# Strength of Materials

## 7. Arches

# Definitions

arc – (geometry) a curved line
arch – a structure or structural element

history:

- a false arch (Egypt)
- the Romans were the first in the world fully to appreciate the advantages of the arch, the vault, and the dome (see many Roman aqueducts scattered all over in Southern Europe)

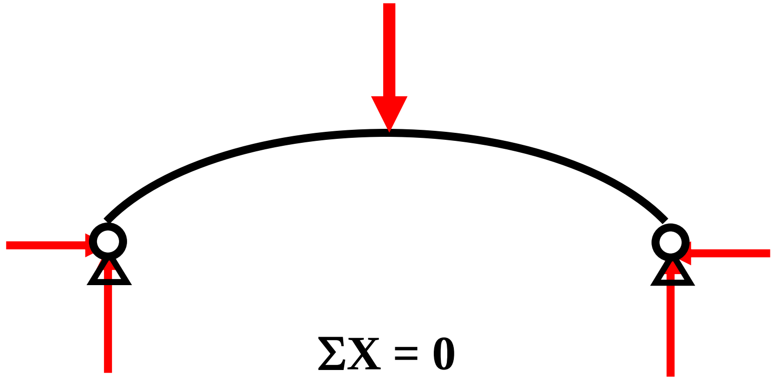
The technology of construction of an arch, its shape and calculations are rather complex. The reasons for the use of an arch are:

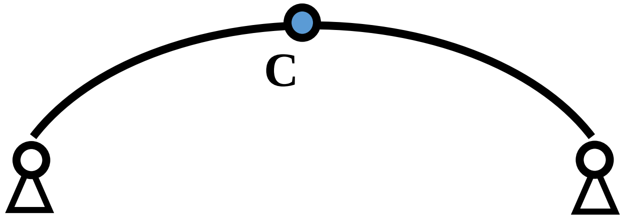
- ceramic materials (the cheapest as stone, bricks or concrete) do not sustain an extension and an arch is loaded mainly in compression
- it is the only way to cap significant area by small elements (except the wooden logs with limited application)
- this type of structure is durable and mechanically resistant

There are three type of arches:

- the fixed arch (in reinforced concrete bridge and tunnels where the spans are short (hyperstatic)
- the two-hinged arch for bridge long spans (hyperstatic)
- the three-hinged arch for medium span structures as large building roofs (statically determinate)

# Arche types



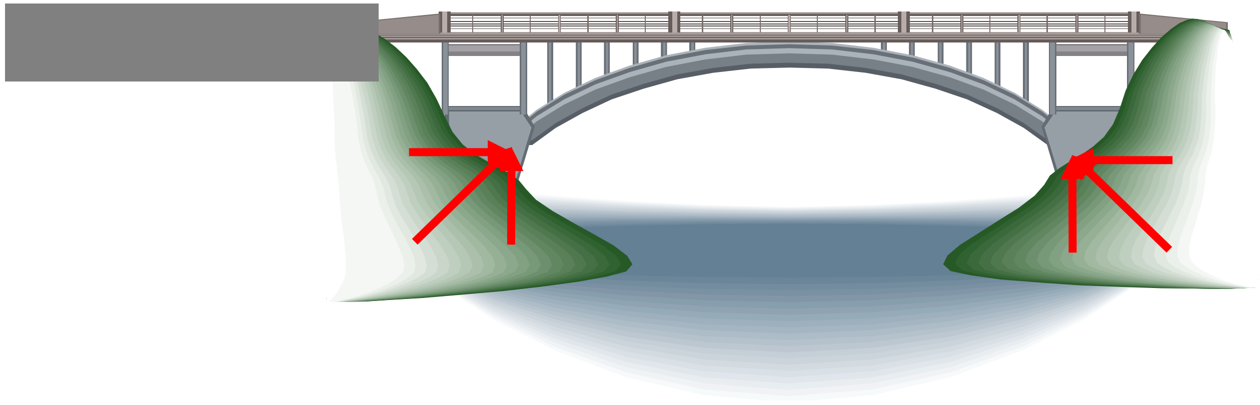



$\Sigma X = 0$

$\Sigma Y = 0$

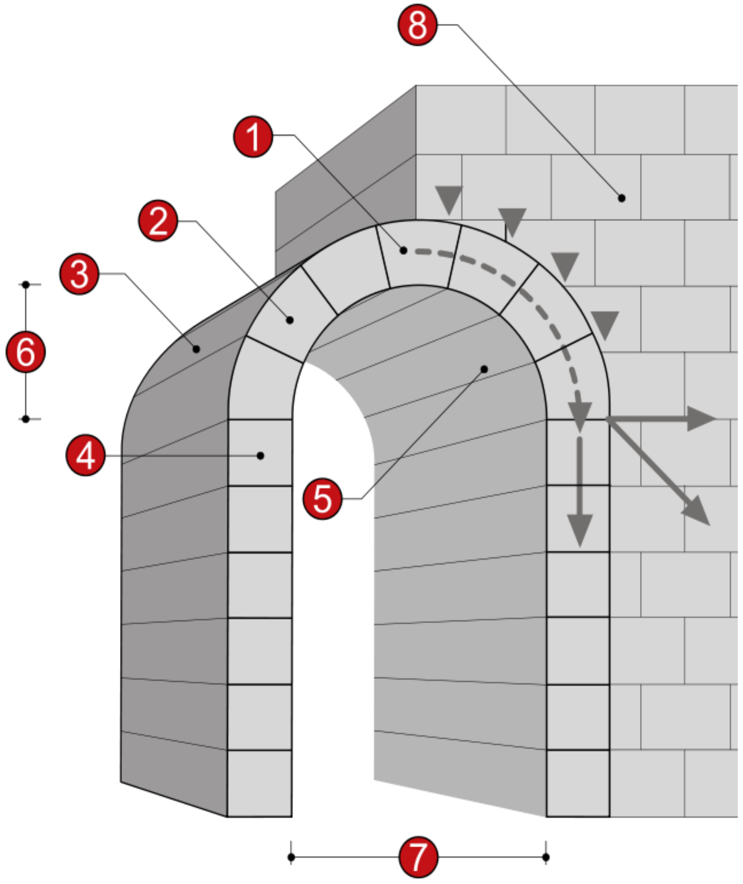
$\Sigma M_K = 0$

$M_c = 0$



thrust – horizontal reaction

# A masonry arch



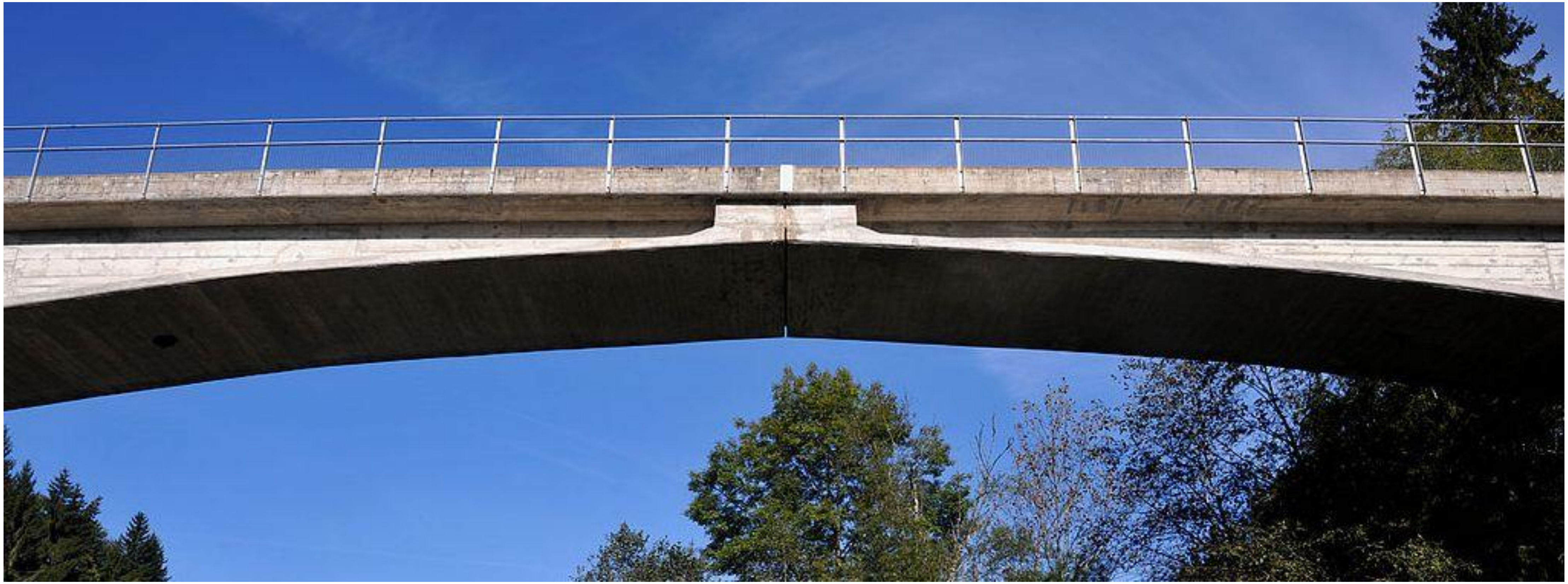

(from: Wiki)

1. Keystone (klucz) – a central stone at the summit of an arch, locking the whole together
2. Voussoir /’vu:swɑ:/ (kliniec) – a wedge-shaped or tapered stone used to construct an arch
3. Extrados /ik’streidɒs/ (grzbiet łuku) – the upper or outer curve of an arch
4. Impost (impost, zwieńczenie filaru nośnego) – the top course of a pillar that support an arch
5. Intrados (podniebienie sklepienia) – the lower or inner curve of an arch, often contrasted with extrados
6. Rise (strzałka łuku) – the vertical height of a step, arch, or incline
7. Clear span (rozpiętość w świetle) – the clear extent from end to end
8. Abutment (wezgłowie) – a structure built to support the lateral pressure of an arch

