Strength of Materials

12. Theory of linear elasticity

Internal balance (Navier's) equations

$$\sigma_{ij,j} + P_i = 0$$

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + P_x = 0$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + P_y = 0$$
$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + P_z = 0$$

σ_x [MPa]

$P \left[\frac{N}{m^3}\right]$

static boundary conditions

$$q_i = \sigma_{ij} n_j$$

$$q_x = \sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z$$
$$q_y = \tau_{xy} n_x + \sigma_y n_y + \tau_{yz} n_z$$
$$q_z = \tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z$$

3 equations
6 unknowns $(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz})$

Problem is statically undetermined

Geometric (Cauchy's) equations

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

$$\varepsilon_x = \frac{\partial u}{\partial x}$$
$$\varepsilon_y = \frac{\partial v}{\partial y}$$
$$\varepsilon_z = \frac{\partial w}{\partial z}$$
$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$
$$\varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$
$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial y}\right)$$

ε_{ij} [1]

Kinematic boundary conditions /Some restrictions (constraints) on u_i or $u_{i,j}$ /

(compatibility equations: $\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0$)

6 equations
9 unknowns $(\varepsilon_x, \varepsilon_y, \varepsilon_z, \varepsilon_{xy}, \varepsilon_{xz}, \varepsilon_{yz})$ and (u, v, w)

Hooke's equations

$$\sigma_{ij} = 2G\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}$$

$$\sigma_x = 2G\varepsilon_x + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z)$$
$$\sigma_y = 2G\varepsilon_y + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z)$$
$$\sigma_z = 2G\varepsilon_z + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z)$$
$$\tau_{xy} = 2G\varepsilon_{xy}$$
$$\tau_{xz} = 2G\varepsilon_{xz}$$
$$\tau_{yz} = 2G\varepsilon_{yz}$$

$$\varepsilon_{ij} = \frac{1}{E}[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}]$$

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$
$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)]$$
$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$
$$\varepsilon_{xy} = \frac{1+\nu}{E}\tau_{xy}$$
$$\varepsilon_{xz} = \frac{1+\nu}{E}\tau_{xz}$$
$$\varepsilon_{yz} = \frac{1+\nu}{E}\tau_{yz}$$

6 equations

$$\sigma_m = 3K\varepsilon_m$$
$$\sigma_{ij} - \sigma_m\delta_{ij} = 2G(\varepsilon_{ij} - \varepsilon_m\delta_{ij})$$

E – elastic (Young) modulus

G – shear (Kirchhoff's) modulus

ν – Poisson's ratio

K – bulk modulus

$$G = \frac{E}{2(1+\nu)}$$

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$ Lamé const.

$$K = \frac{E}{3(1-2\nu)}$$

Boundary value problem

15 variables
15 equations

+ static boundary conditions on ∂S_σ and kinematic boundary conditions on ∂S_v

→ boundary value problem (BVP)

The existence and uniqueness of solution to the BVP has been proved by:

- G. Kirchhoff (1859)
- I. Fredholm
- G. Lauricell
- E. and F. Cosserat
- A. Korn
- L. Lichtenstein
- H. Weyl

The first type of solution methods: elimination of variables (unknowns reduction)

Boundary value problem – cont.

The set of Navier's equations + Cauchy's equations + Hooke's equations consists of 15 linear differential-algebraic equations – and is always the same for any static problem (except of material constants in Hooke's equations).

Individual problems are different only due to different boundary conditions, which define body shape n_i, loading q_i and displacements u_i on the body surface (at the supports). Here is where name Boundary Value Problem of Elasticity comes from.

Analytical methods

Reduction of the unknown fuctions number in exchange for upgrading the differential equations order

a/ Elimination of displacements by transforming Cauchy's eqs. into compatibility equations and the use of Hooke's eqs.; this yields the set of 6 differential equations of the second order for stress components (Beltrami-Mitchell equations):

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad \varepsilon_{ij} = \left\{(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\right\}/E$$

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0$$

b/ Substitution of Cauchy's eqs. to Hooke's eqs. and next to Navier's eqs.; this yields the set of 3 differential equations of the second order for displacements as unknowns (Lamé equations):

$$\left(P_i + \frac{\partial \sigma_{ij}}{\partial x_j}\right) = 0 \qquad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad \sigma_{ij} = 2G\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}$$

