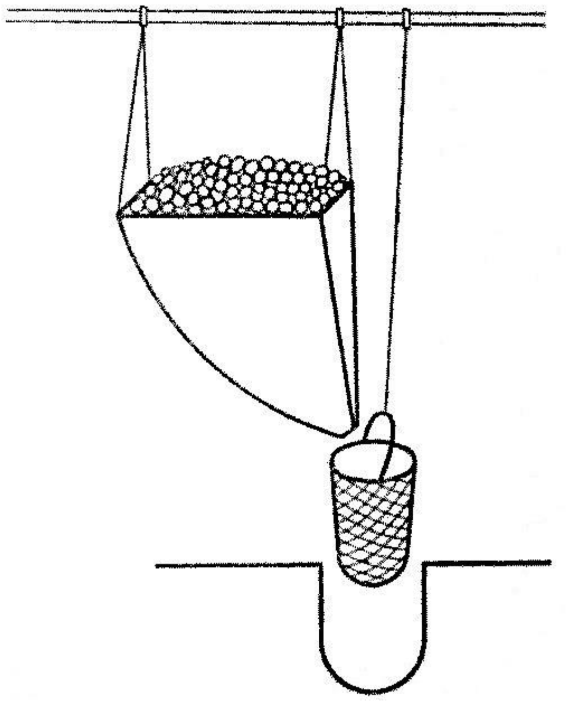
Strength of Materials

1. Tension

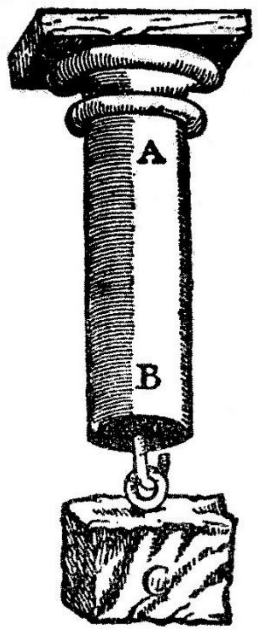
Tension – the first experiments

Ropes: lifting heavy elements/stones; ship tackling, etc.   Wires!

Leonardo da Vinci (XV cent.)

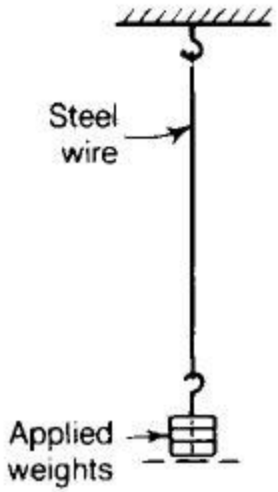


- Domenico Galileo Galilei (XVI-XVII cent.)