# Roman aqueducts

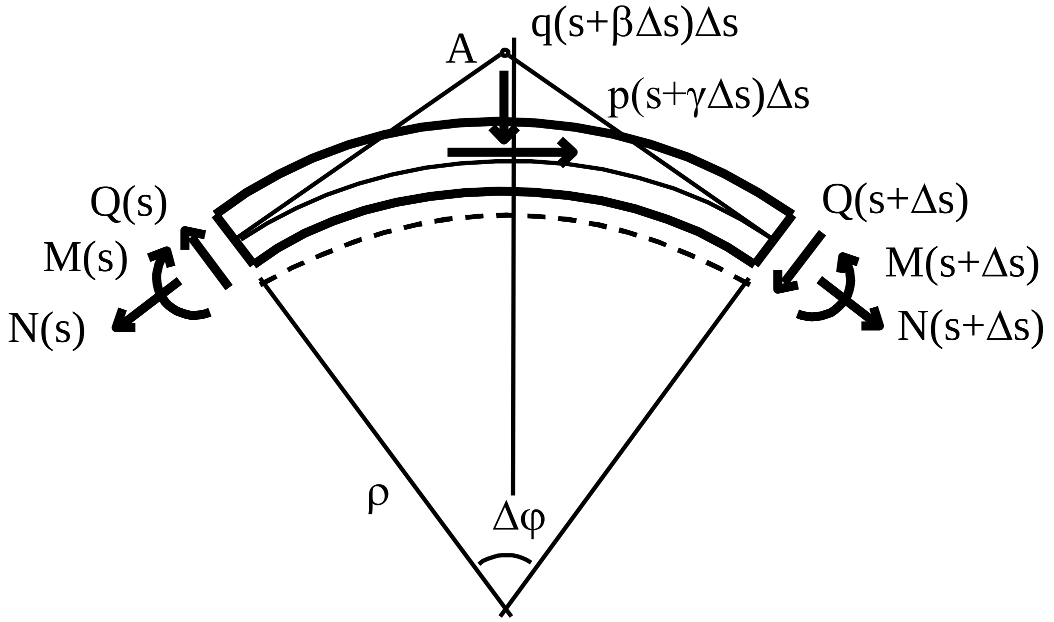
Roman aqueduct, an example of an arcade, employing the circular arch

# Three-hinged arch



Three-hinged arch in Switzerland

# Schwedler – Zhuravski theorem (generalized)




$$\cos\frac{\Delta\varphi}{2}\to 1,\quad \sin\frac{\Delta\varphi}{2}\to\frac{\Delta\varphi}{2},\quad \tan\frac{\Delta\varphi}{2}\to\frac{\Delta\varphi}{2},\quad \frac{\Delta\varphi}{2}\to\frac{\Delta s}{2\rho},$$

$$\text{for}\,\Delta s\to 0 \quad\Rightarrow\quad f(s+\alpha\Delta s)\to f(s)$$

$$p(s+\gamma\Delta s)\Delta s-\left[2Q(s)+\frac{dQ(s+\alpha_Q\Delta s)}{ds}\Delta s\right]\sin\frac{\Delta\varphi}{2}+\frac{dN(s+\alpha_N\Delta s)}{ds}\Delta s\cos\frac{\Delta\varphi}{2}=0,$$

$$-q(s+\beta\Delta s)\Delta s-\frac{dQ(s+\alpha_Q\Delta s)}{ds}\Delta s\cos\frac{\Delta\varphi}{2}-\left[2N(s)+\frac{dN(s+\alpha_N\Delta s)}{ds}\Delta s\right]\sin\frac{\Delta\varphi}{2}=0,$$

$$-\frac{dM(s+\alpha_M\Delta s)}{ds}\Delta s+\left[2Q(s)+\frac{dQ(s+\alpha_Q\Delta s)}{ds}\Delta s\right]\rho\tan\frac{\Delta\varphi}{2}-p(s+\gamma\Delta s)\Delta s\left[\frac{\rho}{\cos\frac{\Delta\varphi}{2}}-\rho\right]=0$$

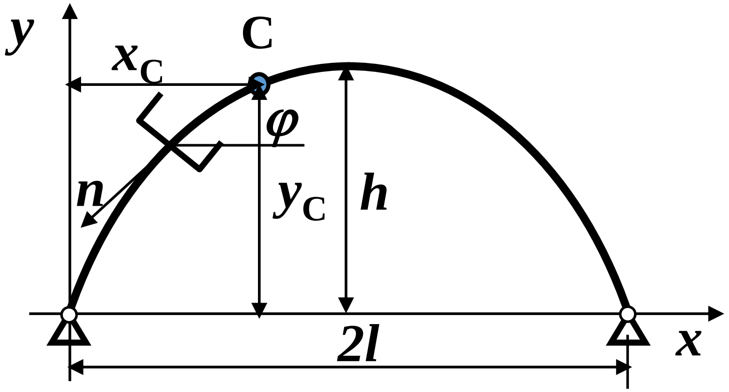
$$\frac{dM(s)}{ds}=Q(s)$$

$$\frac{dQ(s)}{ds}+\frac{N(s)}{\rho}=-q(s)$$

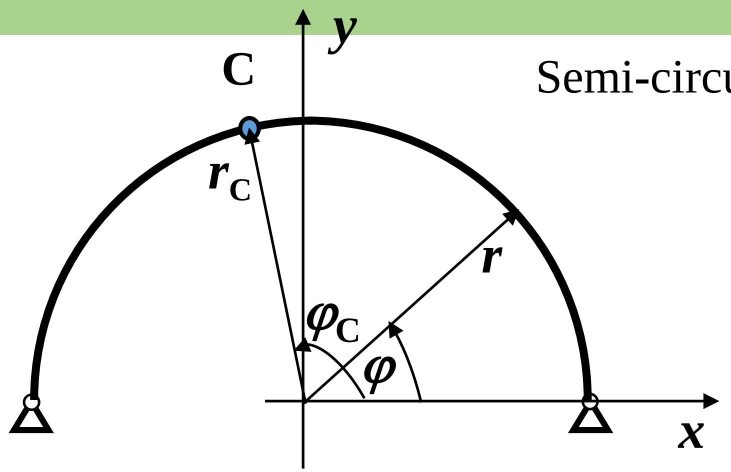
$$\frac{dN(s)}{ds}-\frac{Q(s)}{\rho}=-p(s)$$

# Arch – static calculations

Parabolic arch




Semi-circular arch




To determine reactions we only need to know position and magnitude of loads and position of the hinge and supports

$x_C$ , $y_C$

$r$, $\varphi_C$

But to determine the cross-sectional forces we do need the equation describing shape of the arch: including coordinates of any point and its tangent.

$$y = a + bx + cx^2$$

***a*, *b*, *c*** from: for $x = 0$ $y = 0$

for $x = 2l$ $y = 0$

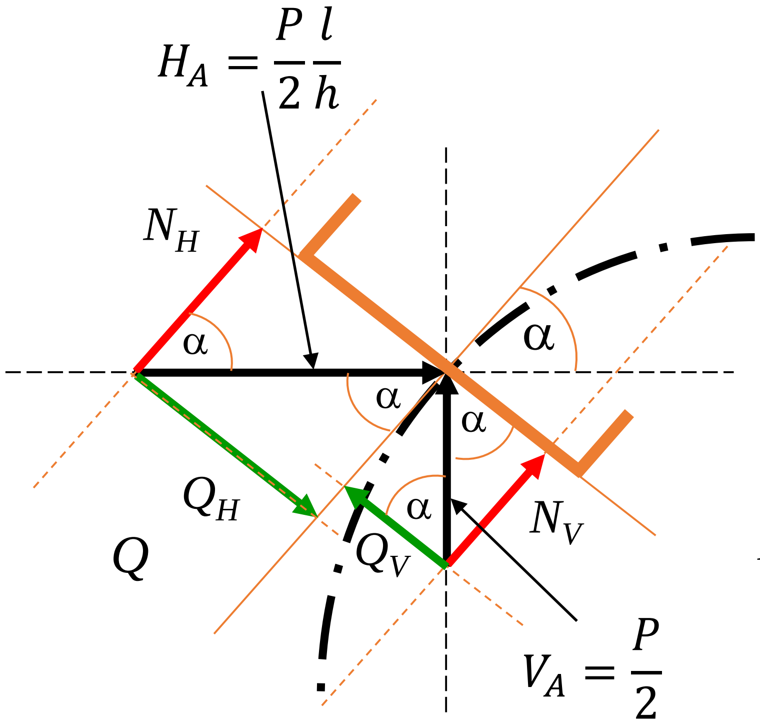
(Symetric arch) for $x = l$ $y = h$

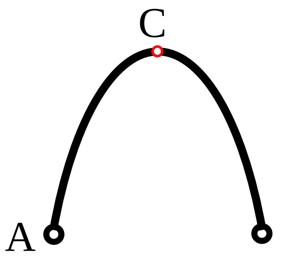
$$\varphi = \text{arctg}\, dy/dx$$

$$x = r \cdot \cos\varphi$$

(in polar coordinates)

# Three-hinged arch – the thrust




$$Q = +Q_V - Q_H = V_A \cos\alpha - H_A \sin\alpha$$

$$Q = (P/2)[\cos\alpha - (l/h)\sin\alpha]$$

$$N = -N_V - N_H = -V_A \sin\alpha - H_A \cos\alpha$$

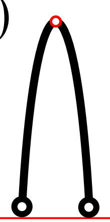
$$N = -(P/2)[\sin\alpha + (l/h)\cos\alpha]$$

**At C: $\alpha = 0$**

$$Q_C = (P/2)$$

$$N_C = -(P/2)(l/h)$$

**At A: $\alpha \neq 0$**

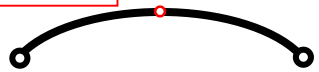


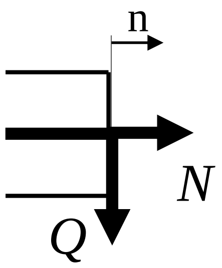
For (l/h)<<1 – steep arch

$$\alpha \cong \pi/2 \Rightarrow \sin\alpha \cong 1, \cos\alpha = 0$$

$$Q_A = (P/2)[0 - (l/h)] \rightarrow (P/2)[0 - 0] = 0$$

$$N_A = -(P/2)[1 + 0] = -(P/2)$$





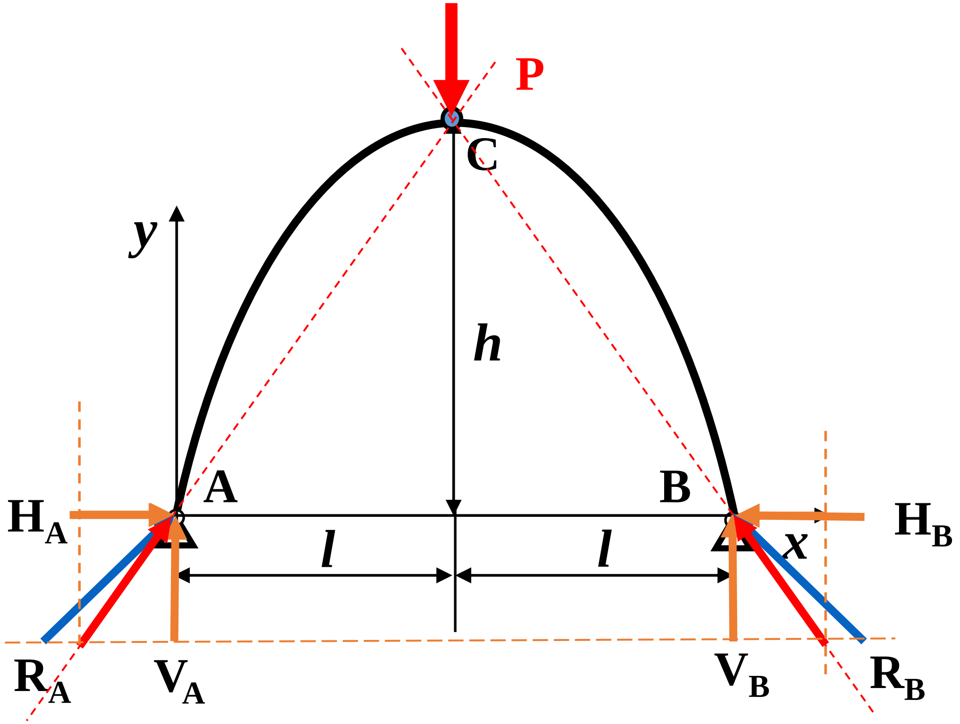

For (l/h)>>1 – shallow arch

$$tg\alpha = \sin\alpha \cong (h/l) \cong 0 \quad \Rightarrow l/h \rightarrow \infty \qquad \cos\alpha \cong 1$$

$$Q_A = (P/2)[1 - (l/h)\sin\alpha] = (P/2)[1 - (1/tg\alpha)\sin\alpha] = (P/2)[1 - 0] = P/2$$

$$N_A = -(P/2)[0 + (l/h)] \quad \Rightarrow N_A \rightarrow -\infty$$

# Example – parabolic arch under point force




$$\Sigma M_A = 0 \rightarrow Pl - V_B \cdot 2l = 0 \rightarrow V_B = \frac{P}{2}$$

