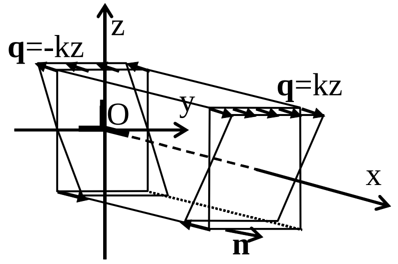
Strength of Materials

2. Bending

Problem formulation



straight, prismatic bar, fixed at a point in origin of coordinate set, loaded on the bottoms with linearly distributed surface continuous loading, $q = kz$, where

(y, z) are principal central inertia axes of the bar cross-section

similarly to the previous problem of tension, we have

15 equations (3+6 partial differential and 6 algebraic) with static and kinematic boundary conditions

15 unknowns (6 stress coordinates, 6 of strains, 3 of displacements)

we apply semi-inverse method of solution

$$T_\sigma \to T_\varepsilon \to u_i$$

Solution to the BVP

static boundary conditions:

a) at the bottoms: $\sigma_x = kz$

b) elsewhere: $\sigma_{ij} = 0$

We try the stress matrix as follows:

$$T_\sigma = \begin{pmatrix} kz & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The stress matrix fulfill the static boundary conditions as well as Navier equations if we put the mass forces $P_i = 0$

We calculate the strain matrix:

$$T_\varepsilon = \begin{pmatrix} \frac{k}{E} z & 0 & 0 \\ 0 & -\upsilon \frac{k}{E} z & 0 \\ 0 & 0 & -\upsilon \frac{k}{E} z \end{pmatrix}$$

The compatibility conditions are fulfilled

Cauchy equations:

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{k}{E} z, \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0,$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = -\nu \frac{k}{E} z, \quad \varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = 0$$

$$\varepsilon_z = \frac{\partial w}{\partial z} = -\nu \frac{k}{E} z, \quad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0$$

general solution = complementary function + particular integral

the complementary functions:

$$u^0 = a + \beta z - \gamma y$$
$$v^0 = b - \alpha z + \gamma x$$
$$w^0 = c - \beta x + \alpha y$$

the particular integral:

$$u^s = \frac{k}{E} xz, v^s = -\nu \frac{k}{E} yz, w^s = \frac{k}{2E}\left(-x^2 + \nu y^2 - \nu z^2\right)$$

$u = u^0 + u^s$ with kinematic boundary conditions:

$$u_i(0) = 0; \;\; u_{i,j}(0) = 0$$

The complementary function vanishes, so

$$u = u^s = \left(\frac{k}{E} xz, -\nu \frac{k}{E} yz, \frac{k}{2E}\left(-x^2 + \nu y^2 - \nu z^2\right)\right)$$

Simple bending – final formulae

stress matrix $T_\sigma = \begin{pmatrix} kz & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

stress state is uniaxial, nonhomogeneous
for $z = 0$ stress vanishes (it is neutral axis, locus in the cross-section where the stress vanishes)

strain matrix $T_\varepsilon = \begin{pmatrix} \frac{k}{E}z & 0 & 0 \\ 0 & -\nu\frac{k}{E}z & 0 \\ 0 & 0 & -\nu\frac{k}{E}z \end{pmatrix}$

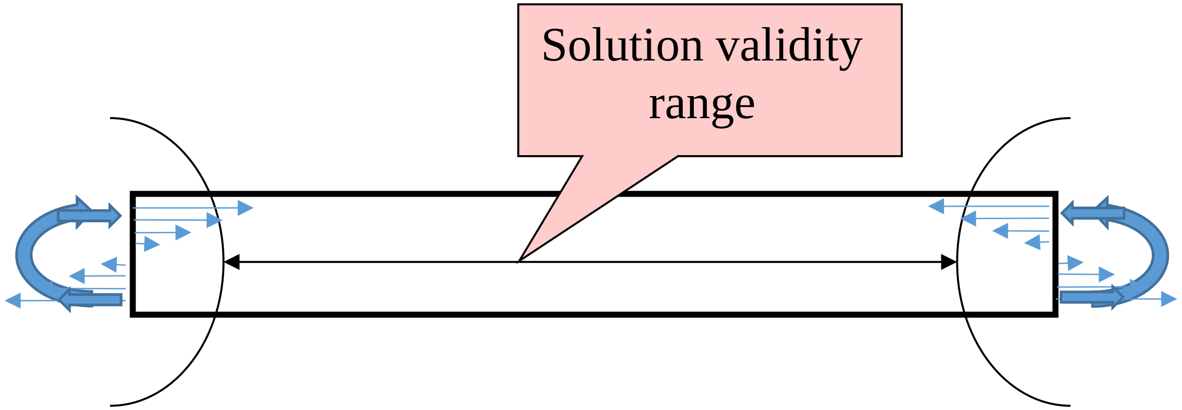
strain state is triaxial and nonhomogeneous

displacements
$u = \frac{k}{E}xz, v = -\nu\frac{k}{E}yz, w = \frac{k}{2E}(-x^2 + \nu y^2 - \nu z^2)$

for x = const: $u = \frac{kx_0}{E}z$, so
plane cross-section remains plane

Pure bending – simple bending

de Saint-Venant principle:



static equivalence: $M_y = \iint_A \sigma_x \cdot z\mathrm{d}A = \iint_A (kz \cdot z)\mathrm{d}A = k \iint_A z^2\mathrm{d}A = kI_y \rightarrow k = \frac{M_y}{I_y}$

$$\sigma_x = \frac{M_y}{I_y} z$$

moreover, from statics: $Q_z = 0 \rightarrow \frac{dM_y}{dx} = 0 \rightarrow M_y = \mathrm{const}$

