Strength of Materials

9. Strain state

The lowdown on tensor calculus

Notation:
- vector,
- matrix, and
- index

example: a velocity vector $\boldsymbol{v}$
- vector notation $\vec{v}$
- matrix notation $\{v_x, v_y, v_z\}$ or $\begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix}$
- index notation (tensor notation) $v_i, i = x, y, z$

example: a transformation matrix A
- vector notation $(\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3})$
- matrix notation $\begin{bmatrix} a_{xx} & a_{xy} & a_{xz} \\ a_{yx} & a_{yy} & a_{yz} \\ a_{zx} & a_{zy} & a_{zz} \end{bmatrix}$
- index notation (tensor notation) $a_{ij},\ i, j = x, y, z$

variables: $x, y, z,$ or $\xi, \eta, \zeta,$ or x_1, x_2, x_3

Tensor calculus – cont.

Einstein's summation convention

Whenever an index is repeated once (and no more) in the same term in equation it implies summation over the specified range of the index. In such cases there is no need to write the expanded form of the sum or use the summation sign.

Example: a set of linear equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + b_1 = 0$$
$$a_{21}x_2 + a_{22}x_2 + a_{23}x_3 + b_2 = 0$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + b_3 = 0$$

$$\Sigma_j a_{1j}x_1 + b_1 = 0$$
$$\Sigma_j a_{2j}x_2 + b_2 = 0$$
$$\Sigma_j a_{3j}x_3 + b_3 = 0$$

$$a_{ij}x_j + b_i = 0, \qquad i, j = x_1, x_2, x_3$$

i – free index, j – dummy index

Dummy indexes and dummy variables

The equations contain two different types of indexes: free indexes that appear once in each term of the equation, and dummy indexes, that are fictitious and appear temporarily.
Similarly, we use dummy variables in the integration:

$$M(x) = \int_0^x q(t) \cdot (x - t)dt = \int_0^x q(\xi) \cdot (x - \xi)d\xi = \int_0^x q(x_1) \cdot (x - x_1)dx_1$$

(t, ξ, x_1 - dummy variables)

differentiation is written using a comma: $\frac{\partial h}{\partial x_i} \equiv h_{,i}$

Tensor: definition

There are several definitions of a tensor.

1. A tensor is just a fancy word for a matrix
2. A tensor is a linear transformation from vectors to vectors
3. A tensor is an ordered set of numbers that transform in a particular way upon a change of basis
4. A tensor of n-rank is a geometric object that projected onto an axis (a direction) gives a tensor of rank $(n-1)$
5. (and others, with dyadic product, for instance)

The transformation rule of a tensor of the second rank

$$t_{ij} = t_{kl} a_{ik} a_{jl}$$

where a_{ij} are elements of the transformation matrix (orthonormal, $a_{ij} \equiv \cos \widehat{x_i, x_j}$)

Our choice of a tensor definition:

A tensor of the second rank is a symmetric matrix that transforms according to the tensor transformation rule

Similarly, we have:

- s, a scalar is a tensor of 0 rank,
- v_i, a vector is a tensor of 1st rank,
- t_{ij}, a tensor of 2nd rank,
- t_{ijk}, a tensor of 3rd rank, (etc.)

Tensor: operations and invariants

summation (allowed only for the terms with the same free index (indexes))

$$c_{ij} = a_{ij} + b_{ij}$$

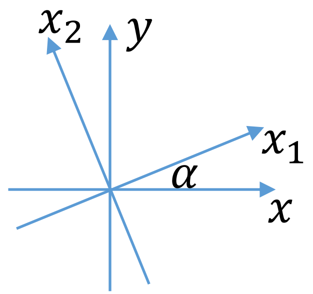
decomposition into an isotropic and deviatoric parts:

$$t_{ij} = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix} = \begin{bmatrix} t_m & 0 & 0 \\ 0 & t_m & 0 \\ 0 & 0 & t_m \end{bmatrix} + \begin{bmatrix} t_{ij} - t_m & t_{12} & t_{13} \\ t_{21} & t_{22} - t_m & t_{23} \\ t_{31} & t_{32} & t_{33} - t_m \end{bmatrix} = \mathrm{A_t} + \mathrm{D_t}$$

