $6 \leq x \leq 8$ m



$M(x) = 40x - 60(x - 4.5) - 45(x - 4) + 30(x - 6); M(6) = 60$ kNm; $M(8) = -10$ kNm

$Q(x) = 40 - 60 - 45 + 30 = -35$ kN = const

$N(x) \equiv 0$

The fourth interval: $8 \leq x \leq 10$ m

$M(x) = 40x - 60(x - 4.5) - 45(x - 4) + 30(x - 6) - 40; M(8) = -50$ kNm; $M(10) = -120$ kNm

$Q(x) = 40 - 105 + 30 = -35$ kN = const

$N(x) \equiv 0$

The fifth interval: $0 \leq x_1 \leq 3$ m (right-to-left)

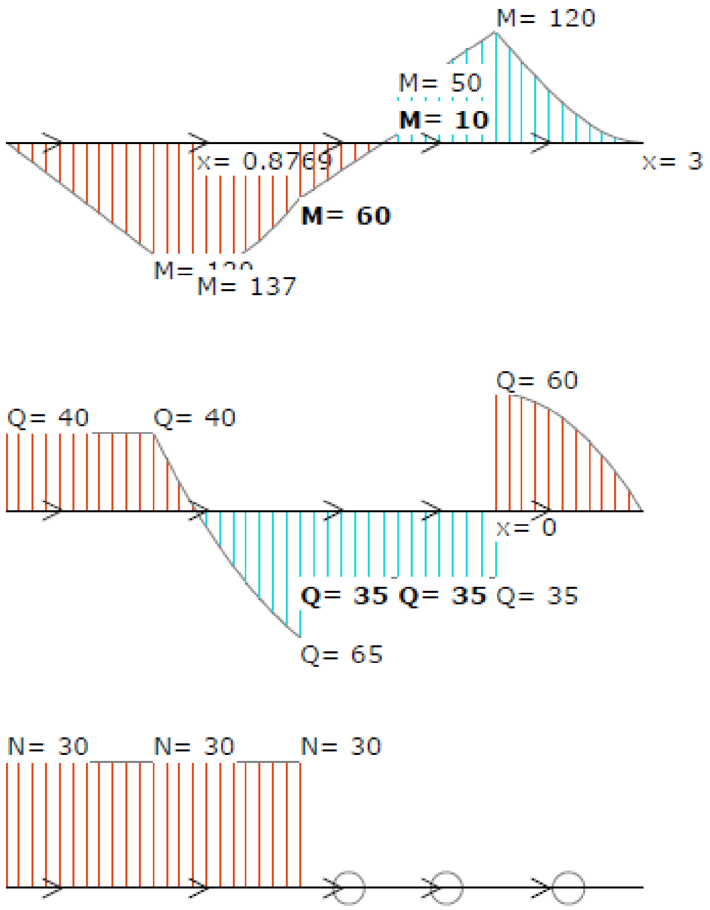
$M(x_1) = -40\frac{x_1^2}{2} + \frac{40}{3}\frac{x_1^3}{6}; M(0) = 0; M(3) = -120$ kNm;

# Example – cont.

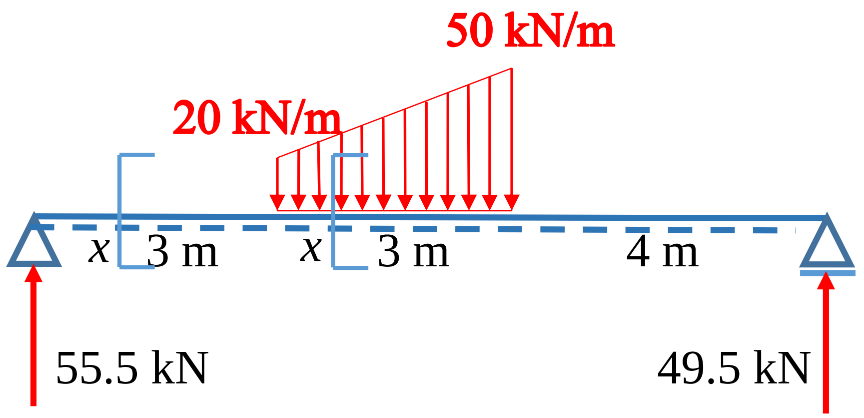
$$Q(x_1) = -40x_1 + \frac{40}{3}\frac{x_1^2}{2}; Q(0) = 9; Q(3) = -60\ \mathrm{kN}$$

$$N(x_1) \equiv 0$$

The cross-sectional forces diagrams



# Trapezoidal load - comments

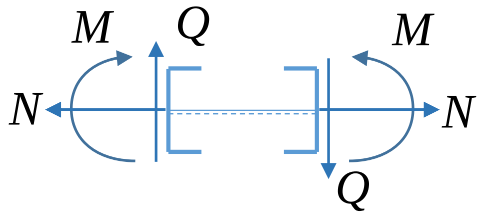


The first interval: $0 \leq x \leq 3\text{m}$

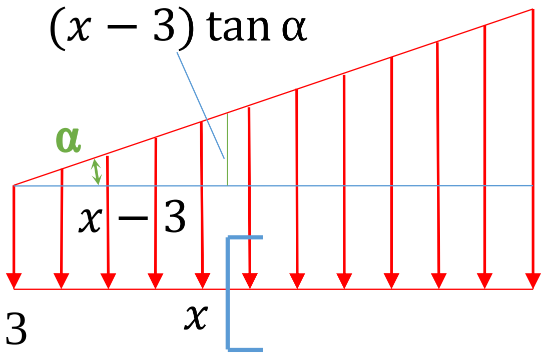
$M(x) = 55.5x; M(0) = 0; M(3) = 166.5\ \text{kNm}$

$Q(x) = 55.5\ \text{kN} = \text{const}$

$N(x) \equiv 0$



$$\tan\alpha = \frac{50-20}{3} = 10$$



the moment: $= \underbrace{\underbrace{\frac{1}{2}(x-3)}_{\text{a half of a base}} \cdot \underbrace{(x-3)\tan\alpha}_{\text{value of intensity at } x}}_{\text{the resultant}} \cdot \underbrace{\frac{(x-3)}{3}}_{\text{arm}}$

The second interval:

$M(x) = 55.5x - 20\frac{(x-3)^2}{2} - 10\frac{(x-3)^3}{6}; M(3) = 166.5\ \text{kNm}; M(6) = 198\ \text{kNm}$

$Q(x) = 55.5 - 20(x-3) - 10\frac{(x-3)^2}{2}; Q(3) = 55.5\ \text{kN}; Q(6) = -49.5\ \text{kN}$ (changes the sign)

# Trapezoidal load – comments cont.

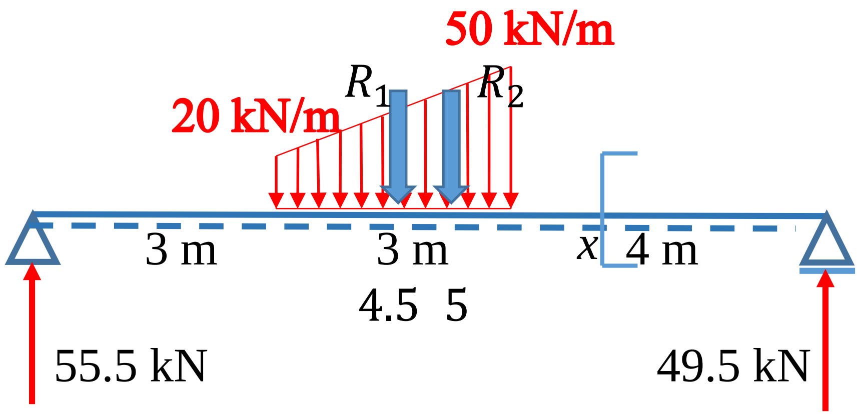
$Q(x_0) = 0 \rightarrow 55.5 - 20x_0 + 60 - 5x_0^2 + 30x_0 - 45 = 0 \rightarrow -5x_0^2 + 10x_0 + 70.5 = 0 \rightarrow x_0 = 4.886$ m

$M(4.886) = 55.5 \cdot 4.886 - 10 \cdot 1.886^2 - 1.667 \cdot 1.886^3 = 224.4$ kNm (extremum)

$N(x) \equiv 0$

The third interval: $6 \leq x \leq 10$ m

(the continuous load can be replaced by its resultant outside the interval of action)



rectangle: $R_! = 20 \cdot 3 = 60$ kN (at $x$ = 4.5 m)

triangle: $R_2 = \frac{1}{2}\, 30 \cdot 3 = 45$ kN (at $x$ = 5 m)