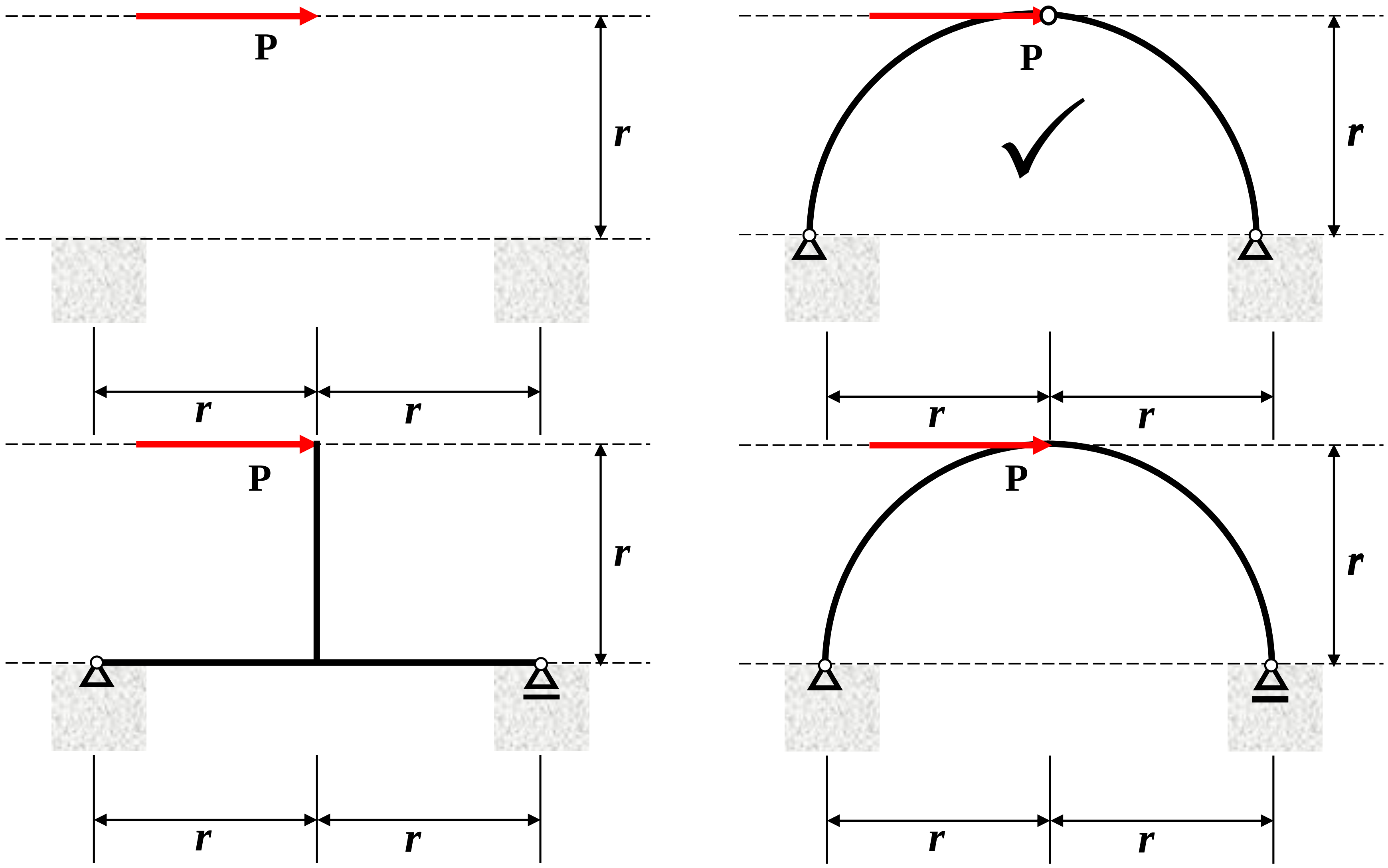
$$\Sigma Y = 0 \rightarrow V_B = V_A = \frac{P}{2}$$

$$\Sigma M_C^L = \Sigma \mathrm{M}_\mathrm{C}^\mathrm{R} = 0 \rightarrow \frac{\mathrm{H_A}}{\mathrm{V_A}} = \frac{\mathrm{H_B}}{\mathrm{V_B}} = \frac{\mathrm{l}}{\mathrm{h}}$$

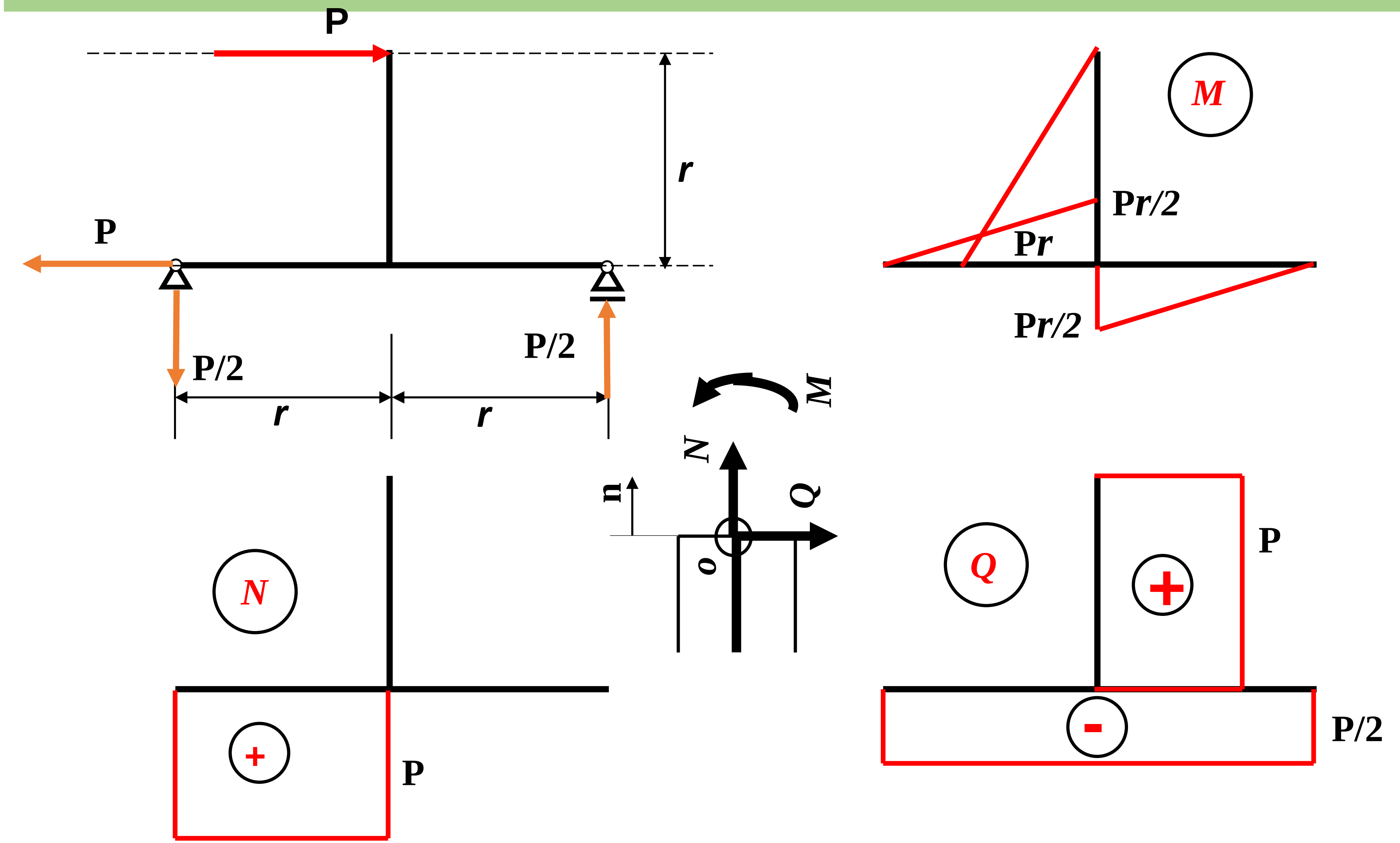
$$\Sigma X = 0 \rightarrow \boxed{H_A = H_B = V_A \frac{l}{h}}$$

the smaller the ratio $\frac{h}{l}$, the higher the value of the horizontal reaction H; the reaction is called a thrust