$$t_m = \frac{1}{3}(t_{11} + t_{22} + t_{33})$$

Multiplication

- scalar product (example: $l^2 = v_x^2 + v_y^2 + v_z^2 = v_i v_i, \; i = x, y, z$)
- contraction (example: $\mathrm{tr}(A) = \mathrm{t}_{11} + t_{22} + t_{33} = t_{ii}, \; i = 1, 2, 3$)
- tensor (cross, dyadic) product (example: $v_i w_j = t_{ij}, \; i = x, y, z$)



transformation from one basis to another (we assume both bases orthogonal)

$$t_{ij} = a_{ik} a_{jl} t_{kl}, \qquad k, l = x, y, z; \quad i, j = 1, 2, 3$$

invariants

invariants of a tensor are independent of the transformation due to rotation of a frame

examples: a vector length, a trace of a matrix (see below)

$$v_i v_i = a_{ix} v_x a_{iy} v_y = a_{ix} a_{iy} v_x v_y = v_x v_x = v_i v_i$$

Kronecker delta and permutation tensor

Kronecker delta

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \qquad \delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{array}{l} \delta_{11} = \delta_{22} = \delta_{33} = 1 \\ \delta_{12} = \delta_{21} = \delta_{13} = \delta_{31} = \delta_{23} = \delta_{32} = 0 \end{array}$$

example of use:

$v_i v_i = a_{ik} v_k a_{il} v_l = a_{ik} a_{il} v_k v_l$ and $v_i v_i = v_k v_k = v_l v_l \rightarrow \boxed{a_{ik} a_{il} = \delta_{kl}}$

$$\mathrm{Tr}(t_{ij}) = t_{11} + t_{22} + t_{33} = t_{ii} = a_{ik} a_{il} t_{kl} = \delta_{kl} t_{kl} = t_{kk} = t_{ll}$$

permutation symbol (alternating tensor or Levi-Civita tensor)

$$\epsilon_{ijk} = \begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix} \text{ if } \; i, j, k \; \text{ are } \begin{Bmatrix} \text{an even} \\ \text{an odd} \\ \text{not a} \end{Bmatrix} \text{ permutation of } 1, 2, 3$$

$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$ (cyclic order: counter clockwise)

$\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1$ (cyclic order: clockwise)

$\epsilon_{111} = \epsilon_{211} = \epsilon_{133} = \cdots = 0$

Eigenvalue problem

Using the method of Lagrange multipliers the extreme values of the tensor can be found as well as the direction of corresponding coordinate set:

$$(t_{ij} - \lambda\delta_{ij})n_j = 0$$

(it is a set of three equations: algebraic, homogeneous and linear)
Non-trivial solution:

$$\det(t_{ij} - \lambda\delta_{ij}) = 0$$

(but in the same time, it means that the equations are not independent)
so:

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$$

the roots are so-called principal values of the tensor (the principal invariants):

$$\lambda_1, \geq \lambda_2, \geq \lambda_3$$

next, solving the equations sets we get the principal directions (the transformation matrix)

The fundamental invariants

$$I_1 = t_{ii} = t_{11} + t_{22} + t_{33}$$

$$I_2 = t_{ij}t_{ji} = \begin{vmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{vmatrix} + \begin{vmatrix} t_{11} & t_{13} \\ t_{31} & t_{33} \end{vmatrix} + \begin{vmatrix} t_{22} & t_{23} \\ t_{32} & t_{33} \end{vmatrix}$$

$$I_3 = t_{ij}t_{jk}t_{ki} = \begin{vmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{vmatrix}$$

Eigenvalue problem – special cases

If two principal values are equal there is a surface where every direction is principal.
If all principal values are equal, any direction in space is equal. Such a tensor is called isotropic.
The mean values part of a tensor is an example of an isotropic tensor.

If $I_3 = 0$, than one principal value is also zero. Such a state is called 2D case.

The canonical equation becomes:

$$t^2 - I_1 t + I_2 = 0$$

Principal values are given as:

$$t_1 = \frac{t_x + t_y}{2} + \sqrt{\left(\frac{t_x - t_y}{2}\right)^2 + t_{xy}^2}$$