$M(x) = 55.5x - 60(x - 4.5) - 45(x - 5); M(6) = 198$ kNm

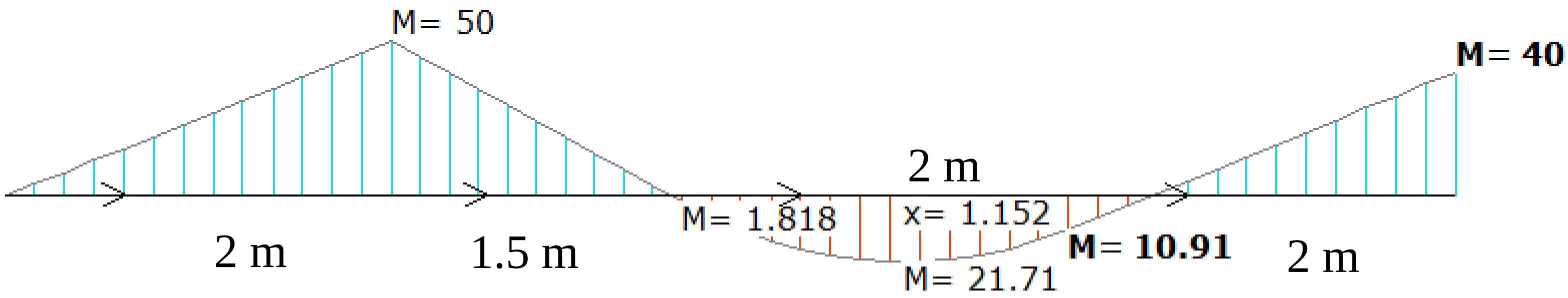
$M(10) = 0$ (OK)

$Q(x) = 55.5 - 60 - 45 = -49.5$ kN = const

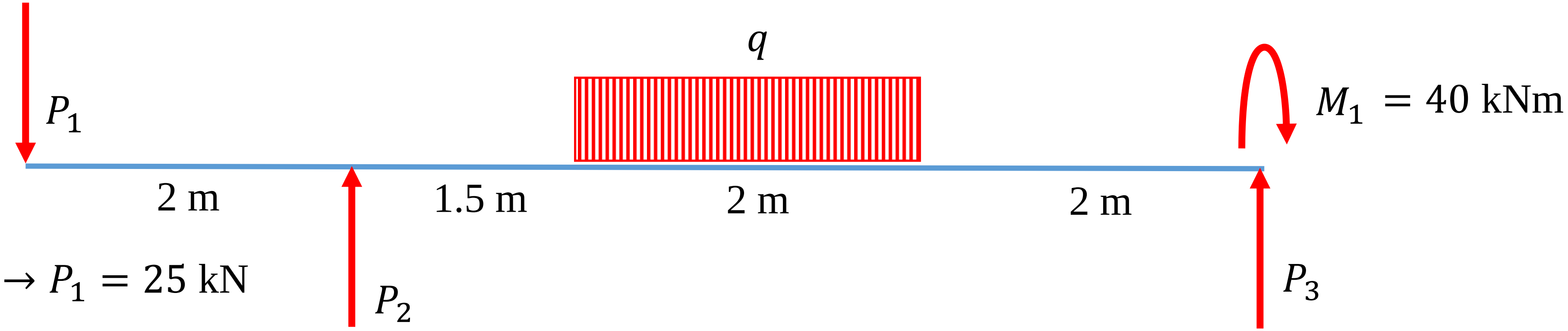
$N(x) \equiv 0$

# Inverse problem

Given bending moment diagram, determine the load applied to the beam.



Solution



$2P_1 = 50 \rightarrow P_1 = 25$ kN

$M(3.5) = 1.818 \text{ kNm} \rightarrow$
$-3.5P_1 + 1.5P_2 = 1.818 \rightarrow P_2 = 59.55$ kN

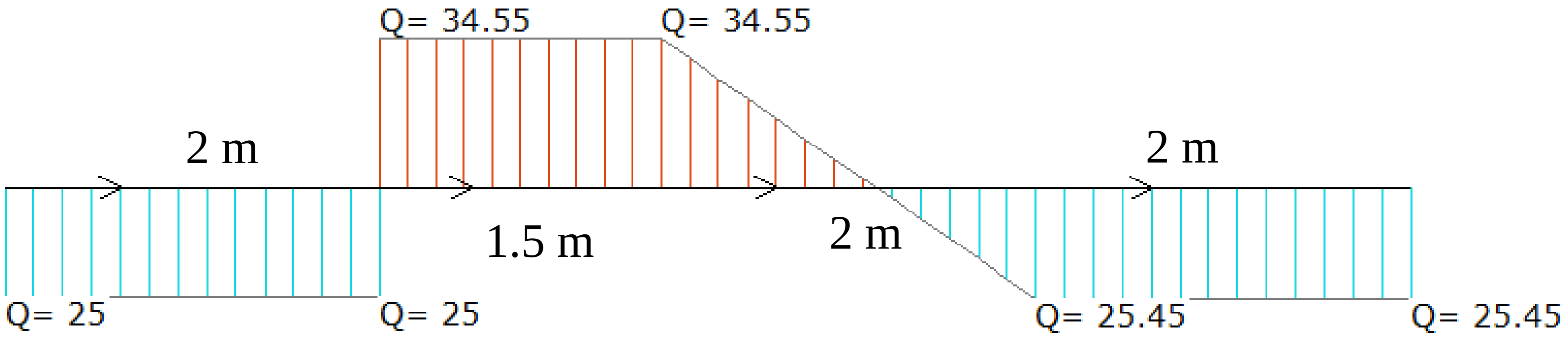
$2P_3 - 40 = 10.91 \rightarrow P_3 = 25.46$ kN

$M(4) = 4P_3 - 40 - 2q \cdot 1 = 1.818 \rightarrow q = 30$ kN/m

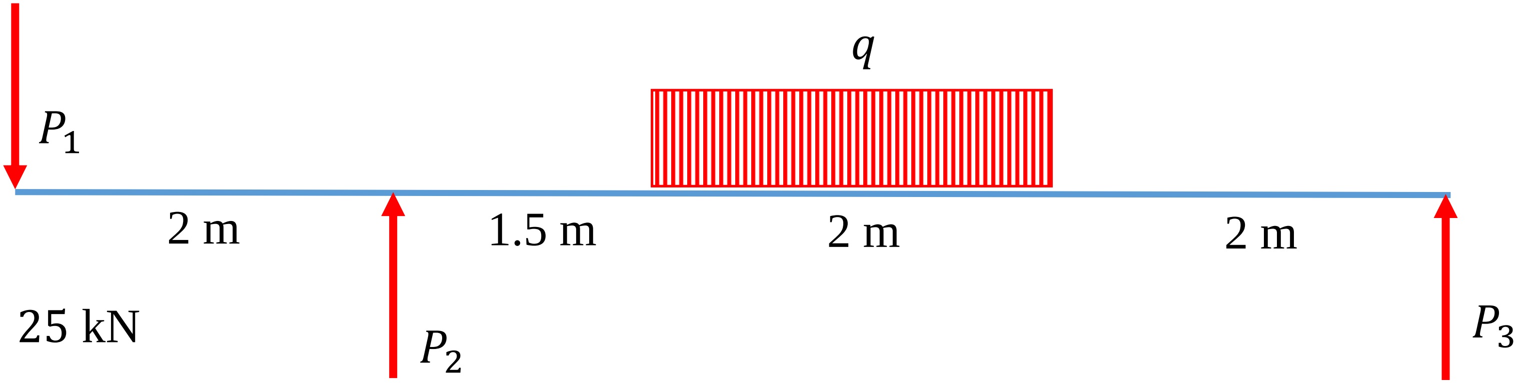
check:
$M(4.652) = -25 \cdot 4.652 + 59.55 \cdot 2.652$
$-15 \cdot 1.152^2 = 21.72$ kNm, OK

# Inverse problem – cont.

Given shear force diagram, determine the load applied to the beam.