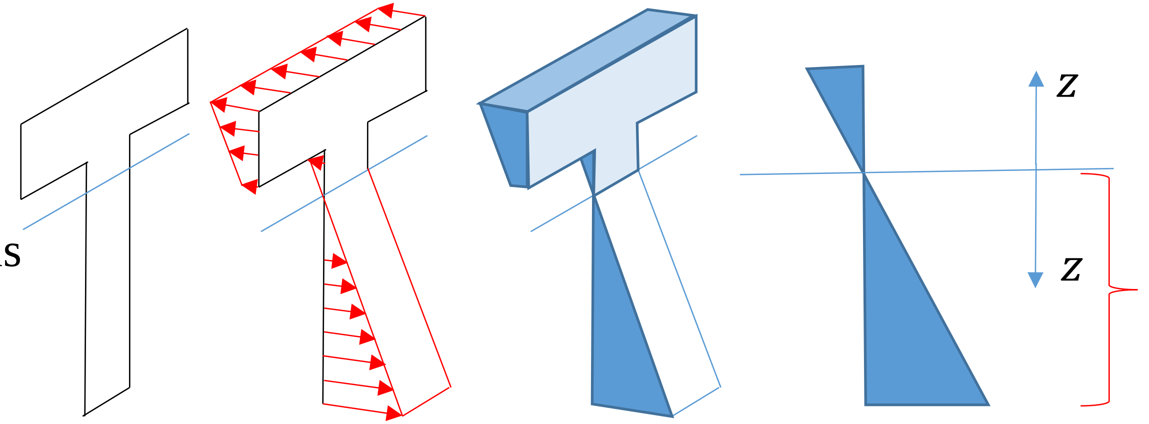
Simple bending:

Direction of the bending moment vector coincides with direction of a principal central axis.

Bending – design

Ultimate limit state: $\max|\sigma_x| \leq R$

neutral axis



The maximum absolute value of the stress is attained at the fibers which are the most distant from the neutral axis

$$\max|\sigma_x| = \frac{|M_y|}{I_y}|z_{\max}|$$

$$W_y \stackrel{\text{def}}{=} \frac{I_y}{|z_{\max}|} \quad [\mathrm{m}^3]$$

(elastic) section modulus

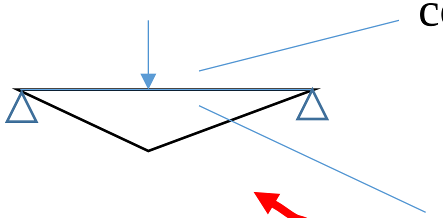
$$\max|\sigma_x| = \frac{|M_y|}{W_y}$$

$$\frac{|M_y|}{W_y} \leq R$$

Bending – signs, signs, and signs…

What is the sign of the stress above/below the neutral axis?

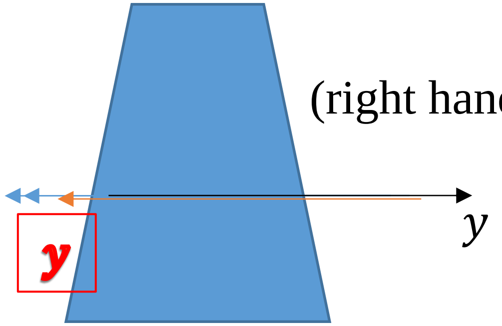
Statics: the bending moments are drawn from the tensioned side!



compression

tension

compression



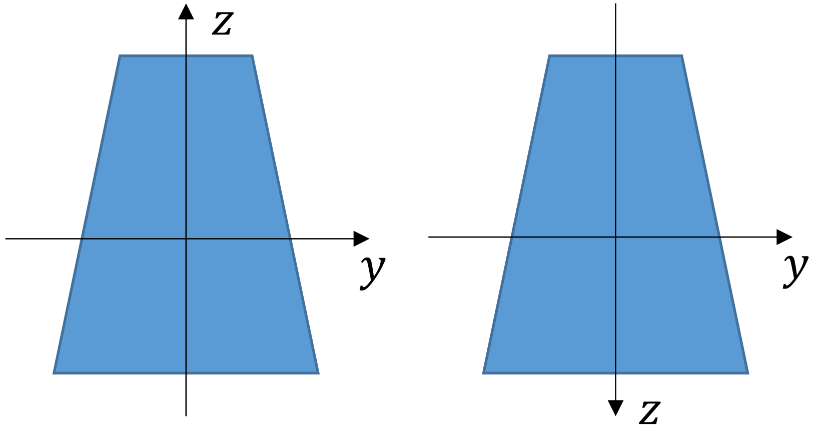
tension

(right hand rule)

$M_y < 0$

$\boldsymbol{M_y > 0}$

Which side is tensioned?



the sign of the stress should be the same!

$$\sigma_x = \frac{M_y}{I_y} z \qquad \sigma_x = -\frac{M_y}{I_y} z$$

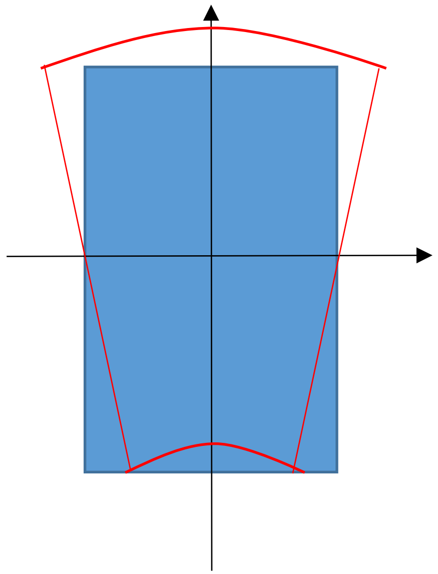
The answer is here!

statics!