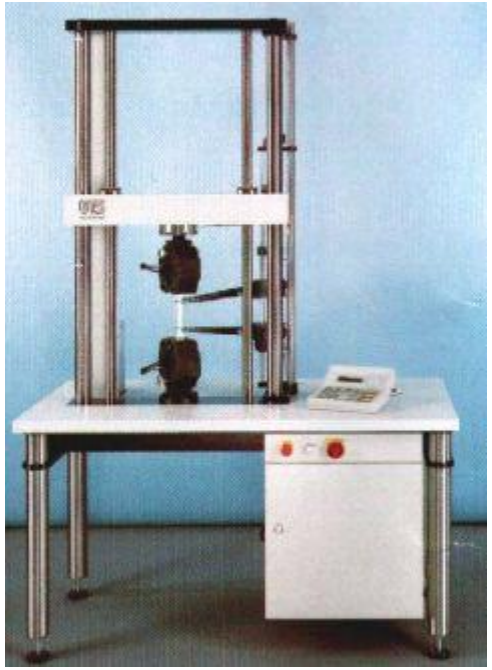


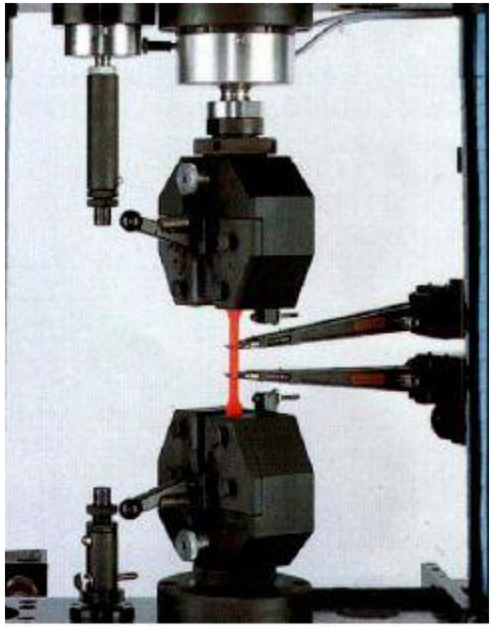
$$\sigma = \frac{P}{A}$$

Tension – contemporary experiments



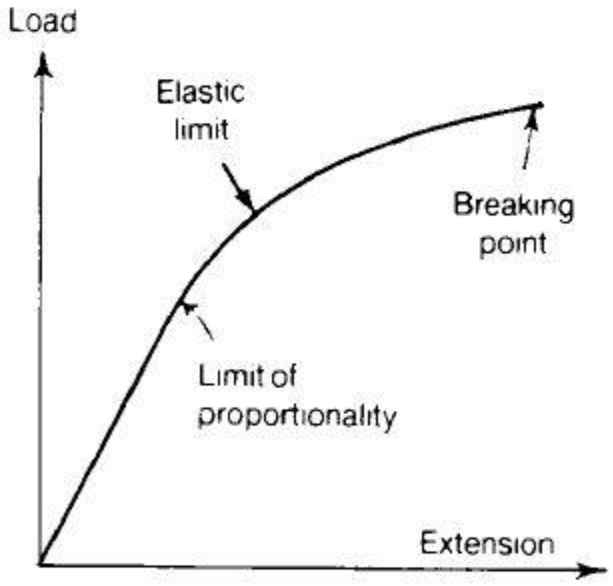
general idea





UTM/UTS

Tension – steel wire specimen

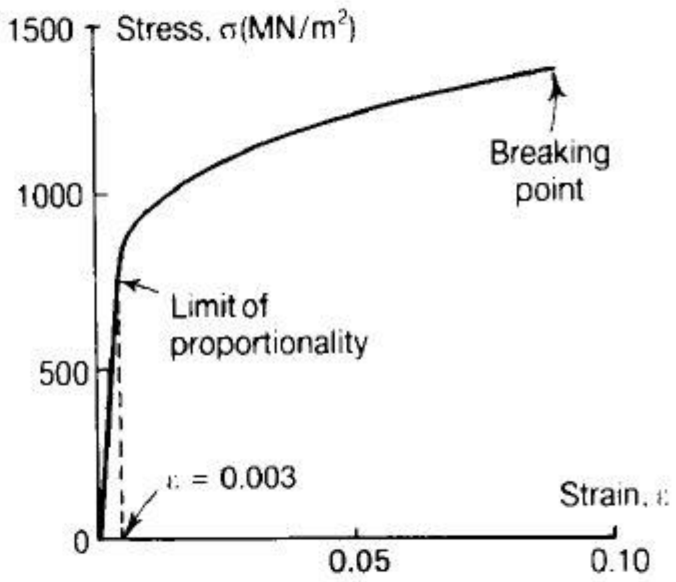


Load-extension curve for a steel wire

several stages:

- proportionality
- elasticity (reversibility)
- plasticity (irreversibility)
- breaking (max load)

Tension – high-strength steel



Tensile stress-strain for a high-strength steel

axis *x* – „engineering" strain measure

strain measures

engineering, Cauchy strain

$$\varepsilon = \frac{l_1 - l_0}{l_0} = \frac{l_1}{l_0} - 1 \rightarrow l_1 = (\varepsilon + 1) l_0$$

$$l = l_0(1+\varepsilon_1)(1+\varepsilon_2) \ =? \ l_0(1+\varepsilon_1) + l_0(1+\varepsilon_2)$$

small strains ($\ll 1$): $\varepsilon_1 = 0.0002, \varepsilon_2 = 0.0003$

$$l = 1.0002 \cdot 1.0003 l_0 = 1.00050006 l_0 \approx$$
$$1.0002 l_0 + 1.0003 l_0 = 1.0005 l_0$$

we observe additivity

finite strains: $\varepsilon_1 = 0.2, \varepsilon_2 = 0.3$

$$l = l_0 \cdot 1.2 \cdot 1.3 = 1.56 l_0$$

$$l = 1.2 l_0 + 1.3 l_0 = 1.5 l_0 \text{ (error 4\%)}$$

strain measures – cont.

true, Hencky strain

$$\varepsilon_H = \ln\frac{l_1}{l_0} = \ln\frac{l_1 - l_0 + l_0}{l_0} = \ln(\varepsilon + 1)$$

$$\varepsilon \ll 1: \quad \varepsilon_1 = 0.0002, \varepsilon_2 = 0.0003$$

$\varepsilon_H = \ln(1.0002) + \ln(1.0003) = \ln(1.0002 \cdot 1.0003) =$0.0004999

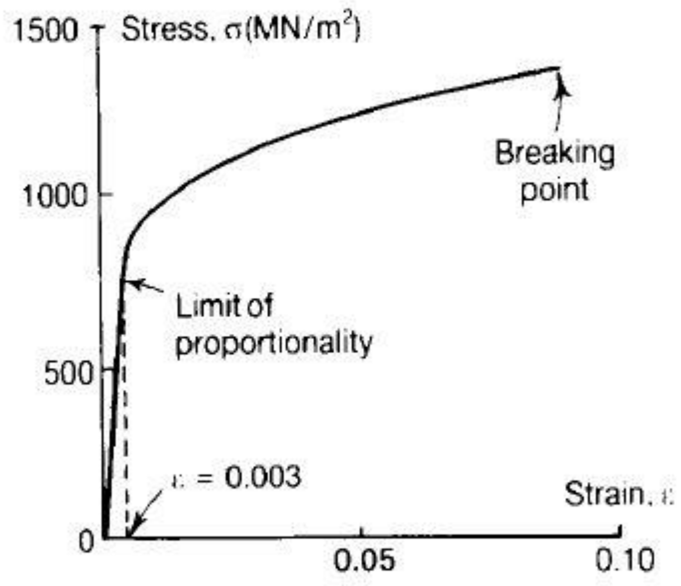
(additivity is obvious)

finite strains:

$$\varepsilon_H = \ln(1.2) + \ln(1.3) = \ln(1.2 \cdot 1.3) = \ln 1.56 = 0.4444685$$

(additivity is still valid)

Tension – high-strength steel



Tensile stress-strain for a high-strength steel

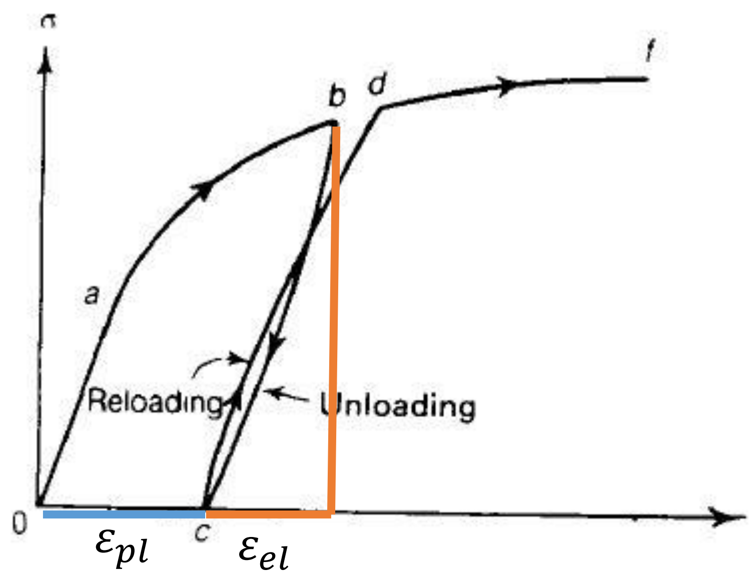
axis *x* – „engineering” strain measure

axis *y* – „engineering” stress (constant area of the cross-section has been used)