Solution



$P_1 = 25\ \text{kN}$

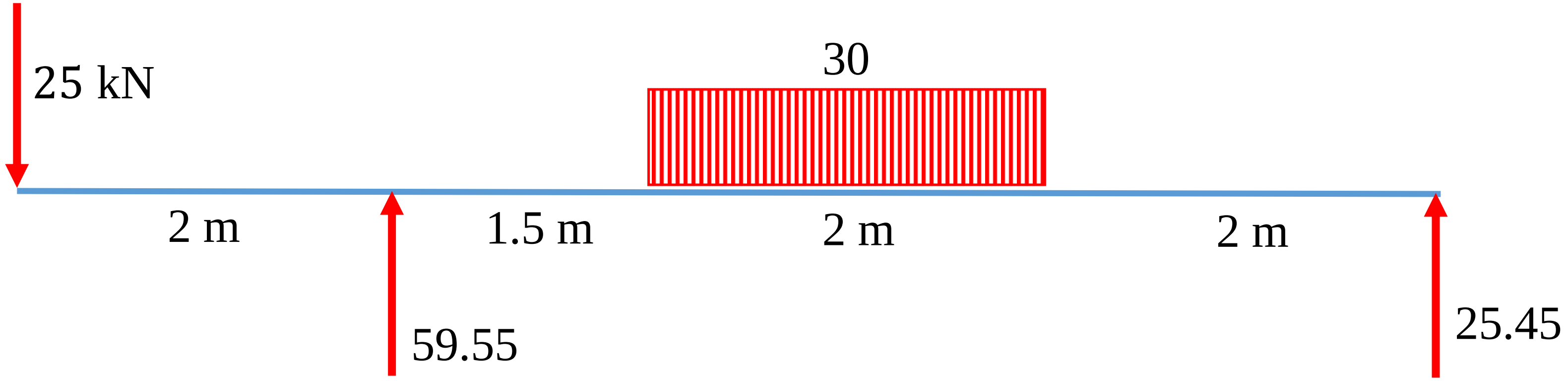
$P_2 = 25 + 34.55 = 59.55\ \text{kN}$

$P_3 = 25.45\ \text{kN}$

$$2q = 34.55 + 25.45 = 60 \rightarrow q = 30\ \text{kN/m}$$

Check: $\Sigma Y = -25 + 59.55 - 60 + 25.45 = 0$, OK

# Inverse problem – cont.

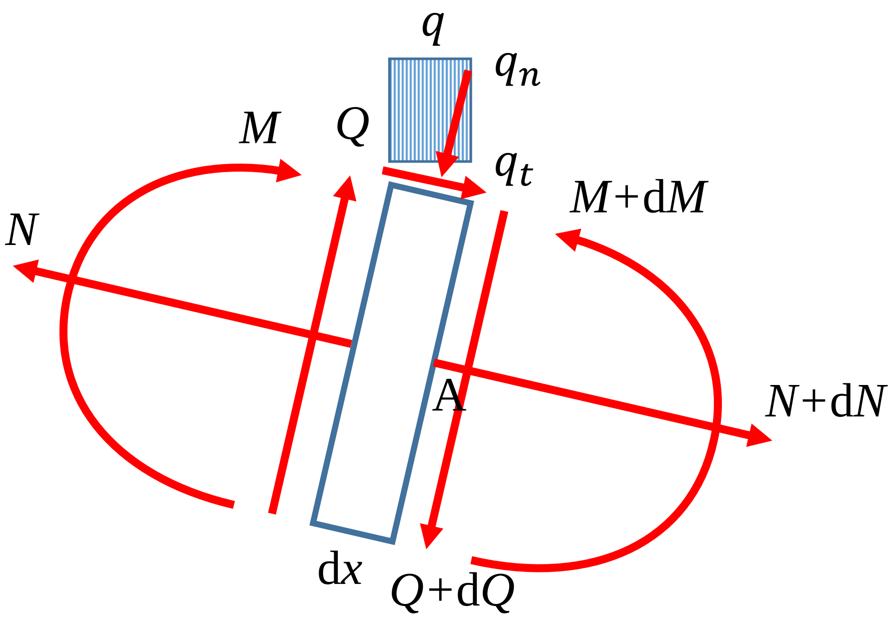


but: $\Sigma M_1 = 59.55 \cdot 2 - 60 \cdot 4.5 + 25.45 \cdot 7.5 = 39.98 \cong 40$, not zero!
and: $\Sigma M_2 = 25 \cdot 2 - 60 \cdot 2.5 + 25.45 \cdot 5.5 = 39.98 \cong 40$, not zero!
and: $\Sigma M_3 = 25 \cdot 7.5 - 59.55 \cdot 5.5 + 60 \cdot 3 = 39.98 \cong 40$, not zero!
The repeatability of the result means that it is not an error, and there is a point moment (or some point moments) applied somewhere, we don't know exactly where.

The inverse problem may be solved completely on the basis of the bending moment diagram. In the case of the shear force diagram, the inverse problem solution is accurate to within point moments applied.

# Slanting beams

Schwedler-Zhuravsky Theorem (generalized)



$q(q_n, q_t)$ – normal and tangent components of load

$$\boxed{\frac{dM}{dx} = Q(x)}$$

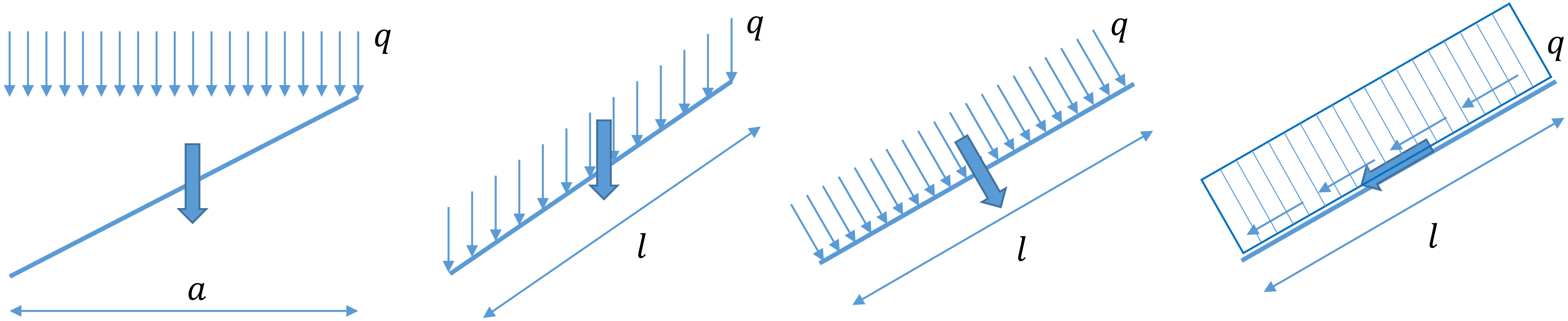
$$\boxed{\frac{dQ}{dx} = -q_n(x)}$$

$$\boxed{\frac{dN}{dx} = -q_t(x)}$$ (not constant)

$$\boxed{\frac{d^2M}{dx^2} = -q_n(x)}$$

# Line load – different possibilities

The intensity of the line (continuous) load can be stated per meter or per running meter. Direction of a line load can be vertical, horizontal, perpendicular to the beam axis or tangent to the axis.