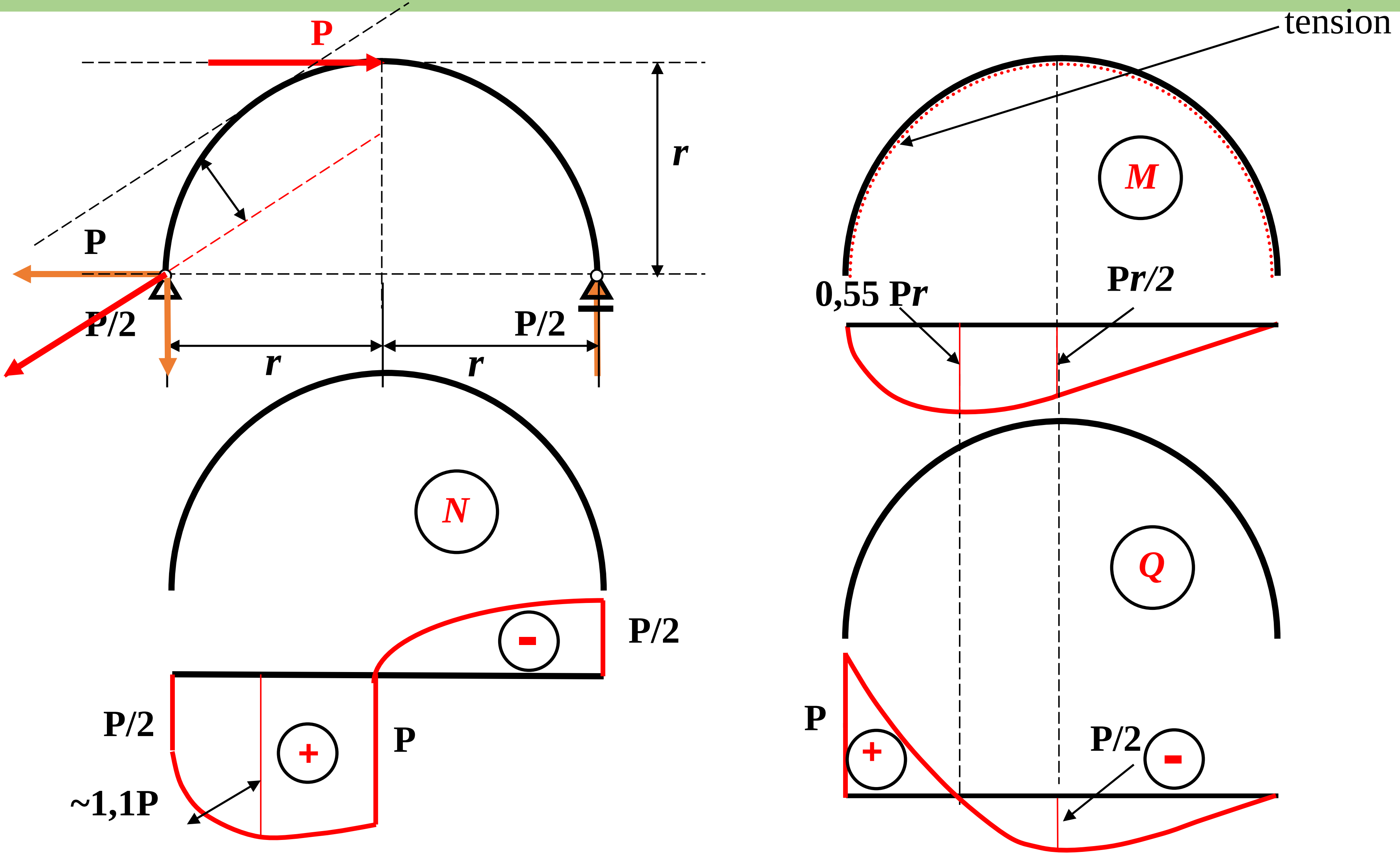
# Comparison of frame, quasi-arch and arch



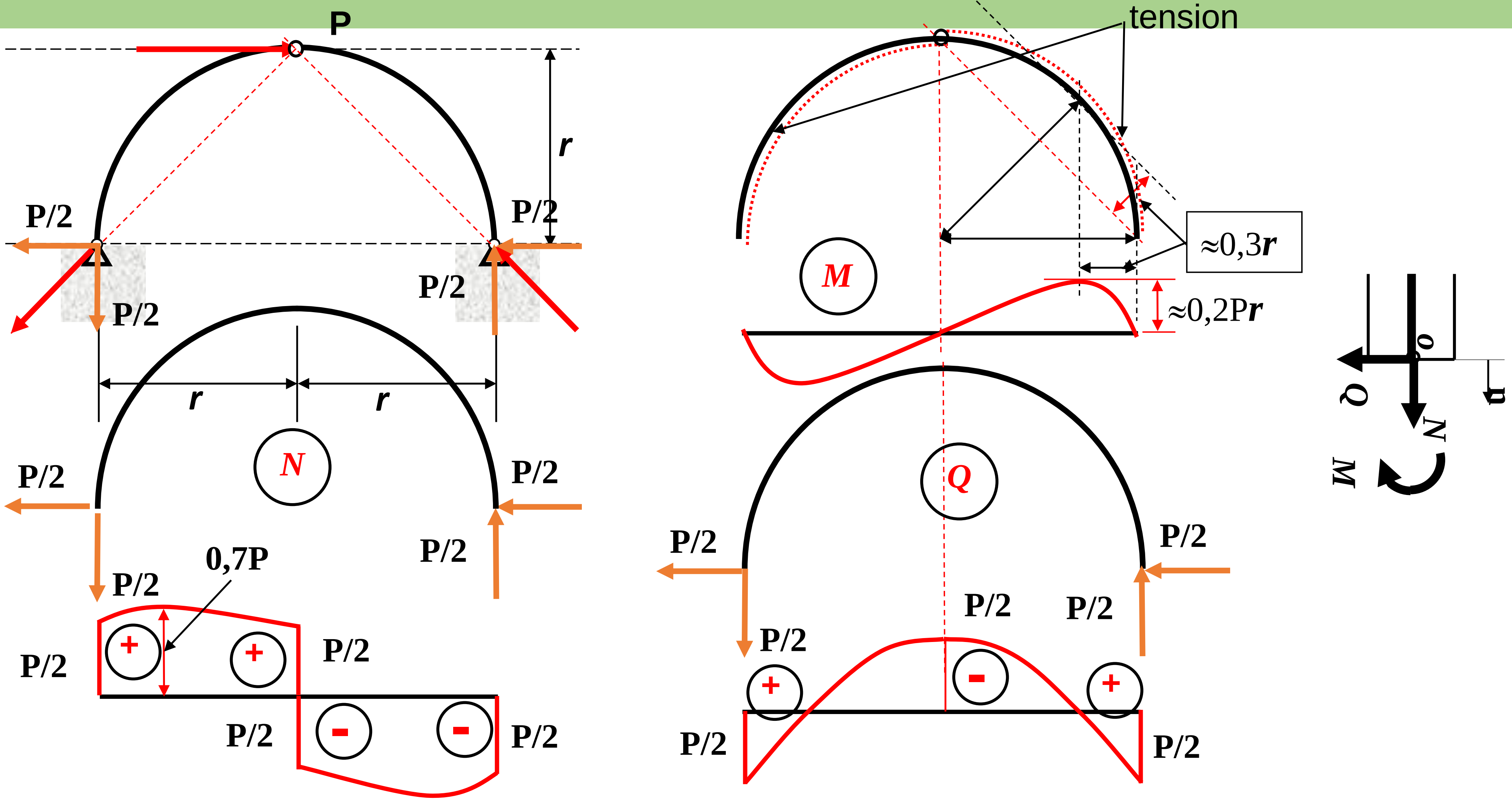
# Comparison: frame



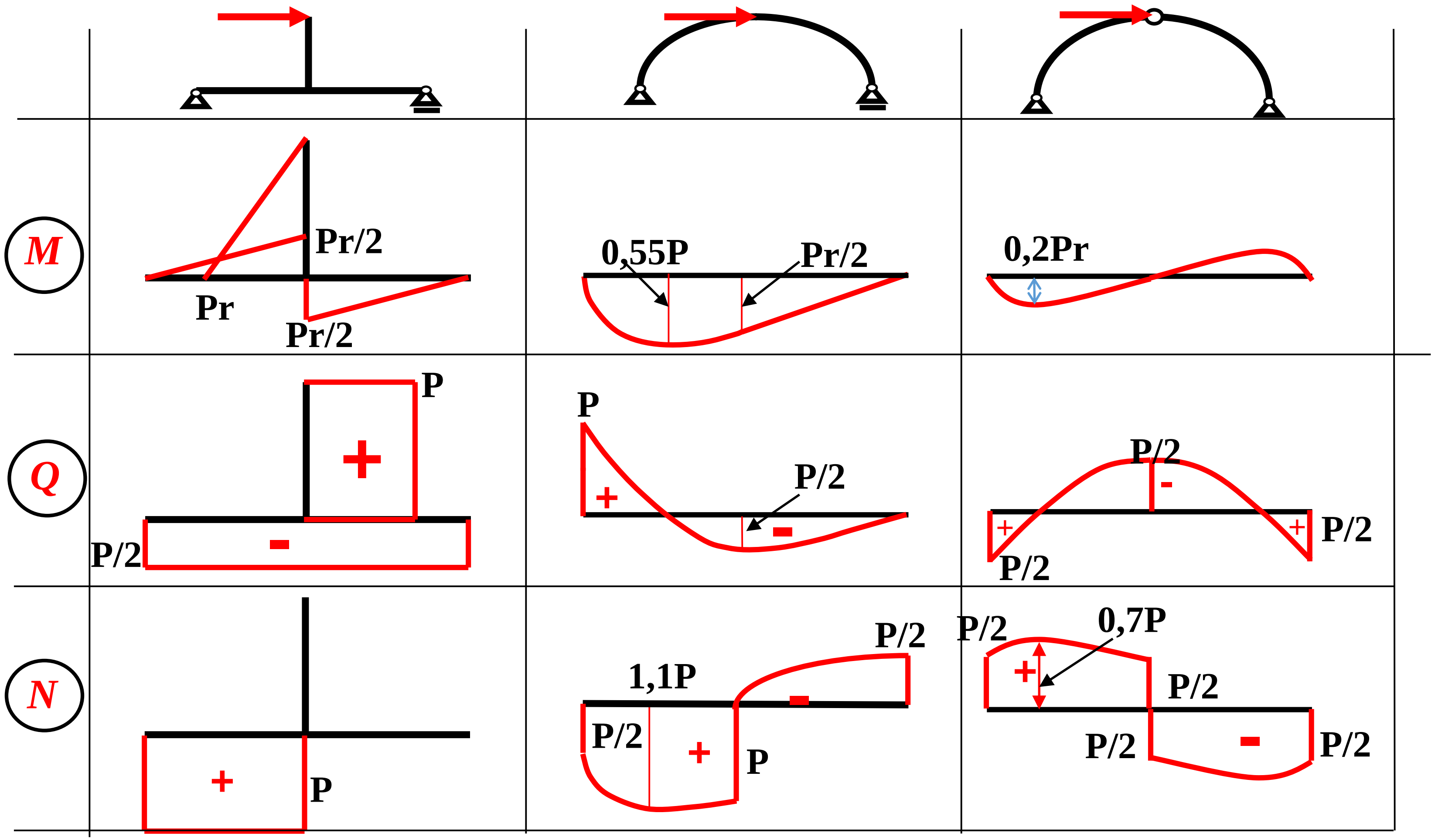
# Comparison: quasi-arch



# Ex.– semi-circular arch under horizontal force

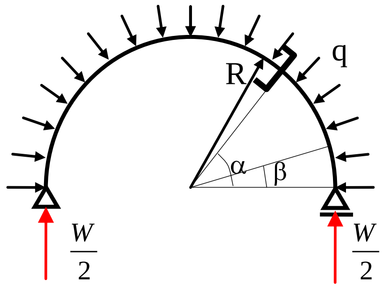


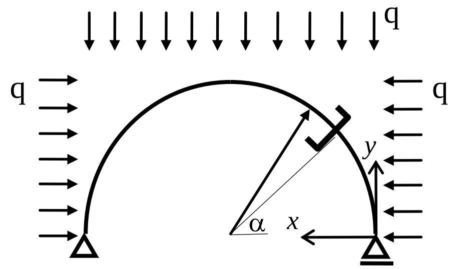
# Comparison - summary



# Arch – rational axis

Example of rational axis of an arch: semi-circular arch under radial loading.