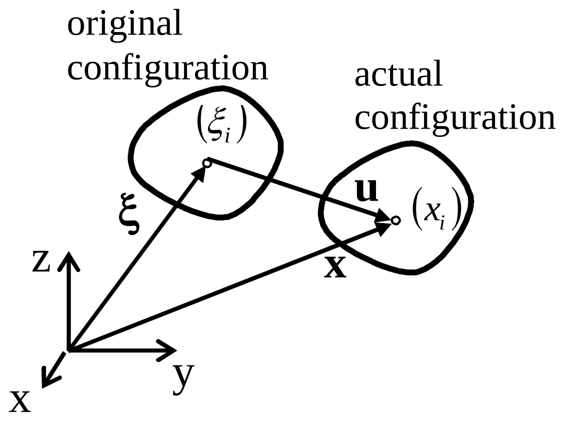
$$t_2 = \frac{t_x + t_y}{2} - \sqrt{\left(\frac{t_x - t_y}{2}\right)^2 + t_{xy}^2}$$

and principal directions as:

$$\tan \alpha_i = \frac{t_i - t_x}{t_{xy}}$$

with the transformation matrix (from original to principal directions) $a_{ij} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

Kinematics of motion



original (initial, unloaded, not deformed) configuration
actual (final, loaded, deformed) configuration
P – a material point
ξ_i - coordinates of the point P in initial configuration (or position vector of P)
x_i - coordinates of the point P in actual configuration (or position vector of P)

displacement vector:

$$u_i = x_i - \xi_i$$

There are two possible ways of deformation description, depending on the choice of the reference configuration:

- if the initial configuration serves as a reference configuration, this is so-called material (or Lagrangian) description

$$x_i = \xi_i + u_i$$

- if the final configuration is chosen as a reference configuration, this is so-called spatial (or Eulerian) description

$$\xi_i = x_i - u_i$$

Another way is applied in hydromechanics: there is no ability to follow material particles over time and the quantities of interest (pressure, velocity, etc.) are described at a fixed position in space.

Lagrangial and Eulerian descriptions

In the structural mechanics, usually, the lagrangian description is used (material coordinates). It means, that we are interested in the change of position of points, chosen in original configuration.
The lagrangian description is totally equivalent to the eulerian description.

Deformation in continuum mechanics is the transformation of a body from a reference configuration to a current configuration.
During the deformation process, a matter (a continuum, a body, a material) undergoes changes in the shape and size. The distances between points as well as angles between directions change.
If the transformation from the original configuration to the actual configuration is continuous and sufficiently smooth, the transition from one configuration to another can be established in both ways: from original to actual configuration and vice versa.
Strain is the change of body shape and size. The *deformation* is more general notion than a strain, including a movement of a rigid body.

To describe the strains of material we need to select a suitable measure for the purpose.

Deformation and strain

The spatial derivatives of one set of coordinates with respect of another set of coordinates, are:

- spatial in regard to material

$$\frac{\partial x_i}{\partial \xi_\alpha} = x_{i,\alpha} = u_{i,\alpha} + \delta_{i\alpha}$$

- material in regard to spatial

$$\frac{\partial \xi_\alpha}{\partial x_i} = \xi_{\alpha,i} = \delta_{\alpha i} - u_{\alpha,i}$$

and their total derivatives, respectively:

$$dx_i = x_{i,\alpha} d\xi_\alpha$$

and

$$d\xi_\alpha = \xi_{\alpha,i} dx_i$$

where the quantities $x_{i,\alpha}$ and $\xi_{\alpha,i}$ are the *deformation gradients*. The deformation gradient is the fundamental measure of body deformation.

Tensors of finite strains

Let's calculate the length of an elementary segment squared:

$$ds_0^2 = d\xi_\alpha d\xi_\alpha = d\xi_\alpha d\xi_\beta \delta_{\alpha\beta} = \xi_{\alpha,i} dx_i \xi_{\beta,j} dx_j \delta_{\alpha\beta}$$

and

$$ds^2 = dx_i dx_i = dx_i dx_j \delta_{ij} = x_{i,\alpha} d\xi_\alpha x_{j,\beta} d\xi_\beta \delta_{ij}$$

Their difference is:

- in material coordinates

$$ds^2 - ds_0^2 = dx_i dx_i - d\xi_\alpha d\xi_\alpha = \left(x_{i,\alpha} x_{j,\beta} \delta_{ij} - \delta_{\alpha\beta}\right) d\xi_\alpha d\xi_\beta$$

- in spatial coordinates

$$ds^2 - ds_0^2 = dx_i dx_i - d\xi_\alpha d\xi_\alpha = \left(\delta_{ij} - \xi_{\alpha,i} \xi_{\beta,j} \delta_{\alpha\beta}\right) dx_i dx_j$$

By definition, the Lagrange strain tensor is:

$$E_{\alpha\beta} \stackrel{\text{def}}{=} \frac{1}{2}\left(x_{i,\alpha} x_{j,\beta} \delta_{ij} - \delta_{\alpha\beta}\right)$$

and the Euler strain tensor is:

$$e_{ij} \stackrel{\text{def}}{=} \frac{1}{2}\left(\delta_{ij} - \xi_{\alpha,i} \xi_{\beta,j} \delta_{\alpha\beta}\right)$$

Tensor character of above objects is obvious as an external product of two vectors.