Bending – cross-section deformation

simple bending

$$u = \frac{M_y}{EI_y} xz, v = -\frac{M_y}{EI_y} yz, w = \frac{M_y}{2EI_y}(-x^2 + \nu y^2 - \nu z^2)$$



for negative bending moment ($M_y < 0$, tension at the lower side)

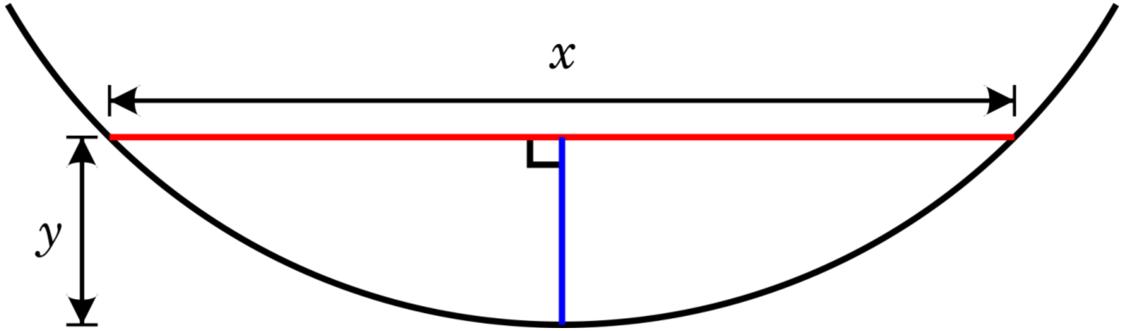
$$u(x_0) = \frac{M_y}{EI_y} x_0 z$$ plane cross-section remains plane

$$v\left(y = \frac{b}{2}\right) = -\nu \frac{M_y \frac{b}{2}}{EI_y} z = -az$$ a straight line

$$v\left(y = -\frac{b}{2}\right) = \nu \frac{M_y \frac{b}{2}}{EI_y} z = az$$ a straight line

$$w\left(x = x_0, z = \frac{h}{2}\right) = \frac{M_y}{2EI_y}\left(-x_0^2 - \nu \frac{h^2}{4} + \nu y^2\right)$$ a parabola 2°

Bending – deformations of the bar axis



x – chord, y – sagitta (versine)

curvature: $\kappa = \frac{1}{\rho} = \frac{|w''|}{\sqrt{1+(w')^2}^3} \cong |w''|$

y/x	r	φ	κ	relative error	span change
1/150	18.75	0.05333	0.9989	0.00107	0.00012
1/250	31.25	0.03200	0.9996	0.00038	0.00004
1/500	62.50	0.01600	0.9999	0.00010	0.00001

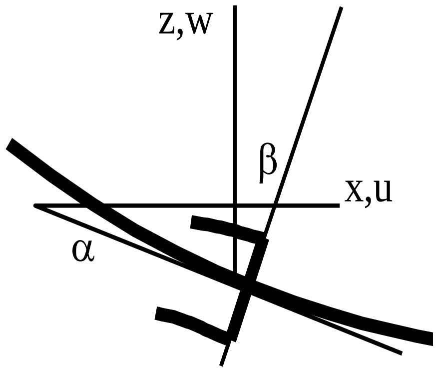
$$w = \frac{M_y}{2EI_y}(-x^2 + \nu y^2 - \nu z^2) \rightarrow w'' = \frac{M_y}{EI_y}$$

$$\kappa = \frac{|M_y|}{EI_y}$$

EI_y - bending stiffness

$M_y = \text{const}, EI_y = \text{const} \rightarrow \kappa = \text{const}$ (circular bending)

Bending – deformations cont.



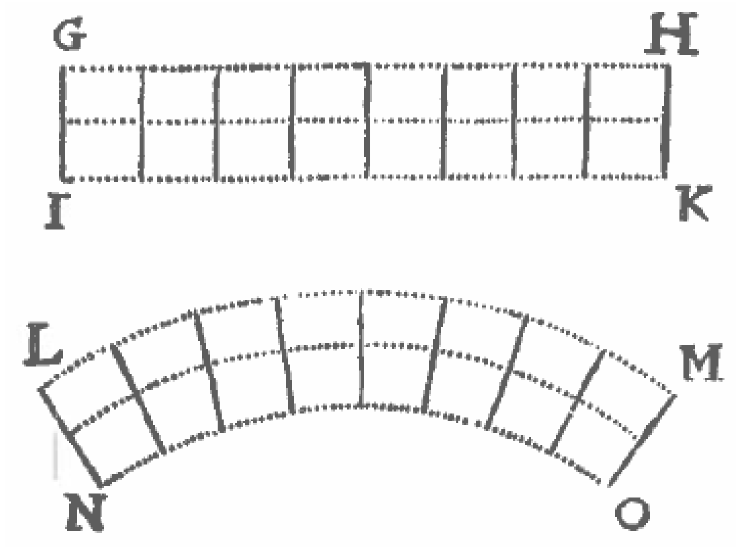
$\mathrm{x} = \mathrm{x}_0$: $\quad \tan\alpha = w'(x_0) = -\frac{M_y}{EI_y} x_0$

the displacements in the same cross-section are

$$u(x_0, y, z) = -\frac{M_y}{EI_y} x_0 z \quad \rightarrow \quad \left.\frac{\partial u}{\partial z}\right|_{x_0} = -\frac{M_y}{EI_y} x_0 = \tan\beta$$

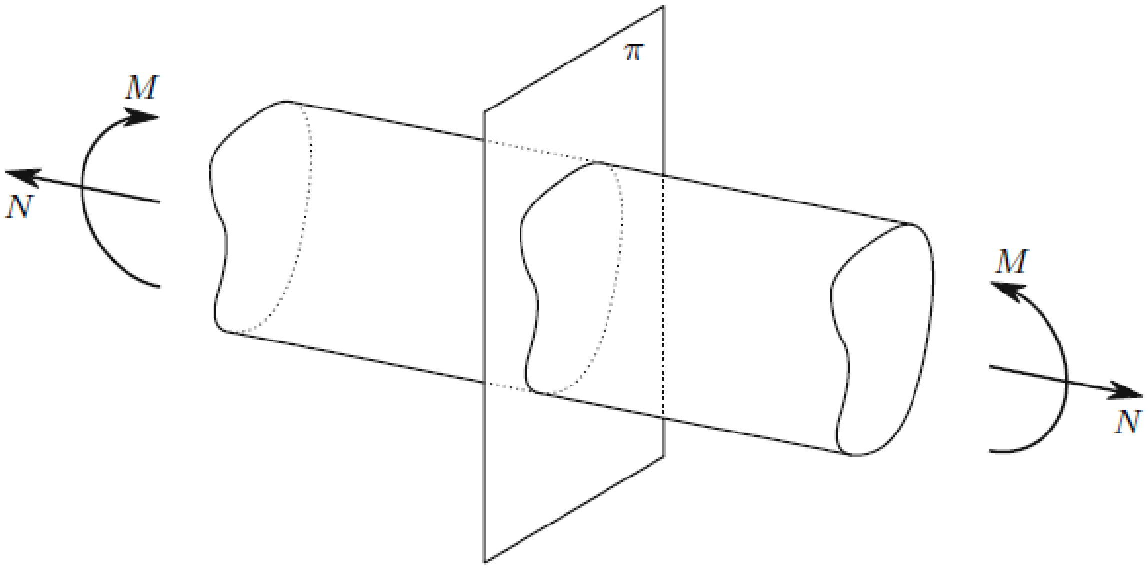
$$\tan\alpha = \tan\beta \rightarrow \quad \alpha = \beta$$

Theorem (Bernoulli):
the cross-sections (plane and perpendicular to the beam axis)
remain plane and perpendicular to the actual beam axis



R. Hooke, *De potentia restitutiva*, 1678

Bending – deformations cont.