Vertical resultant reaction:

$$W = 2\int_0^{\pi/2} qR\sin\alpha \, d\alpha = 2qR(-\cos\alpha)\Big|_0^{\frac{\pi}{2}} = 2qR(0-1) = 2qR$$

Cross-section force at $\alpha - \alpha$ section

cross-section:

bending moment: $$M(a) = \frac{W}{2}R(1-\cos\alpha) - \int_0^{\alpha} qR d\beta\left[\cos\beta(\sin\alpha - \sin\beta) + \sin\beta R(\cos\beta - \cos\alpha)\right] = \ldots$$

$$\ldots = qR^2(1-\cos\alpha) - qR^2\sin\alpha\sin\alpha - qR^2\cos\alpha(\cos\alpha - 1) = \ldots = 0$$

The same (using a different method): $$M(\alpha) = qR^2(1-\cos\alpha) - \frac{qy^2}{2} - \frac{qx^2}{2} = qR^2(1-\cos\alpha) - \frac{qR^2}{2}\sin^2\alpha - \frac{qR^2}{2}(1-\cos\alpha)^2 = \ldots$$

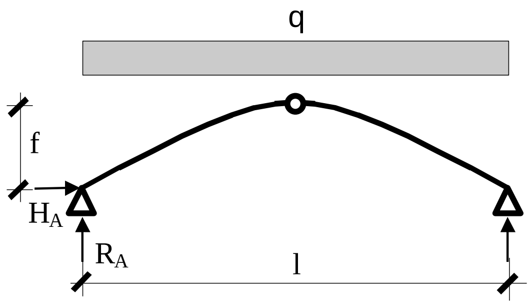
$$\ldots = qR^2(1-\cos\alpha) - qR^2 + qR^2\cos\alpha = \ldots = 0$$

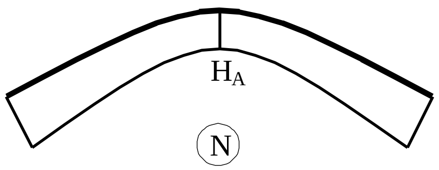
shear force: $Q(\alpha) = -qR\sin\alpha + qx\sin\alpha + qy\cos\alpha = -qR\sin\alpha + qR\sin\alpha(1-\cos\alpha) + qR\sin\alpha\cos\alpha = \ldots = 0$

axial force: $N(x) = -qR\cos\alpha + qx\cos\alpha - qy\sin\alpha = -qR\cos\alpha + qR(1-\cos\alpha)\cos\alpha - qR\sin^2\alpha = \ldots = -qR = \text{const}$

# Rational axis - continued

Example of rational axis of an arch: Parabolic 3-hinges arch under constant continuous loading




We find the equation of arch axis in the form of 2nd order parabola with height *f* and span *l* and calculate reactions for this 3-hinges system:

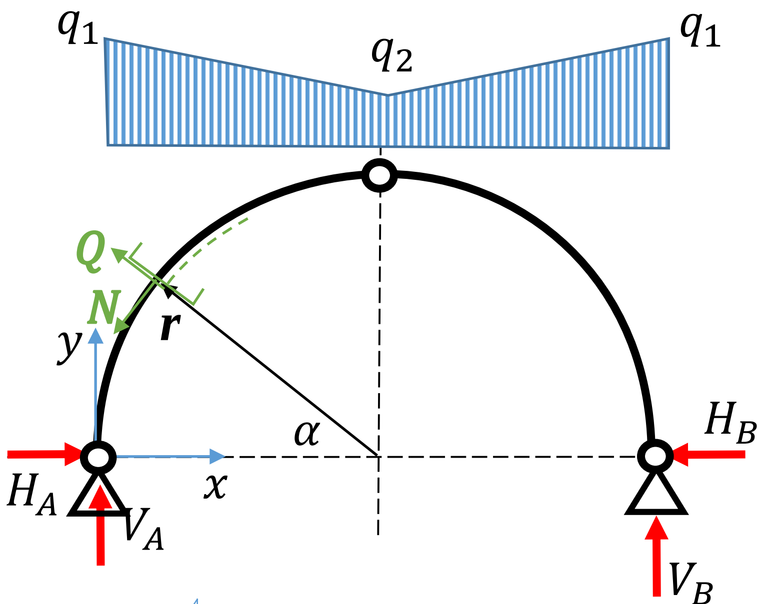
$$y = -\frac{4f}{l^2}\left(x^2 - lx\right), \quad R_A = \frac{ql}{2}, \quad H_A = \frac{ql^2}{8f}$$

The bending moment equation reads:

$$M(x) = R_A x - H_A y - \frac{qx^2}{2} = \frac{ql}{2}x - \frac{ql^2}{8f}\left[-\frac{4f}{l^2}\left(x^2 - lx\right)\right] - \frac{qx^2}{2} = \ldots = 0$$

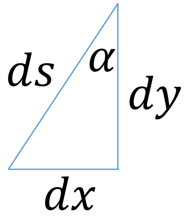
From the bending moment identically equal to zero we see that the shear force is identically equal to zero and the arch works in compression only. So the arch axis is rational.

# Example – circular arch




$$x = r(1 - \cos\alpha)$$
$$y = r\sin\alpha$$

$$V_A = V_B = \frac{q_1+q_2}{2} r$$
$$H_A r + q_2 r \cdot \frac{r}{2} + \frac{q_1-q_2}{2} r \cdot \frac{2}{3} r - \frac{q_1+q_2}{2} r^2 = 0 \rightarrow$$
$$H_A = \frac{q_1+2q_2}{6} r$$

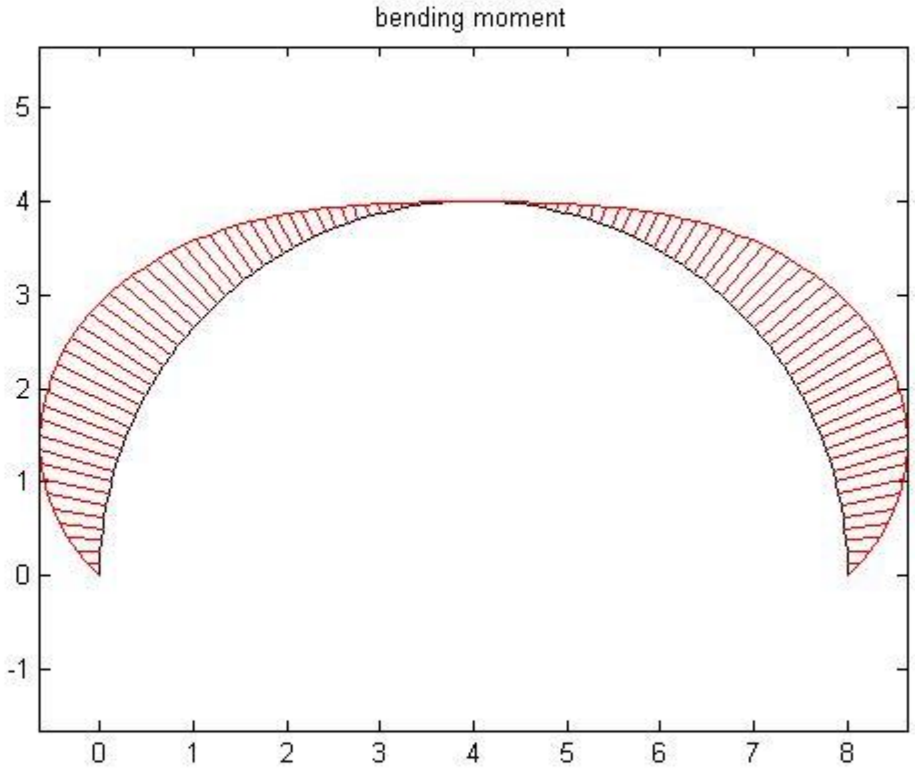


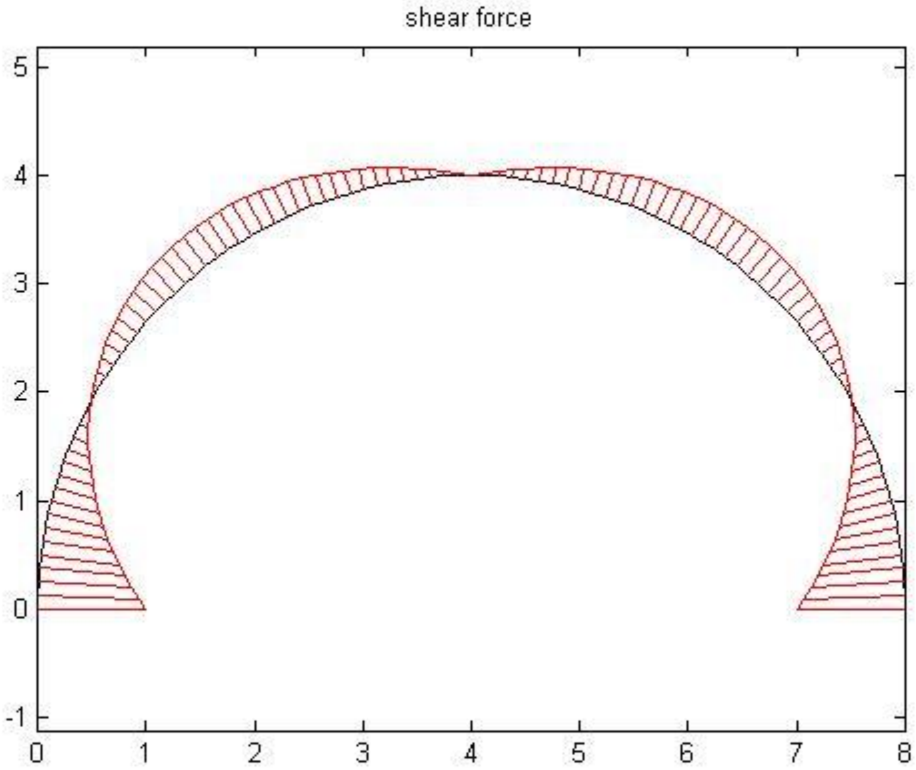

$$M(x) = V_A x - H_A y - \frac{q_1 x^2}{2} + \frac{q_1-q_2}{r} \cdot \frac{x^3}{6}$$
$$Q(x) = \left(V_A - q_1 x + \frac{q_1-q_2}{r} \cdot \frac{x^2}{2}\right) \sin\alpha - H_A \cos\alpha$$
$$N(x) = \left(-V_A + q_1 x - \frac{q_1-q_2}{r} \cdot \frac{x^2}{2}\right) \cos\alpha - H_A \sin\alpha$$

$$\frac{dM}{ds} = \frac{\partial M}{\partial x} \cdot \frac{dx}{ds} = \frac{\partial M}{\partial x} \cdot \sin\alpha = \frac{\partial M}{\partial y} \cdot \cos\alpha =$$
$$= \left(V_A - q_1 x + \frac{q_1 - q_2}{r} \cdot \frac{x^2}{2}\right) \sin\alpha - H_A \cos\alpha$$

