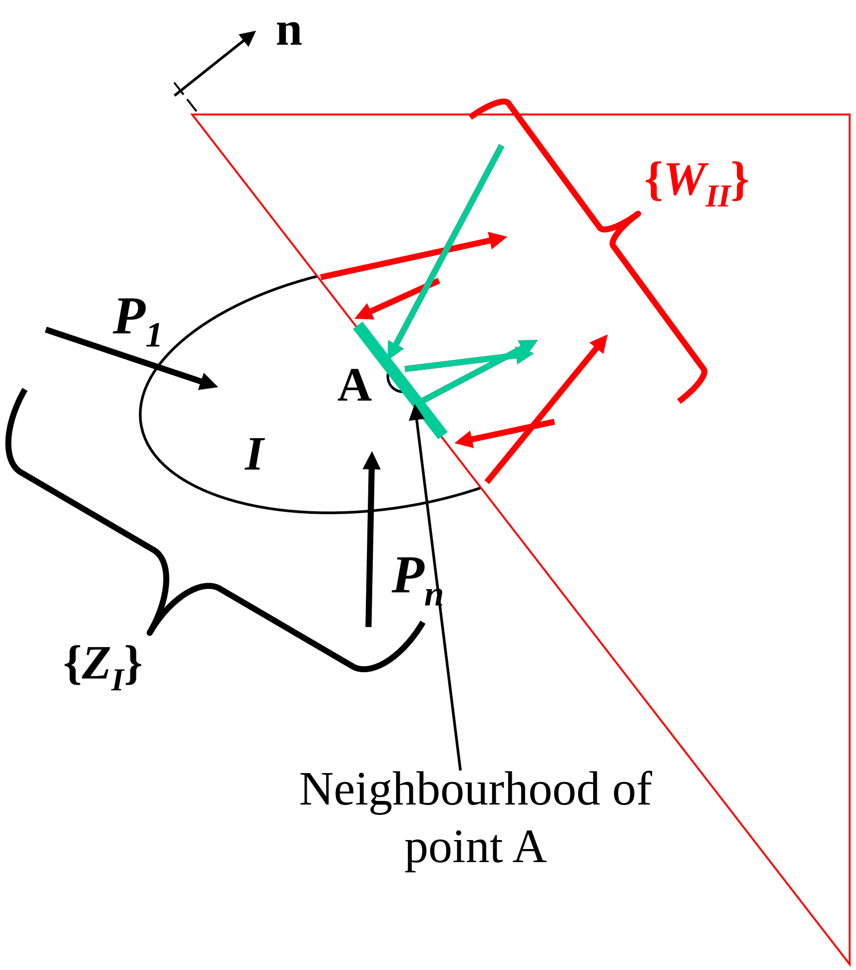
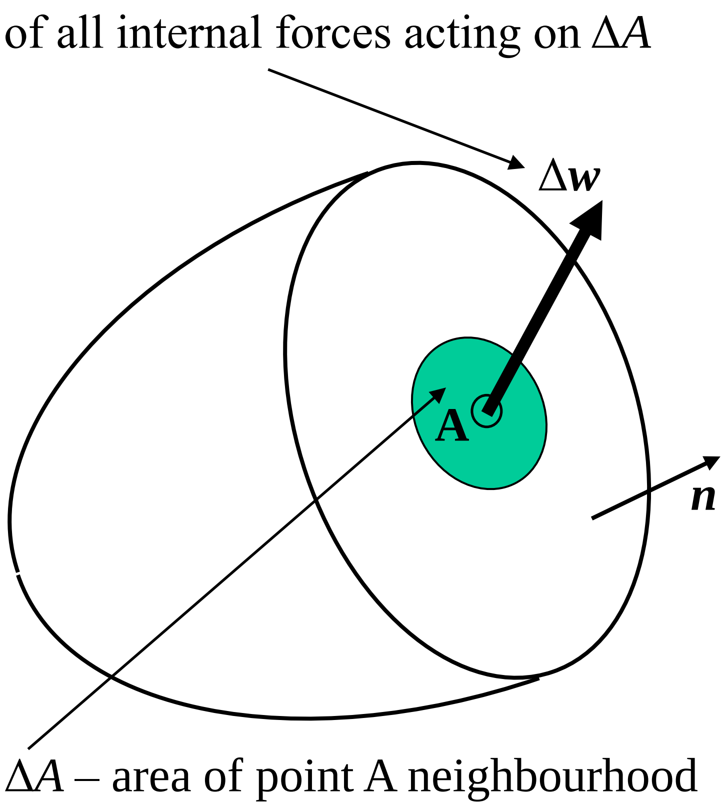
Strength of Materials

10. Stress state

From internal forces to stress





$$\lim_{\Delta A \to 0} \frac{\Delta w}{\Delta A} = \vec{p}_A$$ **stress vector** at A

Stress – definition

$$\lim_{\Delta A \to 0} \frac{\Delta w}{\Delta A} = \vec{p}_A(r_A, n)$$

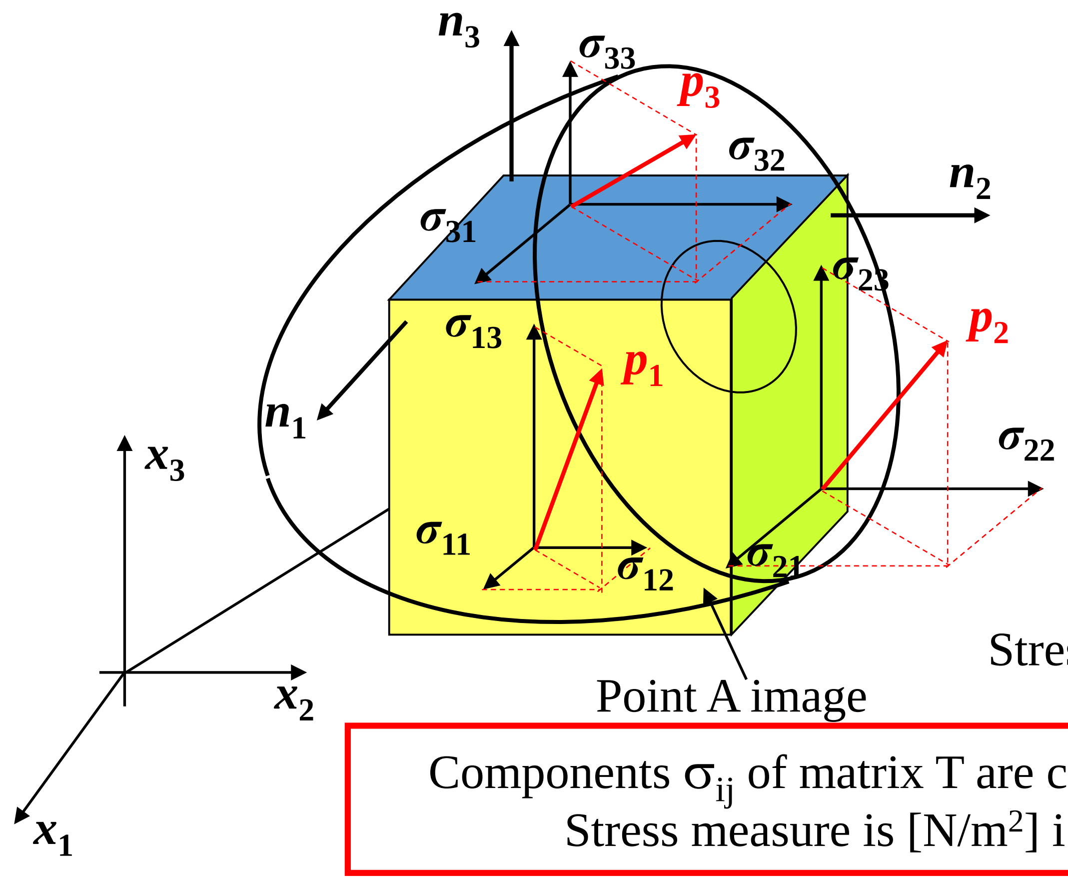
Stress vector is a measure of **internal forces intensity** and depends on the chosen point and cross section

