Strength of Materials

11. Constitutive equations

General considerations

To represent interaction of material and mechanical phenomena, the following principles are respected:

- Balance of linear momentum, moment of momentum, and energy
- The Clausius-Duhem entropy inequality
- Axiom of Casuality
- Axiom of Determinizm
- Axiom of Equipresence
- Axiom of Objectivity
- Axiom of Material Invariance
- Axiom of Neighborhood
- Axiom of Memory
- Axiom of Admissibility

Boundary value problem of elasticity theory

Summary of stress and strain state equations

3 internal equilibrium equations (Navier eq.)

6 unknown functions (stress matrix components)

$$\left(P_i + \frac{\partial \sigma_{ij}}{\partial x_j} \right) = 0$$

$$p_i = \sigma_{ij} n_j$$

boundary conditions (static)

6 kinematics equations (Cauchy eq.),

9 unknown functions (6 strain matrix components, 3 displacements)

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

$$u_i \big|_{S_u} =$$

boundary conditions (kinematics)

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9 equations

15 unknown functions (6 stresses, 6 strains, 3 displacements)

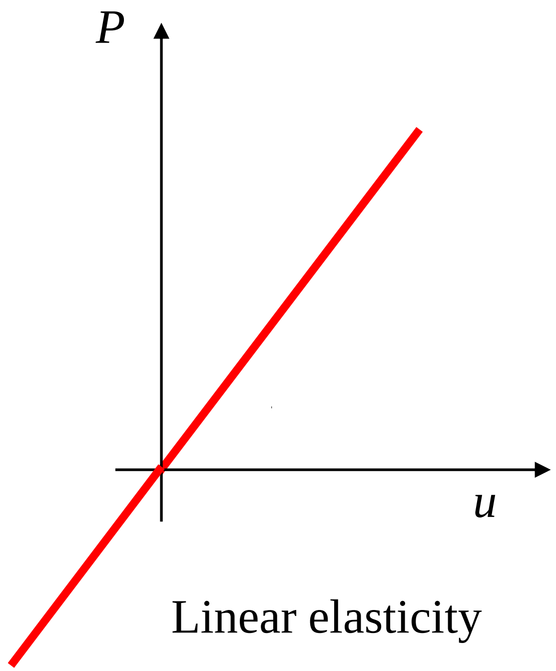
From the formal point of view (mathematics) we are lacking 6 equations

From the point of view of physics – there are no material properties involved

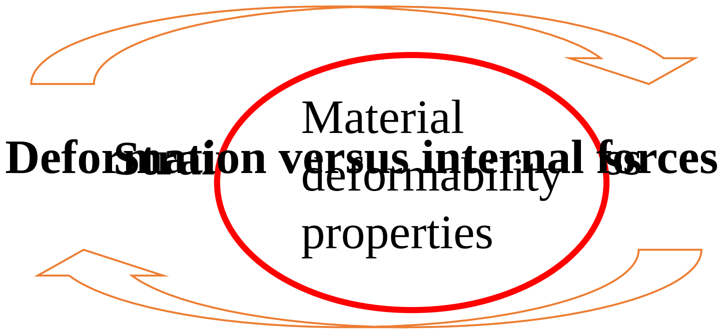
Constitutive equations – preliminaries

An obvious solution is to exploit already noticed interrelation between strains and stresses

General property of majority of solids is elasticity (instantaneous shape memory)



Linear elasticity



This observation was made already in 1676 by Robert Hooke:

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which reads:
„as much the extension as the force is"

$$\vec{P} = k\vec{u}$$

where k is a constant dependent on a material and body shape

Preliminaries – cont.

To make Hooke's law independent of a body shape one has to use state variables characterizing internal forces and deformations in a material point i.e. stress and strain.

$$\vec{P} = k\vec{u}$$

	Dynamics	Kinematics	
Physical quantities (measurable)	$\vec{P}$	$\vec{u}$	Hooke, 1678
Mathematical quantities (non-measurable)	σ_{ij}	ε_{ij}	Navier, 1822

$$\sigma_{ij} = f\left(\varepsilon_{ij}\right)$$

$\boldsymbol{f}$ - linear function of all strain matrix components defining all stress component matrix

Constitutive equations

As constitutive equations set is a relationships between 9 components of the stress state and 9 components of strain state then the number of coefficient in this set is 81 and can be represented as a matrix of 3^4=81 components:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$$

The coefficients of this equation do depend only on the material considered, but not on the body shape.