E (Young modulus) – tangent of the slope angle of the proportionality range line

the modulus indicates the material stiffness

Tension – unloading and reloading



Unloading and reloading of a material in the inelastic range

during unloading the elastic strains vanish

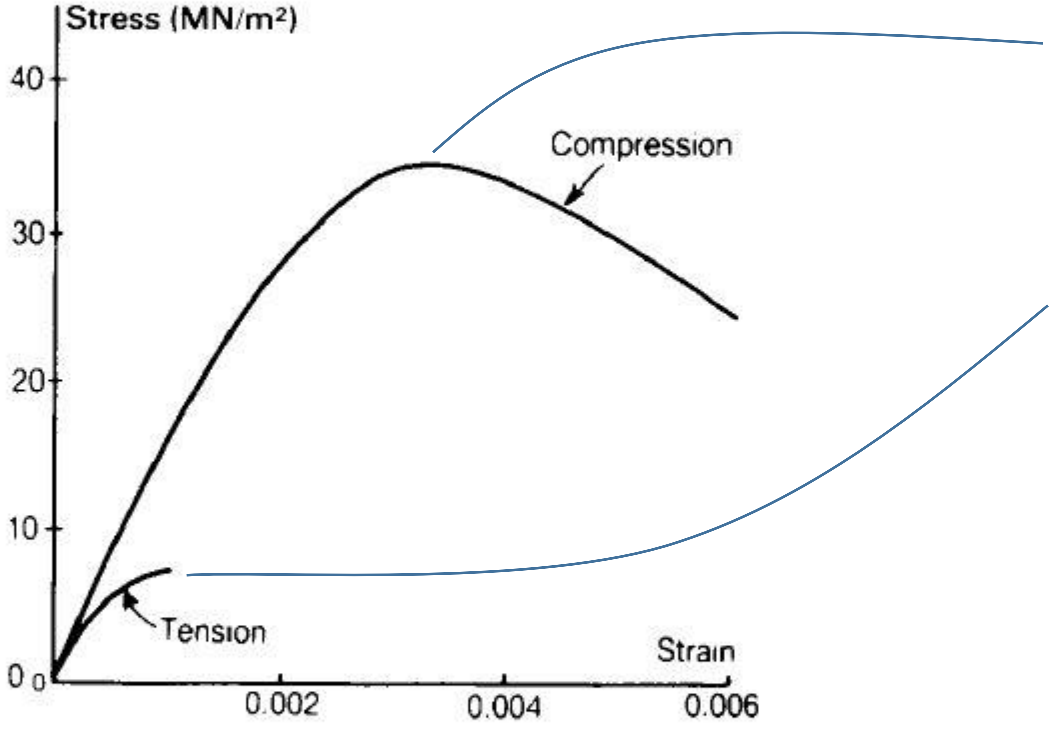
only the plastic strains remains (there are permanent strains)

unloading is (almost) elastic

reloading is (almost) elastic

yielding point changes during loading process – material parameters change

Tension and compression of concrete

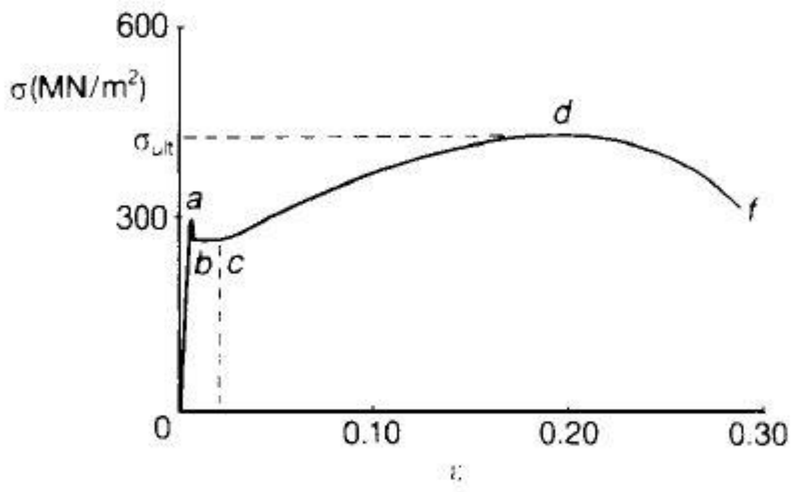


Typical compressive and tensile stress-strains curves for concrete

the compression strength of concrete

the tensile strength of concrete is only about one-tenth of that in compression

Tension of annealed mild steel



Tensile stress-strain curve for an annealed mild steel

slope of 0a is usually about 200 GPa

$\sigma_a \approx 300$ MPa (upper yield point)

$\sigma_b = \sigma_c$ (lower yield point)

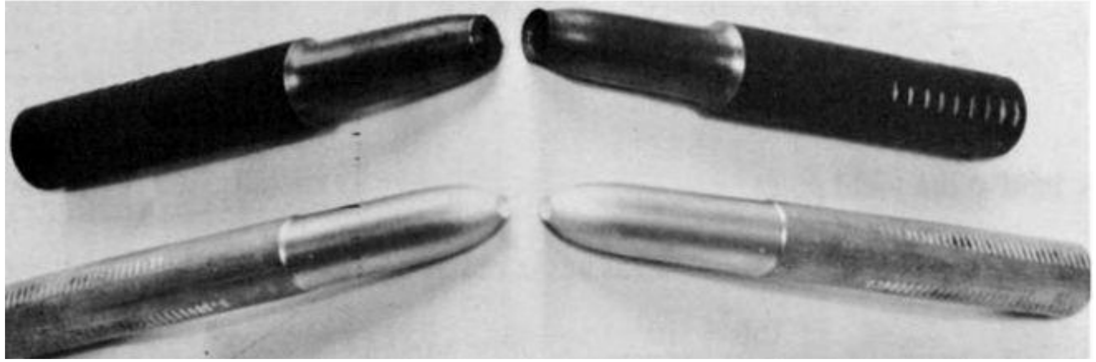
$\varepsilon_{bc} \approx 40\varepsilon_{0a}$