$r = 4$ m, $q_1 = 15$ kN/m, $q_2 = 10$ kN/m

# Circular arch – bending moments & shear forces

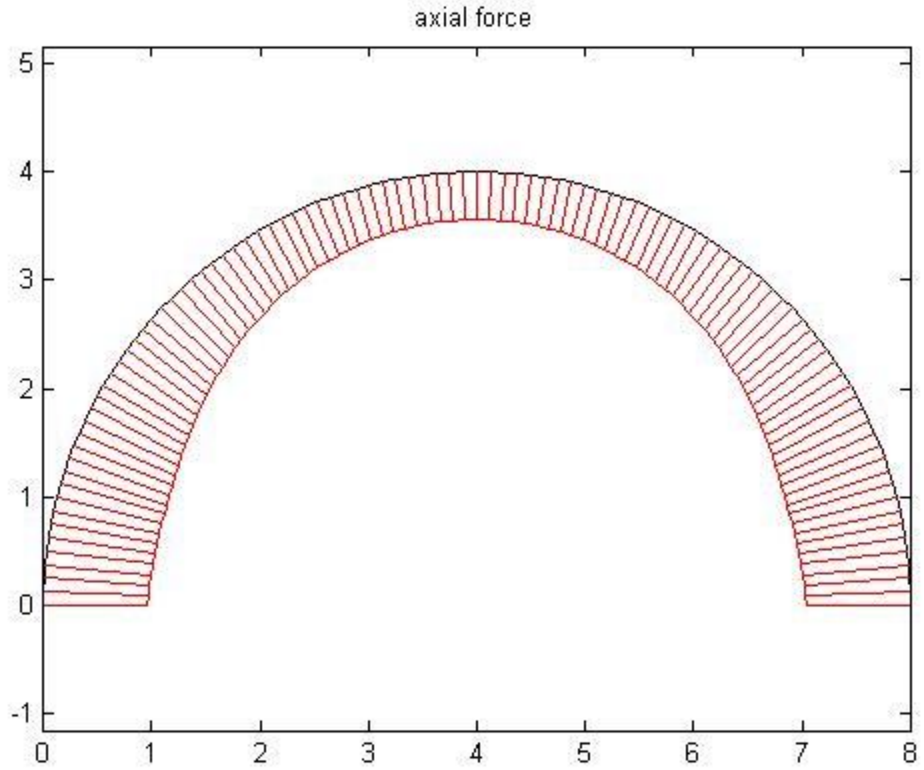


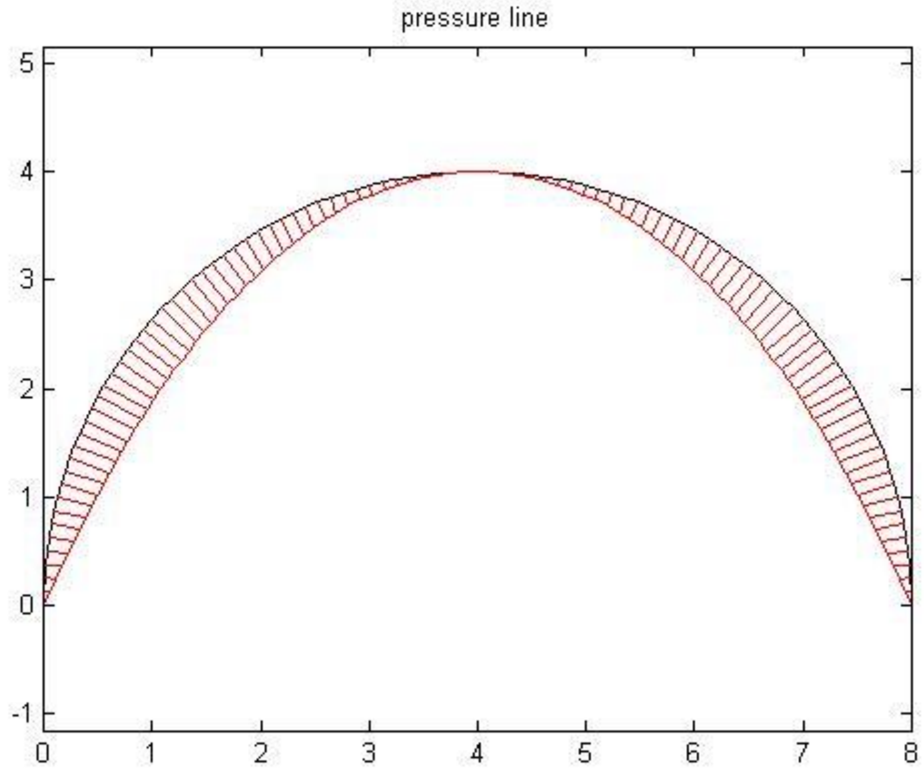




$M_{max} = 22.04$ kNm at $\alpha \approx 28.8°$

$Q_{max} = 23.33$ kN for $\alpha = 0$

# Circular arch – axial forces and pressure line



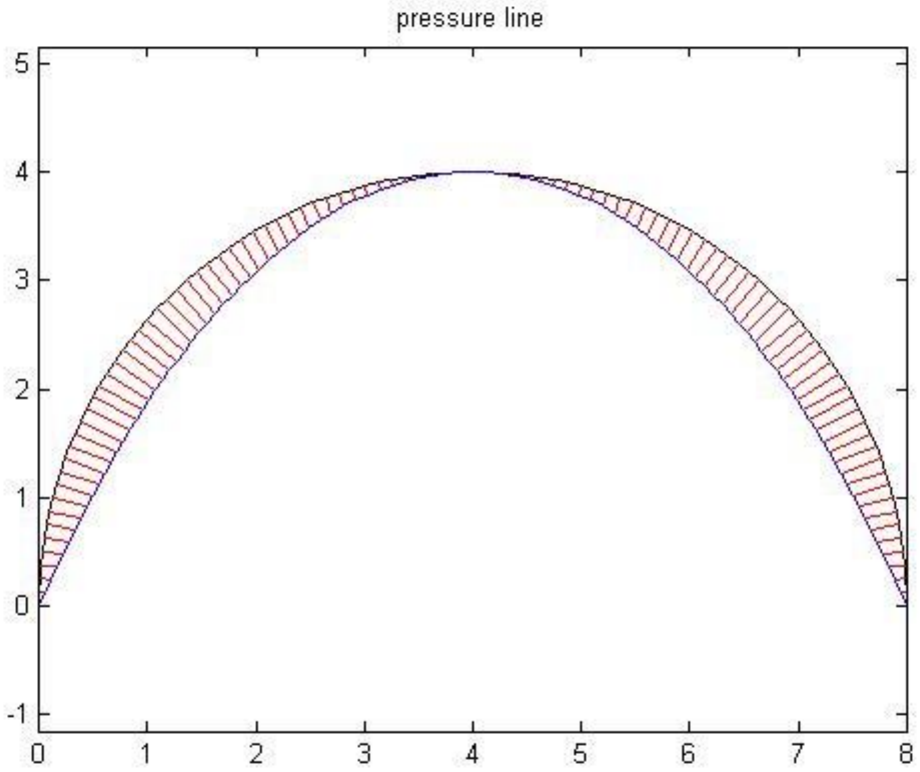

$N_{max} = -52.48$ kN at $\alpha = 12.6°$

$d_{r_{max}} = 0.4638 = 0.116r$

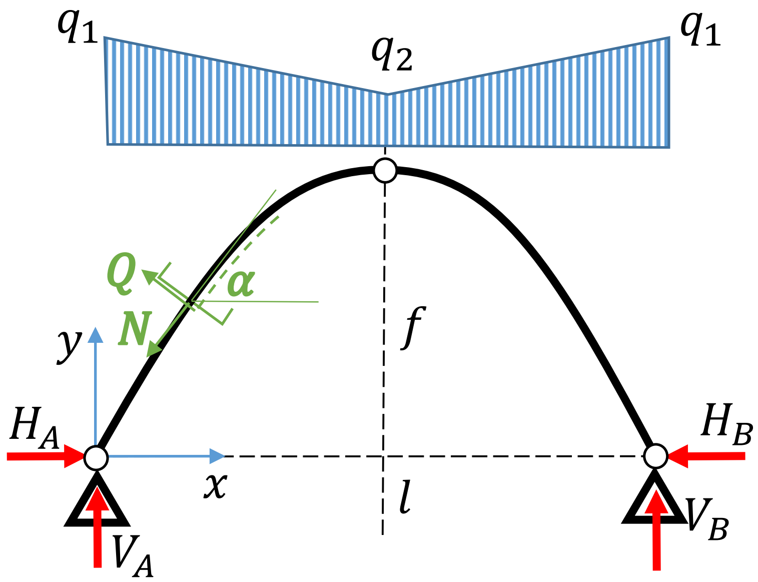
# Circular arch – rational axis

$$M(x) = V_A x - H_A y - \frac{q_1 x^2}{2} + \frac{q_! - q_2}{r} \cdot \frac{x^3}{6}$$

$$M(x) = 0 \rightarrow y = \frac{V_A x - \frac{q_1 x^2}{2} + \frac{q_1 - q_2}{R} \cdot \frac{x^3}{6}}{H_A}$$



# Example – parabolic arch




arch axis equation:

$$y(x) = -\frac{4f}{l^2}(x^2 - lx)$$

$$\tan\alpha = y'(x) = -\frac{4f}{l^2}(2x - l)$$

$$\cos\alpha = \frac{1}{\sqrt{1+(\tan\alpha)^2}}, \ \sin\alpha = \tan\alpha \cdot \cos\alpha$$

$$V_A = V_B = \frac{q_1+q_2}{2}\cdot\frac{l}{2}$$

$$H_A f + q_2 \cdot \frac{l}{2}\cdot\frac{l}{4} + \frac{q_1-q_2}{2}\cdot\frac{l}{2}\cdot\frac{2}{3}\cdot\frac{l}{2} - V_A\cdot\frac{l}{2} = 0 \rightarrow$$

$$H_A = H_B = \frac{l^2}{24f}(q_1 + 2q_2)$$

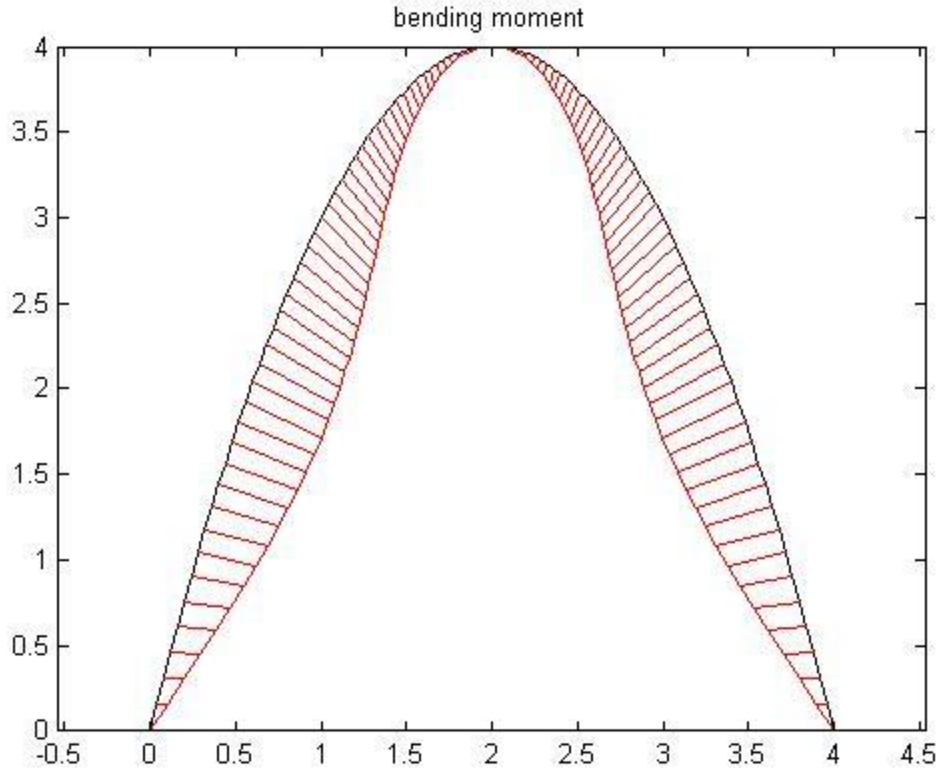
$$M(x) = V_A x - q_1\cdot\frac{x^2}{2} + \frac{q_1-q_2}{\frac{l}{2}}\cdot\frac{x^3}{6} - H_A y$$