Summation over *kl* indices reflects linear character of this constitutive equation.

Universality of linear elasticity follows observation, that for loading below a certain limit (elasticity limit) most of materials exhibit this property.

Nevertheless, the number of assumptions allows for the reduction of the coefficients number: two of them are already inscribed in the formula:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$$

1. For zero valued deformations all stresses vanish: the body in a natural state is free of initial stresses.

2. Coefficients C_{ijkl} do not depend on position in a body – material properties are uniform (homogeneous).

Constitutive equations – cont.

3. Assumption of the existence of elastic potential yields symmetry of group of indices ij - kl thus reducing the number of independent coefficients to
   36 [=(81-9)/2].

4. Symmetry of material inner structure allows for further reductions. In a general case of lacking any symmetry (anisotropy) the number of independent coefficients is 21 [=(36-6)/2+6].

5. In the simplest and the most frequent case of structural materials (except the composite materials) – the number of coefficients is 2 (isotropy):

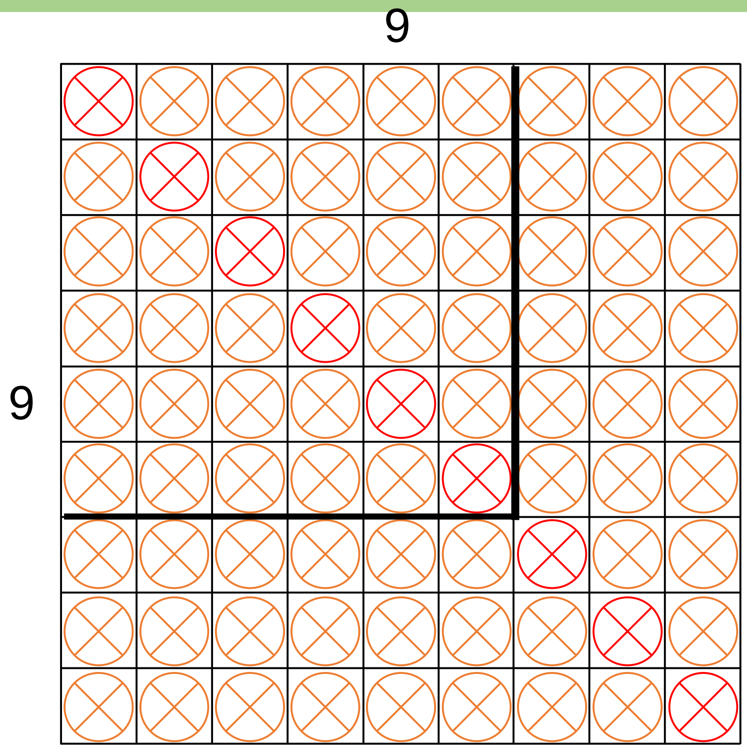
$$\boxed{\sigma_{ij} = 2G\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}}$$

or in an inverse form:

$$\boxed{\varepsilon_{ij} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\right]}$$

The pairs of coefficients G,λ i E, ν are mutually dependent so there are really only two material independent constants.