intensity per meter

$$R = q \times a$$

intensity per running meter

$$R = q \times l$$

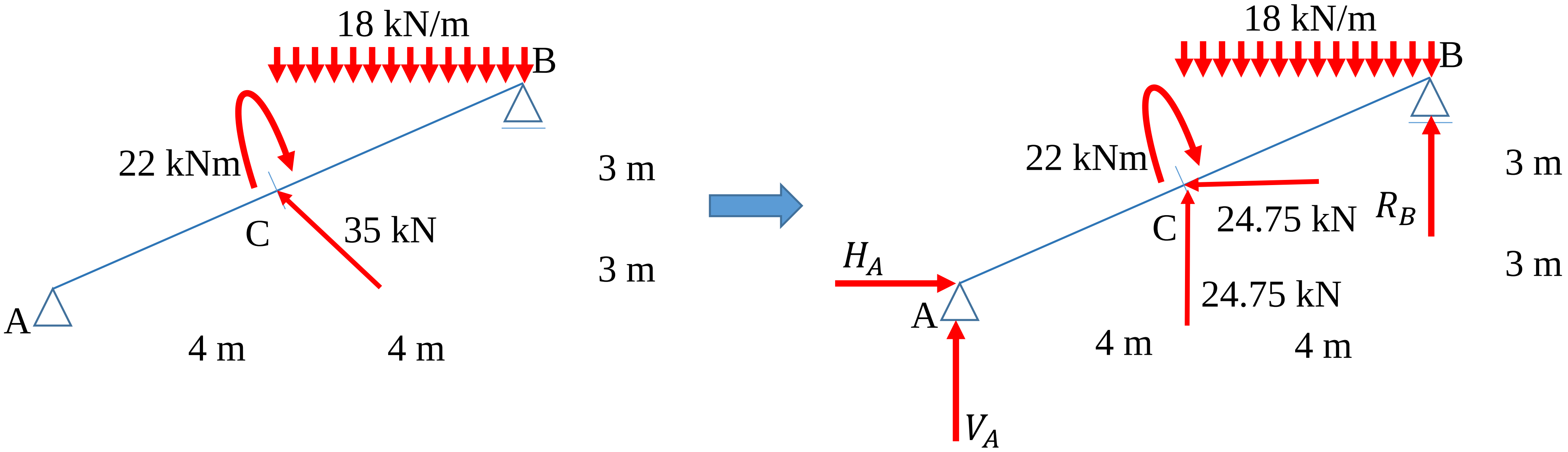
intensity per running meter

$$R = q \times l$$

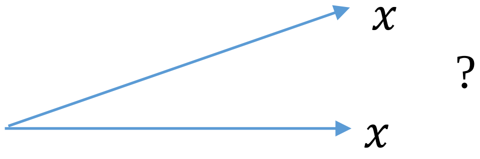
intensity per running meter

$$R = q \times l$$

# Slanting beam – an example

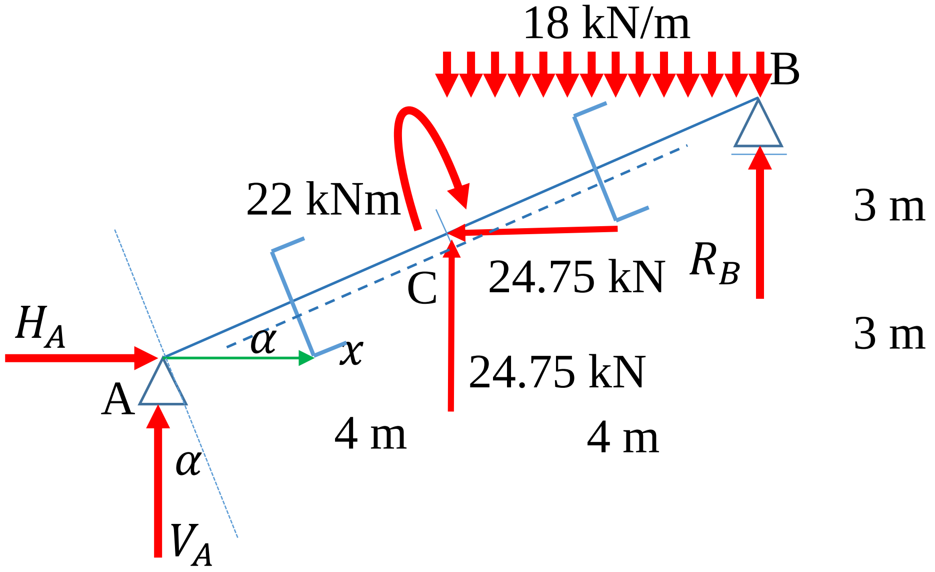


There are three characteristic points: $A, C,$ and $C$, and two characteristic intervals” AC and CB.



There are several possibilities to
adopt a suitable coordinate set.
The best choice is the most natural one.

# Example – cont.



Reactions:

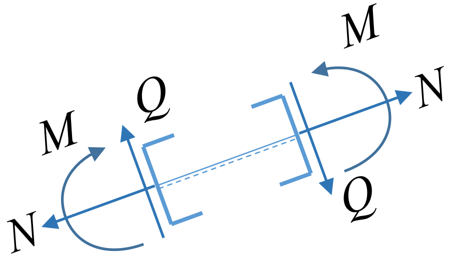
$V_A = 12.16$ kN

$H_A = 24.75$ kN

$R_B = 35.09$ kN

$\cos\alpha = 0.8$

$\sin\alpha = 0.6$



The first interval: $0 \le x \le 4$ m

$$M(x) = V_A x - H_A y = 12.16x - 24.75 \cdot \frac{3}{4}x;\ M(0) = 0;$$

$$M(4) = -25.63 \text{ kNm}$$

$$Q(x) = 12.16\cos\alpha - 24.75\sin\alpha = -5.125 \text{ kN}$$

$$N(x) = -12.16\sin\alpha - 24.75\cos\alpha = -27.09 \text{ kN}$$

The second interval: $4 \le x \le 8$ m

$$M(x) = 12.16x - \frac{3}{4}\,24.75x + 22 + 24.75(x-4) + 24.75 \cdot \frac{3}{4}(x-4) - 18\,\frac{(x-4)^2}{2};\ M(4) = -3.63 \text{ kNm};$$

$$M(8) = 0$$

$$Q(x) = (12.16 + 24.75 - 18(x-4)) \cdot 0.8 + (-24.75 + 24.75) \cdot 0.6 = \big(36.91 - 18(\mathrm{x} - 4)\big) \cdot 0.8;$$

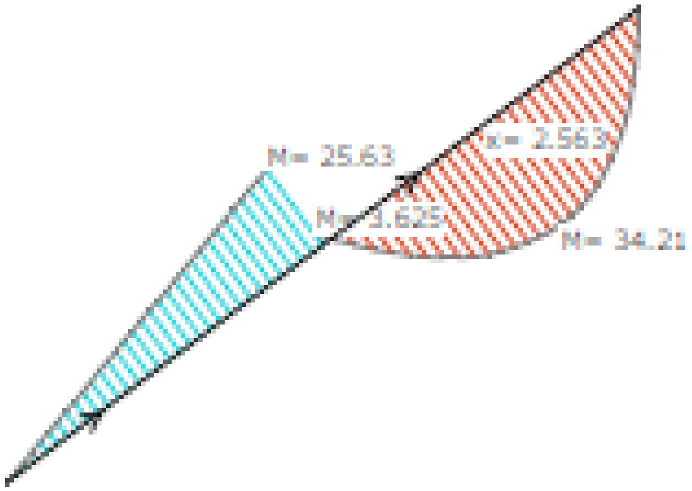
$$Q(4) = 29.52 \text{ kN};\ Q(8) = 28.08 \text{ kN (sign change)}$$

$$Q(x_0) = 0 \rightarrow x_0 = 6.051 \text{ m};\ M(6.051) = 34.21 \text{ kNm; (extremum)}$$

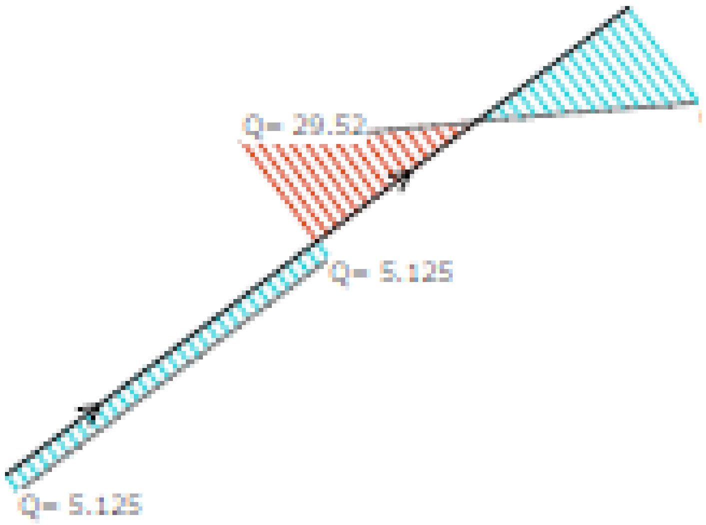
$$N(x) = \big(-12.16 - 24.75 + 18(x-4)\big) \cdot 0.6 + (-24.75 + 24.75) \cdot 0.8 = (-36.91 + 18(x-4)) \cdot 0.6$$