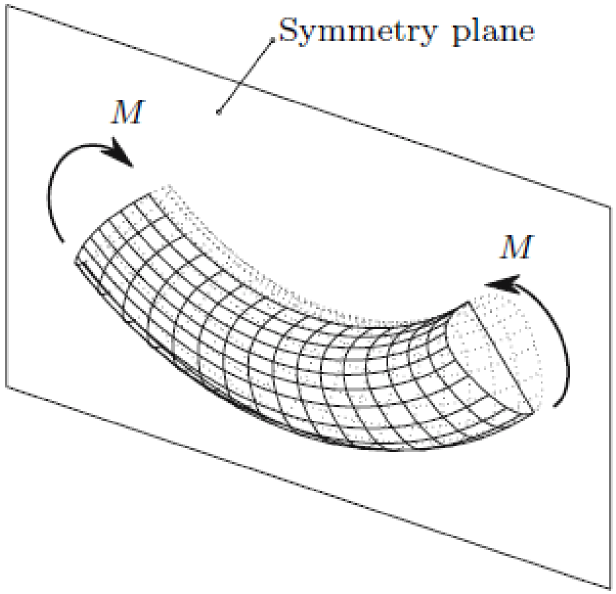
- prismatic bar
- isotropic material
- constant axial force and bending moment

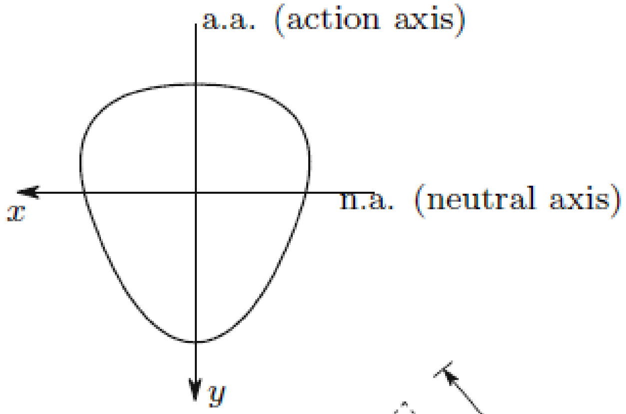
we observe a symmetry plane (symmetry of geometry and forces)

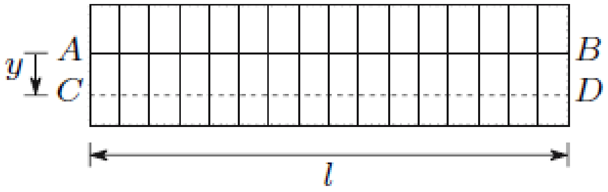
J. Bernoulli (1705): in a prismatic bar under constant axial force and constant bending moment, the cross-sections remain plane and perpendicular to the axis of the bar during the deformation.
The above statement was formulated as a hypothesis.
It is not valid for non-symmetrical internal forces resultants, such as varying axial force and bending moment, shear force and torsional moment.

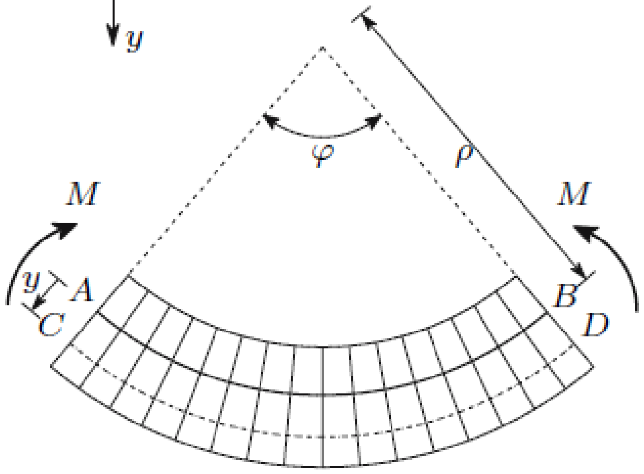
Bending – deformations cont.

da Silva:

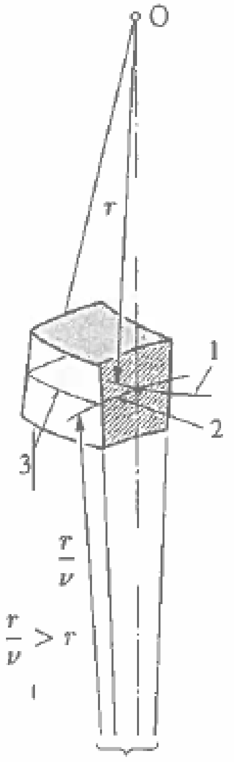








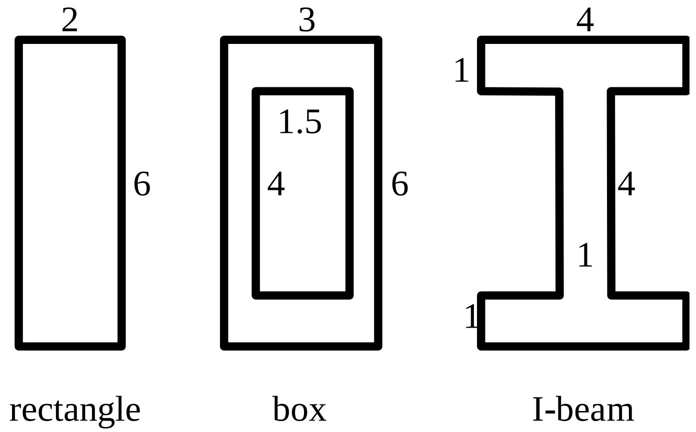
Frey:



1. beam axis, 2. neutral axis, 3. neutral surface

Bending – example

There are three cross-sections with the same height and area. Which section will be the most useful for the simple bending? Assume the same value of bending moment.