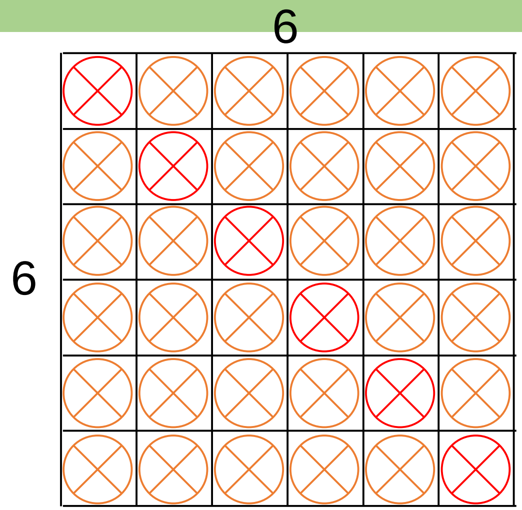
Stiffness matrix



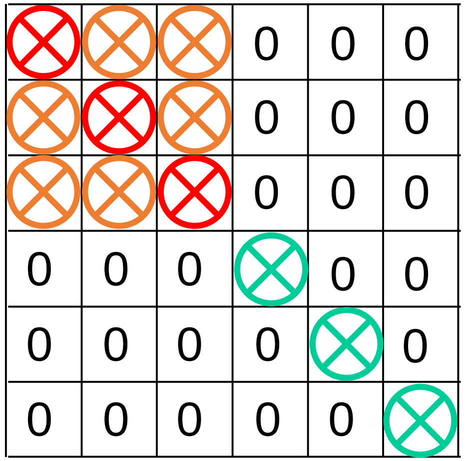
81 components of C_{ijkl}

$$\sigma_{ij} = \sigma_{ji}$$

$$\varepsilon_{ij} = \varepsilon_{ji}$$



General anisotropy: 15+6=21 constants

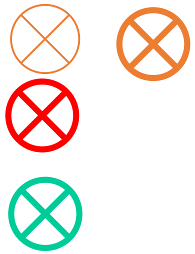


Isotropy: 2 constants

Symmetrical components

Identical components

Components dependent on other components



Linear and angular strain relationships

$$\boxed{\sigma_{ij} = 2G\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}}$$

$\boxed{G, \lambda}$ Lamé constants [Pa]

Summation obeys !

Kronecker's delta

This equation consists of two groups:

Normal stress and normal strain dependences

$$\begin{cases} \sigma_{11} = 2G\varepsilon_{11} + \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) \\ \sigma_{22} = 2G\varepsilon_{22} + \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) \\ \sigma_{33} = 2G\varepsilon_{33} + \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) \end{cases}$$

Shear stress and shear strain dependences

$$\begin{cases} \sigma_{12} = 2G\varepsilon_{12} \\ \sigma_{23} = 2G\varepsilon_{23} \\ \sigma_{31} = 2G\varepsilon_{31} \end{cases}$$

Material constants

$$\boxed{\varepsilon_{ij} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\right]}$$

Summation obeys !

Kronecker's delta

$\boxed{E}$ Young modulus [Pa]

$\boxed{\nu}$ Poisson modulus [0]

$$\varepsilon_{11} = \frac{1}{E}\left[(1+\cancel{\nu)\sigma_{11}} - \nu(\cancel{\sigma_{11}} + \sigma_{22} + \sigma_{33})\right]$$

Normal stress and normal strain dependences

$$\begin{cases} \varepsilon_{11} = \left[\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})\right]/E \\ \varepsilon_{22} = \left[\sigma_{22} - \nu(\sigma_{11} + \sigma_{33})\right]/E \\ \varepsilon_{33} = \left[\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})\right]/E \end{cases}$$

Shear stress and shear strain dependences

$$\begin{cases} \varepsilon_{12} = (1+\nu)\sigma_{12}/E \\ \varepsilon_{23} = (1+\nu)\sigma_{23}/E \\ \varepsilon_{31} = (1+\nu)\sigma_{31}/E \end{cases}$$

$$\sigma_{31} = 2G\varepsilon_{31} \qquad \varepsilon_{31} = \sigma_{31}/2G$$

$$\Rightarrow \quad \boxed{G = E/2(1+\nu)}$$

Volumetric change relationship

$\boxed{i = j = l}$ $$\sigma_{ij} = 2G\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}$$

$$\sigma_{ll} = 2G\varepsilon_{ll} + \lambda\varepsilon_{kk}\delta_{ll}$$

$\boxed{=3}$

$$\sigma_{ll} = \sigma_{11} + \sigma_{22} + \sigma_{33} = 3\sigma_m$$

$\sigma_m = \sigma_{ll}/3$ Mean stress

$\varepsilon_m = \varepsilon_{kk}/3$ Mean strain

$$\cancel{3}\sigma_m = 2G \cdot \cancel{3}\varepsilon_m + \lambda 3\varepsilon_m \cdot \cancel{3}$$

$$\sigma_m = (2G + 3\lambda)\varepsilon_m$$

$$\boxed{\sigma_m = 3K\varepsilon_m}$$

$$3K = 2G + 3\lambda$$

Volume change law

$$\sigma_m \cdot \delta_{ij} = 3K\varepsilon_m \cdot \delta_{ij}$$

$$\begin{Bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{Bmatrix} = 3K \begin{Bmatrix} \varepsilon_m & 0 & 0 \\ 0 & \varepsilon_m & 0 \\ 0 & 0 & \varepsilon_m \end{Bmatrix}$$