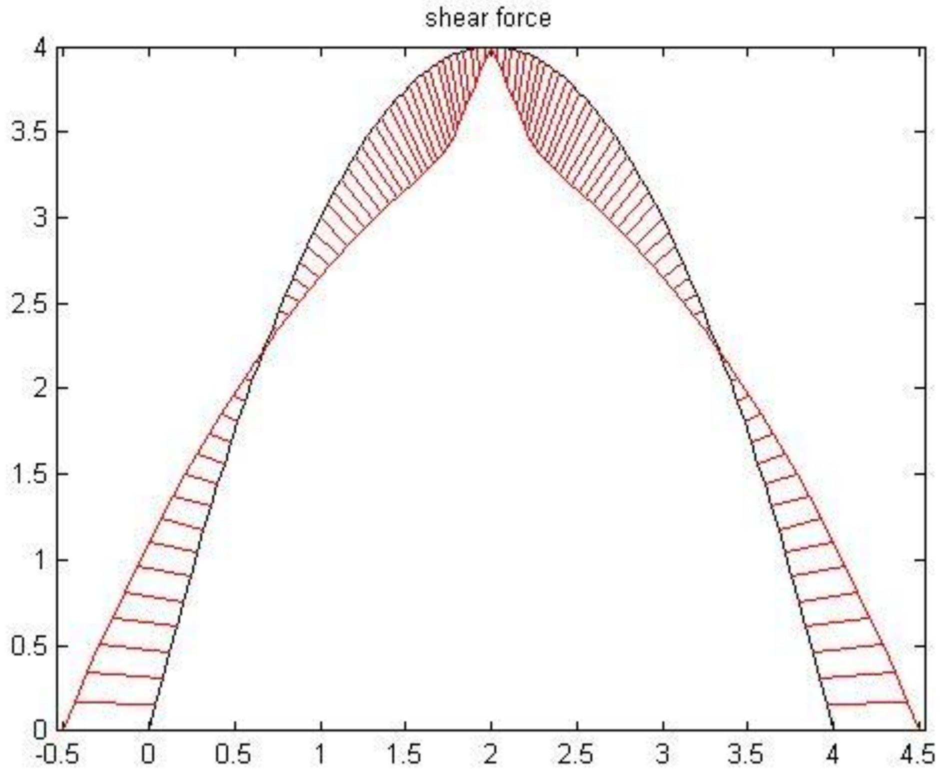
$$Q(x) = \left(V_A - q_1 x + \frac{q_1-q_2}{\frac{l}{2}}\cdot\frac{x^2}{2}\right)\cos\alpha - H_A\sin\alpha$$

$$N(x) = \left(V_A - q_1 x + \frac{q_1-q_2}{\frac{l}{2}}\cdot\frac{x^2}{2}\right)\sin\alpha - H_A\cos\alpha$$

$l = f = 4$ m, $q_1 = 15$ kN/m, $q_2 = 10$ kN/m

# Parabolic arch – moments & shear forces

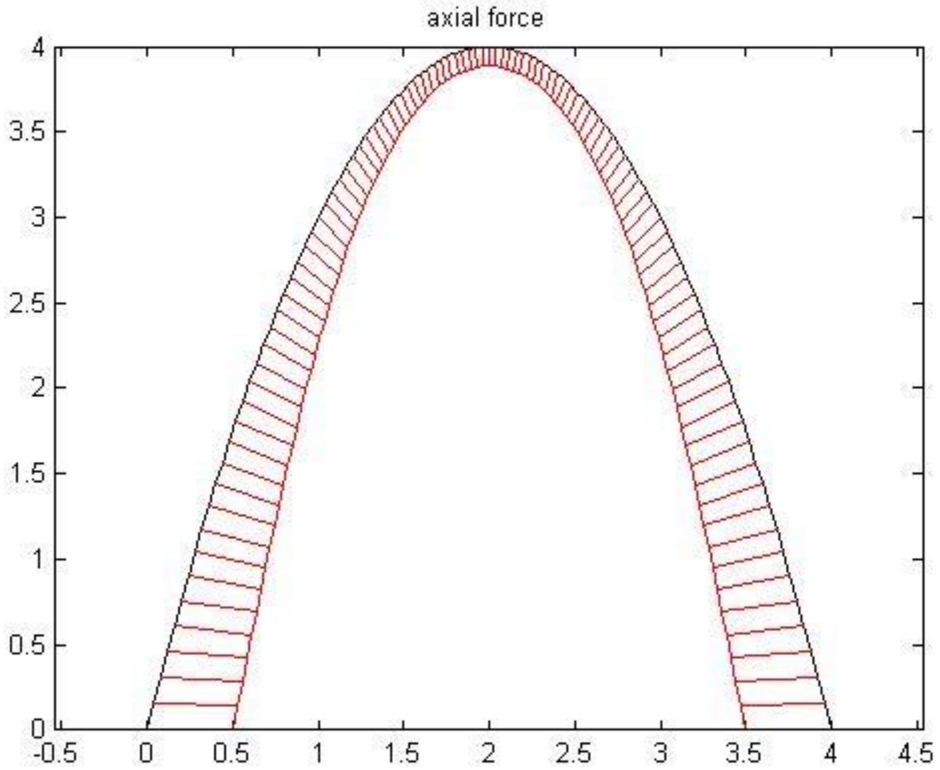


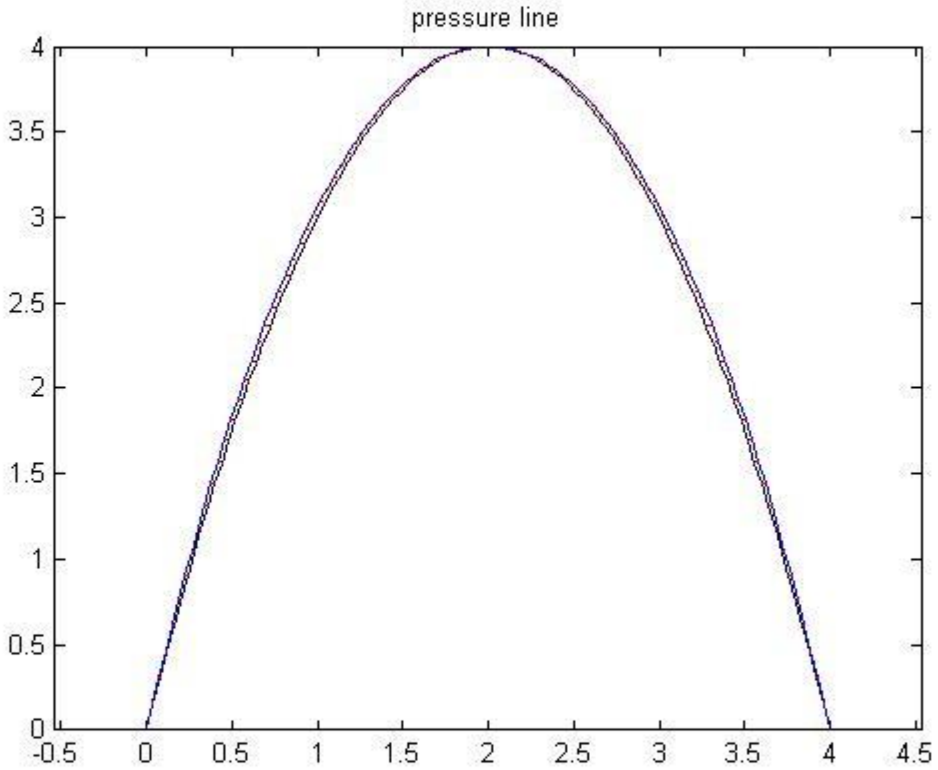




$M_{max} = 0.4937$ kNm at $x = 0.68$ m

$Q_{max} = 0.4042$ kN at $x = 0$

# Parabolic arch – axial forces & pressure line

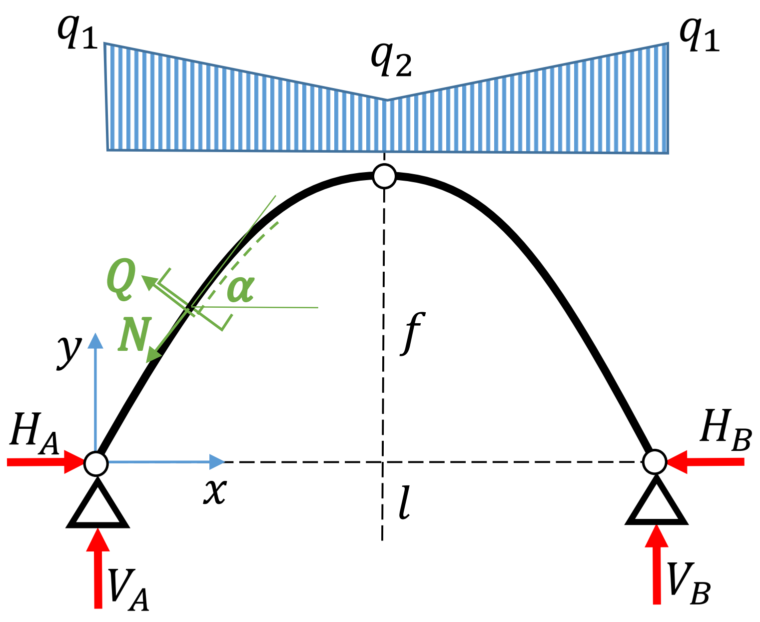



$N_{max} = -25.67$ kN at $x = 0$

$d_{max} = 0.033$ m at $x = 1$ m

# Example – sinusoidal arch




arch axis equation:

$$y(x) = f \sin\frac{\pi x}{l}$$

$$\tan\alpha = y' = \frac{f\pi}{l}\cos\frac{\pi x}{l}$$

$$\cos\alpha = \frac{1}{\sqrt{1+(\tan\alpha)^2}},\ \sin\alpha = \tan\alpha \cdot \cos\alpha$$