cd stage is a strain-hardening

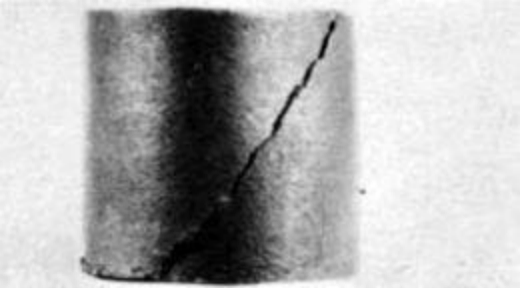
point d is the ultimate stress point

df stage is a necking

Tension – testing specimens failure



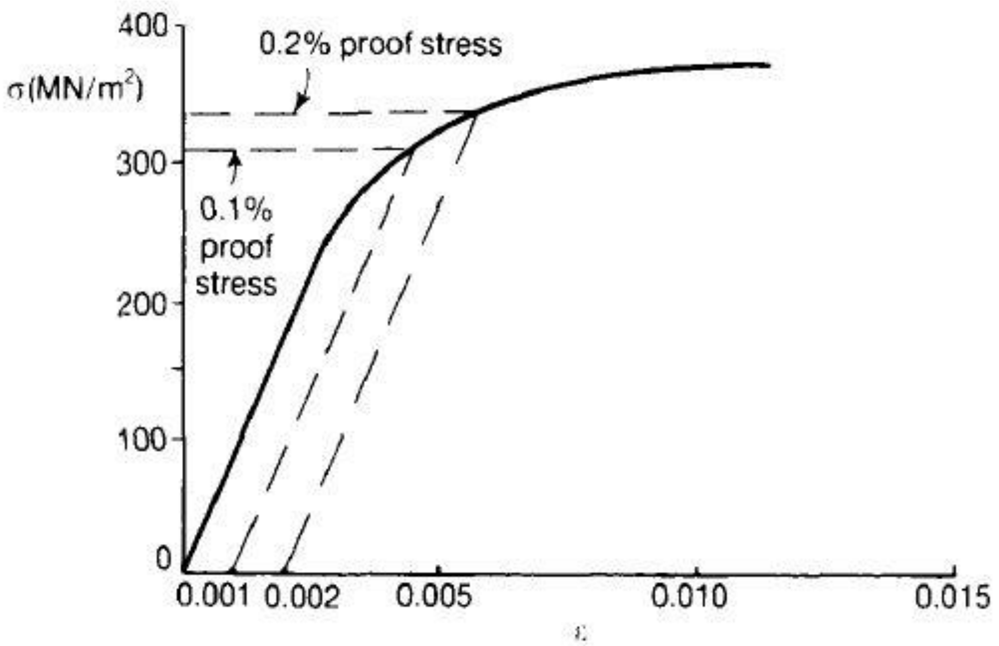
Necking in tensile failures of ductile materials: mild-steel specimen with „cup and cone" (upper) and aluminum-alloy specimen with double „cup" type of failure



Failure on a diagonal plane

Barrel-like failure of compressed mild-steel specimen

Tension – proof stresses



Proof stresses of an aluminum-alloy material

The proportionality limit is difficult to determine experimentally.

To overcome this problem the proof stress is defined on the basis of a specific level of the permanent strains.

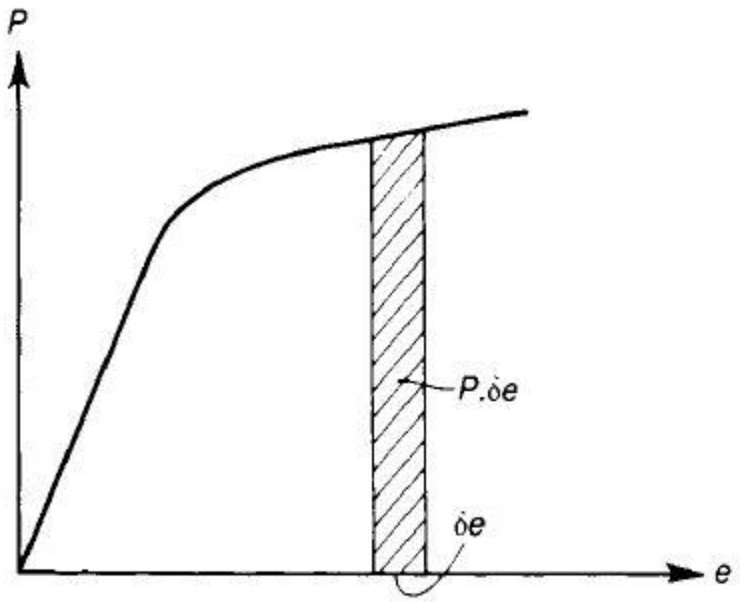
Tension – ductility measurement, toughness