$$N(4) = -22.14 \text{ kN};\ N(8) = 21.06 \text{ kN}$$

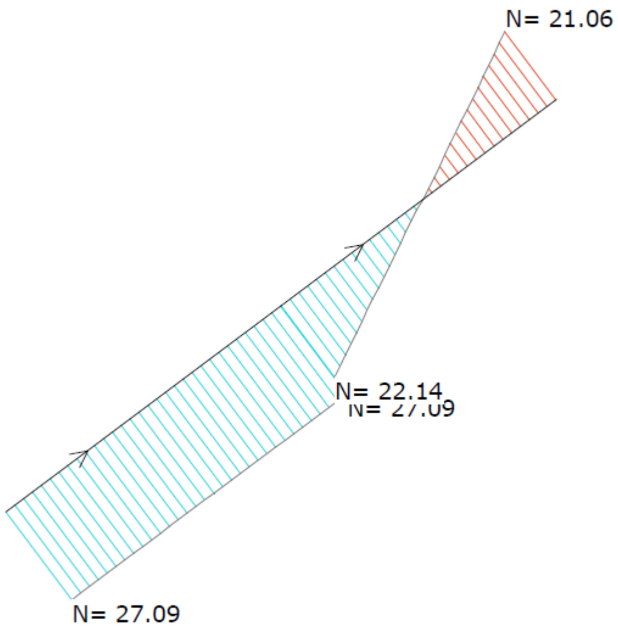
# Example - diagrams



Bending moments diagram

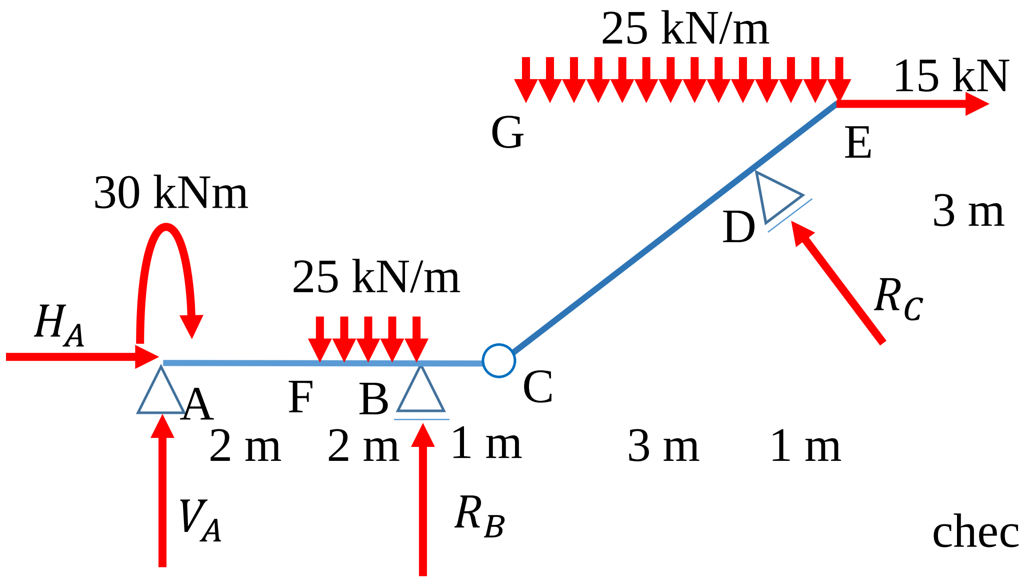


Shear forces diagram



Axial forces diagram

# Hinged beams – an example



Constraints reactions:

a hinge equation:

$$\Sigma M_C^R = 0 \rightarrow \frac{3}{4}\,5R_C = 15\cdot 3 + 25\cdot 4\cdot 2 \rightarrow R_C = 65.33\text{ kN}$$

$$\begin{cases} \Sigma M_C^L = 0 \rightarrow 5V_A + 30 - 25\cdot 2\cdot 2 + R_B = 0 \\ \Sigma Y = 0 \rightarrow V_A + R_B + 0.8R_C - 25\cdot 2 - 25\cdot 4 = 0 \end{cases}$$

$V_A = -6.93$ kN, $R_B = 104.7$ kN

$\Sigma X = 0 \rightarrow H_A + 15 - 0.6R_C = 0 \rightarrow H_A = 24.2$ kN

check: $\Sigma M_G = 0 \rightarrow -5\cdot 6.93 - 3\cdot 24.2 - 25\cdot 2\cdot 2 + 30 + 104.7 + 25\cdot 4\cdot 2 - 0.8\cdot 3\cdot 65.33 + 0.6\cdot\frac{3}{4}\cdot 65.33 = 0.06 \cong 0$

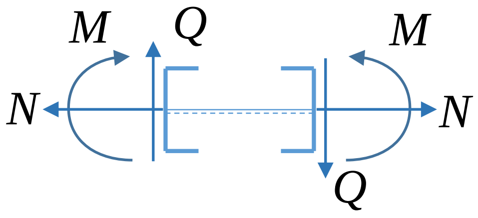
There are six characteristic points: $A, B, C, D, E$, and $F$, and five characteristic intervals: AF, FB, BC, CD, and DE.

The interval AF:

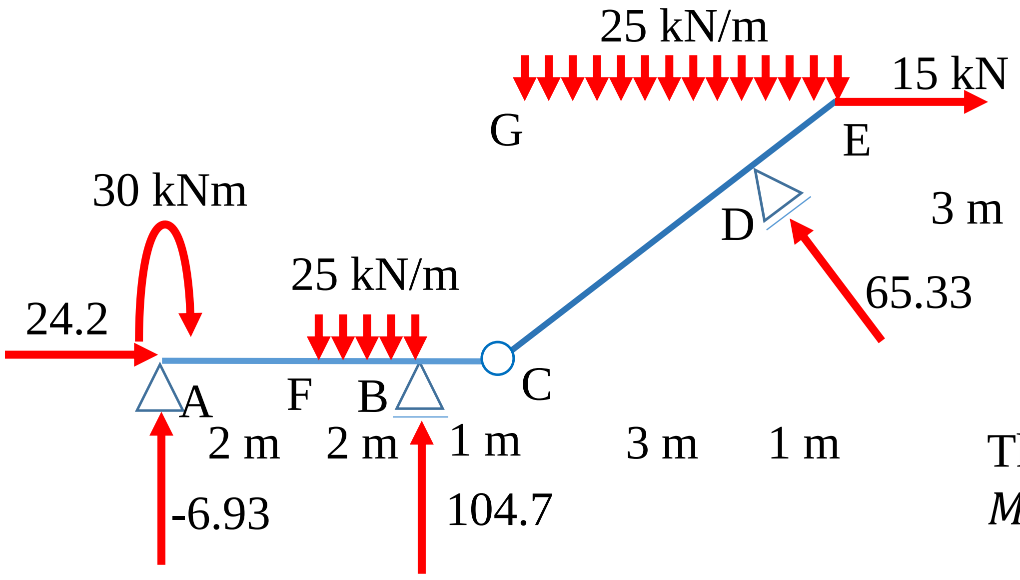
$M(x) = -6.93x + 30$; $M(0) = 30$ kNm; $M(2) = 16.13$ kNm

$Q(x) = -6.93$ kN = const

$N(x) = -24.2$ kN = const



# Example – cont.



The interval FB:

$M(x) = -6.93x + 30 - 25\frac{(x-2)^2}{2}$; $M(2) = 16.14$ kNm

$M(4)$ =-47.73 kNm

$Q(x) = -6.93 - 25(x-2)$; $Q(2) = -6.93$;

$Q(4) = -56.93$ kN

$N(x) = -24.2$ kN

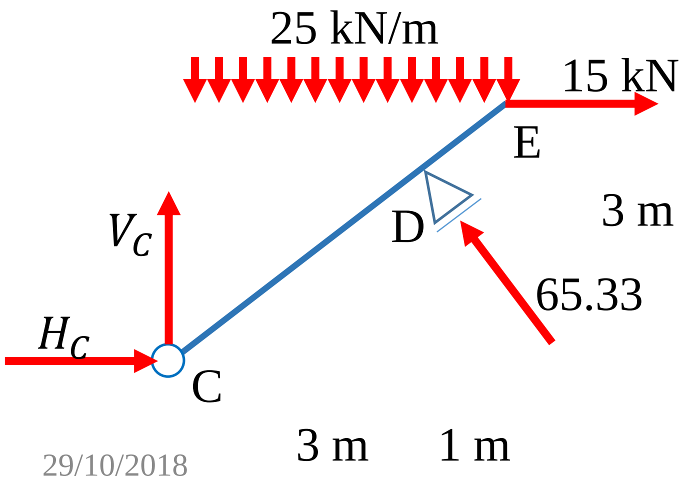
The interval BC:

$M(x) = -6.93x + 30 - 50(x-3) + 104.7(x-4)$;

$M(4) = -47.73$ kNm; $M(5) = 0.05 \cong 0$

$Q(x) = -6.93 - 50 + 104.7 = 47.77$ kN = const

$N(x) = -24.2$ kN = const



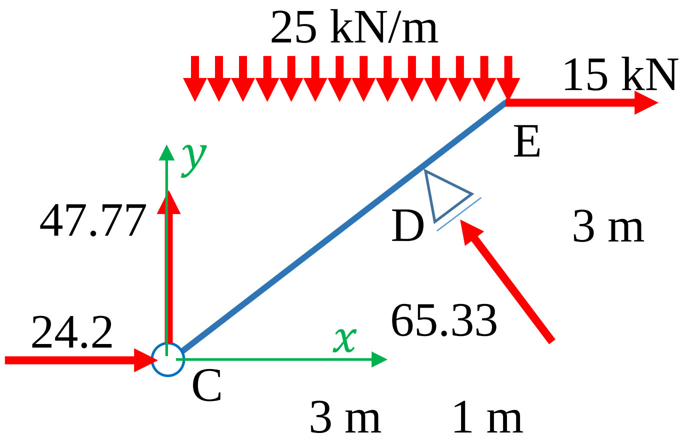
The right side subset (for section through the hinge *C* ):

reactions $V_C$ and $H_C$ are statically equivalent to the left side forces set, so:

$V_C = -6.93 - 50 + 104.7 = 47.77$ kN

$H_C = 24.2$ kN

# Example – cont.



The interval CD:

$M(x) = 47.77x - 24.2y - 25\frac{x^2}{2}$; $M(0) = 0$; $M(3) = -23.64$ kNm

$Q(x) = (47.77 - 25x) \cdot 0.8 - 24.2 \cdot 0.6$; $Q(0) = 23.64$ kN;