Tensor decomposition – isotropic part and not

Decomposition of symmetric matrix (tensor) into deviator and volumetric part (mean values)

$$T(t_{ij}) \qquad t_{ij} = t_{ij}\underbrace{\qquad\qquad}_{\text{deviator}}\;\underbrace{\qquad\qquad}_{\text{isotropic part}} \qquad t_m = t_{kk}/3$$

$$\left\{\begin{matrix} t_{11} & t_{12} & t_{13} \\ t_{22} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{matrix}\right\} = \left\{\begin{matrix} t_{11} - t_m & t_{12} & t_{13} \\ t_{22} & t_{22} - t_m & t_{23} \\ t_{31} & t_{32} & t_{33} - t_m \end{matrix}\right\} + \left\{\begin{matrix} t_m & 0 & 0 \\ 0 & t_m & 0 \\ 0 & 0 & t_m \end{matrix}\right\}$$

$$T_t = D_t + A_t$$

$$D_t(t_{ij} - t_m \delta_{ij}) \qquad A_t(t_m \delta_{ij})$$

Forms of the Hooke's law

$$\sigma_m \delta_{ij} = 3K\varepsilon_m \delta_{ij} \qquad \begin{Bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{Bmatrix} = 3K \begin{Bmatrix} \varepsilon_m & 0 & 0 \\ 0 & \varepsilon_m & 0 \\ 0 & 0 & \varepsilon_m \end{Bmatrix}$$

$$\boxed{A_\sigma = 3KA_\varepsilon}$$

Volume change law

$$\sigma_{ij} - \sigma_m\delta_{ij} = 2G\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij} - 3K\varepsilon_m\delta_{ij} =$$

$$= 2G\varepsilon_{ij} - (3K\varepsilon_m - 3\lambda\varepsilon_m)\delta_{ij} =$$

$$3K = 2G + 3\lambda$$

$$= 2G\varepsilon_{ij} - ([2G + 3\cancel{\lambda}]\varepsilon_m - 3\cancel{\lambda}\varepsilon_m)\delta_{ij} =$$

$$= 2G\varepsilon_{ij} - (2G\varepsilon_m)\delta_{ij} = 2G(\varepsilon_{ij} - \varepsilon_m\delta_{ij})$$

$$\boxed{D_\sigma = 2GD_\varepsilon}$$

Distortion law

Volume and shape change – cont.

$$D_\sigma = 2GD_\varepsilon \qquad A_\sigma = 3KA_\varepsilon \qquad \boxed{T_\sigma = 3KA_\varepsilon + 2GD_\varepsilon}$$

$$0 \leftarrow D_\varepsilon = D_\sigma / 2G \qquad A_\varepsilon = A_\sigma / 3K \qquad \boxed{T_\varepsilon = A_\sigma / 3K + D_\sigma / 2G}$$

$$\boxed{\sigma_{ij} = 2G\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}}$$

$$\boxed{\varepsilon_{ij} = \{(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\}/E}$$

-1		ν		1/2
∞		G		E/3
E/9		K		∞
- ∞		λ		∞

0

No shape change

No volume change: incompressible (stiff) material

Constants cross-relations:

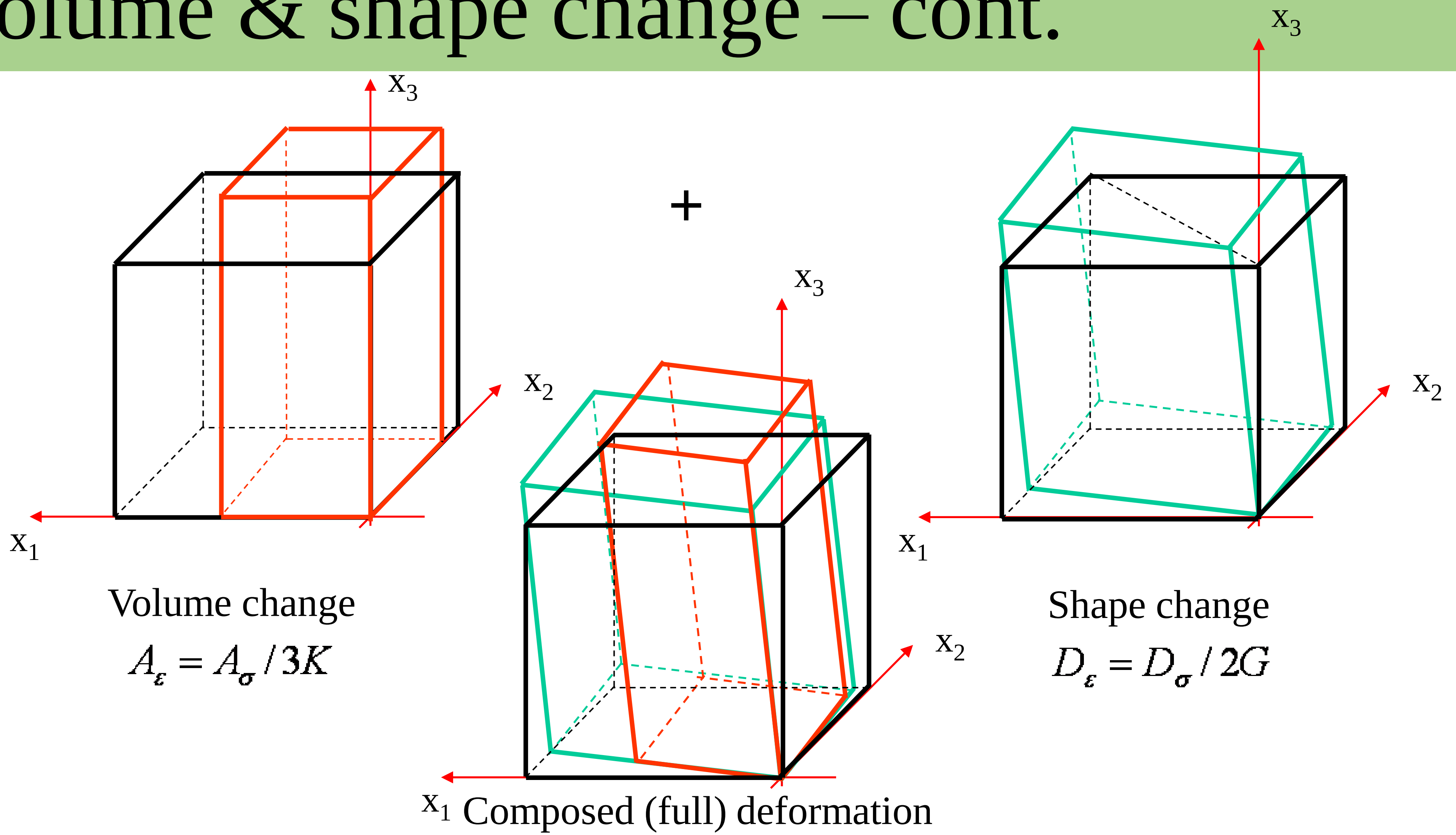
$$G = E/2(1+\nu)$$

$$K = E/3(1-2\nu)$$

$$3K = 2G + 3\lambda$$

$$\lambda = E\nu/(1+\nu)(1-2\nu)$$

Volume & shape change – cont.



Volume change

$$A_\varepsilon = A_\sigma / 3K$$

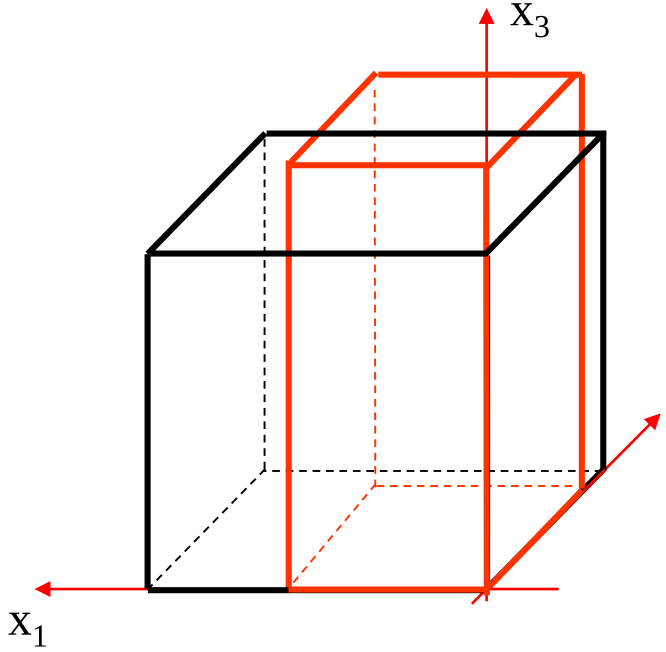
Shape change

$$D_\varepsilon = D_\sigma / 2G$$

Composed (full) deformation

$$T_\varepsilon = A_\sigma / 3K + D_\sigma / 2G$$

Volume & shape change – cont.