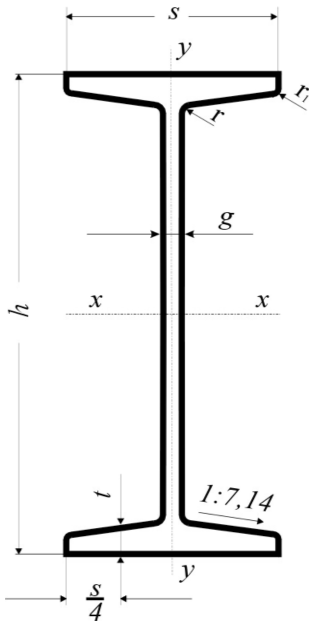


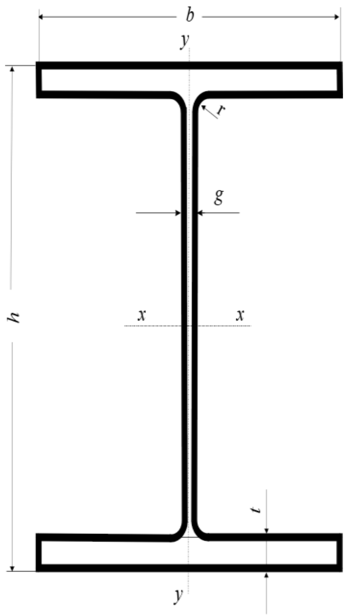
$$\max(\sigma_x) = \frac{M_y}{W_y}$$

area	height	inertia m.	s.modulus	s.m. in %	max σ_x in %
12	6	36	12	100	100
12	6	46	15.33	128	78
12	6	56	18.67	156	64

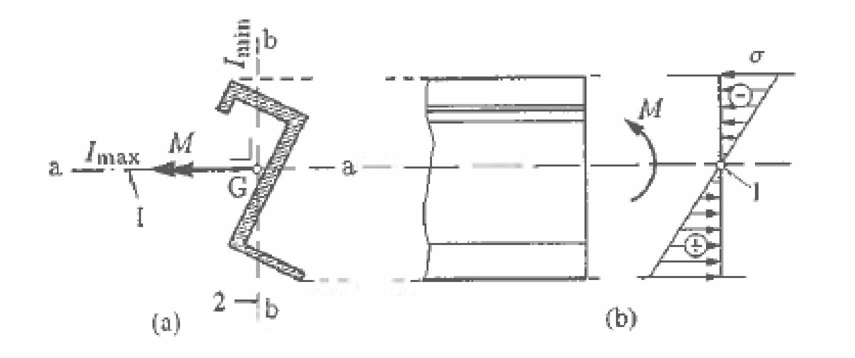
rectangle flatwise: $A = 12, h = 2, I_y = 4, W_y = 4, 33.33\%, 300\%$

Bending – standard I-beam, wide flange W-beam



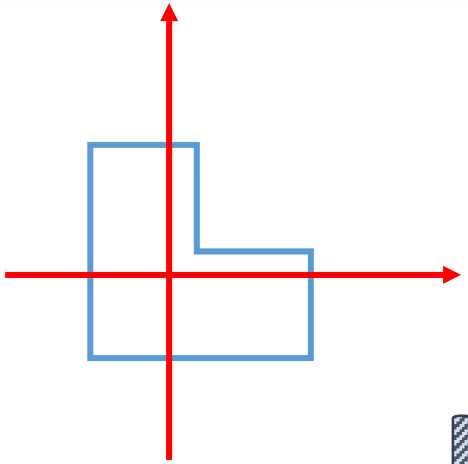


Bending – principal central inertia axes

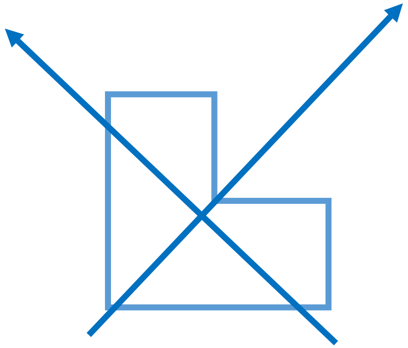


bending of a beam with asymmetric cross-section (thin-walled profile)

a) slice (1) – neutral axis, 2) – load plane)
b) elevation and stresses (1) – neutral axis)



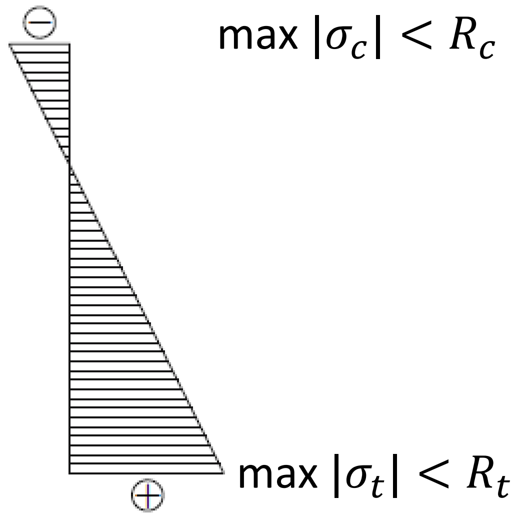
HORROR !