Tensors of finite strain – cont.

The tensors may be expressed in terms of the displacement:

- Langrange tensor

$$E_{\alpha\beta}=\frac{1}{2}\left[\left(u_{i,\alpha}+\delta_{i\alpha}\right)\left(u_{j,\beta}+\delta_{j\beta}\right)\delta_{ij}-\delta_{\alpha\beta}\right]=\cdots=\frac{1}{2}\left(u_{\alpha,\beta}+u_{\beta,\alpha}+u_{k,\alpha}u_{k,\beta}\right)$$

- Euler tensor

$$e_{ij}=\frac{1}{2}\left[\delta_{ij}-\left(\delta_{\alpha,i}-u_{\alpha,i}\right)\left(\delta_{\beta j}-u_{\beta,j}\right)\delta_{\alpha\beta}\right]=\cdots=\frac{1}{2}\left(u_{i,j}+u_{j,i}-u_{\alpha,i}u_{\alpha,j}\right)$$

non-linear terms

Measures of deformation

The primitive notions in the continuum mechanics are *motion* and *material*.

The one particular measure of strain may be the *relative elongation*:

$$\lambda = \frac{l}{l_0}$$

The above measure is not suitable for the structural materials, because the results would be within the range:

$$\lambda \in (0.999 \div 1.001)$$

Another possible choice of the function $\varepsilon = f(\lambda)$, is $f(\lambda = 1) = 0$. Developing the function in Taylor series about 1, we get:

$$f(\lambda) = f(1) + \frac{\lambda - 1}{1!} f'(1) + \frac{(\lambda - 1)^2}{2!} f''(1) + \cdots + \frac{(\lambda - 1)^n}{n!} f^{(n)}(1) + \cdots$$

with the conditions:

$$f(1) = 0, f'(1) = 1 \text{ and } \forall\lambda > 0\text{:}\ \ f'(0) > 0 \text{ (function monotonicity)}$$

we get general Hill form:

$$\varepsilon(n) = f(\lambda) = \frac{\lambda^{2n} - 1}{2n}$$

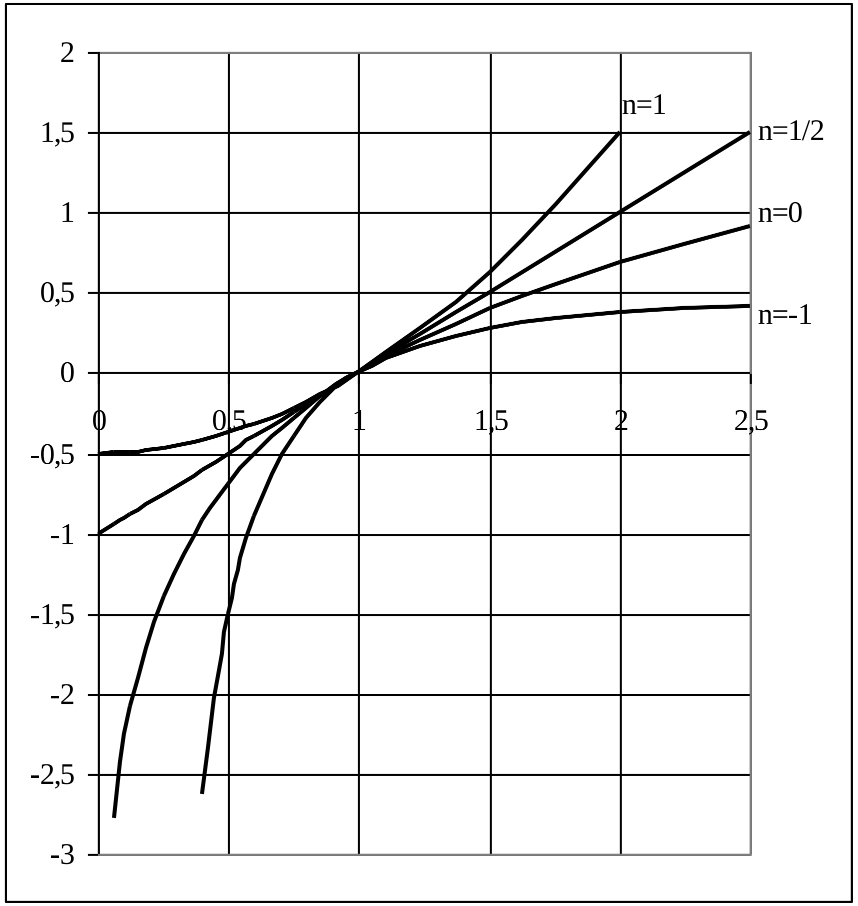
Measures of deformation – cont.