Volume change

$$A_\varepsilon = A_\sigma / 3K$$

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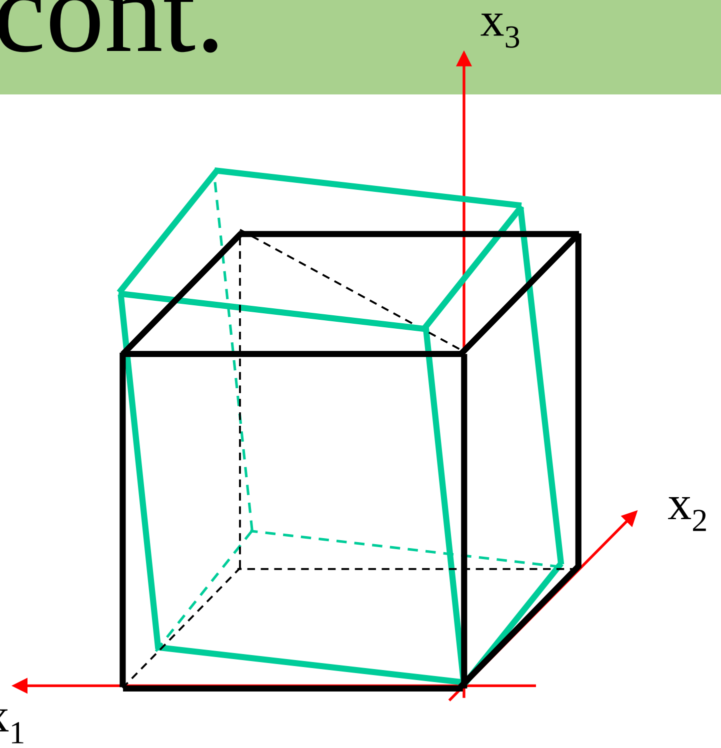
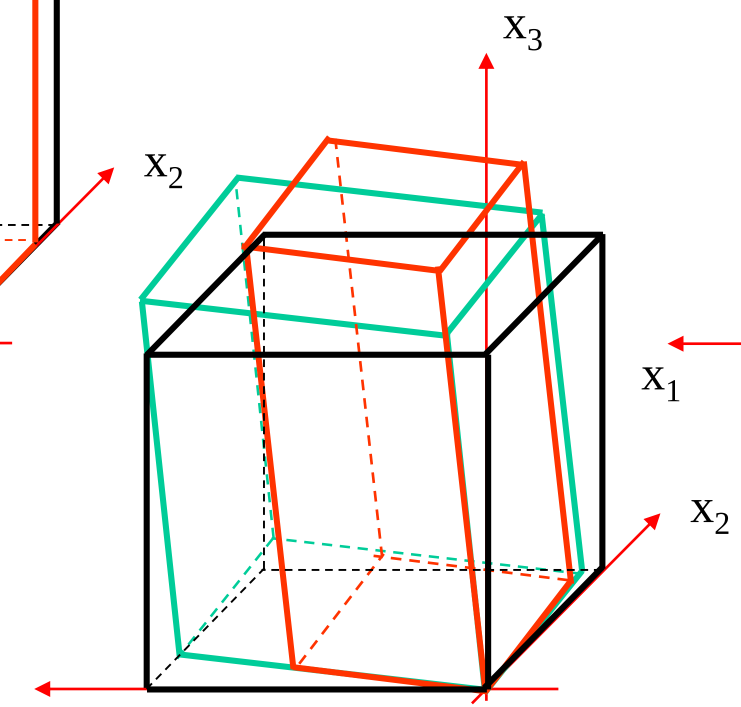


Shape change

$$D_\varepsilon = D_\sigma / 2G$$



Composed (full) deformation

$$T_\varepsilon = A_\sigma / 3K + D_\sigma / 2G$$

Thank you for your attention!