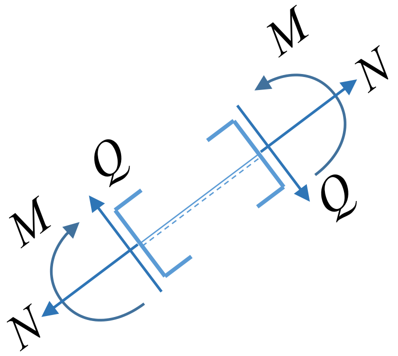
$Q(3) = -36.36$ kN (sign change)

$Q(x_0) = 0 \rightarrow x_0 = 1.185$ m $\rightarrow M(1.185) = 17.55$ kNm (extremum)

$N(x) = (-47.77 + 25x) \cdot 0.6 - 24.2 \cdot 0.8$; $N(0) = -48.0$ kN,

$N(3) = -3$ kN

(round off errors)



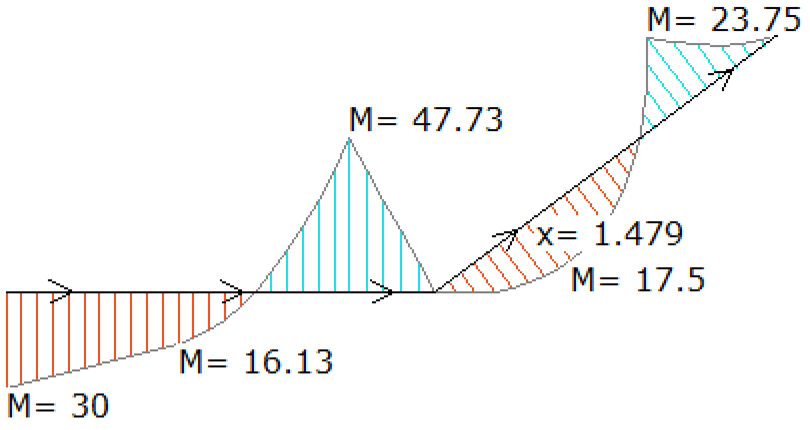
The interval ED (right to left)

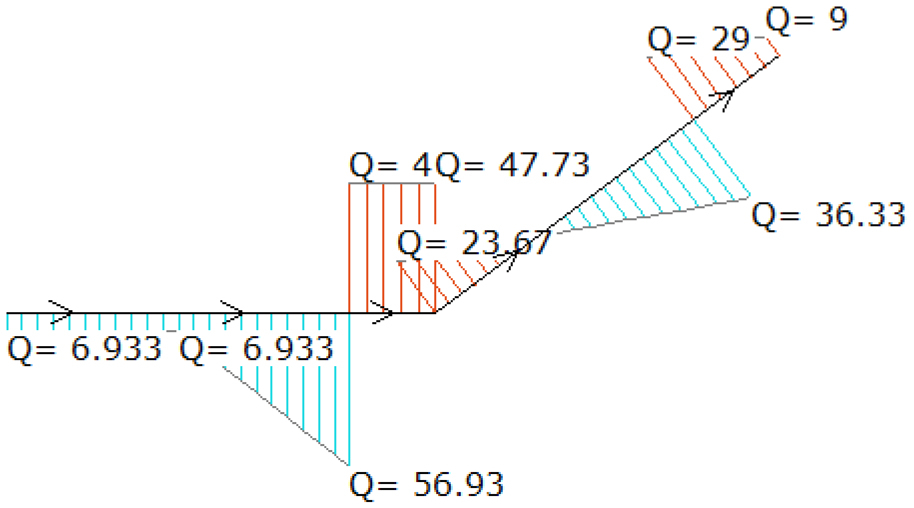
$M(x_1) = -15\frac{3}{4}x_1 - 25\frac{x_1^2}{2}$; $M(0) = 0$; $M(1) = -23.75$ kNm

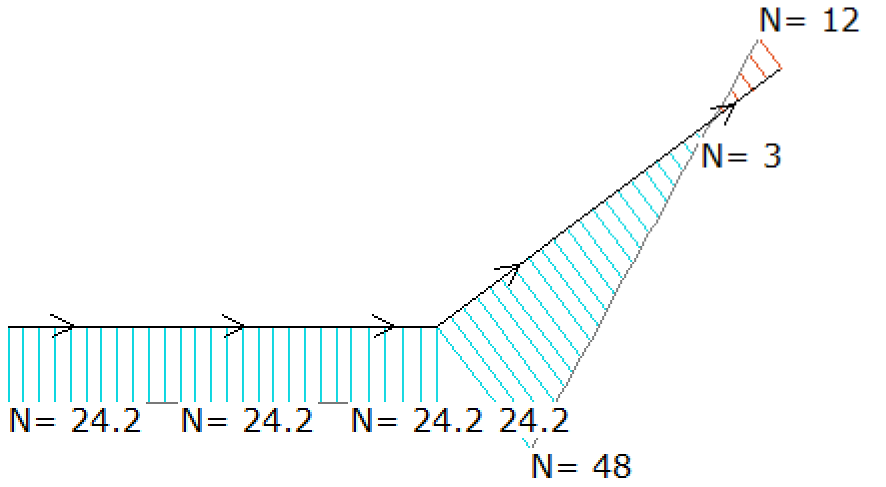
$Q(x_1) = 15 \cdot 0.6 + 25x_1 \cdot 0.8$; $Q(0) = 9$ kN; $Q(1) = 29$ kN