for $n = \frac{1}{2} \to \ \varepsilon = \lambda - 1 = \frac{l - l_0}{l_0}$ (Cauchy (infinitesimal) tensor)

for $n = 1 \to \ \varepsilon = \frac{\lambda^2 - 1}{2} = \frac{1}{2}\frac{l^2 - l_0^2}{l_0^2}$ (Green (eulerian) tensor)

for $n = -1 \to \ \varepsilon = \frac{\lambda^2 - 1}{-2} = \frac{1}{2}\frac{l^2 - l_0^2}{l^2}$ (Almansi (lagrangian) tensor)

for $n = 0 \to \ \varepsilon = \lim\limits_{n \to 0} \left(\frac{\lambda^{2n} - 1}{2n}\right)' = \ln \xi = \ln \frac{l}{l_0}$ (Hencky measure)



Small derivatives of displacements

we assume:

$$\frac{\partial u_i}{\partial x_j}, \qquad \frac{\partial u_\alpha}{\partial \xi_\beta} \ll 1$$

the nonlinear terms in the Lagrange and Euler tensor vanish
moreover, we equate the actual configuration with the initial configuration

Cauchy tensor of infinitesimal strains:

$$\boxed{\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)}$$

In engineering notation, the Cauchy equations (the geometric equations) are:

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \qquad \varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\varepsilon_z = \frac{\partial w}{\partial z} \qquad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

Infinitesimal strain tensor (Cauchy)

The strain matrix is an image of state of strain

$$T_\varepsilon = \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_z \end{bmatrix}$$

(in an old engineering notation):

$$T_\varepsilon = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_z \end{bmatrix}$$

The strain matrix is symmetric:

$$\varepsilon_{ij} = \varepsilon_{ji}$$

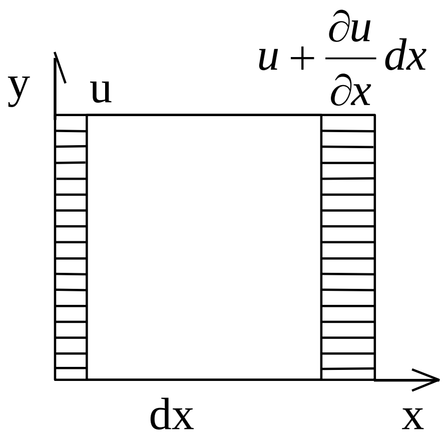
Strain is dimensionless. All values in the above matrix are less than

$$|\varepsilon_{ij}| < 0.001$$

$\varepsilon_x, \varepsilon_y, \varepsilon_z$ – linear strains in direction x, y, z, respectively

$\varepsilon_{xy}, \varepsilon_{xz}, \varepsilon_{yz}$ – angular strains between directions xy, xz, yz, respectively

Interpretation of Cauchy tensor of strains



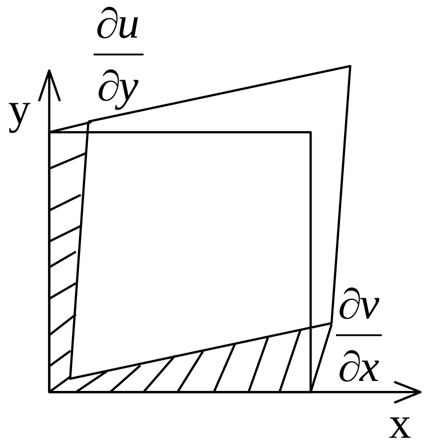
relative elongation; in the direction of axis *x*:

$$\frac{\partial u}{\partial x} \frac{dx}{dx} = \frac{\partial u}{\partial x} = \varepsilon_x$$

it is a ratio of an elongation to total length calculated for a segment with direction *x*

these elements are on the main diagonal of the strain matrix

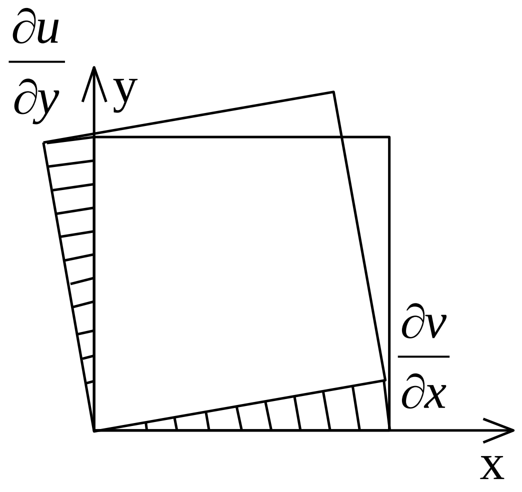
Interpretation – cont.



change of shape – the angular strains
it is a half of change of the right angle between two axes

$$\frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)=\varepsilon_{xy}$$

Interpretation – cont.



the rigid rotation

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \Rightarrow \quad \varepsilon_{xy} \equiv 0 \quad \Rightarrow \quad \varepsilon_{ij} \equiv 0.$$

there is no strain, rigid body movement only

Strain matrix

The strain matrix as an image of the state of strain at a point and can be determined in an arbitrary coordinate set:

- in (x, y, z) Cartesian set

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ & \varepsilon_y & \varepsilon_{yz} \\ & & \varepsilon_z \end{bmatrix}$$

- in (ξ, η, ζ) Cartesian set

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_\xi & \varepsilon_{\xi\eta} & \varepsilon_{\xi\zeta} \\ & \varepsilon_\eta & \varepsilon_{\eta\zeta} \\ & & \varepsilon_\zeta \end{bmatrix}$$

- in (x_1, x_2, x_3) Cartesian set

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_1 & \varepsilon_{12} & \varepsilon_{13} \\ & \varepsilon_2 & \varepsilon_{23} \\ & & \varepsilon_3 \end{bmatrix}$$

The above matrix describe the same strain state at the same material point
A fundamental question arises: what is a coordinate set that gives extreme values of strains?

Strain eigenvalues problem

The strain matrix is a tensor, so one can use the transformation rule to determine the strain matrix in an arbitrary coordinate set, using the transformation matrix. The transformation matrix is ortho-normal (orthogonal and normalized).

In the principal coordinate set, the strain matrix reads:

$$T_\varepsilon = \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ & \varepsilon_y & \varepsilon_{yz} \\ & & \varepsilon_z \end{bmatrix} \rightarrow \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}$$

with the principal directions, given by the transformation matrix

$$a_{ij} = \begin{bmatrix} a_{1x} & a_{1y} & a_{1z} \\ a_{2x} & a_{2y} & a_{2z} \\ a_{3x} & a_{3y} & a_{3z} \end{bmatrix}$$

The first invariant of the strain matrix: has an interpretation of a volume change.

Proof:

$$\Delta V = (1+\varepsilon_x)\left(1+\varepsilon_y\right)(1+\varepsilon_z) - 1 = 1 + \varepsilon_x + \varepsilon_y + \varepsilon_z + \varepsilon_x\varepsilon_y + \varepsilon_x\varepsilon_z + \varepsilon_y\varepsilon_z + \varepsilon_x\varepsilon_y\varepsilon_z - 1 \cong \varepsilon_x + \varepsilon_y + \varepsilon_z$$