Method of forces

Compatibility equations → Hooke's equations → 6 equations in stress components

Beltrami-Mitchell equations (1892, 1900)

$$\nabla^2\sigma_x + \frac{1}{1+\nu}\frac{\partial^2\sigma_{kk}}{\partial x^2} = -\frac{\nu}{1-\nu}\left(\frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}\right) - 2\frac{\partial P_x}{\partial x}$$

$$\nabla^2\sigma_y + \frac{1}{1+\nu}\frac{\partial^2\sigma_{kk}}{\partial y^2} = -\frac{\nu}{1-\nu}\left(\frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}\right) - 2\frac{\partial P_y}{\partial y}$$

$$\nabla^2\sigma_z + \frac{1}{1+\nu}\frac{\partial^2\sigma_{kk}}{\partial z^2} = -\frac{\nu}{1-\nu}\left(\frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}\right) - 2\frac{\partial P_z}{\partial z}$$

$$\nabla^2\tau_{xy} + \frac{1}{1+\nu}\frac{\partial^2\sigma_{kk}}{\partial x\partial y} = -\left(\frac{\partial P_x}{\partial y} + \frac{\partial P_y}{\partial x}\right)$$

$$\nabla^2\tau_{xz} + \frac{1}{1+\nu}\frac{\partial^2\sigma_{kk}}{\partial x\partial z} = -\left(\frac{\partial P_z}{\partial x} + \frac{\partial P_x}{\partial z}\right)$$

$$\nabla^2\tau_{yz} + \frac{1}{1+\nu}\frac{\partial^2\sigma_{kk}}{\partial z\partial y} = -\left(\frac{\partial P_z}{\partial y} + \frac{\partial P_y}{\partial z}\right)$$

Method of displacements

Cauchy's equations → Hooke's equations → Navier's equations

3 equations in displacements: Lamé equations

$$\nabla^2 u + \frac{1}{1-2\nu}\frac{\partial \varepsilon_{kk}}{\partial x} + \frac{P_x}{G} = 0$$
$$\nabla^2 v + \frac{1}{1-2\nu}\frac{\partial \varepsilon_{kk}}{\partial y} + \frac{P_y}{G} = 0$$
$$\nabla^2 w + \frac{1}{1-2\nu}\frac{\partial \varepsilon_{kk}}{\partial z} + \frac{P_z}{G} = 0$$

BVP solution

Direct methods:
- analytical (very complex, wide-spread math apparatus needed, like complex variables methods etc.)
- numerical (FEM, FDM, BIM), usually based on variational principles

Much easier but restricted in use:

Semi-inverse methods:
- static approach
- kinematic approach

Static approach

We guess the stress matrix, it should fulfill:

- the Navier's (internal balance) equations as well as
- the static boundary equations

The algorithm is as follows:

Calculate the strain matrix from Hooke's equations using the estimated stress matrix

Check the compatibility equations

Get displacements by integration of the Cauchy's (geometric) equations

Check the kinematic boundary conditions

If the displacements fulfill the kinematic boundary conditions, the assumed stress matrix, resulting strain matrix and the displacements are solution to the BVP.

Kinematic approach

The algorithm is as follows:

We guess the displacement vector functions, continuous and differentiable, that fulfill the kinematic boundary conditions

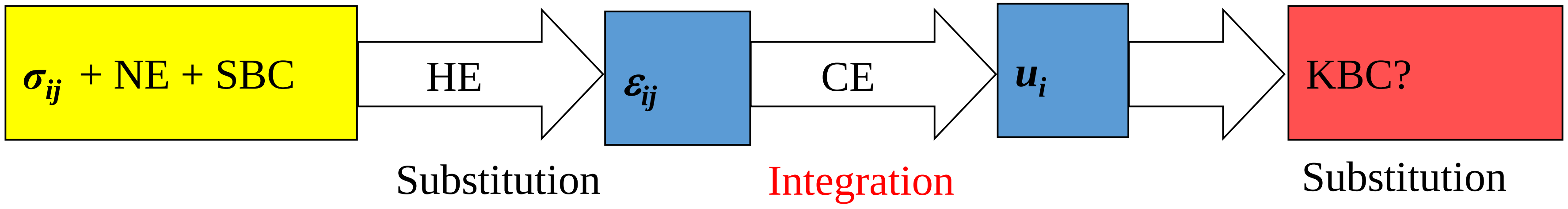
From Cauchy's equation we calculate the strain matrix (obviously, it fulfills the compatibility equations due to displacements continuity)