percentage reduction in area

$$\frac{A_0 - A_1}{A_0} \cdot 100\%$$

percentage increase in length

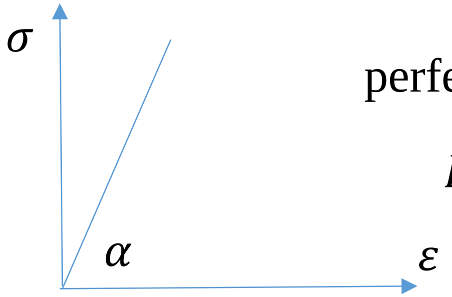
$$\frac{L_1 - L_0}{L_0} \cdot 100\%$$



Work done in stretching a bar through a small extension

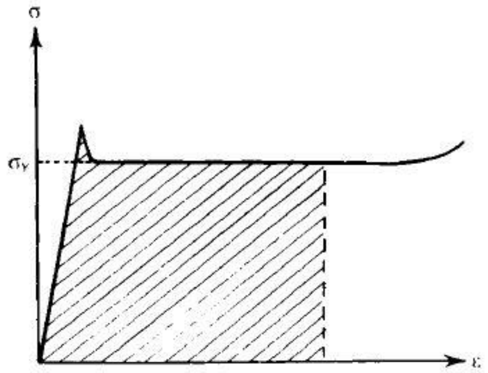
Tension - schematizations

Hooke – material stiffness

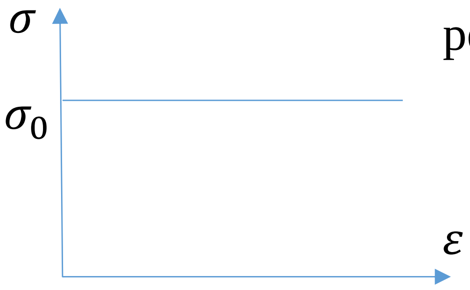


perfectly elastic material

$$E = \tan \alpha$$

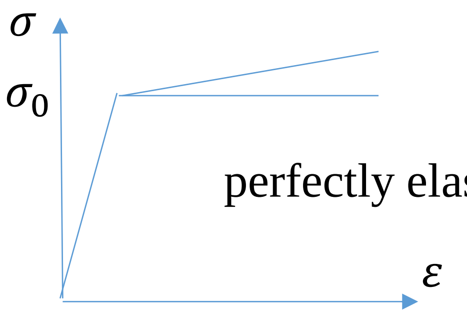


Levy-Mises – material strength



perfectly plastic material

Prandtl – material toughness

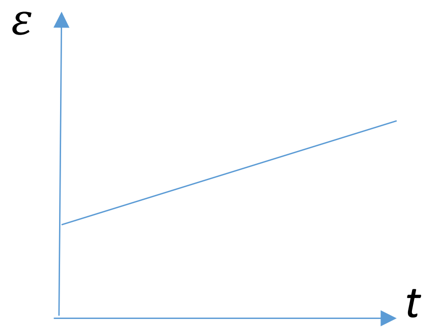


perfectly elastic-plastic material

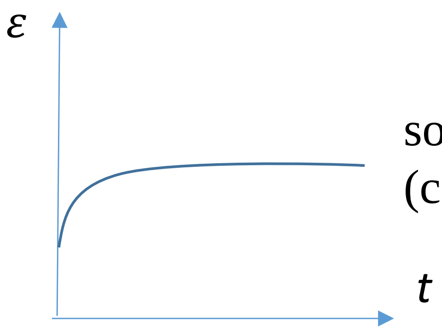
Tension - creep

creep – slow increase of strains in time (high stress level, high temperature)

constant stress loading

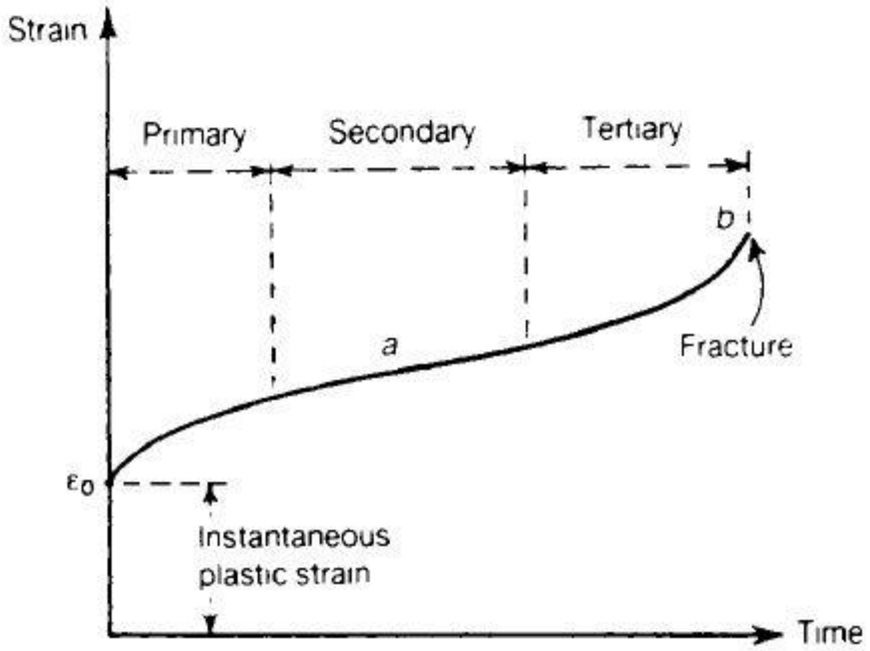


fluid type, unrestricted (asphalt)

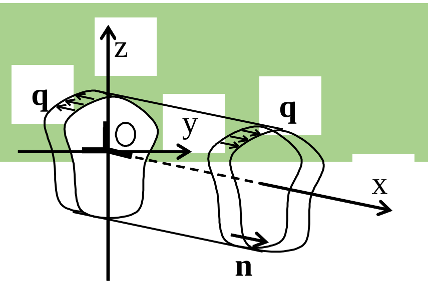


solid body type (concrete)

creep curve for a material in inelastic range



Tension – BVP solution



Problem formulation

A straight prismatic bar of an arbitrary cross-section shape, fixed at one point on extremity (coordinate set origin), loaded by continuous surface loading with constant intensity

We have 15 algebraic/differential equations, we seek 15 unknown functions.

Semi-inverse approach – static approach:

We guess the stress matrix that fulfill Navier's equations (of internal static balance) and static boundary conditions. Next, we calculate (from Hooke's equations) the strain matrix. We check the compatibility equations to make sure there is solution for displacements in the class of continuous functions (because we assume continuum). Using Cauchy geometrical equations we find displacements, and if the displacements fulfill kinematic boundary conditions we have exact mathematical solution to the problem.

$$T_\sigma \;\rightarrow\; T_\varepsilon \;\rightarrow\; \boldsymbol{u}$$

Tension – BVP solution

static boundary conditions:

a) at the bottoms: $\sigma_x = q$

b) elsewhere: $\sigma_{ij} = 0$

We try the stress matrix as follows:

$$T_\sigma = \begin{pmatrix} q & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The stress matrix fulfill the static boundary conditions as well as Navier equations if we put the mass forces $P_i = 0$

We calculate the strain matrix:

$$T_\varepsilon = \begin{pmatrix} \frac{q}{E} & 0 & 0 \\ 0 & -\upsilon\frac{q}{E} & 0 \\ 0 & 0 & -\upsilon\frac{q}{E} \end{pmatrix}$$

The compatibility conditions are fulfilled.

Cauchy equations:

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{q}{E}, \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0,$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = -\upsilon\frac{q}{E}, \quad \varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = 0$$

$$\varepsilon_z = \frac{\partial w}{\partial z} = -\upsilon\frac{q}{E}, \quad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0$$

the complementary functions:

$$u^0 = a + \beta z - \gamma y$$
$$v^0 = b - \alpha z + \gamma x$$
$$w^0 = c - \beta x + \alpha y$$

the particular integral:

$$u^s = \frac{q}{E}x, v^s = -\upsilon\frac{q}{E}y, w^s = -\upsilon\frac{q}{E}z$$

$$u = u^0 + u^s \quad \text{with} \quad u_i(0) = 0;\ u_{i,j}(0) = 0$$

Tension – BVP final solution

stress matrix

$$T_\sigma = \begin{pmatrix} q & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

strain matrix

$$T_\varepsilon = \begin{pmatrix} \frac{q}{E} & 0 & 0 \\ 0 & -\upsilon\frac{q}{E} & 0 \\ 0 & 0 & -\upsilon\frac{q}{E} \end{pmatrix}$$

displacement vector

$$u = \frac{q}{E}x, v = -\upsilon\frac{q}{E}y, w = -\upsilon\frac{q}{E}z$$

The stress state is uniaxial and uniform (homogeneous).