Stress vectors:

$p_1[\sigma_{11}, \sigma_{12}, \sigma_{13}]$

$p_2[\sigma_{21}, \sigma_{22}, \sigma_{23}]$

$p_3[\sigma_{31}, \sigma_{32}, \sigma_{33}]$



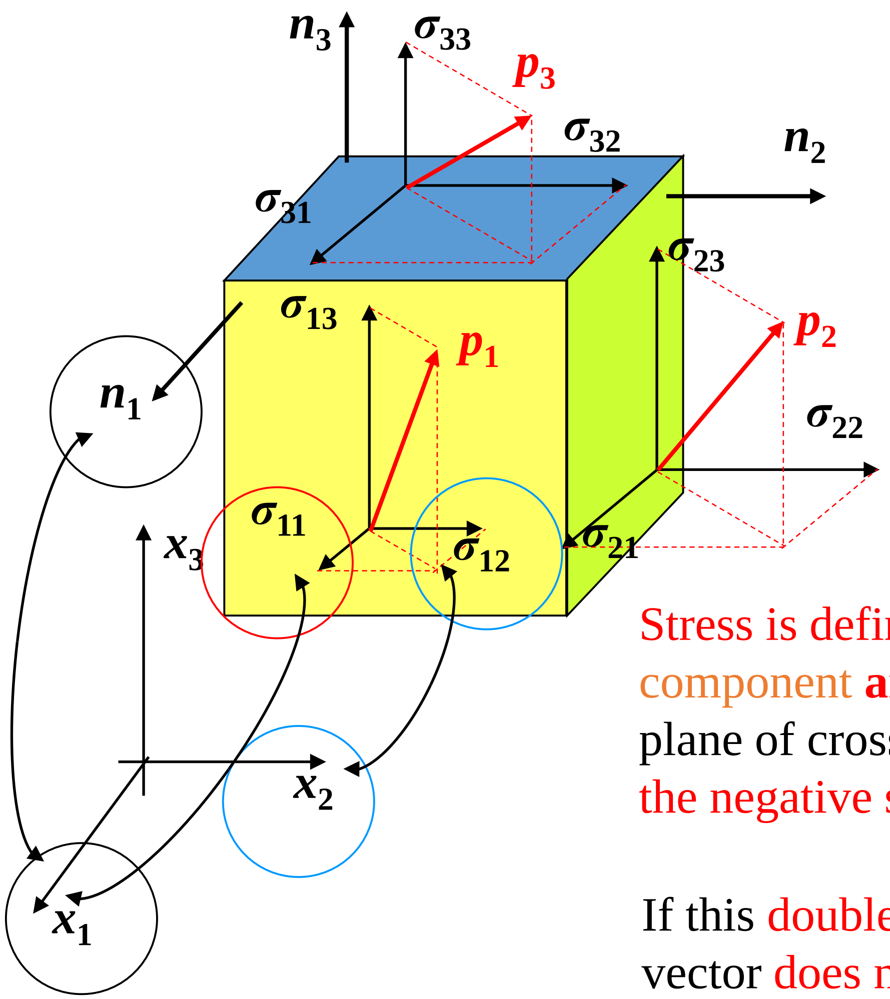
$$T_\sigma \begin{pmatrix} \sigma_{11}, \sigma_{12}, \sigma_{13} \\ \sigma_{21}, \sigma_{22}, \sigma_{23} \\ \sigma_{31}, \sigma_{32}, \sigma_{33} \end{pmatrix}$$

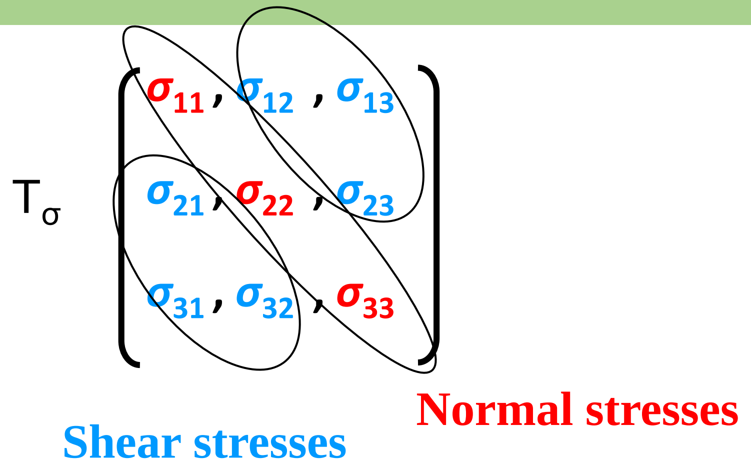
Stress matrix

Components σ_{ij} of matrix T are called **stresses**.
Stress measure is $[N/m^2]$ i.e. [Pa]