$N(x_1) = 15 \cdot 0.8 - 25x_1 \cdot 0.6$; $N(0) = 12$ kN; $N(1) = -3$ kN

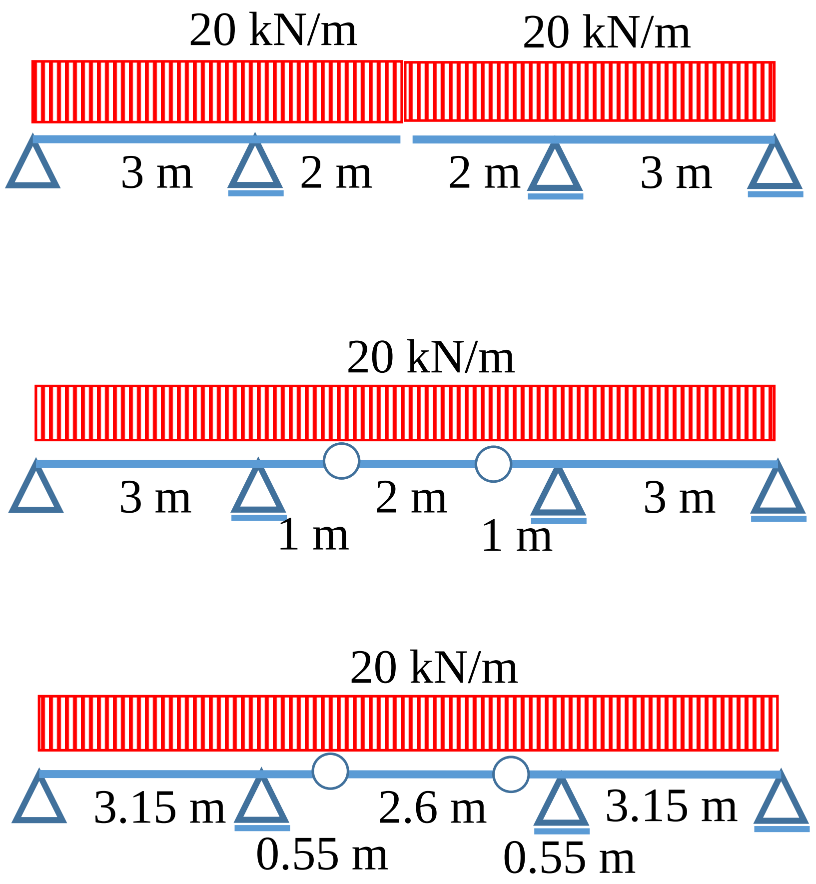
# Example – diagrams

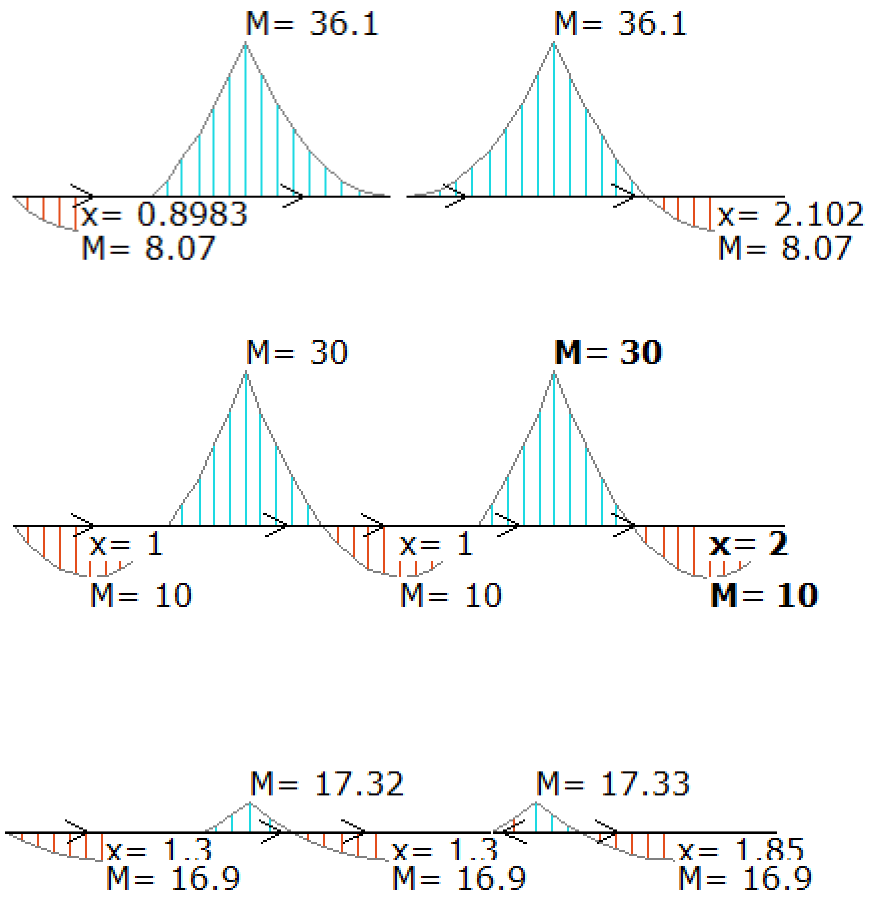




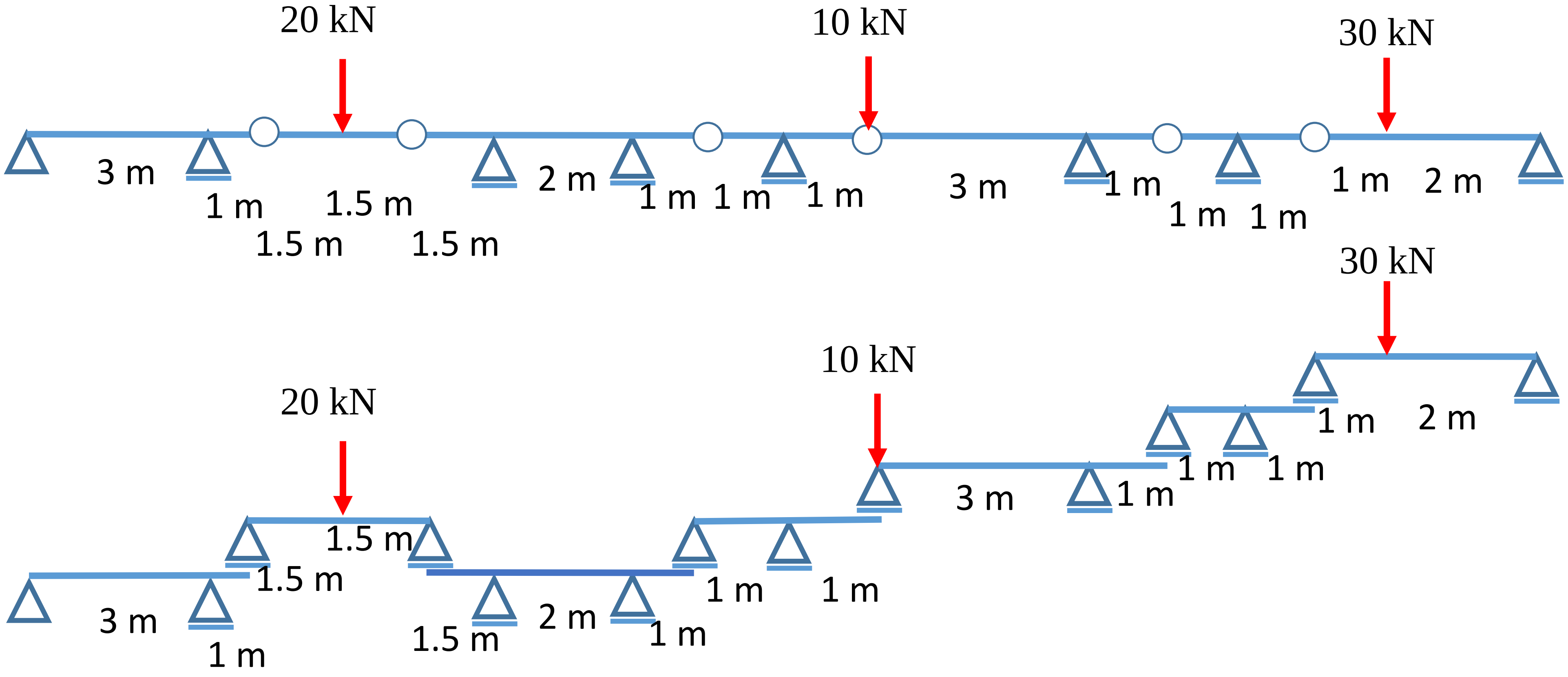


# Horizontal hinged beams (Gerber's beams)

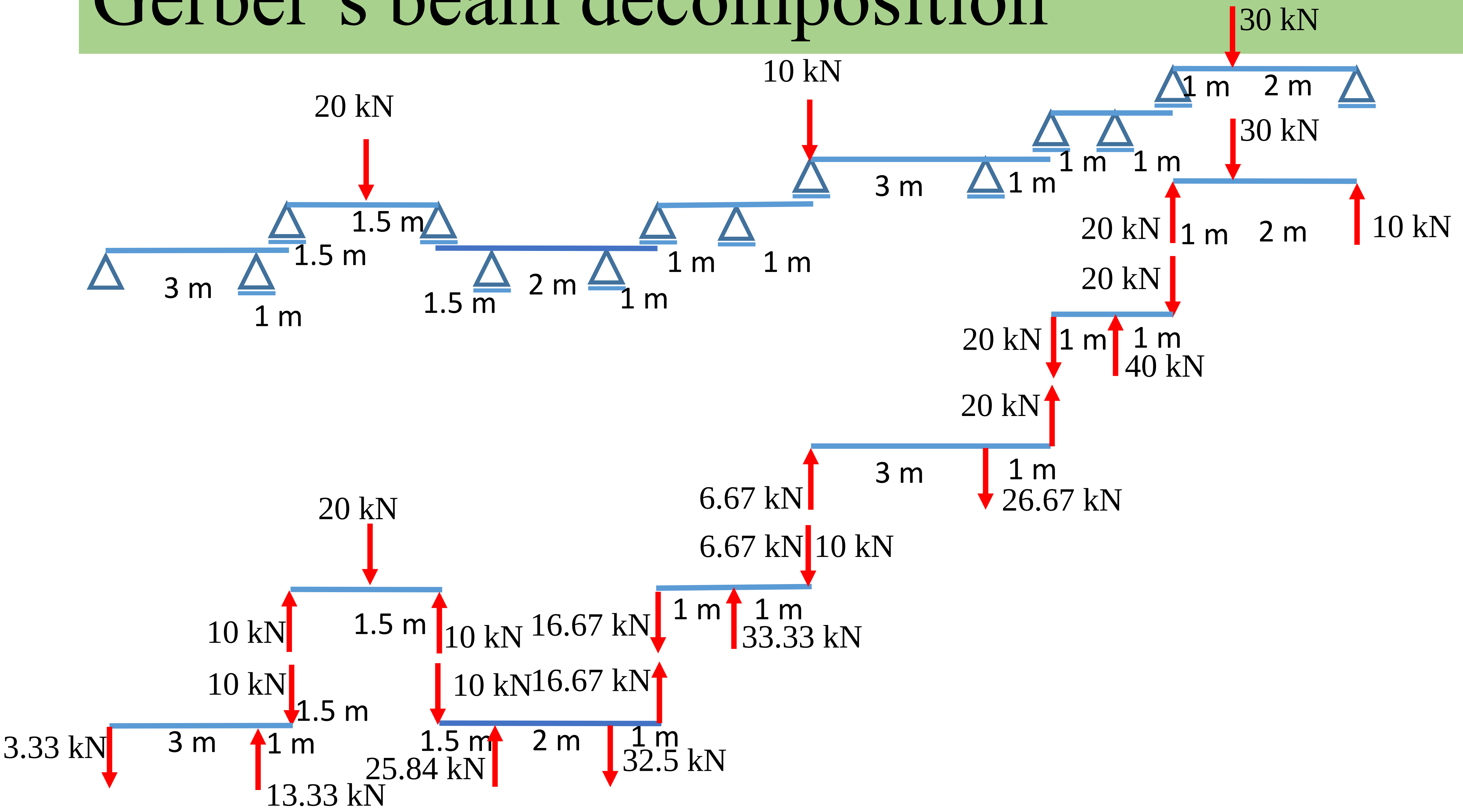




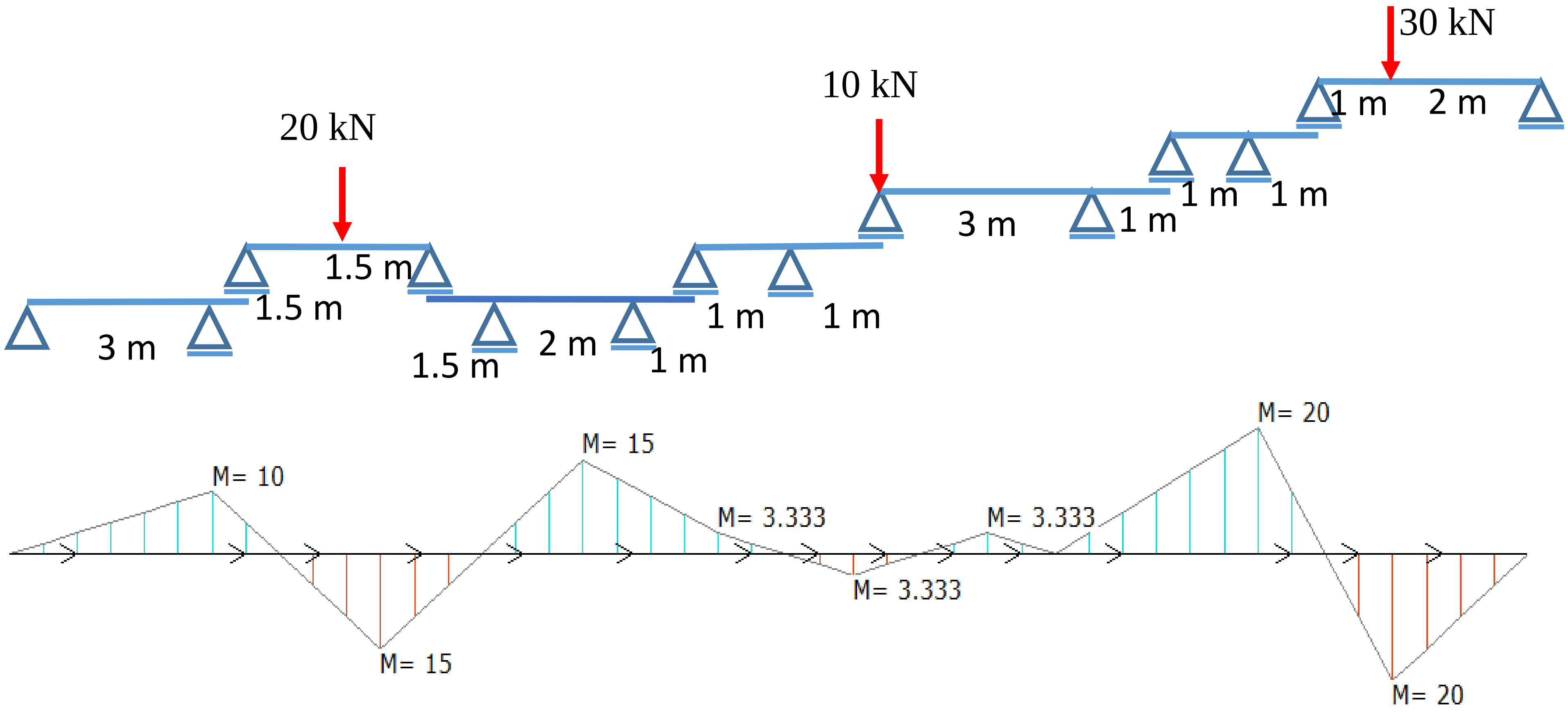
# Gerber's beams – cont.



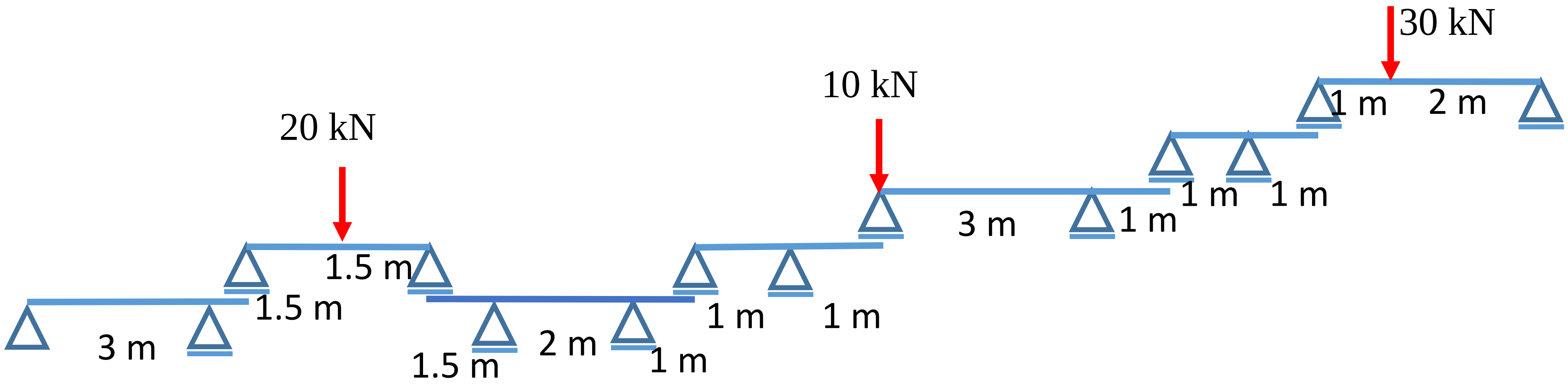