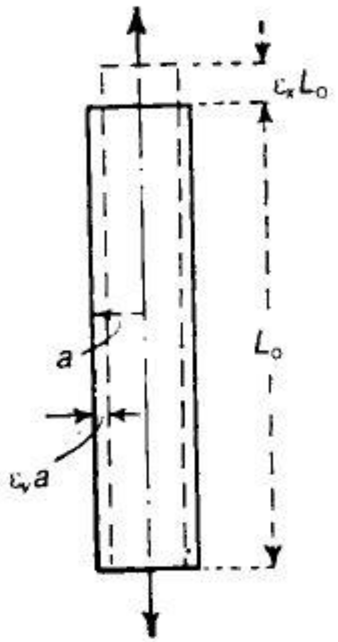
The strain state is triaxial and uniform (homogeneous).

The elongation of a bar is proportional to the bar length.

The Poisson's coefficient is the ratio between axial elongation and lateral shrink:

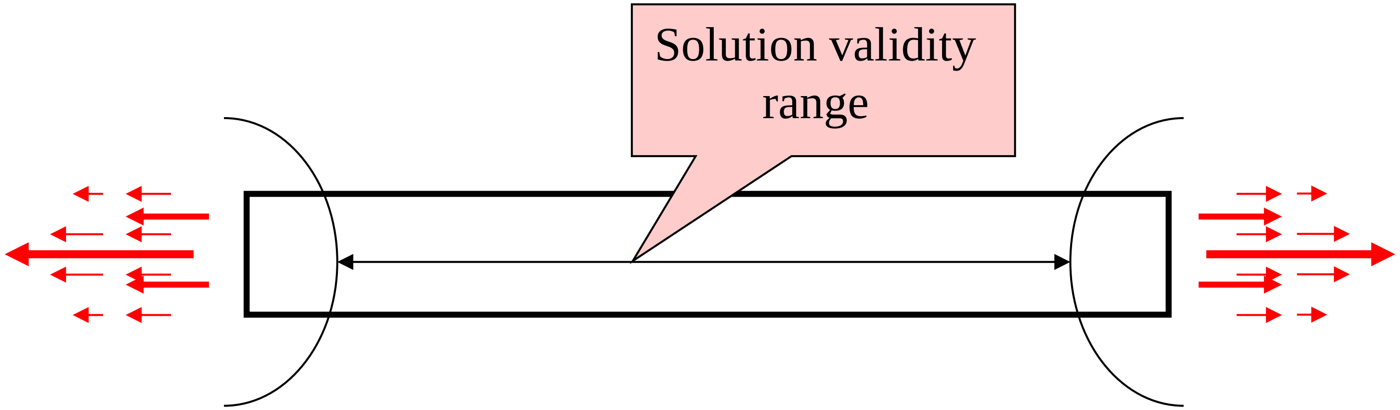
$$\upsilon = -\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\varepsilon_z}{\varepsilon_x}$$

Plane cross-section remains plane.



Pure tension versus simple tension

from de Saint-Venant hypothesis:



instead of q we can put: $q = \sigma_x = \frac{N}{A}$ (Galileo's finding)

Tension – formulae ready to use

In the case of constant axial force, N=const

- stress, $\sigma_x = \frac{N}{A}$ (in [MPa])
- strain, $\varepsilon_x = \frac{N}{EA}$, EA – tension stiffness (in [N])
- elongation, $\Delta l = \frac{Nl}{EA}$

Compression?

Yes, if the member is „sufficiently thick”. The rule of thumb: if the slenderness, $\lambda = \frac{l}{i_{min}}$ is less than 10 we can safely use the above formulas with the opposite sign.

Tension – design conditions

There are two main requirements: of the strength and of the usability.

The first requirement means that every structure has to sustain the applied load.

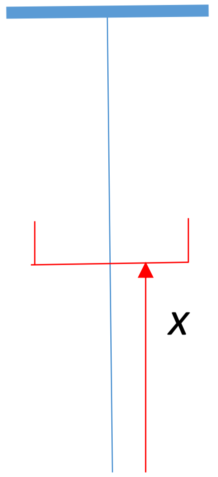
The second one consists in several demands of durability, rigidity, resistance to severe weather conditions, and so on.

From the point of view of the strength of materials course, two principal design conditions should be listed:

- the ultimate limit state, $\sigma_x \leq R$
- the serviceability limit state, usually the stiffness of the structure; for the tension it can be: $\Delta l \leq \Delta l_{accessible}$

Tension – breaking length (self support length)

What is the maximum length of a vertical column of the material (assuming a fixed cross-section) that could suspend its own weight when supported only at the top?



axial force (changes with elevation): $N(x) = A \cdot x \cdot \gamma$

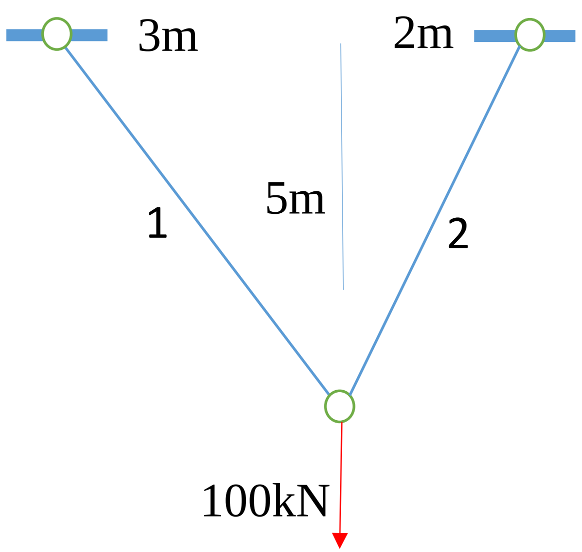
normal stress: $\sigma_x = \frac{N}{A} = \gamma \cdot x$ (doesn't depend on the cross-section area!)