$T_\sigma(\sigma_{ij})$ i,j = 1,2,3

Stress sign



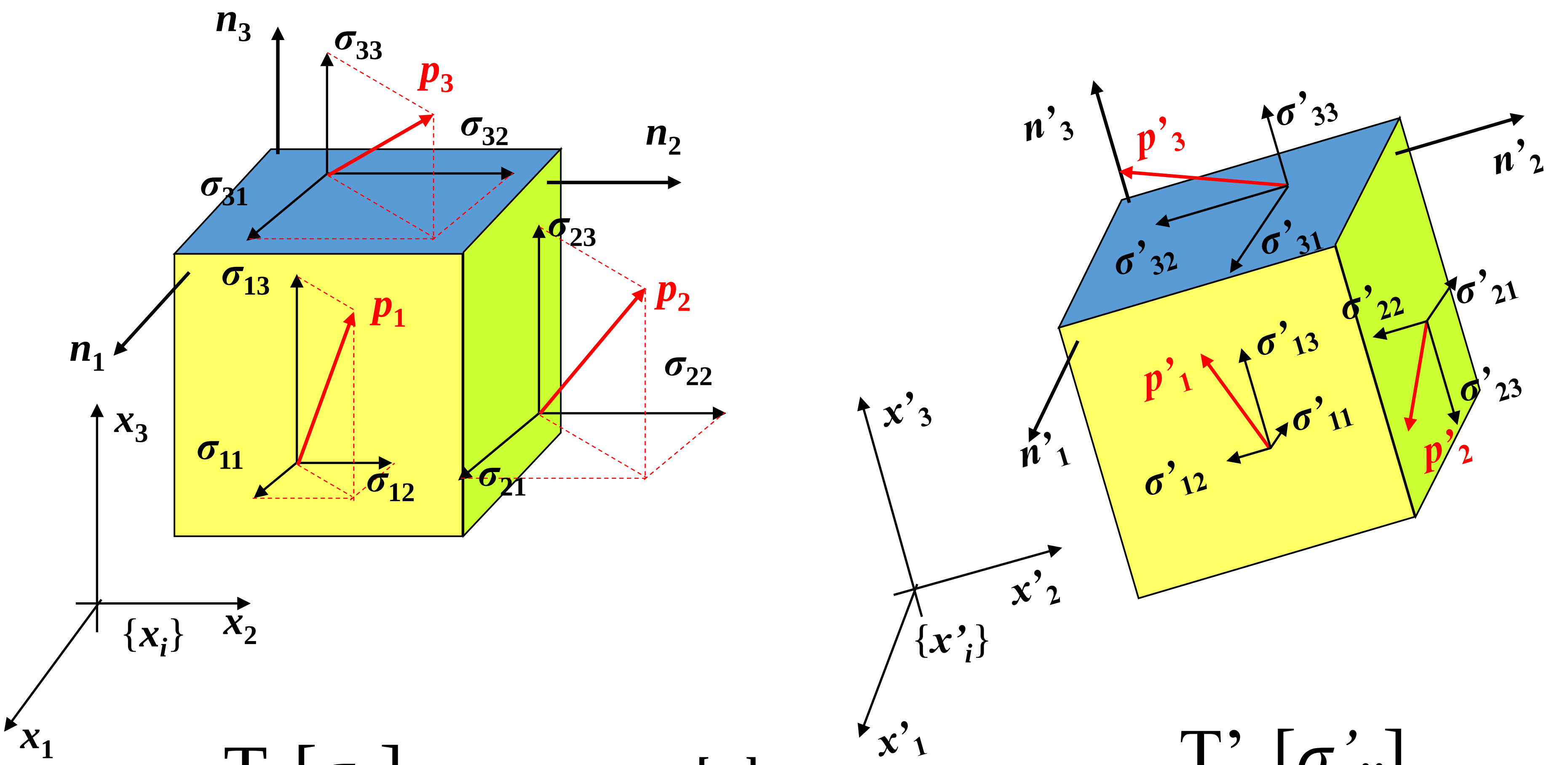


Positive and negative stresses

Stress is defined as **positive** when the direction of stress vector component **and** the direction of the outward normal to the plane of cross-section are both in the positive sense **or** both in the negative sense in relation to the co-ordinate axes.

If this double conjunction of stress component and normal vector does not occur – the stress component is **negative** one.

Stress transformation

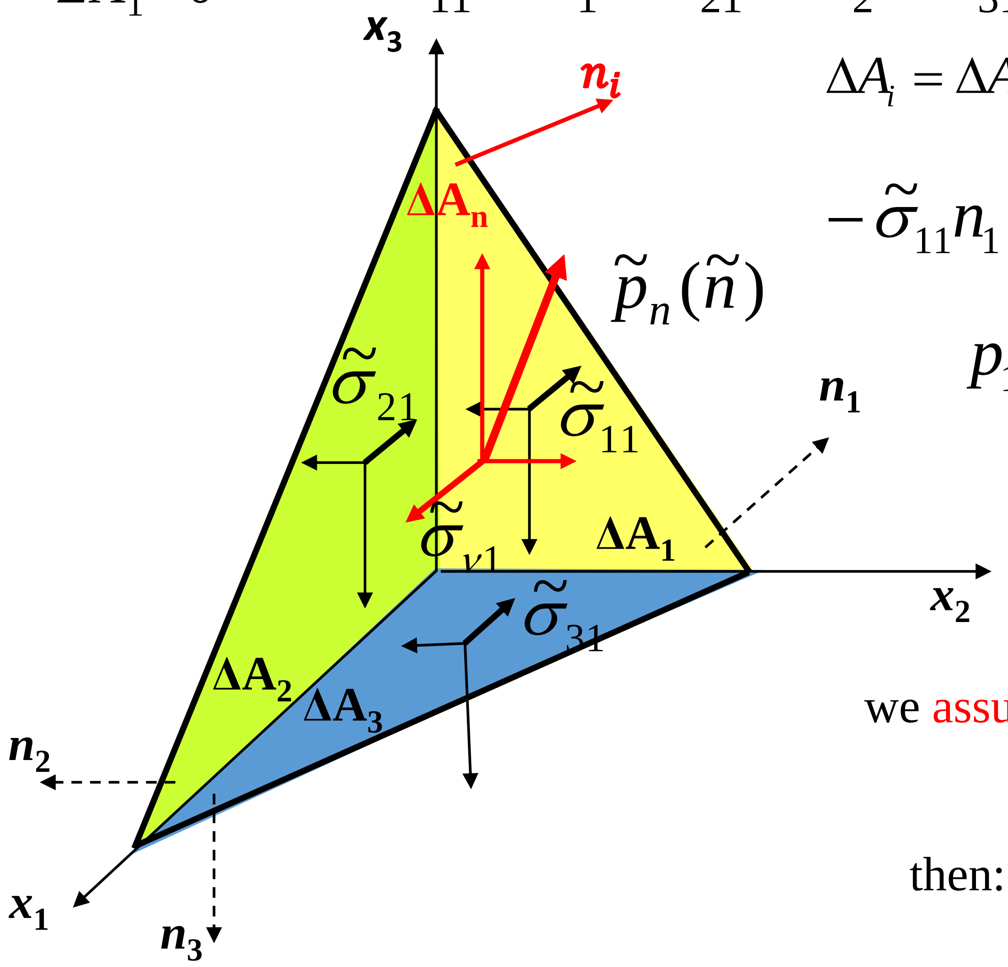


$T_\sigma[\sigma_{ij}]$ → $[a_{ij}]$ → $T'_\sigma[\sigma'_{ij}]$

→ → $\sigma'_{ij} = a_{ik} a_{jl} \sigma_{kl}$

Stress on inclined plane

$\Sigma X_1 = 0$ $$-\tilde{\sigma}_{11}\cdot\Delta A_1 - \tilde{\sigma}_{21}\cdot\Delta A_2 - \tilde{\sigma}_{31}\cdot\Delta A_3 + \tilde{p}_1\cdot\Delta A_n = 0 \quad \Big\} \; \frac{1}{\Delta A_n}$$



$$\Delta A_i = \Delta A_n n_i \qquad \Delta A_1/\Delta A_n = n_1 \qquad \ldots, \quad \ldots$$

$$-\tilde{\sigma}_{11}n_1 - \tilde{\sigma}_{21}n_2 - \tilde{\sigma}_{31}n_3 + \tilde{p}_1 = 0 \quad \Big\} \; \lim_{\Delta A\to 0}$$

$$p_1 = \sigma_{11}n_1 + \sigma_{21}n_2 + \sigma_{31}n_3$$

$$p_1 = \sigma_{i1}n_i$$

$$p_i = \sigma_{ji}n_j$$

we assume: $\sigma_{ij} = \sigma_{ji}$

then: $$p_i = \sigma_{ij}n_j$$

Stress on inclined plane – cont.