Bending – different tension/compr. strength

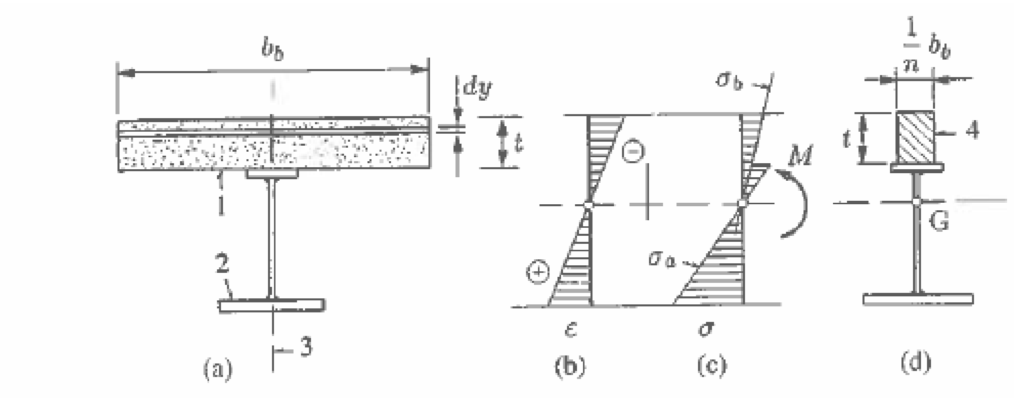
When $R_t \neq R_c$

- don't use the section modulus (it a little confuses the issue)
- determine tensioned/compressed side (statics!)
- verify separately safety of both sides
  (at extreme fibers)
- when dimensioning adopt greater area
- when assessing bearing capacity adopt lower value
- general rule: always adopt solution from safer side



Bending – different elastic materials

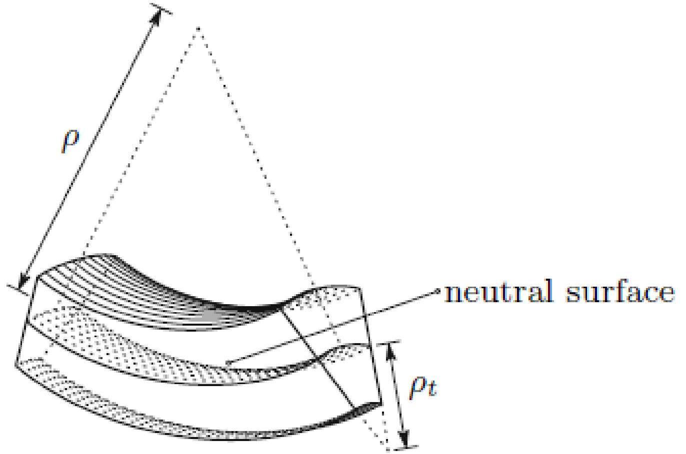
when the cross-section is composed from two different materials, both with geometry that does not violate geometric symmetry, the Bernoulli principle of plane cross-sections remains valid
the strain distribution is linear in the cross-section
the stress distribution is piecewise linear in the cross-section

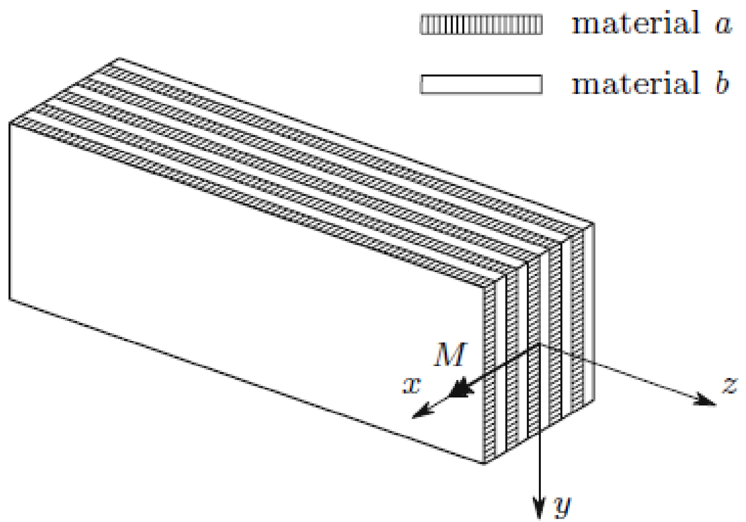


Composite cross-section: a) reinforced concrete plate and steel girder,
b) strain distribution, c) stress distribution, d) homogenized cross-section

Homogenization:
let: $\frac{E_1}{E_2} = n$, when we admit homogenized cross-section, made from material *1*, we overstate the stresses *n* times in material *2*, so we can correct this making width *n* times smaller in the overestimated region

Bending – composite material



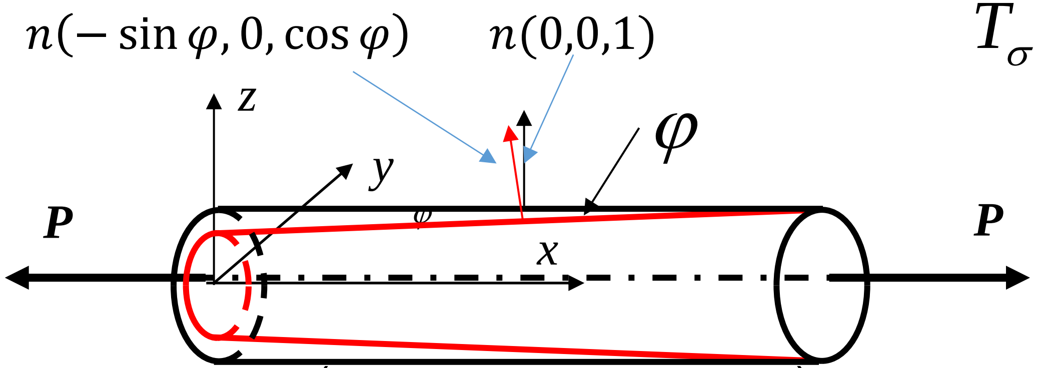


The neutral surface has an anticlastic shape (saddle shape), and $\frac{1}{\rho_t} = -\upsilon\frac{1}{\rho}$. So, if the Poisson coefficient is not the same in both materials, compatibility conditions should be revised and taken into account. It means that other components of the stress matrix are non-zero.