# Gerber’s beam decomposition

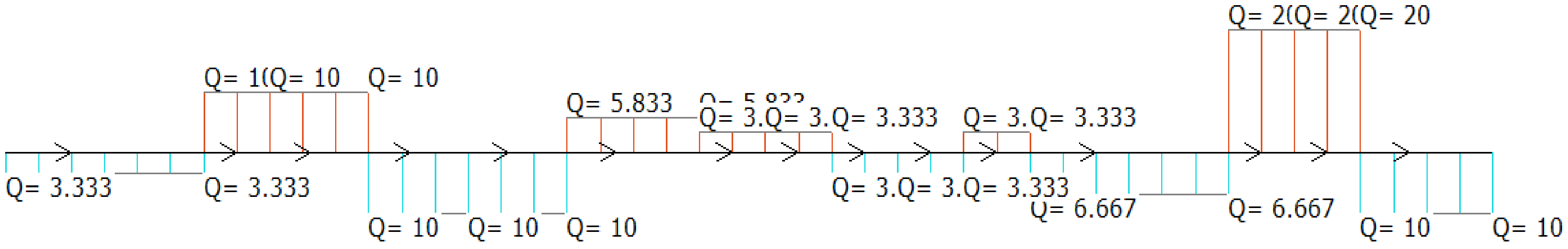


# Gerber’s beam – bending moments



# Gerber's beam – shear forces



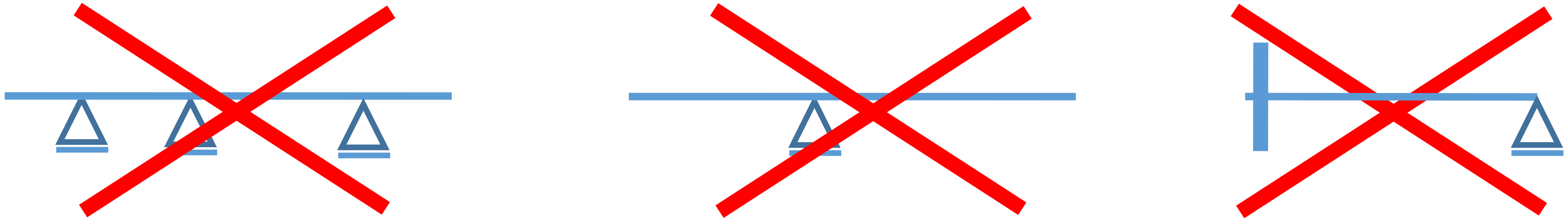


# Gerber’s beam – summary

well done

two supports or fixed end

wrong



three supports, fixed end and support (hyperstatic) or only one support (unstable)

# Thank you for your attention!