$$p_i = \sigma_{ij} n_j$$

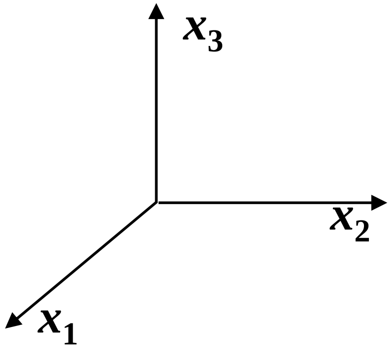
$$\vec{p}_n = T_\sigma \vec{n}$$

$$\vec{p}_n(\sigma_{ni}) = ?$$

$$[p_1, p_2, p_3] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

i=1, i=2, i=3

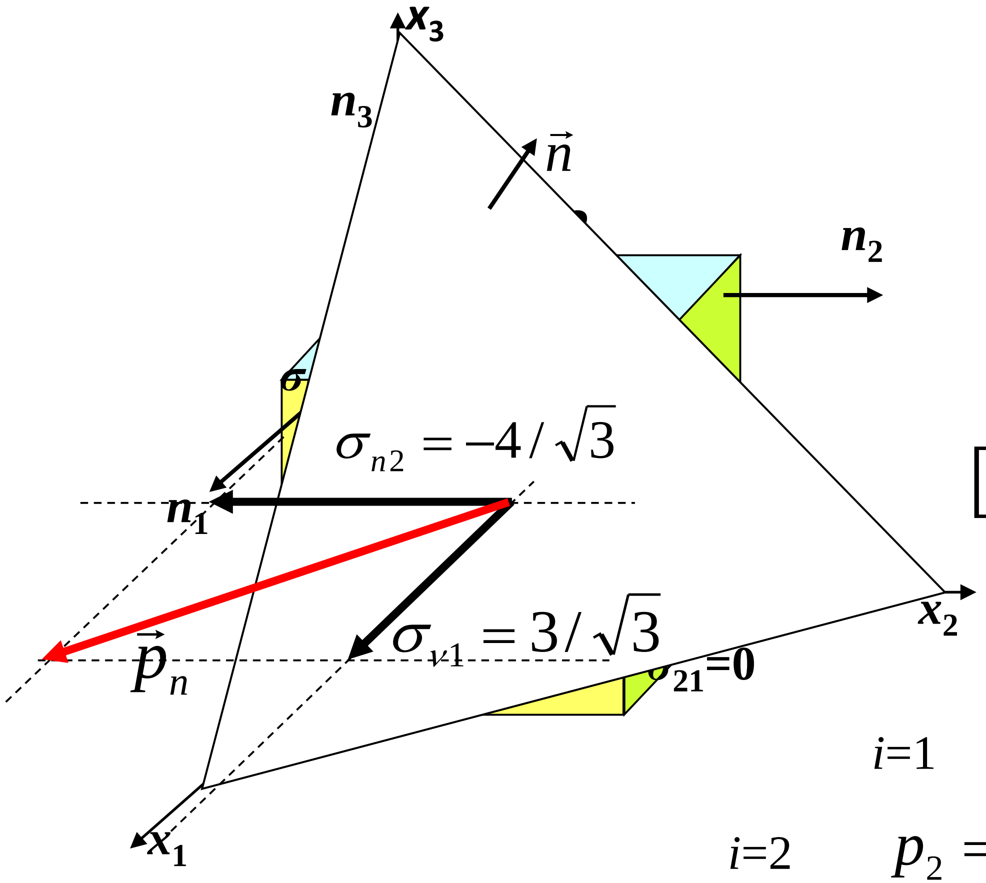
j=1, j=2, j=3



i=1 $\qquad \sigma_{n1} = p_1 n_1 + p_2 n_2 + p_3 n_3$

j=1, j=2, j=3

Stress on inclined plane



$$T_\sigma = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & -3 \\ 1 & -3 & 2 \end{bmatrix} \qquad \vec{n} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$\vec{p}_n(p_i) = ? \qquad p_i = \sigma_{ij} n_j$$

$$[p_1, p_2, p_3] = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & -3 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \quad \begin{matrix} i=1 \\ i=2 \\ i=3 \end{matrix}$$

$j=1 \quad j=2 \quad j=3$

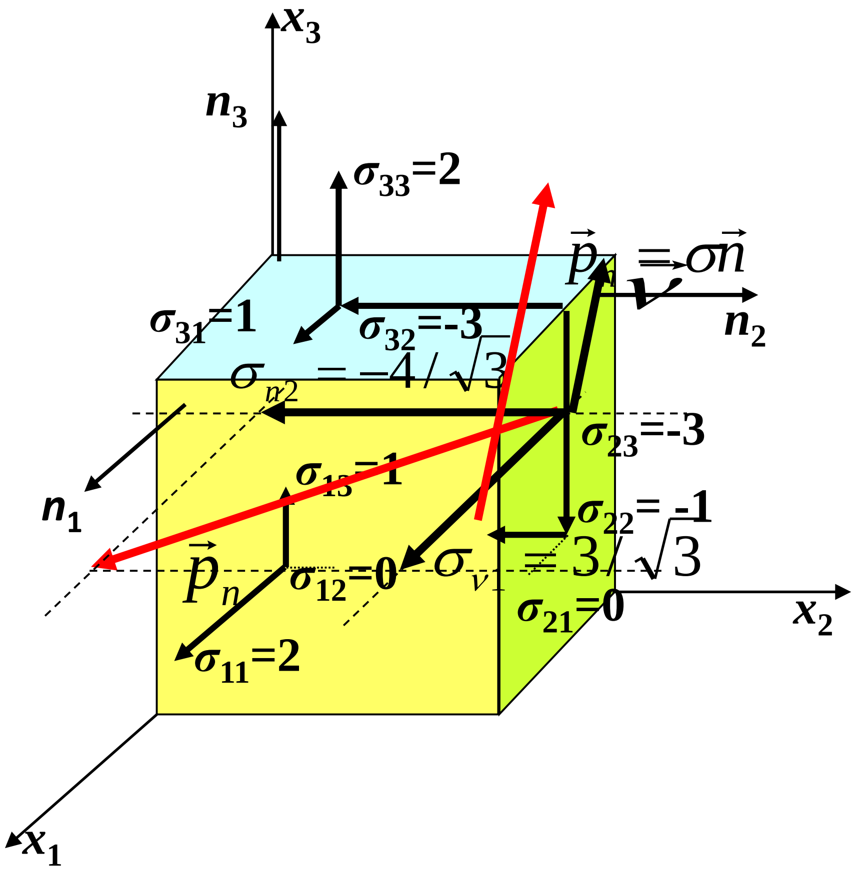
$i=1 \quad p_1 = 2/\sqrt{3} + 0/\sqrt{3} + 1/\sqrt{3} = 3/\sqrt{3}$

$i=2 \quad p_2 = 0/\sqrt{3} + \left(-1/\sqrt{3}\right) + \left(-3/\sqrt{3}\right) = -4/\sqrt{3}$

$i=3 \quad p_3 = 1/\sqrt{3} + \left(-3/\sqrt{3}\right) + 2/\sqrt{3} = 0$

$$\vec{p}_n(3/\sqrt{3}, -4/\sqrt{3}, 0)$$

Stress on inclined plane – cont.



$$T_\sigma = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & -3 \\ 1 & -3 & 2 \end{bmatrix} \qquad \vec{n} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$\vec{p}_n(3/\sqrt{3}, -4/\sqrt{3}, 0)$$

On this plane none of the vector $\vec{p}_n$ components are perpendicular nor parallel to the plane.