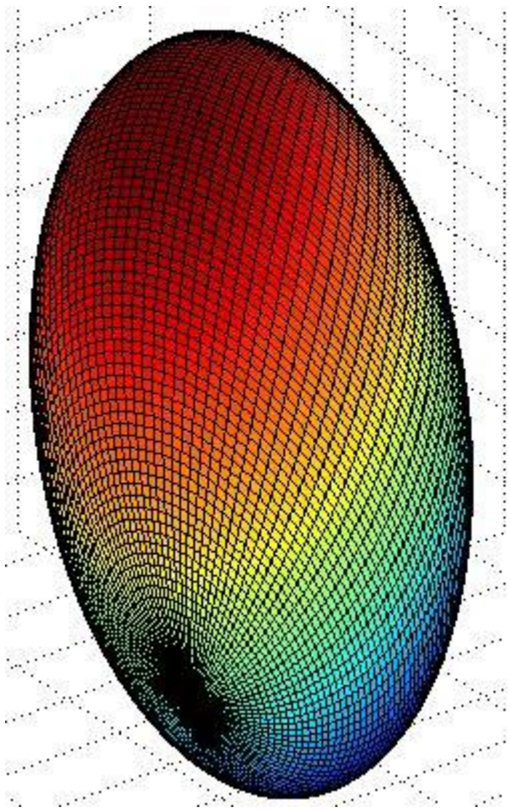
The change of volume is called *dilatation.*

Fundamental theorem on deformation

Any strain state at a point is composed from extensions/shortenings only in three perpendicular directions.

The fundamental theorem on deformation
Any deformation consists of a rigid body translation and rotation of the principal axes and of elongations and/or shortenings along these axes

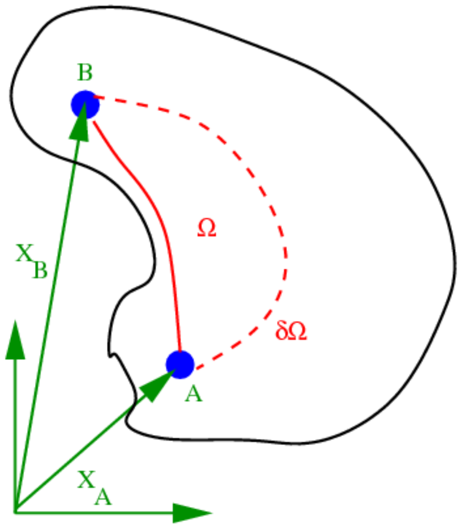
Strain ellipsoid – graphical presentation of the strain state
The principal axes of the strain ellipsoid are the principal extensions.



Compatibility equations

We assumed that the transformation from initial configuration to the actual configuration is continuous and sufficiently smooth:

- curved lines remain curved lines (without a sharp crease nor break)
- every points with their neighborhood transfers with its neighborhood
- no new points arise during the process
- no point vanishes during the process



The above restrictions on the continuity of material have mathematical expression by so-called the compatibility equations:

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0$$

(81 equations, but only 6 independent)

Every symmetric matrix may be a strain matrix if and only if the matrix fulfills the compatibility conditions.

The Cauchy geometric equations are the set of differential equations with partial derivatives. With the (kinematic) boundary conditions they form a boundary value problem (BVP), called in mathematics the Cauchy's problem.

Strain gage rosettes

Strain gages – a measurement of strain along some direction (in 2D)

the strain state in 2D: $\varepsilon_{ij} = \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} \\ \varepsilon_{xy} & \varepsilon_y \end{bmatrix}$

can be the strain state coordinates restored from measurement of linear strain in three different directions?

the transformation rule for arbitrary directions (a, b, c) inclined by an angle (α, β, γ):

$$\left.\begin{aligned} \varepsilon_a &= a_{\alpha x}^2 \varepsilon_x + 2a_{\alpha x} a_{\alpha y} \varepsilon_{xy} + \mathrm{a}_{\mathrm{ay}}^2 \varepsilon_y \\ \varepsilon_b &= a_{bx}^2 \varepsilon_x + 2a_{bx} a_{by} \varepsilon_{xy} + \mathrm{a}_{\mathrm{by}}^2 \varepsilon_y \\ \varepsilon_c &= a_{cx}^2 \varepsilon_x + 2a_{cx} a_{cy} \varepsilon_{xy} + \mathrm{a}_{\mathrm{cy}}^2 \varepsilon_y \end{aligned}\right\} \rightarrow \left.\begin{aligned} \varepsilon_x &= \varepsilon_x(\varepsilon_a, \varepsilon_b, \varepsilon_c) \\ \varepsilon_y &= \varepsilon_y(\varepsilon_a, \varepsilon_b, \varepsilon_c) \\ \varepsilon_{xy} &= \varepsilon_{xy}(\varepsilon_a, \varepsilon_b, \varepsilon_c) \end{aligned}\right\} \rightarrow \varepsilon_1, \varepsilon_2, \alpha$$

so, a measurement of an angular strain is not needed, however such gages also exist

Thank you for your attention!