$$V_A = V_B = \frac{q_1+q_2}{2}\cdot\frac{l}{2}$$

$$H_A f + q_2\cdot\frac{l}{2}\cdot\frac{l}{4} + \frac{q_1-q_2}{2}\cdot\frac{l}{2}\cdot\frac{2}{3}\cdot\frac{l}{2} - V_A\cdot\frac{l}{2} = 0 \rightarrow$$

$$H_A = H_B = \frac{l^2}{24f}(q_1 + 2q_2)$$

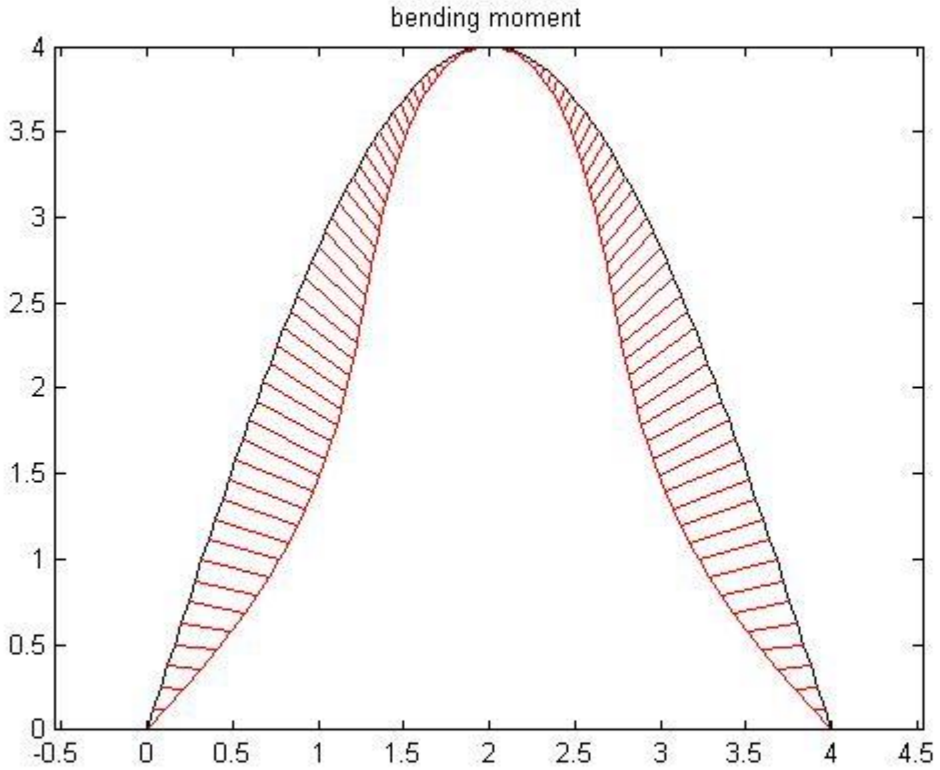
$$M(x) = V_A x - q_1\cdot\frac{x^2}{2} + \frac{q_1-q_2}{\frac{l}{2}}\cdot\frac{x^3}{6} - H_A y$$

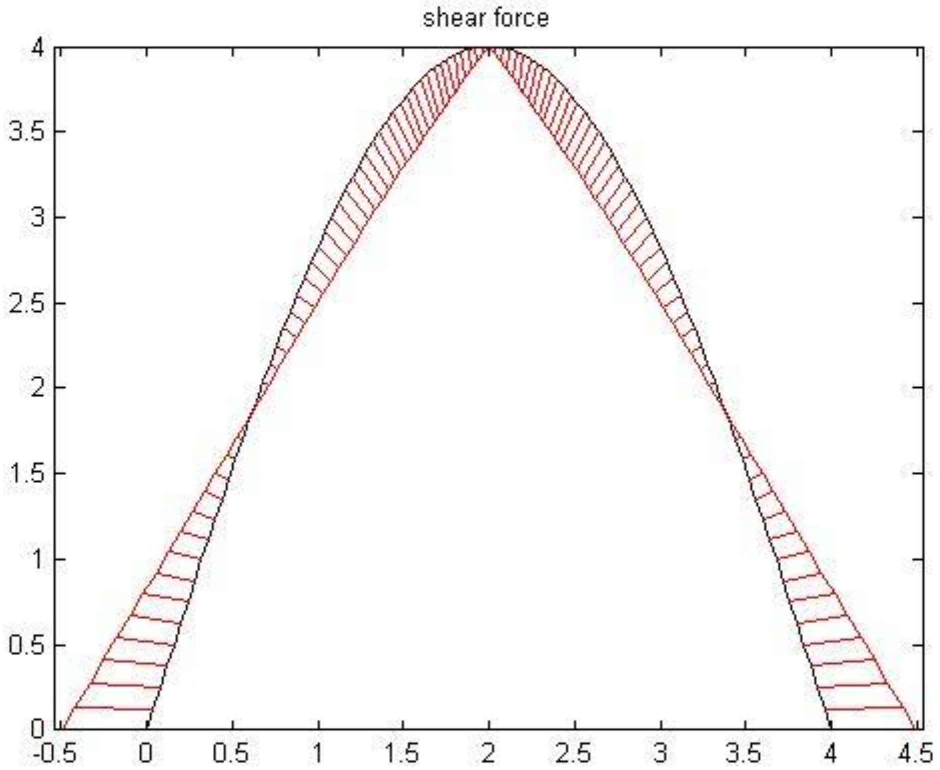
$$Q(x) = \left(V_A - q_1 x + \frac{q_1-q_2}{\frac{l}{2}}\cdot\frac{x^2}{2}\right)\cos\alpha - H_A\sin\alpha$$

$$N(x) = \left(V_A - q_1 x + \frac{q_1-q_2}{\frac{l}{2}}\cdot\frac{x^2}{2}\right)\sin\alpha - H_A\cos\alpha$$

$l = f = 4$ m, $q_1 = 15$ kN/m, $q_2 = 10$ kN/m

# Sinusoidal arch – moments & shear forces

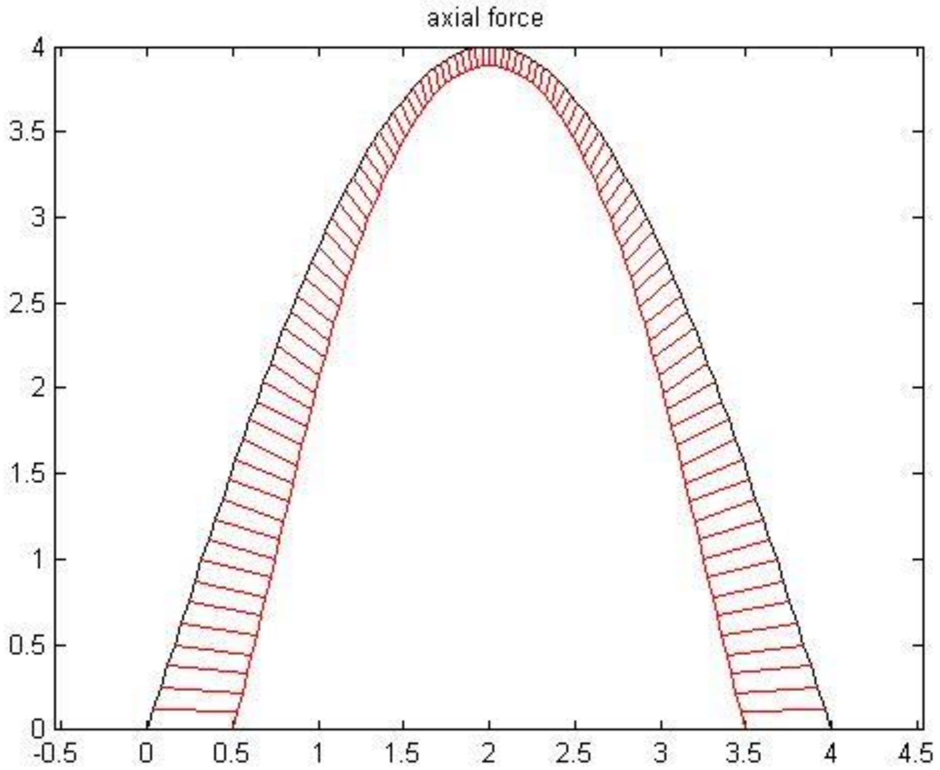


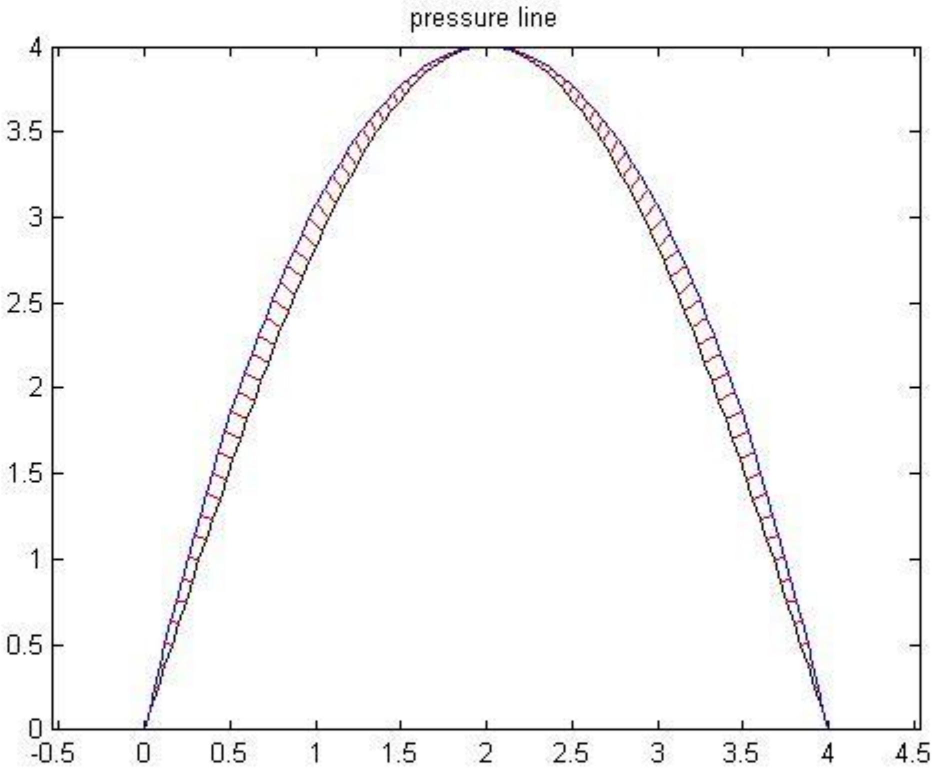




$M_{max} = 1.80$ kNm at $x = 0.6$ m

$Q_{max} = 2.02$ kN at $x = 0$

# Sinusoidal arch – axial forces & pressure line




$N_{max} = -25.6$ kN at $x = 0$

$d_{max} = 0.1143$ m at $x = 0.88$ m

# Thank you for your attention!