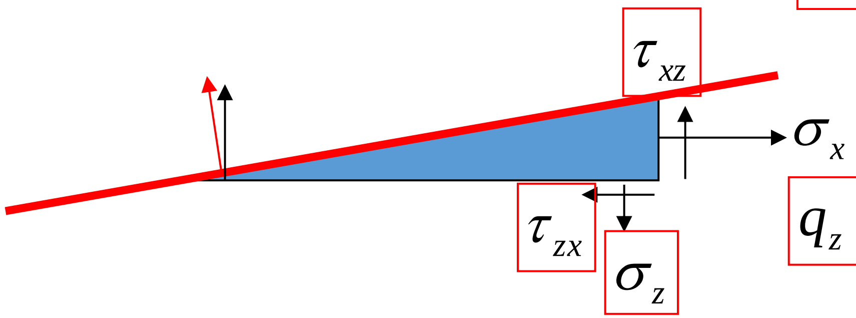
The error reported for the stresses and curvatures doesn't exceed 6.7% (for extreme case of 0 and 0.5 values of Poisson coefficients) and reduces to 0.6% for the curvatures and 1.8% for stresses (for more usual values 0.15 and 0.3).

Usually, the compatibility conditions are not considered and the „normal" simple solution is used.

Bending – non-prismatic members



$$q\left(q_x = 0 = q_y = q_z\right)$$



$$T_\sigma = \begin{bmatrix} \sigma_x & 0 & \tau_{xz} \\ 0 & 0 & 0 \\ \tau_{zx} & 0 & \sigma_z \end{bmatrix}$$

$$\sigma_x(x) \overset{?}{=} \frac{M_y}{I_y(x)} z$$

$$q_i = n_j \sigma_{ij}$$ **SBC**

$$q_x = 0 = \sigma_x \cdot 0 + 0 \cdot 0 + 0 \cdot 1 = 0$$

$$q_y = 0 = \sigma_x(-\sin\varphi) + 0 \cdot 0 + 0 \cdot \cos\varphi \neq 0$$

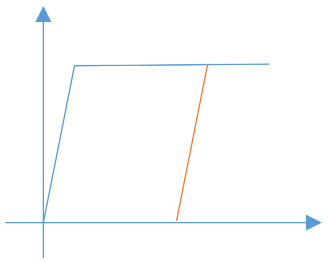
$$\tau_{xz} = \sigma_x \tan\varphi$$ **!**

$$q_z = 0 = \tau_{zx}(-\sin\varphi) + 0 \cdot 0 + \sigma_z \cdot \cos\varphi \neq 0$$

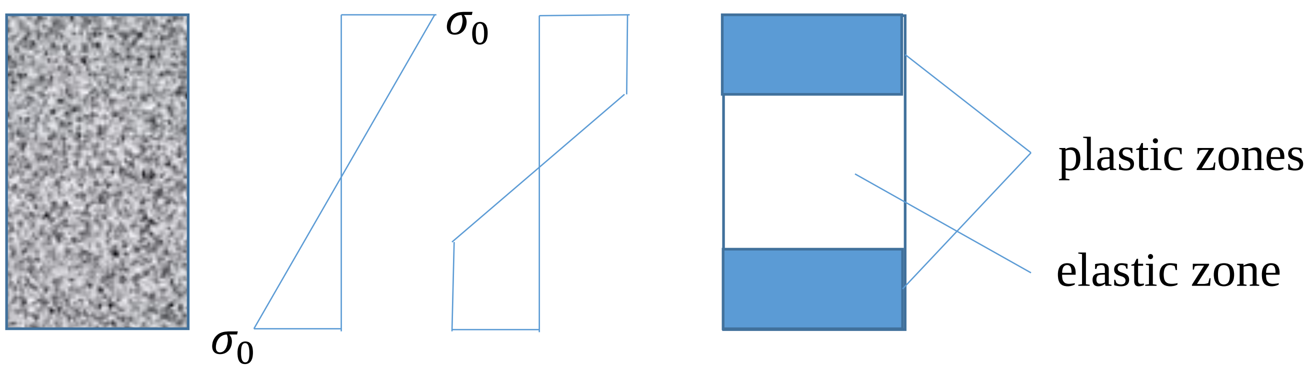
$$\sigma_z = \tau_{zx} \tan\varphi = \sigma_x \tan^2\varphi$$

Bending – residual stresses

Prandtl's model



when $M > M_{el}$ the material response is partially elastic and partially plastic

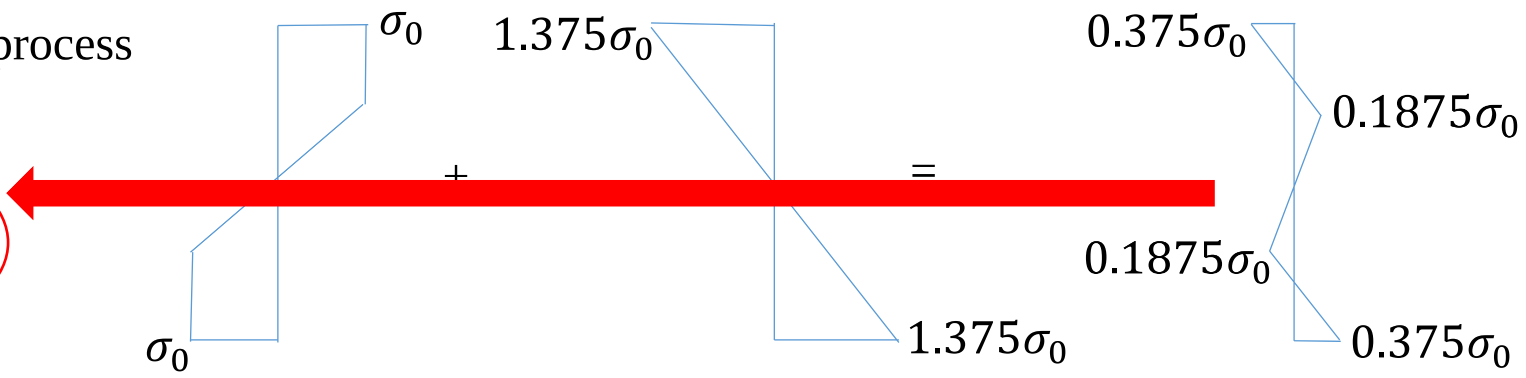


max elastic bending moment: $\frac{M_{el}}{W_y} = \sigma_0 \rightarrow M_{el} = \sigma_0 W_y = \frac{bh^2}{6}\sigma_0$