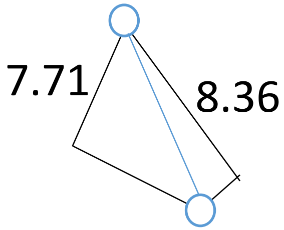
$$\max \sigma_x < R_m \quad \rightarrow \quad x = \frac{R_m}{\gamma}$$

some typical values: low carbon steel 4.73 km, nylon 7.04 km, aluminum alloy 11.7 km, oak 12-13 km, titanium 20 km, balsa 53.2 km, spider silk 109 km, kevlar 256 km, carbon nanotube 4716 km, fundamental (theoretical) limit $9.2 \cdot 10^{12}$ km

Tension – bar structure deformability

Problem: Determine needed cross-sectional area of the bars in the truss below. Determine a new position of the bottom node and the change of bar angles. Assume E = 210 GPa and R = 300 MPa.





Solution:

from statics: $N_1 = 46.65$ kN, $N_2 = 64.62$ kN

needed area of the bars:

$$A_1 = \frac{N_1}{R} = \frac{46.65 \cdot 10^3}{300 \cdot 10^3} = 1.56 \cdot 10^{-4}\ \mathrm{m}^2 = 1.56\ \mathrm{cm}^2$$

$$A_2 = \frac{N_2}{R} = \frac{64.62 \cdot 10^3}{300 \cdot 10^3} = 2.16 \cdot 10^{-4}\ \mathrm{m}^2 = 2.16\ \mathrm{cm}^2$$

elongations of the bars:

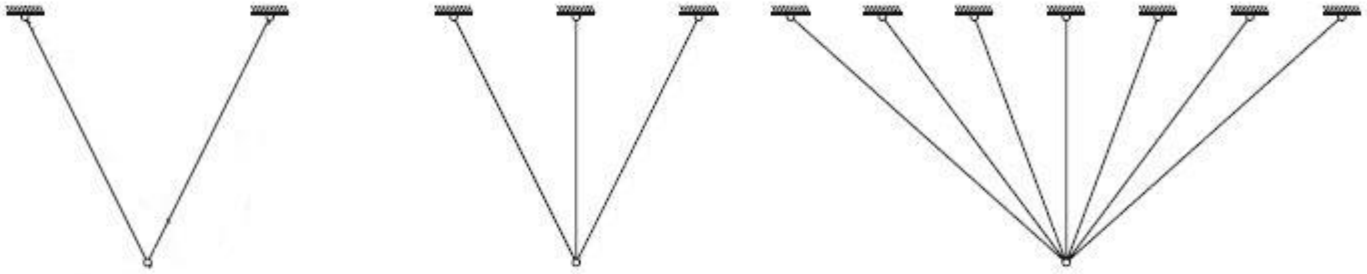
$$\Delta l_1 = \frac{N_1 l_1}{EA_1} = \frac{46.65 \cdot 10^3 \cdot \sqrt{34}}{210 \cdot 10^9 \cdot 1.56 \cdot 10^{-4}} = 8.36 \cdot 10^{-3}\ \mathrm{m} = 8.36\ \mathrm{mm}$$

$$\Delta l_2 = \frac{N_2 l_2}{EA_2} = \frac{64.62 \cdot 10^3 \cdot \sqrt{29}}{210 \cdot 10^9 \cdot 2.16 \cdot 10^{-4}} = 7.71 \cdot 10^{-3}\ \mathrm{m} = 7.71\ \mathrm{mm}$$

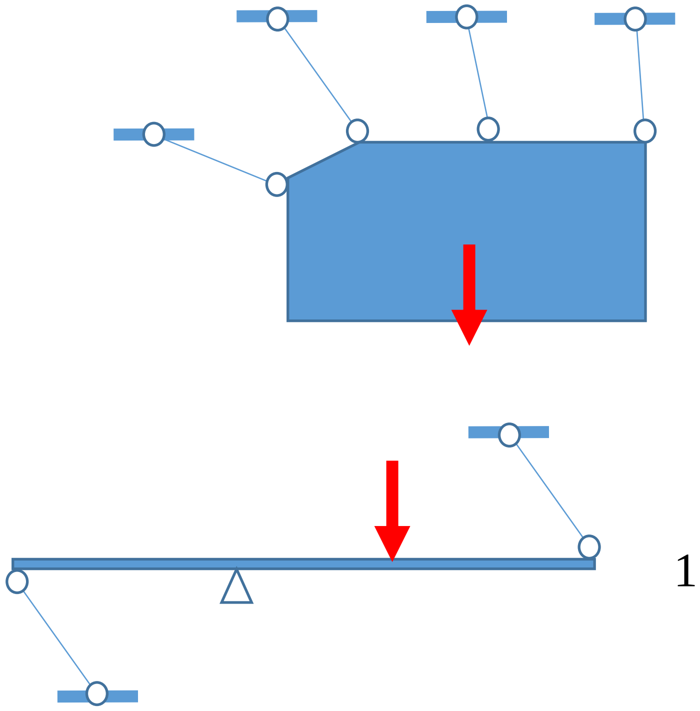
it can be proved from geometric calculation that angle changes are:

$\Delta\alpha_1 = 0.033°$, $\Delta\alpha_2 = 0.049°$ and can be neglected

Statically indeterminate structures



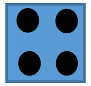
structure number	1	2	3
degree of static indeterminacy	0	1	5
degree of kinematic indeterminacy	2	2	2
number of degrees of freedom	2	2	2



The key is the number of degrees of freedom!

7 unknowns => 7 equations => 7 – 2 static eq.
=> we need 5 additional compatibility equations

Tension – composed members

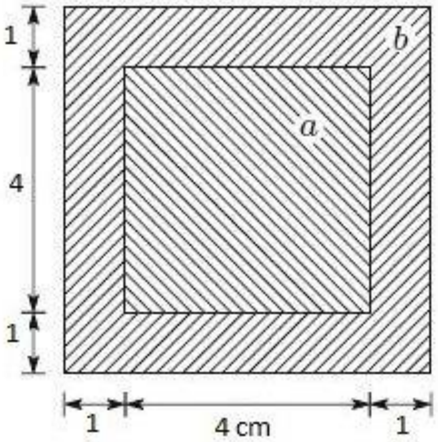


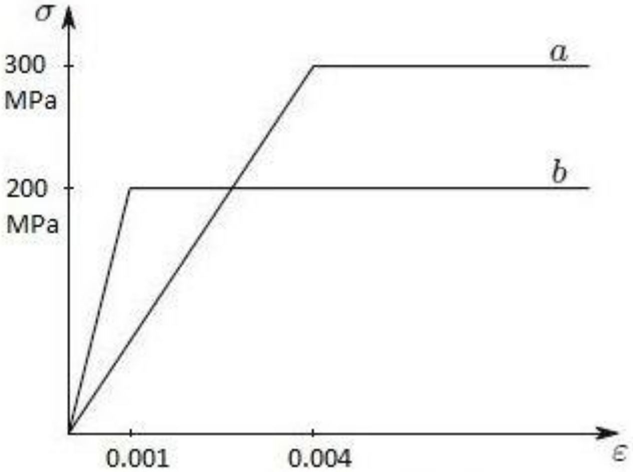
If the bottoms are plane, every cross-section remains plane.