We will look for such a plane to which vector $\vec{p}_n$ will be perpendicular, thus having no shear components.

$$\left.\begin{aligned} \vec{p}_n &= \sigma\vec{n} \\ \vec{p}_n &= T_n\vec{n} \end{aligned}\right\} \Rightarrow \sigma\vec{n} = T_n\vec{n}$$

$$p_i = \sigma_{ij} n_j \qquad \sigma_{ij} n_j - \sigma n_i = 0$$

Principal stress

$$\sigma_{ij} n_j - \sigma n_i = 0 \qquad \delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{or} \qquad \delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

We will use Kronecker's delta to renumber normal vector components n_i

$$n_i = \delta_{ij} n_j = \delta_{i1} n_1 + \delta_{i2} n_2 + \delta_{i3} n_3$$

$$n_1 = \delta_{1j} n_j = \delta_{11} n_1 + \delta_{12} n_2 + \delta_{13} n_3 = 1 \cdot n_1 + 0 + 0 = n_1$$ etc…

$$\sigma_{ij} n_j - \sigma n_j \delta_{ij} = 0$$

$$\left(\sigma_{ij} - \sigma \delta_{ij}\right) n_j = 0$$ three equations

Seeking are :

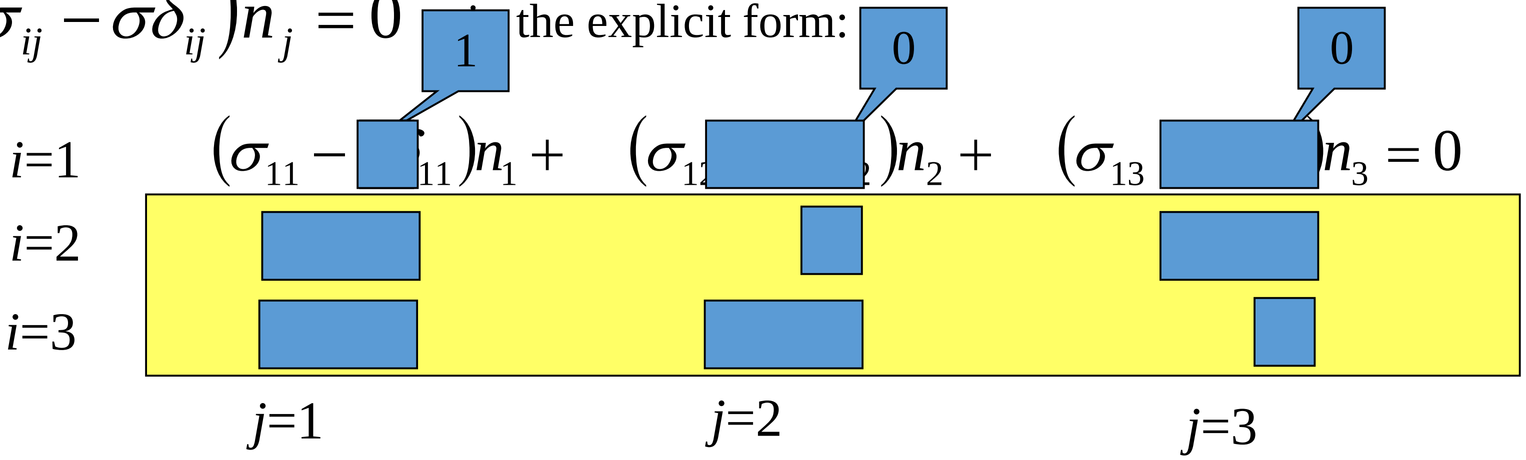
3 components of normal vector : $\vec{n}$ n_1, n_2, n_3

and a vector p_n

} 4 unknowns

Principal stress – cont.

$(\sigma_{ij} - \sigma\delta_{ij})n_j = 0$ the explicit form:



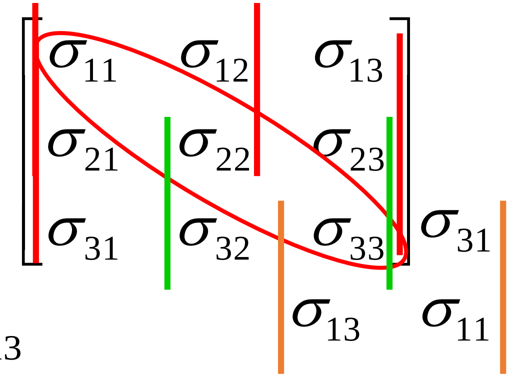
$$
\begin{aligned}
(\sigma_{11}-\sigma)n_1 + \quad & \sigma_{12}n_2 + \quad & \sigma_{13}n_3 = 0 \\
\sigma_{21}n_1 + \quad & (\sigma_{22}-\sigma)n_2 + \quad & \sigma_{23}n_3 = 0 \\
\sigma_{31}n_1 + \quad & \sigma_{32}n_2 + \quad & (\sigma_{33}-\sigma)n_3 = 0
\end{aligned}
$$

The above is set of 3 linear equations with respect to 3 unknowns α_{ni} with zero-valued constants . The necessary condition for non-zero solution is vanishing of matrix main determinant composed of the coefficients of the unknowns.

Principal stress – cont.

$$\begin{vmatrix} \sigma_{11}-\sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma \end{vmatrix} = 0 \quad \Rightarrow \quad \sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

where invariants $\boldsymbol{I_1}$ **,** $\boldsymbol{I_2}$ **,** $\boldsymbol{I_3}$ are following determinants of σ_{ij} matrix



$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

$$I_2 = \sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21} + \sigma_{22}\sigma_{33} - \sigma_{23}\sigma_{32} + \sigma_{33}\sigma_{11} - \sigma_{31}\sigma_{13}$$

$$I_3 = \sigma_{11}(\sigma_{22}\sigma_{33} - \sigma_{23}\sigma_{32}) + \sigma_{12}(\sigma_{23}\sigma_{31} - \sigma_{21}\sigma_{33}) + \sigma_{13}(\sigma_{21}\sigma_{32} - \sigma_{22}\sigma_{31})$$

Solution of this algebraic equation of the 3rd order yields 3 roots being real numbers due to symmetry of σ_{ij} matrix

$$\sigma_1, \sigma_2, \sigma_3$$

These roots being eigenvalues of matrix σ_{ij} are called principal stresses

Plane stress state

In the special case of plane stress state

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

$$I_1 = \sigma_{11} + \sigma_{22} \qquad I_2 = \sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21} \qquad I_3 = 0$$

$$\sigma^3 - I_1\sigma^2 + I_2\sigma = 0 \qquad \left(\sigma^2 - I_1\sigma + I_2\right)\sigma = 0 \;\longrightarrow\; \sigma = 0$$

and:

$$\sigma_{1,2} = \frac{1}{2}\left[I_1 \pm \sqrt{\left(I_1^2 - 4I_2\right)}\right] = \ldots$$

$$\ldots\sigma_{1,2} = \frac{\sigma_{11} + \sigma_{22}}{2} \pm \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2}$$

Principal directions

Now, from the set of equations:

$$\begin{cases} (\sigma_{11}-\sigma\delta_{11})n_1+ & (\sigma_{12}-\sigma\delta_{12})n_2+ & (\sigma_{13}-\sigma\delta_{13})n_3=0 \\ (\sigma_{21}-\sigma\delta_{21})n_1+ & (\sigma_{22}-\sigma\delta_{22})n_2+ & (\sigma_{23}-\sigma\delta_{23})n_3=0 \\ (\sigma_{31}-\sigma\delta_{31})n_1+ & (\sigma_{32}-\sigma\delta_{32})n_2+ & (\sigma_{33}-\sigma\delta_{33})n_3=0 \end{cases}$$

$$\vec{n}^{(1)}\left(n_1^{(1)},n_2^{(1)},n_3^{(1)}\right) \qquad \vec{n}^{(2)}\left(n_1^{(2)},n_2^{(2)},n_3^{(2)}\right) \qquad \vec{n}^{(3)}\left(n_1^{(3)},n_2^{(3)},n_3^{(3)}\right)$$

These vectors are normal to three perpendicular planes. The stress vectors on these planes are also perpendicular to them and no shear components of stress vector exist, whereas normal stresses are equal principal stresses.