$$M_{pl} = \frac{bh^2}{4}\sigma_0 = 1.5M_{el}$$

when plastic zone area is half of the c-s: $M_{0.5} = 2 \cdot \left(\frac{bh}{4} \cdot \frac{3}{4} \cdot \frac{h}{2} \cdot \sigma_0 + \frac{1}{2} \cdot \frac{bh}{4} \cdot \frac{2}{3} \cdot \frac{h}{4} \cdot \sigma_0\right) = \frac{22}{96} \cdot bh^2 = 1.375M_{el}$

unloading is an elastic process



residual stresses:
self-equilibrated set
of internal forces

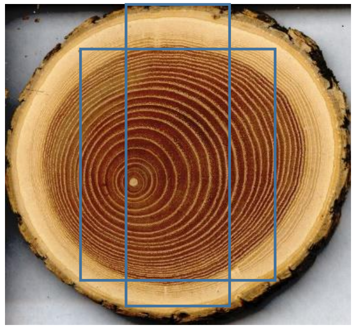
Bending – possible problems

basic formulae: $\sigma_x = \frac{M_y}{I_y} z$, $\max|\sigma_x| < R$, $\kappa = \frac{M_y}{EI_y}$

1) stress distribution (given: bending moment, cross-section inertia moment; adopt *z* axis)
2) stress at extreme fibers (given: bending moment, c-s inertia moment; adopt *z* axis)
3) bearing capacity (given: material *R*, c-s inertia moment; calculate M_y)
4) cross-section dimensioning (given: material *R*, bending moment M_y; calculate inertia moment needed)
    1) profile from the catalogue
    2) characteristic parameter of the cross-section geometry
    3) just the value of the inertia moment needed
5) curvature (given: bending moment, bending stiffness)
6) bending stiffness (given: curvature, bending moment)
7) material (very rarely, given: bending moment, cross-section geometry)

Bending – examples

(da Silva)



A tree trunk with circular cross-section of diameter d is to be cut to a rectangular cross-section of base b and height h. Determine the dimensions b and h, in order to maximize: a) the bending stiffness, b) the bending strength.

Solution

a) The bending stiffness depends on the moment of inertia. To maximize the stiffness, the inertia moment should be maximized. Because: $h^2 + b^2 = d^2$, we get: $I = \frac{bh^3}{12} = \frac{b\left(b^2 - d^2\right)^{\frac{3}{2}}}{12}$, the extreme is: $\frac{dI}{db} = 0 \rightarrow d^2 - 4b^2 = 0 \rightarrow b = \frac{d}{2} \rightarrow h = \frac{\sqrt{3}}{2} d \rightarrow \frac{h}{b} = \sqrt{3} \approx 1.732$

b) To obtain the maximum bending strength, the section modulus must be maximized. $W_y = \frac{bh^2}{6} = \frac{b\left(d^2 - b^2\right)}{6}$, and $\frac{dW_y}{db} = 0 \rightarrow d^2 - 3b^2 = 0 \rightarrow b = \frac{d}{\sqrt{3}} \rightarrow h = \frac{\sqrt{2}}{\sqrt{3}} d \rightarrow \frac{h}{b} = \sqrt{2} \approx 1.414$

Bending – examples cont.

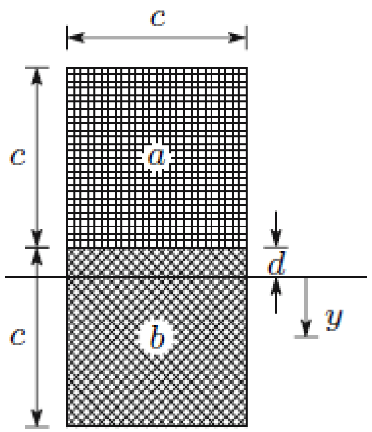
(da Silva)

The prismatic bar with the cross-section made of two materials, *a* and *b*, undergoes a uniform temperature increase Δt. The materials have linear elastic behavior defined by the parameters:

$$E_a = E, E_b = 2E, \alpha_a = \alpha, \alpha_b = 2\alpha.$$

a) Determine the elongation and the curvature introduced by Δt (the bar has length *l*)

b) Determine the distribution of stresses in the cross-section.



Bending – examples cont.

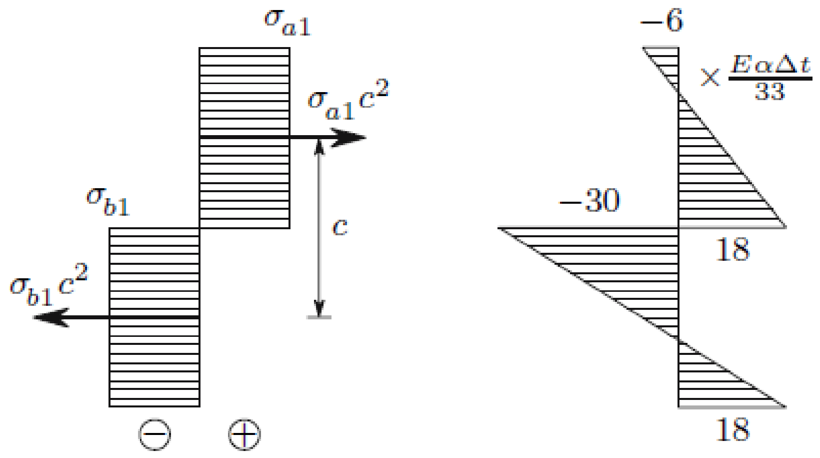
Outline of solution

In the problem, the conditions of symmetry haven't been violated, so the Bernoulli's hypothesis is still valid. We can put: $\varepsilon = \varepsilon_0 + \kappa z$. On the other hand, we have: $\varepsilon = \frac{\sigma_x}{E} + \alpha \Delta t$, so, we arrive at the set of two linear equations

$$\varepsilon = \frac{\sigma_a}{E_a} + \alpha_a \Delta t = \frac{\sigma_b}{E_b} + \alpha_b \Delta t$$

$$N = \sigma_a A_a + \sigma_b A_b = 0$$

with two unknowns σ_a and σ_b.



That’s all, folks!