It means that the strain is constant in the cross-section and the kinematic indeterminacy is one, regardless of the number of the materials.

The static indeterminacy is equal to the number of materials minus one.

Example of two materials: a) Compute the maximum axial force that may be applied in the elastic regime.
b) Compute the axial force which causes yielding of the bar.





Solution

$A_a = 16\text{cm}^2, A_b = 20\ \text{cm}^2, E_a = 75\ \text{GPa}, E_b = 200\ \text{GPa}$

$a)\ \ \varepsilon_a = \varepsilon_b = 0.001$

$N = N_a + N_b = \sigma_a A_a + \sigma_b A_b = 0.001(E_a A_a + E_b A_b) =$

$= 0.001(75 \cdot 10^9 \cdot 16 \cdot 10^{-4} + 200 \cdot 10^9 \cdot 20 \cdot 10^{-4}) =$

$= (120 + 400) \cdot 10^3\ \text{N} = 520\ \text{kN}$

b) $N = 300 \cdot 10^6 \cdot 16 \cdot 10^{-4} + 200 \cdot 10^6 \cdot 20 \cdot 10^{-4} =$

$= (400 + 480) \cdot 10^3\ N = 880\ \text{kN}$

Tension – prestressing

Let's consider the same materials and the cross-section as in the previous example.
To maximize the tensile loading capacity in the elastic regime, the bar is prestressed, by applying a tensile force to the material *a* before the connection between the two materials is established. This force is removed after the connection's bonding.
Compute the value of the prestressing force, which leads to the simultaneous yielding of the two materials.

Solution
The strains in each of materials are different, the final values are: $\varepsilon_a = 0.004, \varepsilon_b = 0.001$.
After the connection's bonding the changes of the strains were equal, but the values of the strains not:
$\varepsilon_b = 0, \varepsilon_a =$? The strain in the material *b* was smaller by 0.001, so the same change was in the material *a:*
$\varepsilon_a = 0.004 - 0.001 = 0.003$. This strain resulted from the prestressing force:

$$\varepsilon_a = \frac{\sigma_a}{E_a} = \frac{N_a}{E_a A_a} = 0.003 \quad \rightarrow \quad N_a = 0.003 \cdot 75 \cdot 10^9 \cdot 16 \cdot 10^{-4} = 360000 \text{ N} = 360 \text{ kN}$$

This prestressing force raises the maximum load in the elastic regime from 520 kN to the value of 880 kN and the yielding stress in both materials.

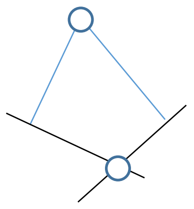
Tension – temperature variations

Dimensions of the body can change due to:

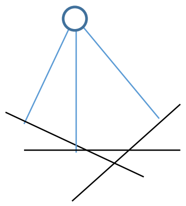
- mechanical loading, $\Delta l = \frac{Nl}{EA}$
- temperature variations, $\Delta l = \alpha l \Delta t$
- humidity variations (wood)
- other factors…

In the case of statically determined structures non-mechanical factors don't change stresses. This is not true for statically indeterminate structures.

Tension – temperature variations cont.



Statically determinate structure:
the unconstrained elongations match
Statically indeterminate structure:
the unconstrained elongations don’t match
compatibility conditions require different node position
the constrained elongations result from balanced set of forces
these forces stem from thermal stresses

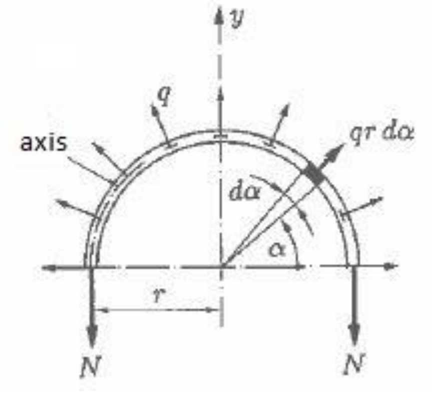


Useful applications of thermal stresses :

- riveted joints (permanent assembly)
- heat shrink assembly (steel tire assembly on a wheel of many rail vehicles, ball bearing assembly)

Is any relation between tension and a ring? Yes!

Tension – rings and boilers



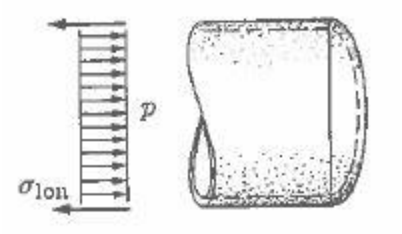
The rotational symmetry of the problem is evident.

From projection on the vertical axis we get: $2\int_0^\pi qr\sin\alpha\,d\alpha - 2N = 2qr - 2N = 0$

If the ring thickness is small in comparison to the radius we may assume the normal stress constant (the wall is almost in tension). The error is of the order of 5% for $h \le r/10$.

$\sigma \cong \text{const} = \frac{N}{A} = \frac{qr}{A} \quad \rightarrow \quad \varepsilon = \frac{qr}{EA}$, so the axis length increases by $2\pi r\varepsilon$ and radial displacement is $u_r = \varepsilon r = \frac{qr^2}{EA}$

The problem with the heat shrink assembly of two rings is a one DoF problem.



The above solution can be adopted for the problem of a boiler with internal pressure.

The circumferential (hoop) stress is the same as in a ring, $\sigma_{cir} = \frac{pr}{t}$

The longitudinal stress may be calculated from the projection onto the horizontal axis:

$$\sigma_{lon} = \frac{N}{A} = \frac{\pi r^2 p}{2\pi r t} = \frac{pr}{2t} = \frac{1}{2}\sigma_{cir}$$

This is the cause that a sausage bursts alongside.

Thank you for your attention!