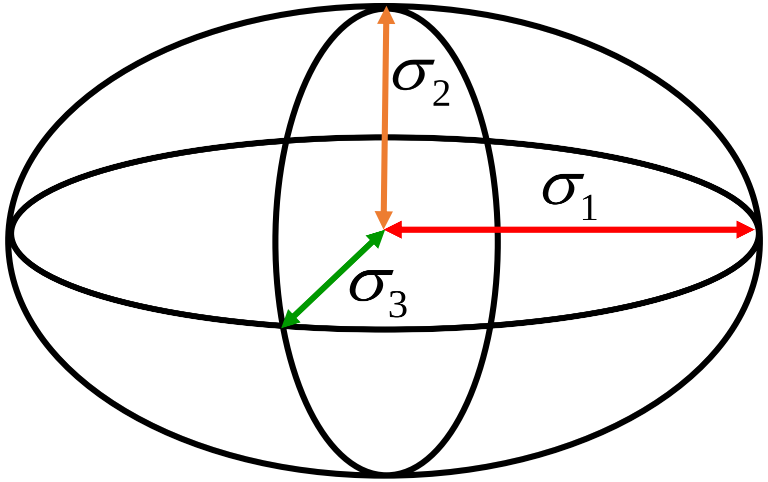
$$\vec{n}^{(1)} \Leftrightarrow \sigma_1 \qquad \vec{n}^{(2)} \Leftrightarrow \sigma_2 \qquad \vec{n}^{(3)} \Leftrightarrow \sigma_3$$

Stress ellipsoid

It can be proved that $\sigma_1, \sigma_2, \sigma_3$ are extreme values of normal stresses (stresses on a main diagonal of stress matrix) . Customary, these values are ordered as follows

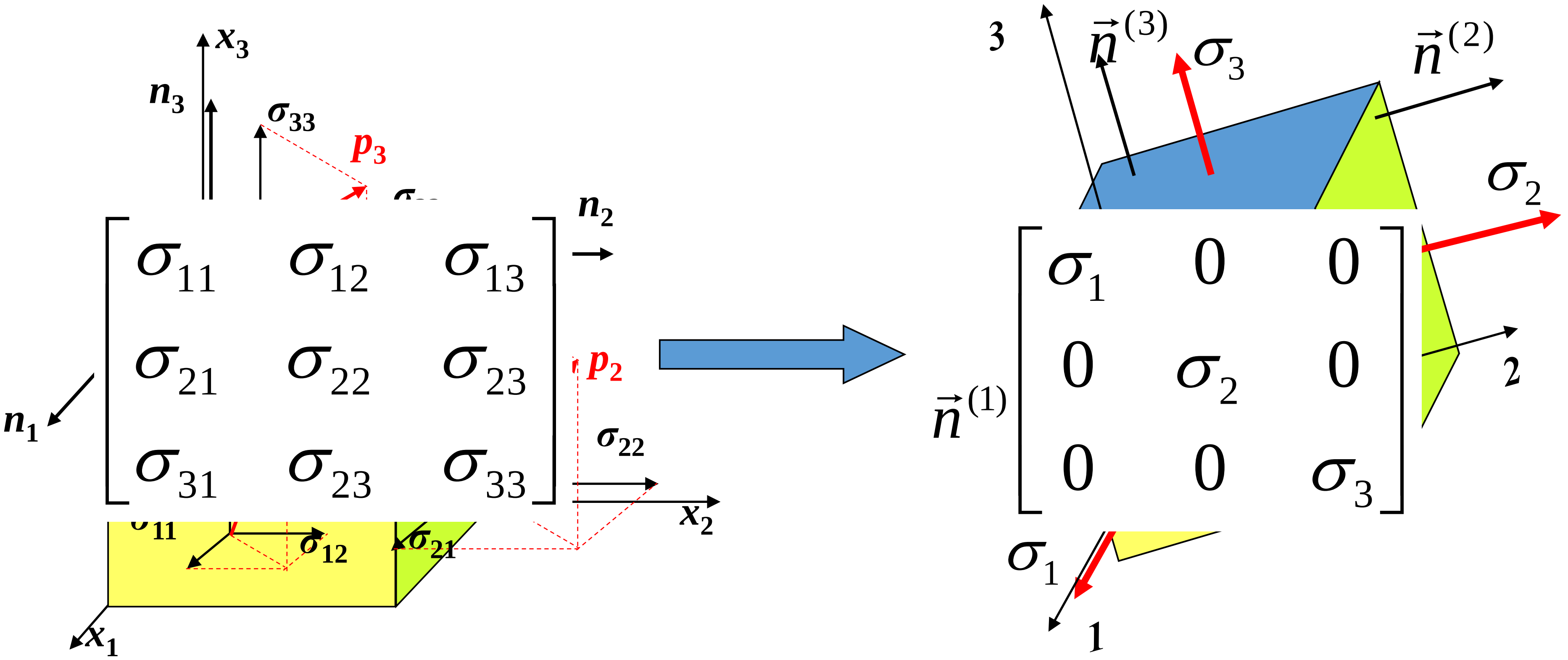
$$\sigma_1 \geq \sigma_2 \geq \sigma_3$$

Surface of an ellipsoid with semi-axis (equatorial radii) equal to the values of principal stresses represents all possible stress vectors in the chosen point and under given loading.



Principal stress – cont.

With given stress matrix in the chosen point and given loading one can find 3 perpendicular planes such that stress vectors (principal stresses) have only normal components (no shear components).
The coordinate system defined by the directions of principal stresses is called system of principal axes.

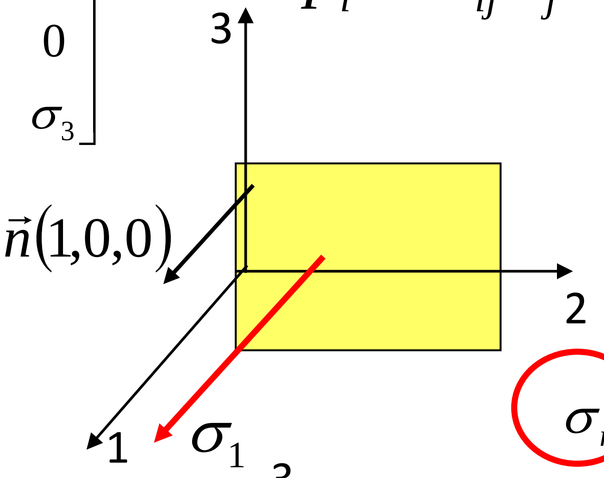


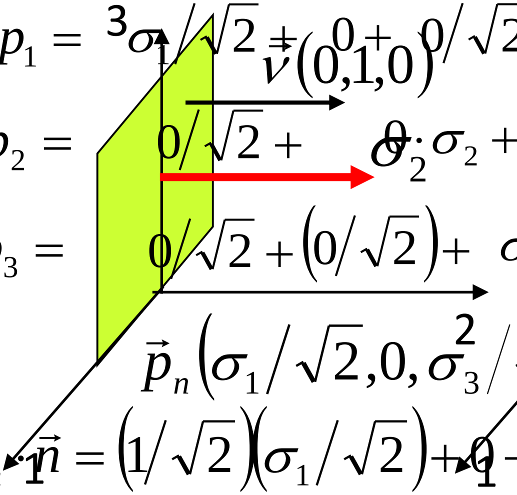
$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{23} & \sigma_{33} \end{bmatrix} \Rightarrow \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