Using the Hooke's equations we determine stress matrix

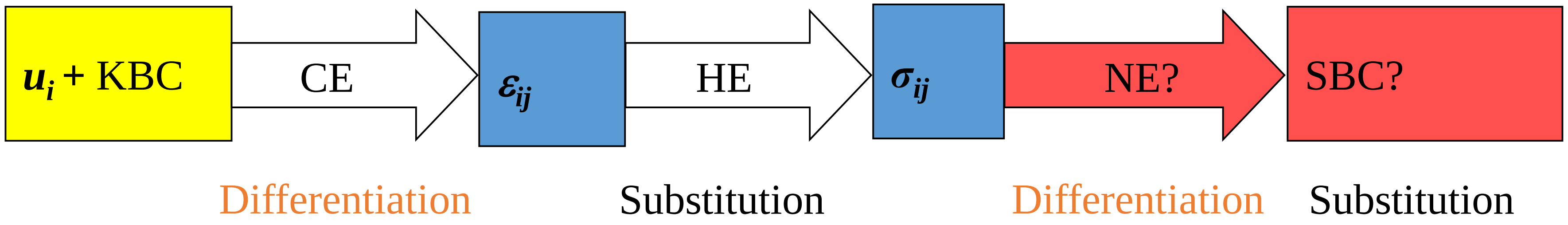
If the stress matrix fulfill the Navier's equations with the static boundary conditions, the solution has been found.

Comparison of the semi-inverse methods

a/ Stress approach:



b/ Kinematic approach



Out of these two semi-inverse methods, the kinematic approach seems to be superior as it requires only three displacements to be guessed which are physical quantities and can be measured experimentally. Moreover, only two operations to be performed are substitution and differentiation, the latter being much easier than integration required by stress approach.

The price to be paid in the kinematic approach is a necessity of checking Navier's Equation of equilibrium and Static Boundary Conditions.

Numerical methods

Numerical Methods features space discretization and application of one of numerous methods: development of all functions sought into power series, finite differences, finite elements, boundary integrals, meshless methods etc.

Numerical methods are discussed in detail as a separate subject of curriculum and will not be dealt with here. However, it is worthwhile to emphasize that numerical methods allow for overcoming of the fundamental problem of theory of elasticity which is solving problems with singular boundary conditions (sharp edges of structures, concentrated loadings etc.)

Superposition principle

a loading $q_i^{(1)}, P_i^{(1)}$ with a solution $\sigma_{ij}^{(1)}, \varepsilon_{ij}^{(1)}, u_i^{(1)}$

a loading $q_i^{(2)}, P_i^{(2)}$ with a solution $\sigma_{ij}^{(2)}, \varepsilon_{ij}^{(2)}, u_i^{(2)}$

then for the loading $q_i^{(1)} + q_i^{(2)}, P_i^{(1)} + P_i^{(2)}$

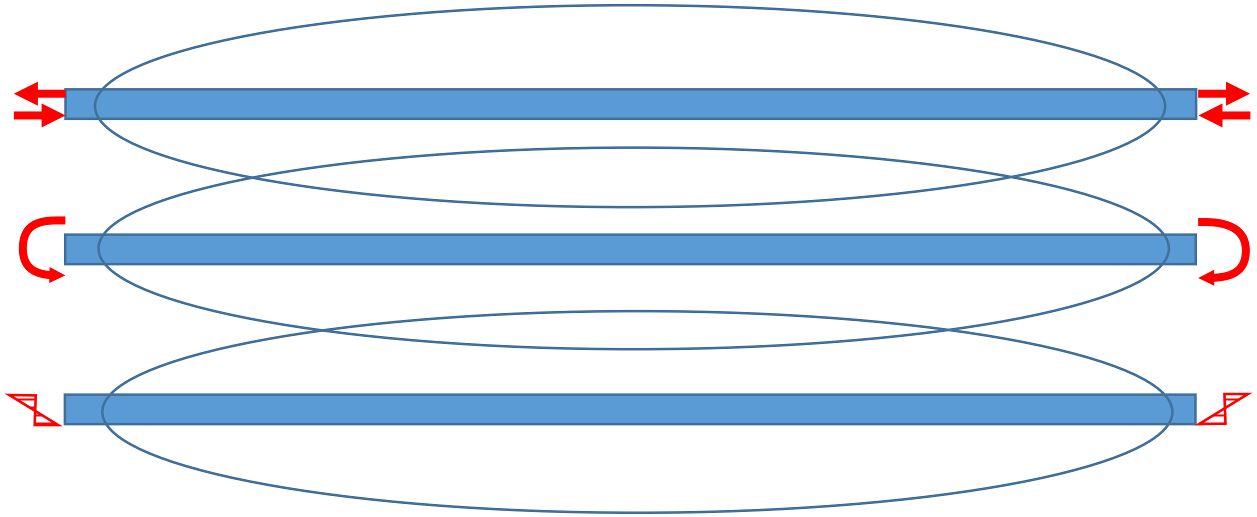
we have:

$$\sigma_{ij} = \sigma_{ij}^{(1)} + \sigma_{ij}^{(2)} \qquad \varepsilon_{ij} = \varepsilon_{ij}^{(1)} + \varepsilon_{ij}^{(2)} \qquad u_i = u_i^{(1)} + u_i^{(2)}$$

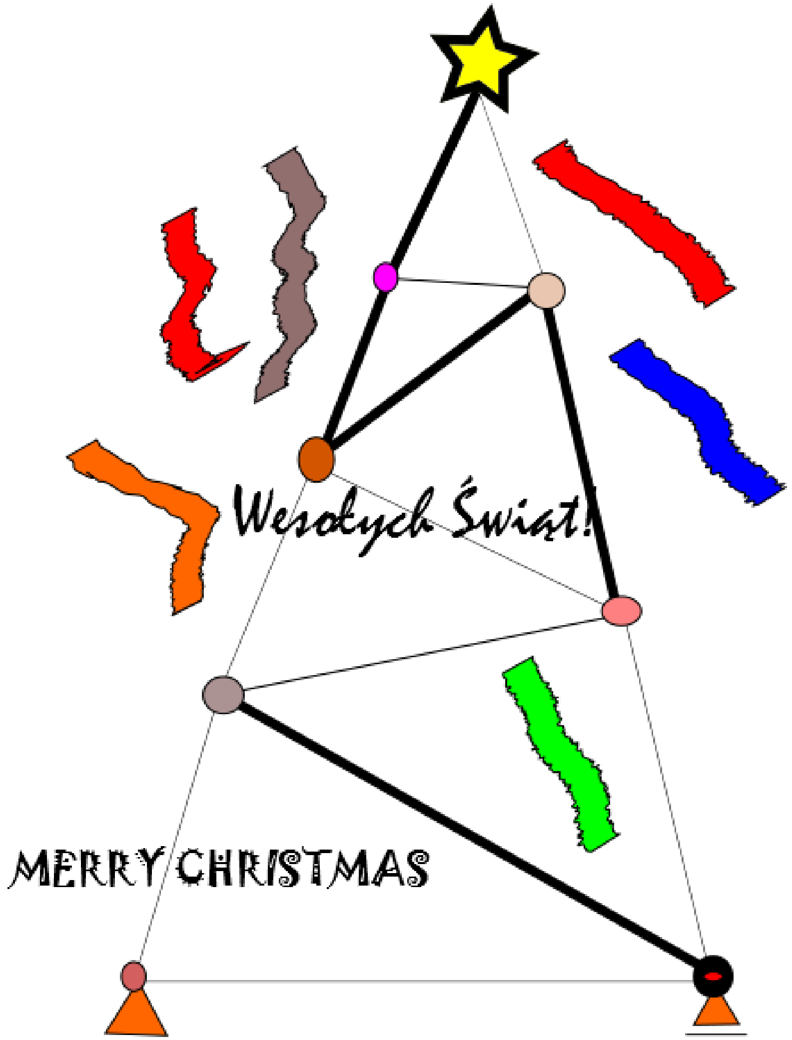
de Saint-Venant's principle

The difference between the effects of two different but statically equivalent loads (applied on sufficiently small part of the boundary) becomes very small at sufficiently large distances from load

Explanation: a bar element loadings at the ends



(in the great majority of the body the states of stress, strains and displacements are the same for equivalent loads)


Wesołych Świąt!
MERRY CHRISTMAS

Thank you for your attention!