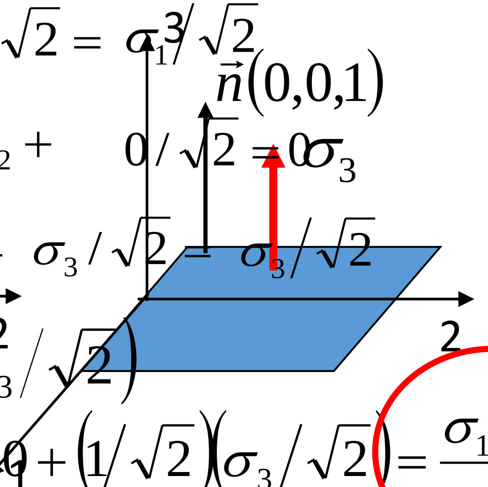
Stress on characteristic planes

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

$$p_i = \sigma_{ij} n_j$$







$$p_1 = \sigma_1/\sqrt{2} + 0 + 0/\sqrt{2} = \sigma_1/\sqrt{2}$$

$$p_2 = 0/\sqrt{2} + 0 \cdot \sigma_2 + 0/\sqrt{2} = 0$$

$$p_3 = 0/\sqrt{2} + \left(0/\sqrt{2}\right) + \sigma_3/\sqrt{2} = \sigma_3/\sqrt{2}$$

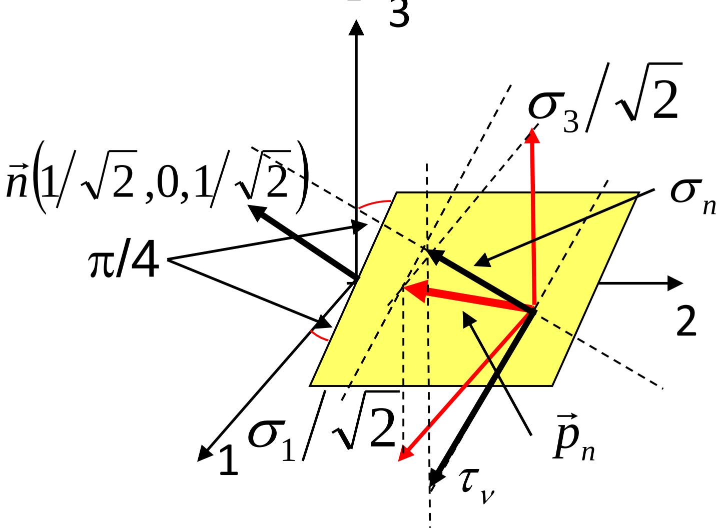
$$\vec{p}_n\left(\sigma_1/\sqrt{2}, 0, \sigma_3/\sqrt{2}\right)$$

$$\sigma_n = \vec{p}_n \cdot \vec{n} = \left(1/\sqrt{2}\right)\left(\sigma_1/\sqrt{2}\right) + 0 + \left(1/\sqrt{2}\right)\left(\sigma_3/\sqrt{2}\right) = \frac{\sigma_1 + \sigma_3}{2}$$

$$\tau_n^2 + \sigma_n^2 = \left|\vec{p}_n\right|^2 \qquad \left|\vec{p}_n\right|^2 = \sigma_1^2/2 + \sigma_3^2/2$$

$$\tau_n^2 = \left|\vec{p}_n\right|^2 - \sigma_n^2 = \sigma_1^2/2 + \sigma_3^2/2 - \left(\frac{\sigma_1 + \sigma_3}{2}\right)^2$$

$$\tau_n^2 = \sigma_1^2/4 - \frac{1}{2}\sigma_1\sigma_3 + \sigma_3^2/4 = \frac{1}{4}\left(\sigma_1 - \sigma_3\right)^2$$



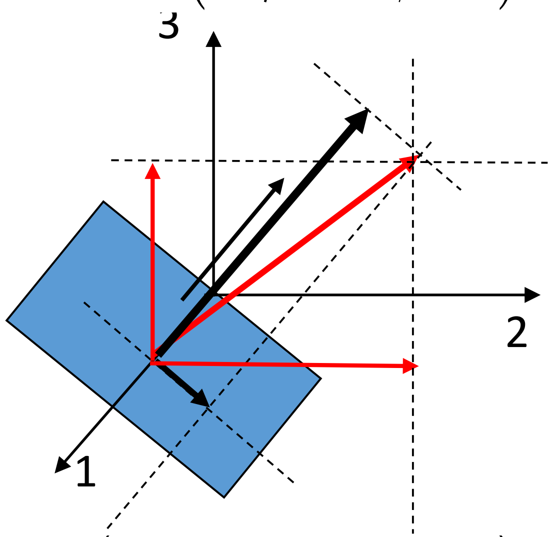
$$\tau_n = \left|\frac{\sigma_1 - \sigma_3}{2}\right|$$

Extreme normal and shear stress

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

$$p_i = \sigma_{ij} n_j$$

$\vec{n}\left(0, 1/\sqrt{2}, 1/\sqrt{2}\right)$

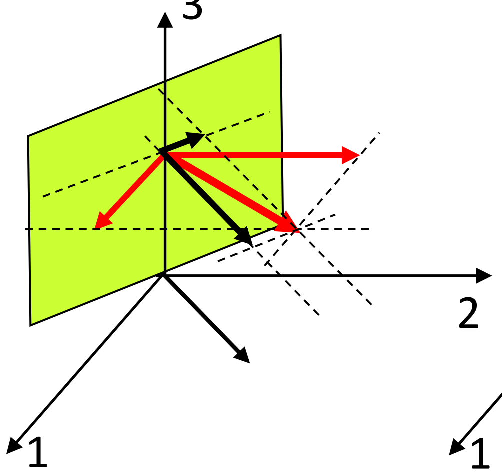


$\vec{p}_n\left(0, \sigma_2/\sqrt{2}, \sigma_3/\sqrt{2}\right)$

$$\sigma_n = \frac{\sigma_2 + \sigma_3}{2}$$

$$\tau_n = \frac{\sigma_2 - \sigma_3}{2}$$

$\vec{n}\left(1/\sqrt{2}, 1/\sqrt{2}, 0\right)$



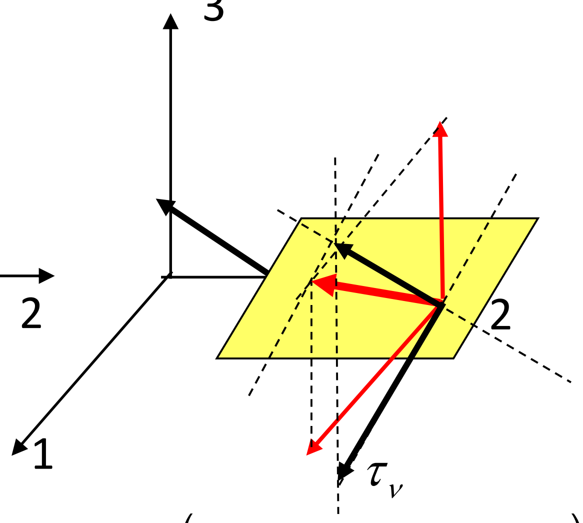
$\vec{p}_n\left(\sigma_1/\sqrt{2}, \sigma_2/\sqrt{2}, 0\right)$

max σ

$$\sigma_n = \frac{\sigma_1 + \sigma_2}{2}$$

$$\tau_n = \frac{\sigma_1 - \sigma_2}{2}$$

$\vec{n}\left(1/\sqrt{2}, 0, 1/\sqrt{2}\right)$



$\vec{p}_n\left(\sigma_1/\sqrt{2}, \sigma_2/\sqrt{2}, 0\right)$

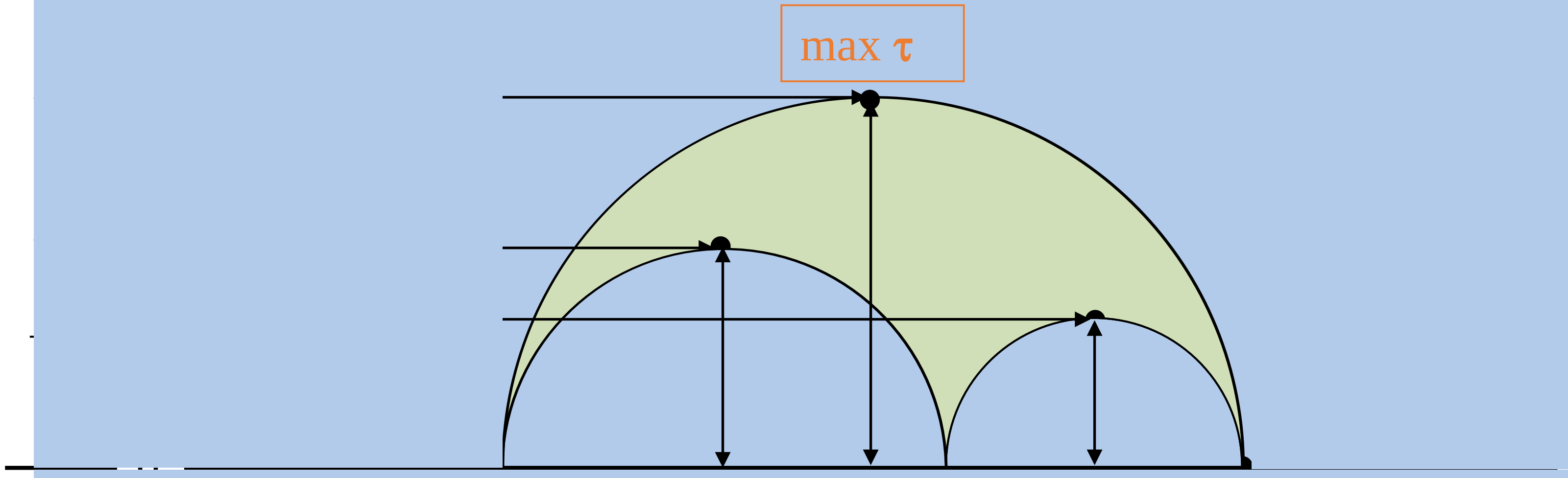
$$\sigma_n = \frac{\sigma_1 + \sigma_3}{2}$$

max τ

$$\tau_n = \frac{\sigma_1 - \sigma_3}{2}$$

Mohr circles

Mohr circles
– represent 3D state of stress in a given point – on the plane of normal and shear stresses



Mohr circles – cont.

The stress state in the cross-section is given by the stress matrix: $T_\sigma = \begin{bmatrix} \sigma_x & 0 \\ 0 & 0 \end{bmatrix}$. Determine the stress matrix at the same point for a section inclined by 30°.

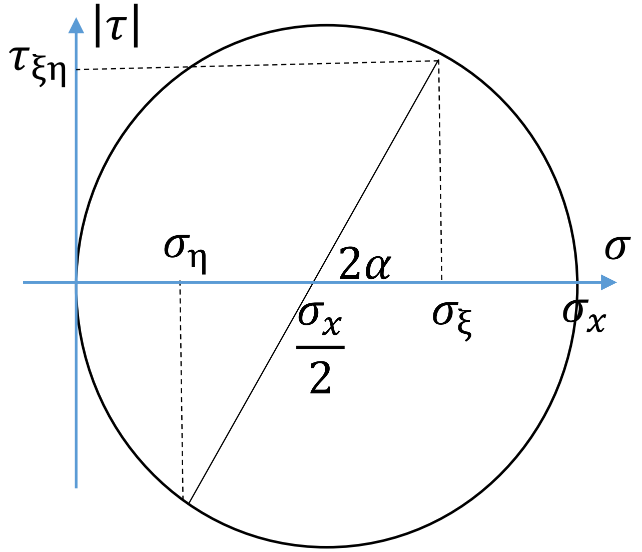
Solution

the transformation matrix: $a_{ij} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

$$\sigma_\xi = \cos^2(\xi, x)\,\sigma_x + 0 + 0 = 0.75\sigma_x$$
$$\sigma_\eta = \sin^2(\eta, x)\,\sigma_x + 0 + 0 = 0.25\sigma_x$$
$$\sigma_{\xi\eta} = \cos(\xi, x)\sin(\eta, x)\,\sigma_x + 0 + 0 = 0.433\sigma_x$$

the same with the use of Mohr circles

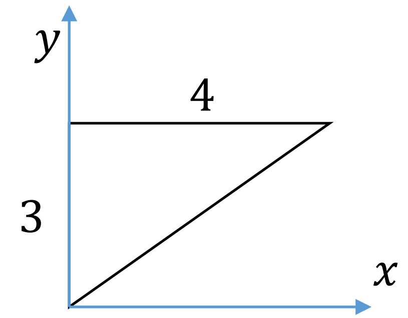
$$\sigma_\xi = \frac{\sigma_x}{2}(1 + 2\cos 2\alpha) = 0.75\sigma_x$$
$$\sigma_\eta = \frac{\sigma_x}{2}(1 - 2\cos 2\alpha) = 0.25\sigma_x$$
$$\tau_{\xi\eta} = \frac{\sigma_x}{2}\sin 2\alpha = 0.433\sigma_x$$



Example – static boundary condition

For given stress matrix in the shield, determine the volume forces and the loading on the boundary.

$$T_\sigma = \begin{bmatrix} -5x & -3x \\ -3x & 2y \end{bmatrix}$$



Solution

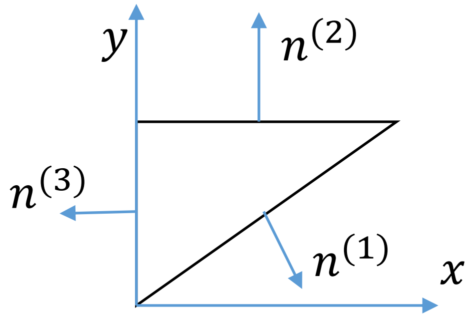
from the internal balance (Navier's) equations:

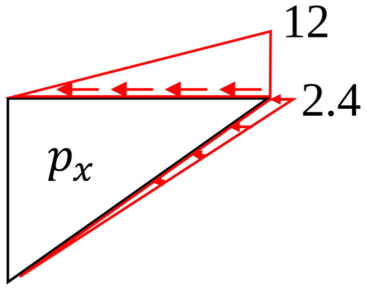
$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + P_x = 0 \rightarrow P_x = 5$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + P_y = 0 \rightarrow -3 + 2 + P_y = 0 \rightarrow P_y = 1$$

$$\rightarrow \vec{P}(5{,}1)$$

static boundary conditions: $p_i = \sigma_{ij} n_j$





boundary part (1): $\vec{n}(0.6, -0.8)$

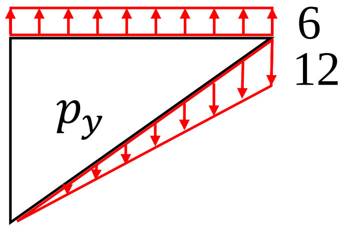
$p_x = 0.6(-5x) - 0.8(-3x) = -0.6x$

$p_y = 0.6(-3x) - 0.8(2y) = -1.8x - 1.6y$

boundary part (2): $\vec{n}(0{,}1)$

$p_x = 0(-5x) + 1 \cdot (-3x) = -3x$

$p_y = 2y = 6$



boundary part (3): $\vec{n}(-1{,}0)$

$p_x = 5x = 0$

$p_y = -3x = 0$

body forces — length

check: $\Sigma X = 5 \cdot 6 - 12 \cdot 0.5 \cdot 4 - 2.4 \cdot 0.5 \cdot 5 = 0 \quad \Sigma Y = \cdots = 0 \quad \Sigma M_0 = \cdots = 0$

Thank you for your attention!